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A matrix approach to generalized Bernoulli–Fibonacci polynomials of order *m* and applications

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Abstract
We know that the matrices provide a flexible framework to study combinatorial structures. In fact, the generalized Fibonacci matrices allow us to develop the applications to coding theory. In the beginning of this work, a new family of generalized Bernoulli–Fibonacci polynomials of order m is introduced followed by investigating various properties associated with this polynomial class, as well as its relationships with other polynomial families and numbers. These include explicit relations, difference equations, summation formulae, linear and differential recurrence relations. Furthermore, we focus on matrix approach associated with this family by providing the generalized Fibo–Bernoulli polynomials matrix, Fibo–Pascal polynomial matrix and other important matrices. Some product and inverse formulae for the generalized Fibo–Bernoulli polynomials matrix involving other matrices are also derived at the end.

Keywords Generalized Bernoulli–Fibonacci polynomials of order m · Generalized Pascal matrix · Fibonacci matrix · Fibo–Pascal matrix and Fibo–Bernoulli matrix

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1 Introduction

Many mathematicians have recently explored and studied various forms of matrices and their analogs, which are obtained by using numbers and polynomials such as the Pascal, Bernoulli, Euler, q–Bernoulli, and q-Euler *et cetera*, for this see [4, 5, 12, 13, 15, 16, 23, 24, 26, 27]. The matrix representations of various numbers and polynomials offer a powerful tool to obtain new or classical identities. In particular, the Pascal type matrices have been used to obtain some new and interesting combinatorial identities involving Fibonacci and Lucas sequences. In this study, we are interested in matrices whose entries are the Bernoulli–Fibonacci numbers and Bernoulli–Fibonacci polynomials, which involves the use of Fibonacci number sequence F_n.

Fibonacci numbers appear unexpectedly often in mathematics. Applications of Fibonacci numbers include computer algorithms such as the Fibonacci search technique and the Fibonacci heap data structure, and graphs called Fibonacci cubes used for interconnecting parallel and distributed systems. They also appear in biological settings, such as branching in trees, the arrangement of leaves on a stem, the fruit sprouts of a pineapple, the flowering of an artichoke, and the arrangement of a pine cone's bracts, though they do not occur in all species. Fibonacci numbers are also strongly related to the golden ratio. Fibonacci numbers are also closely related to Lucas numbers, which obey the same recurrence relation and with the Fibonacci numbers form a complementary pair of Lucas sequences. The Fibonacci sequence is one of the simplest and earliest known sequences defined by a recurrence relation, and specifically by a linear difference equation. We provide the following mathematical notations and some basic definitions [10, 11].

The Fibonacci sequence $F_{n\geq 0}$ is defined by (see, [22, p. 1]):

$$F_n = \begin{cases} F_{n+2} = F_{n+1} + F_n, \\ F_0 = 0, \quad F_1 = 1. \end{cases}$$

The F-factorial is given by

$$F_n! = F_n F_{n-1} F_{n-2} \cdots F_1, \qquad F_0! = 1.$$

The Fibonomial coefficients are defined as (cf. [22, p. 2]):

$$\binom{n}{k}_F = \frac{F_n!}{F_{n-k}!F_k!}, \quad n \geq k \geq 1 \quad \text{and} \quad \binom{n}{k}_F = 0 \quad \text{for} \quad n < k,$$

which satisfy the following properties:

$$\binom{n}{k}_F = \binom{n}{n-k}_F$$

and

$$\binom{n}{k}_F \binom{k}{j}_F = \binom{n}{j}_F \binom{n-j}{k-j}_F.$$

The binomial theorem for the F-analog is given by (see, [22, p. 2 Eq. (1)]):

$$(x \ +_F \ y)^n = \sum_{k=0}^{n} \binom{n}{k}_F x^k y^{n-k}.$$

The first and second type F-exponential functions e_F^t and E_F^t are defined as (see [22, p. 2]):

$$e_F^t = \sum_{n=0}^{\infty} \frac{t^n}{F_n!} \qquad E_F^t = \sum_{n=0}^{\infty} (-1)^{n(n-1)/2} \frac{t^n}{F_n!}, \tag{1}$$

with

$$e_{-F}^t \equiv E_F^t.$$

The Golden derivative operator D_F^x acts on any arbitrary function $f(x)$ is given by

$$D_F^x[f(x)] = \frac{f(\varphi x) - f(\bar{\varphi} x)}{(\varphi - \bar{\varphi})x},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$ are two conjugate roots of Fibonacci sequence $F_n = \lambda^n$ such that

$$F_n := \frac{\varphi^n - \bar{\varphi}^n}{(\varphi - \bar{\varphi})}.$$

In view of this, we have

$$D_F^x[x^n] = \frac{(\varphi x)^n - (\bar{\varphi} x)^n}{(\varphi - \bar{\varphi})x} = \frac{(\varphi)^n - (\bar{\varphi})^n}{(\varphi - \bar{\varphi})} x^{n-1} := F_n \; x^{n-1}$$

and

$$D_F^x[e^x] = \sum_{n=0}^{\infty} F_{n+1} \frac{x^n}{(n+1)!}.$$

The Golden derivative of first and second type Golden exponential functions are given as

$$D_F^x[e_F^{kx}] = k \; e_F^{kx} \qquad D_F^x[E_F^{kx}] = k \; E_F^{-kx}.$$

The Golden Leibnitz rule is given as

$$\begin{aligned} D_F^x[f(x)g(x)] &= D_F^x(f(x))g(\varphi x) + f(\bar{\varphi} x) D_F^x(g(x)) \\ &= D_F^x(f(x))g(\bar{\varphi} x) + f(\varphi x) D_F^x(g(x)). \end{aligned}$$

In [25], a new family of generalized Bernoulli polynomials of order m, $R_n^{(m)}(x)$ is introduced and their properties are studied. For the parameters $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, the generalized Bernoulli polynomials $R_n^{(m)}(x)$ of order m are generated by the function

$$\left(\frac{z}{2}\right)^m \left(\frac{z}{e^z - 1}\right)^m = \left(\frac{z^2}{2e^z - 2}\right)^m e^{xz} = \sum_{n=0}^{\infty} R_n^{(m)}(x) \frac{z^n}{n!}, \quad |z| < 2\pi$$

and

$$R_n(x) := R_n^{(1)}(x),$$

where $R_n(x)$ are generalized Bernoulli polynomials.

For broad information on old literature and new research trends about these classes of polynomials and for the matrix approach to other classes of special polynomials, we recommend to the interested reader (see [1–3, 6–9, 14, 18–21]).

Recently, researchers have shown their interest to obtain important and interesting results concerning with the F-polynomials and their analogs, which involves the Fibonacci numbers and their associated matrices [22, 24]. This is a remarkable step towards extracting helpful results in matrix theory related with the special polynomials. In [22], the n-th Bernoulli–Fibonacci (or Bernoulli–F) polynomials $B_n^F(x)$ are introduced and their connections with the n-th Euler-Fibonacci polynomials $E_n^F(x)$ are established.

For all $n \in \mathbb{N}_0$, the nth degree Bernoulli–Fibonacci polynomials $B_n^F(x)$ are defined by the exponential generating function

$$\left(\frac{z}{e_F^z - 1}\right) e_F^{zx} = \sum_{n=0}^{\infty} B_n^F(x) \frac{z^n}{F_n!}, \quad |z| < \frac{2\pi}{\ln |e_F|},$$

where $B_n^F := B_n^F(0)$ are the n-th Bernoulli–Fibonacci numbers.

Motivated by the previous works on matrix approach of the polynomials, in this article, we focus on introducing the generalized Bernoulli–Fibonacci polynomials and associated matrices. Certain properties comprising explicit and recurrence relations, difference equations, summation formulae are derived for these polynomials. The generalized Fibo–Bernoulli polynomials matrix is established and some product formulae are obtained involving Fibo–Pascal polynomial matrix and other matrices. Inverse formula for the generalized Fibo–Bernoulli polynomials matrix is also provided.

2 The generalized Bernoulli–Fibonacci polynomials $R_n^{(m)}(x; F)$ of order m

In this section, we introduce the generalized Bernoulli–Fibonacci polynomials and establish some properties related to these polynomials.

Definition 2.1 For the parameters $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, the generalized Bernoulli–Fibonacci polynomials $R_n^{(m)}(x; F)$ of order m are defined by the following exponential generating function:

$$\left(\frac{z}{2}\right)^m \left(\frac{z}{e_F^z - 1}\right)^m = \left(\frac{z^2}{2e_F^z - 2}\right)^m e_F^{xz} = \sum_{n=0}^{\infty} R_n^{(m)}(x; F) \frac{z^n}{F_n!}, \qquad |z| < \frac{2\pi}{\ln |e_F|}, \tag{2}$$

where $R_n^{(m)}(F) := R_n^{(m)}(0; F)$ are generalized Bernoulli–Fibonacci numbers defined by the following exponential generating function:

$$\left(\frac{z^2}{2e_F^z - 2}\right)^m = \sum_{n=0}^{\infty} R_n^{(m)}(F) \frac{z^n}{F_n!}, \qquad |z| < \frac{2\pi}{\ln |e_F|}.$$

Lemma 2.1 *Let $R_n^{(m)}(x; F)$ be the sequence of generalized Bernoulli–Fibonacci polynomials of order m. Then the following relation holds for every $n \in \mathbb{N}_0$:*

$$R_n^{(0)}(x; F) = x^n. \tag{3}$$

Proof Taking $m = 0$ in (2), we obtain

$$\begin{aligned} e_F^{xz} &= \sum_{n=0}^{\infty} R_n^{(0)}(x; F)\frac{z^n}{F_n!}. \\ \sum_{n=0}^{\infty} \frac{(xz)^n}{F_n!} &= \sum_{n=0}^{\infty} R_n^{(0)}(x; F)\frac{z^n}{F_n!} \\ \sum_{n=0}^{\infty} x^n \frac{z^n}{F_n!} &= \sum_{n=0}^{\infty} R_n^{(0)}(x; F)\frac{z^n}{F_n!}. \end{aligned}$$

□

Comparing the coefficients of z^n, we get result (3).

Lemma 2.2 *The sequence of generalized Bernoulli–Fibonacci polynomials $R_n^{(m)}(x; F)$ of order m satisfy the following relation for every $n \in \mathbb{N}$:*

$$D_F^x(R_n^{(m)}(x; F)) = F_n R_{n-1}^{(m)}(x; F), \quad n \in \mathbb{N}. \tag{4}$$

Proof Operating D_F^x on both sides of equation (2), we get

$$\begin{aligned} D_F^x \left\{ \sum_{n=0}^{\infty} R_n^{(m)}(x; F)\frac{z^n}{F_n!} \right\} &= D_F^x \left\{ \left(\frac{z^2}{2e_F^z - 2} \right)^m e_F^{zx} \right\} = z \left(\frac{z^2}{2e_F^z - 2} \right)^m e_F^{zx} \\ &= \sum_{n=0}^{\infty} R_n^{(m)}(x; F)\frac{z^{n+1}}{F_n!} \\ &= \sum_{n=0}^{\infty} R_{n-1}^{(m)}(x; F)\frac{z^n}{F_{n-1}!} \\ &= \sum_{n=0}^{\infty} F_n \, R_{n-1}^{(m)}(x; F)\frac{z^n}{F_n!}, \end{aligned}$$

which on equating the coefficients of z^n yields assertion (4). □

Theorem 2.1 *For the sequence of generalized Bernoulli–Fibonacci polynomials $R_n^{(m)}(x, F)_{n \geq 0}$ of order m, the following relation is satisfied:*

$$R_n^{(m+p)}(x + y; F) = \sum_{k=0}^{n} \binom{n}{k}_F R_k^{(m)}(x; F) R_{n-k}^{(p)}(y; F). \tag{5}$$

Proof In view of equation (2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} R_n^{(m+p)}(x+y;F)\frac{z^n}{F_n!} &= \left(\frac{z^2}{2e_F^z-2}\right)^{(m+p)} e_F^{(x+y)z}\\ &= \sum_{n=0}^{\infty} R_n^{(m)}(x;F)\frac{z^n}{F_n!}\sum_{n=0}^{\infty} R_n^{(p)}(y;F)\frac{z^n}{F_n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} R_k^{(m)}(x;F)\frac{z^k}{F_k!}R_{n-k}^{(p)}(y;F)\frac{z^{n-k}}{F_{n-k}!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_F R_k^{(m)}(x;F)R_{n-k}^{(p)}(y;F)\frac{z^n}{F_n!}.\end{aligned}$$

□

Corollary 2.1 *Let $R_n^{(m)}(x;F)$ be the sequence of generalized Bernoulli–Fibonacci polynomials of order m. Then the following summation formulae hold true:*

$$R_n^{(m)}(x+y;F) = \sum_{k=0}^{n}\binom{n}{k}_F R_k^{(m)}(y;F)x^{n-k}, \tag{6}$$

$$R_n^{(p)}(x+y;F) = \sum_{k=0}^{n}\binom{n}{k}_F R_{n-k}^{(p)}(y;F)x^{n}, \tag{7}$$

$$R_n(x+y;F) = \sum_{k=0}^{n}\binom{n}{k}_F R_k(y;F)x^{n-k}, \tag{8}$$

$$R_n(x;F) = \sum_{k=0}^{n}\binom{n}{k}_F R_{n-k}(F)x^{n}. \tag{9}$$

Proof Let $p=0$ in (5), we find

$$R_n^{(m)}(x+y;F) = \sum_{k=0}^{n}\binom{n}{k}_F R_k^{(m)}(x;F)R_{n-k}^{(0)}(y;F).$$

Exchanging x for y and using $R_{n-k}^{(0)}(x;F)=x^{n-k}$, we get assertion (6). Other parts can be proved similarly by making simple substitutions. Thus we omit. □

For the parameter $m=1$, we deduce the following:

Definition 2.2 The generalized Bernoulli–Fibonacci polynomials $R_n(x;F)$ in variable x are defined by the following generating function:

$$\left(\frac{z^2}{2e_F^z-2}\right)e_F^{zx} = \sum_{n=0}^{\infty} R_n(x;F)\frac{z^n}{F_n!},\quad |z|<\frac{2\pi}{\ln|e_F|}. \tag{10}$$

Thus, we have the so-called generalized Bernoulli–Fibonacci numbers $R_n(F)$ generated by

$$\left(\frac{z^2}{2e_F^z-2}\right) = \sum_{n=0}^{\infty} R_n(F)\frac{z^n}{F_n!},\quad |z|<\frac{2\pi}{\ln|e_F|} \tag{11}$$

and satisfy

$$R_n(F) = -\sum_{k=1}^{n-1} \binom{n}{k}_F R_k(F), \qquad R_1(F) = \frac{1}{2}.$$

Theorem 2.2 *Let $\{R_n(x;F)\}_{n\geq 0}$ be the sequence of the generalized Bernoulli–Fibonacci polynomials. Then the following are satisfied:*

(a) *Explicit formula:*

$$R_n(x;F) = \sum_{k=0}^{n} \binom{n}{k}_F R_k(F)x^{n-k}, \quad R_0(F) = 0.$$

(b) *Difference formula:*

$$R_n(x+1;F) - R_n(x;F) = \sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1} \binom{n-k-1}{l}_F \frac{R_l(F)\, x^{n-k-l-1} F_n!}{F_{n-k-1}!\, F_{k+1}!}.$$

(c) *Recurrence formula for $R_n(F)$:*

$$\sum_{k=0}^{n} \binom{n+1}{k}_F R_k(F) = 0, \quad n \geq 3; \;\; R_0(F) = 0, \; R_1(F) = \frac{1}{2}, \; R_2(F) = -\frac{1}{2}.$$

Proof (a) Using expansions (1) and (11) in l.h.s of generating function (10), we have

$$\left(\frac{z^2}{2e_F^z - 2}\right) e_F^{xz} = \sum_{n=0}^{\infty} R_n(F)\frac{z^n}{F_n!} \cdot \sum_{n=0}^{\infty} x^n \frac{z^n}{F_n!} = \sum_{n=0}^{\infty} R_n(x;F)\frac{z^n}{F_n!},$$

which on applying the Cauchy product gives

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} R_k(F)x^{n-k}\frac{z^k}{F_k!}\frac{z^{n-k}}{F_{n-k}!} = \sum_{n=0}^{\infty} R_n(x;F)\frac{z^n}{F_n!}.$$

On equating the coefficients of z^n yields assertion (a).
(b) We know that

$$\begin{aligned}
\sum_{n=0}^{\infty}(R_n(x+1;F) - R_n(x;F))\frac{z^n}{F_n!} &= \left(\frac{z^2}{2e_F^z - 2}\right) e_F^{zx} \left(\sum_{n=0}^{\infty} \frac{z^n}{F_n!} - 1\right) \\
&= \left(\sum_{n=0}^{\infty} R_n(x;F)\frac{z^n}{F_n!}\right)\left(\sum_{k=0}^{\infty} \frac{z^{k+1}}{F_{k+1}!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n-1} R_{n-k-1}(x;F)\frac{z^n}{F_{n-k-1}!\, F_{k+1}!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n-1} R_{n-k-1}(x;F)\frac{z^n}{F_{n-k-1}!\, F_{k+1}!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1} \binom{n-k-1}{l}_F \frac{R_l(F)\, x^{n-k-l-1} z^n}{F_{n-k-1}!\, F_{k+1}!}.
\end{aligned}$$

On equating the coefficients of z^n yields assertion (b).

(c) Consider equation (11) such that

$$\begin{aligned}
z^2 &= 2e_F^z \sum_{n=0}^{\infty} R_n(F)\frac{z^n}{F_n!} - 2\sum_{n=0}^{\infty} R_n(F)\frac{z^n}{F_n!} \\
&= 2\sum_{n=0}^{\infty} \frac{z^n}{F_n!} \sum_{k=0}^{\infty} R_k(F)\frac{z^k}{F_k!} - 2\sum_{n=0}^{\infty} R_n(F)\frac{z^n}{F_n!} \\
&= 2\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}_F R_k(F) - R_n(F) \right) \frac{z^n}{F_n!} \\
z^2 &= 2\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n+1}{k}_F R_k(F) \right) \frac{z^n}{F_n!},
\end{aligned}$$

which on equating the terms of z yields assertion (c). □

We provide the first few expressions for the generalized Bernoulli–Fibonacci polynomials $R_n(x;F)$ as follows:

$$\begin{aligned}
R_0(x;F) &= 0, \\
R_1(x;F) &= \frac{1}{F_{n+2}!} = \frac{1}{2}, \\
R_2(x;F) &= x\frac{F_2}{F_3F_1!} - \frac{F_2}{F_3F_2!} = \frac{x}{2} - \frac{1}{2}, \\
R_3(x;F) &= x^2\frac{F_3}{F_3} - x\frac{F_3}{(F_3F_1!)} + \frac{F_3}{F_2!F_3} - \frac{F_3}{F_3F_3} = x^2 - x + \frac{1}{2}, \\
R_4(x;F) &= x^3\frac{F_4}{F_3} - x^2\left(\frac{F_4F_3!}{F_3F_2!F_2!}\right) + x\left(\frac{F_4F_3!}{(F_3F_1!F_2!F_2!)} - \frac{F_4}{F_3F_1!}\right) \\
&\quad + \left(2\frac{F_4}{F_3F_2!} - \frac{F_4}{F_3F_4} - \frac{F_4F_3}{F_3}\right) \\
&= \frac{3}{2}x^3 - 3x^2 + \frac{3}{2}x - \frac{1}{2}.
\end{aligned}$$

Theorem 2.3 *Let $R_n^{(m)}(x;F)$ be the sequence of generalized Bernoulli–Fibonacci polynomials of order m. Then the following recurrence formula hold true:*

$$\left(1 - \frac{2m}{F_{n+1}}\right) R_{n+1}^{(m)}(x;F) = xR_n^{(m)}(x;F) - 2\sum_{s=0}^{n+2} \binom{n}{s}_F \frac{R_s^{(m)}(x;F)R_{n-s+2}(x;F)}{F_{n-s+2}\,F_{n-s+1}}. \tag{12}$$

Proof Applying D_F^z on both sides of generating function (2), we find

$$D_F^z\left\{\sum_{n=0}^{\infty} R_n^{(m)}(x;F)\frac{z^n}{F_n!}\right\} = D_F^z\left\{\left(\frac{z^2}{2e_F^z-2}\right)^m e_F^{xz}\right\}$$

$$\sum_{n=0}^{\infty} R_{n+1}^{(m)}(x;F)\frac{z^n}{F_n!} = \left(\frac{z^2}{2e_F^z-2}\right)^m e_F^{xz}\left\{x+\frac{2m}{z}-2\frac{z^2e_F^{xz}}{2e_F^z-2}\frac{1}{z^2}\right\}$$

$$= \sum_{n=0}^{\infty} R_n^{(m)}(x;F)\frac{z^n}{F_n!}\left\{x+2mz^{-1}-2\sum_{n=0}^{\infty} R_n(x;F)\frac{z^{n-2}}{F_n!}\right\}$$

$$= x\sum_{n=0}^{\infty} R_n^{(m)}(x;F)\frac{z^n}{F_n!}+2m\sum_{n=0}^{\infty} R_n^{(m)}(x;F)\frac{z^{n-1}}{F_n!}$$

$$-2\sum_{n=0}^{\infty}\sum_{s=0}^{\infty} R_s^{(m)}(x;F)R_n(x;F)\frac{z^{n+s-2}}{F_n!\,F_s!}$$

$$\sum_{n=0}^{\infty} R_{n+1}^{(m)}(x;F)\frac{z^n}{F_n!} = x\sum_{n=0}^{\infty} R_n^{(m)}(x;F)\frac{z^n}{F_n!}+\frac{2m}{F_{n+1}}\sum_{n=0}^{\infty} R_{n+1}^{(m)}(x;F)\frac{z^n}{F_n!}$$

$$-2\sum_{n=0}^{\infty}\sum_{s=0}^{n+2} R_s^{(m}(x;F)R_{n-s+2}(x;F)\frac{z^n}{F_{n-s+2}!\,F_s!}$$

$$\sum_{n=0}^{\infty}\left(1-\frac{2m}{F_{n+1}}\right)R_{n+1}^{(m)}(x;F)\frac{z^n}{F_n!}$$

$$= \sum_{n=0}^{\infty}\left(xR_n^{(m)}(x;F)-2\sum_{s=0}^{n+2}\frac{R_s^{(m)}(x;F)R_{n-s+2}(x;F)F_n!}{F_{n-s+2}!\,F_s!}\right)\frac{z^n}{F_n!},$$

which on simplifying and equating the coefficients of z^n, we are led to assertion (12). □

Theorem 2.4 *Let $R_n^{(m)}(x;F)$ be the sequence of generalized Bernoulli–Fibonacci polynomials of order m. Then the following recurrence formula hold true:*

$$\left(1-\frac{2m}{F_{n+1}}\right)R_{n+1}^{(m)}(x;F) = xR_n^{(m)}(x;F)-2\sum_{s=0}^{n+2}\sum_{l=0}^{n-s+2}\binom{n}{s}_F\binom{n-s+2}{l}_F \frac{R_s^{(m)}(x;F)R_l(F)x^{n-s+2-l}}{F_{n-s+2}\,F_{n-s+1}}. \tag{13}$$

Proof Recurrence formula (12) in view of explicit formulas for $R_n(x;F)$ can be expressed as (13). □

Theorem 2.5 *Let $R_n^{(m)}(x;F)$ be the sequence of generalized Bernoulli–Fibonacci polynomials of order m. Then the following recurrence formula hold true:*

$$\left(1-\frac{2m}{F_{n+1}}\right)R_{n+1}^{(m)}(x;F) = xR_n^{(m)}(x;F)-2\sum_{s=0}^{n}\sum_{p=0}^{s+2}\frac{F_n!\,R_{n-s}^{(m)}(x;F)R_{s-p+2}(F)x^p}{F_{s-p+2}!\,F_p!\,F_{n-s}!}. \tag{14}$$

Proof By making some other rearrangements of terms in equation (31), we are let to assertion (14). □

Theorem 2.6 *For the sequence of generalized Bernoulli–Fibonacci polynomials $R_n^{(m)}(x;F)$ of order m,*

$$\Theta_{n,F}^{(-1)} := \Psi_{n,F}^{-} = \frac{1}{F_n} D_F^x \tag{15}$$

and

$$\Theta_{n,F}^{(-k)} := \prod_{m=n-k+1}^{n} \Psi_{m,F}^{-} = (\Psi_{n-k+1,F}^{-} . \Psi_{n-k+2,F}^{-} \cdots \Psi_{n,F}^{-}) = \frac{F_{n-k}!}{F_n!} \, {}_{(k)}D_F^x \tag{16}$$

are the lowering and k-times lowering operators and ${}_{(k)}D_F^x$ is the k-th order Golden derivative operator [11] given by

$${}_{(k)}D_F^x[f(x)] = \frac{f(\varphi^k x) - f(\bar{\varphi}^k x)}{(\varphi^k - \bar{\varphi}^k)x}, \qquad k \in \mathbb{Z}.$$

Proof Operating ${}_{(k)}D_F^x$ on both sides of equation (2), we get

$$\begin{aligned} {}_{(k)}D_F^x \left\{ \sum_{n=0}^{\infty} R_n^{(m)}(x;F) \frac{z^n}{F_n!} \right\} &= {}_{(k)}D_F^x \left\{ \left(\frac{z^2}{2e_F^z - 2} \right)^m e_F^{zx} \right\} = z^k \left(\frac{z^2}{2e_F^z - 2} \right)^m e_F^{zx} \\ &= \sum_{n=0}^{\infty} R_{n-k}^{(m)}(x;F) \frac{z^n}{F_{n-k}!} \\ &= \sum_{n=0}^{\infty} \frac{F_n!}{F_{n-k}!} R_{n-k}^{(m)}(x;F) \frac{z^n}{F_n!} \\ {}_{(k)}D_F^x (R_n^{(m)}(x;F)) &= \frac{F_n!}{F_{n-k}!} R_{n-k}^{(m)}(x;F), \quad n, k \geq 1. \end{aligned}$$

Since the operator $\Psi_{n,F}^{-} = \frac{1}{F_n} D_F^x$ satisfies the relation $\Psi_{n,F}^{-} R_n^{(m)}(x;F) = R_{n-1}^{(m)}(x;F)$, therefore the lowering operator is given by equation (15) and

$$R_{n-k}^{(m)}(x;F) = (\Psi_{n-k+1,F}^{-} \cdots \Psi_{n,F}^{-})\{R_n^{(m)}(x;F)\} = \frac{F_{n-k}!}{F_n!} \, {}_{(k)}D_F^x \{R_n^{(m)}(x;F)\}. \tag{17}$$

This proves the demonstration. □

Theorem 2.7 *Let $R_n^{(m)}(x;F)$ be the sequence of generalized Bernoulli–Fibonacci polynomials of order m. Then the following difference equation hold true:*

$$\begin{aligned} &(2m - F_{n+1}) R_n^{(m)}(x;F) + (\bar{\varphi}x) D_F^x (R_n^{(m)}(x;F)) + \Omega((\bar{\varphi}x^{n-s+2-l})_{(n-s+1)} D_F^x (R_n^{(m)}(x;F))) \\ &+ R_n^{(m)}(\varphi x;F) + \Omega(F_{n-s-l+2} \, x^{n-s-l+1} \, {}_{(n-s)}D_F^x (R_n^{(m)}(\varphi x;F))) = 0, \end{aligned} \tag{18}$$

where

$$\Omega := -2 \sum_{s=0}^{n+2} \sum_{l=0}^{n-s+2} \binom{n}{s}_F \binom{n-s+2}{l}_F \frac{R_l(F) F_s!}{F_{n-s+2} \, F_{n-s+1} F_n!}.$$

Proof We know that

$$R_{n-1}^{(m)}(x;F) = \frac{1}{F_n} D_F^x (R_n^{(m)}(x;F)),$$

which on taking $n \to n+1$ becomes

$$R_n^{(m)}(x;F) = \frac{1}{F_{n+1}} D_F^x (R_{n+1}^{(m)}(x;F)).$$

Use of equation (13) in r.h.s. of above equation gives

$$R_n^{(m)}(x;F)=\frac{1}{F_{n+1}}\left(1-\frac{2m}{F_{n+1}}\right)^{-1}D_F^x\left\{xR_n^{(m)}(x;F)-2\sum_{s=0}^{n+2}\sum_{l=0}^{n-s+2}\binom{n}{s}_F\right.$$
$$\left.\binom{n-s+2}{l}_F\frac{R_s^{(m)}(x;F)R_l(F)x^{n-s+2-l}}{F_{n-s+2}\,F_{n-s+1}}\right\}$$
$$(F_{n+1}-2m)R_n^{(m)}(x;F)=D_F^x\left\{xR_n^{(m)}(x;F)\right\}-2\sum_{s=0}^{n+2}\sum_{l=0}^{n-s+2}\binom{n}{s}_F\binom{n-s+2}{l}_F$$
$$\frac{R_l(F)}{F_{n-s+2}\,F_{n-s+1}}D_F^x\left\{R_s^{(m)}(x;F)x^{n-s+2-l}\right\}$$
$$(F_{n+1}-2m)R_n^{(m)}(x;F)=(\bar{\varphi}x)D_F^x(R_n^{(m)}(x;F))+R_n^{(m)}(\varphi x;F)-2\sum_{s=0}^{n+2}\sum_{l=0}^{n-s+2}\binom{n}{s}_F$$
$$\binom{n-s+2}{l}_F\frac{R_l(F)}{F_{n-s+2}\,F_{n-s+1}}\left((\bar{\varphi}x^{n-s+2-l})D_F^x(R_s^{(m)}(x;F))+F_{n-s-l+2}x^{n-s-l+1}R_s^{(m)}(\varphi x;F)\right).$$

Now applying the following formula for $R_s^{(m)}(x;F)$ (obtained by taking $n-1=s$ in equation (17) with $k=1$)

$$R_s^{(m)}(x;F)=\frac{F_s!}{F_{n!}}{}_{(n-s)}D_F^x(R_n^{(m)}(x;F)).$$

We have

$$(F_{n+1}-2m)R_n^{(m)}(x;F)=(\bar{\varphi}x)D_F^x(R_n^{(m)}(x;F))+R_n^{(m)}(\varphi x;F)-2\sum_{s=0}^{n+2}\sum_{l=0}^{n-s+2}\binom{n}{s}_F$$
$$\binom{n-s+2}{l}_F\frac{R_l(F)F_s!}{F_{n-s+2}\,F_{n-s+1}F_n!}\left((\bar{\varphi}x^{n-s+2-l})_{(n-s+1)}D_F^x(R_n^{(m)}(x;F))\right.$$
$$\left.+F_{n-s-l+2}\,x^{n-s-l+1}\;{}_{(n-s)}D_F^x(R_n^{(m)}(\varphi x;F))\right).$$

This completes assertion (18). □

In the next section, we introduce the matrices associated with the generalized Bernoulli–Fibonacci polynomials $R_n^{(m)}(x;F)$.

3 The generalized Fibo–Bernoulli polynomials matrix $[r_{ij}^{(m)}(x;F)]$

Here, we establish the matrix associated with the generalized Bernoulli–Fibonacci polynomials $R_n^{(m)}(x;F)$, called as the generalized Fibo–Bernoulli polynomials matrix. The properties to be derived for the Fibo–Bernoulli polynomials matrix require the use of several other matrices such as Fibo–Pascal matrix, Fibo–Lucas matrix *et cetera*. Let us first recall the following basic matrices:

Let $M_{n+1}(\mathbb{R})$ be the set of all $(n+1)$-square matrices over the real field. Also, for any nonnegative integers i, j, we have

$$\binom{i}{j}=0,\ \text{whenever}\ j>i.$$

Let x be any nonzero real number. The generalized Pascal matrix of first kind $P_n[x]$, is an $(n+1)\times(n+1)$ matrix whose entries are given by (see, [13, Definition 1]):

$$p_{i,j}(x) := \begin{cases} \binom{i}{j}x^{i-j}, & i \geq j, \\ 0, & \text{otherwise}. \end{cases}$$

The Fibonacci matrix $\mathscr{F} = [f_{i,j}]$ $(i, j = 0, 1, 2, ..., n)$ is an $(n+1)\times(n+1)$ matrix whose entries are given by (see, [26, Eq. (16)]):

$$f_{i,j} := \begin{cases} F_{i-j+1}, & if \quad i-j+1 \geq 0, \\ 0, & if \quad i-j+1 < 0, \end{cases}$$

where F_n be the n-th Fibonacci number.

The Lucas matrix $\mathscr{L} = [l_{i,j}]$ is an $(n+1)\times(n+1)$ matrix whose entries are given by (see, [27, Eq. (2)]):

$$l_{i,j} := \begin{cases} L_{i-j+1}, & if \quad i-j \geq 0, \\ 0, & if \quad i-j < 0. \end{cases}$$

where L_n be the n-th Lucas number such that $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$ with initial conditions $L_1 = 1$ and $L_2 = 3$.

For broad information on old literature and new research trends about these classes of matrices, we strongly recommend to the interested reader (see, [4, 13, 26, 27]). Now, we provide the definition for the new family of generalized Fibo–Bernoulli polynomaisl matrix and other matrices.

Definition 3.1 Let $R_n^{(m)}(x, F)$ be the generalized Bernoulli–Fibonacci polynomials. Then, the associated $(n+1)\times(n+1)$ generalized Fibo–Bernoulli polynomials matrix, $\mathscr{R}_n^{(m)}(x; F) = [r_{ij}^{(m)}(x; F)]$; $i, j = 0, 1, 2, ..., n$ is defined as follows:

$$r_{ij}^{(m)}(x; F) = \begin{cases} \dfrac{\binom{i+1}{j+1}_F}{F_m!\binom{i-j+m}{m}_F} R_{i-j+m}^{(m)}(x; F), & i \geq j, \\ 0, & \text{otherwise}. \end{cases}$$

For $m = 1$, $\mathscr{R}_n^{(1)}(x; F) := \mathscr{R}_n(x; F)$ are called the Fibo–Bernoulli polynomials matrix and $\mathscr{R}_n(0; F) = \mathscr{R}_n(F)$ is the Fibo–Bernoulli number matrix.

For a particular choice of $n = 3$. It follows from Definition 3.1 that $\mathscr{R}_3(x; F)$ is give by

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2}x - \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ x^2 - x + \frac{1}{2} & x - 1 & \frac{1}{2} & 0 \\ \frac{3}{2}x^3 - 3x^2 + \frac{3}{2}x - \frac{1}{2} & 3x^2 - 3x + \frac{3}{2} & \frac{3}{2}x - \frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

Definition 3.2 The $(n+1)\times(n+1)$ Fibo–Pascal polynomial matrix, $\mathscr{P}_n[x](F) = [p_{ij}(x;F)]$ $0 \le i, j \le n$ is defined by

$$p_{i,j}(x;F) = \begin{cases} \binom{i+1}{j+1}_F x^{i-j}, & i \ge j, \\ \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.3 Let $(n+1)\times(n+1)$ be the matrix $\mathfrak{E}_n(F) = [e_{ij}(F)],\ \ 0 \le i, j \le n$ whose entries are given by

$$e_{i,j}(F) = \begin{cases} \dfrac{2}{F_{i-j+1}}\binom{i+1}{j+1}_F, & i \ge j, \\ \\ 0, & \text{otherwise,} \end{cases}$$

where F_n are the Fibonacci numbers.

Theorem 3.1 *The Fibo–Bernoulli polynomials matrix, $\mathscr{R}_n(x;F)$ satisfies the following product formulae:*

$$\mathscr{R}_n(x+y;F) = \mathscr{P}_n[x](F)\mathscr{R}_n(y;F) = \mathscr{P}_n[y](F)\mathscr{R}_n(x;F). \tag{19}$$

Particularly,

$$\mathscr{R}_n(x;F) = \mathscr{P}_n[x](F)\mathscr{R}_n(F). \tag{20}$$

Proof By use of Definition 3.1, we have

$$\begin{aligned}
\mathscr{R}_n(x+y;F) &= \binom{i}{j} R_{i-j}(x+y;F) \\
&= \sum_{k=j}^{i} \binom{i+1}{k+1}_F x^{i-k} \frac{\binom{k+1}{j+1}_F}{F_{k-j+1}} R_{k-j+1}(y;F) \\
&= \sum_{k=j}^{i} \frac{\binom{i+1}{j+1}_F \binom{i-j}{k-j}_F}{F_{k-j+1}} x^{i-k} R_{k-j+1}(y;F) \\
&= \frac{\binom{i+1}{j+1}_F}{F_{i-j+1}} \sum_{k=1}^{i-j+1} \frac{F_{i-j+1}}{F_k}\binom{i-j}{k-1} x^{i-j+1-k} R_k(y;F) \\
&= \frac{\binom{i+1}{j+1}_F}{F_{i-j+1}} \sum_{k=1}^{i-j+1} \frac{(F_{i-j+1})(F_{i-j})!}{(F_k)(F_{k-1})!(F_{i-j-k+1})!} x^{i-j+1-k} R_k(y;F) \\
&= \frac{\binom{i+1}{j+1}_F}{F_{i-j+1}} \sum_{k=1}^{i-j+1} \frac{(F_{i-j+1})!}{(F_k)!(F_{i-j-k+1})!} x^{i-j+1-k} R_k(y;F) \\
&= \frac{\binom{i+1}{j+1}_F}{F_{i-j+1}} \sum_{k=1}^{i-j+1} \binom{i-j+1}{k}_F x^{i-j+1-k} R_k(y;F), \\
&= \frac{\binom{i+1}{j+1}_F}{F_{i-j+1}} R_{i-j+1}(x+y;F),
\end{aligned}$$

which implies

$$\mathscr{R}_n(x+y;F)=\mathscr{P}_n[x](F)\mathscr{R}_n(y;F).$$

Similarly, it can be shown that $\mathscr{R}_n(x+y;F)=\mathscr{P}_n[y](F)\mathscr{R}_n(x;F)$.

Finally, by taking $y=0$ in (19), we obtain assertion (20), which ends the demonstration. □

Theorem 3.2 *For the generalized Fibo–Bernoulli polynomials matrix $\mathscr{R}_n^{(m)}(x;F)$, the following formulae hold:*

$$\begin{aligned}\mathscr{R}_n^{(m+p)}(x+y;F)&=\mathscr{R}_n^{(m)}(x;F)\mathscr{R}_n^{(p)}(y;F)\\&=\mathscr{R}_n^{(m)}(y;F)\mathscr{R}_n^{(p)}(x;F).\end{aligned}$$

Proof Taking $i>j$ and from Definition 3.1, we have

$$\begin{aligned}&\mathscr{R}_n^{(m+p)}(x+y;F)\\&=\left\{\sum_{k=j}^{i}\frac{\binom{i+1}{k+1}_F}{F_m!\binom{i-k+m}{m}_F}R_{i-k+m}^{(m)}(x;F)\,\frac{\binom{k+1}{j+1}_F}{F_p!\binom{k-j+p}{p}_F}R_{k-j+p}^{(p)}(y;F)\right.\\&=\binom{i+1}{j+1}_F\sum_{k=j}^{i}\frac{\binom{i-j}{k-j}_F R_{i-k+m}^{(m)}(x;F)R_{k-j+p}^{(p)}(y;F)}{F_m!\binom{i-k+m}{m}_F F_p!\binom{k-j+p}{p}_F}\\&=\frac{\binom{i+1}{j+1}_F}{F_{(m+p)}!\binom{i-j+m+p}{m+p}_F}\sum_{k=j}^{i}\frac{\binom{i-j}{k-j}_F\frac{F_{(m+p)}!F_{(i-j+m+p)}!}{F_{(i-j)}!F_{(m+p)}!}}{\frac{F_{(i-k+m)}!F_{(k-j+p)}!}{F_{(i-k)}!F_{(k-j)}!}}\\&\quad\times R_{i-k+m}^{(m)}(x;F)R_{k-j+p}^{(p)}(y;F),\end{aligned}$$

simplifying, we get

$$\begin{aligned}&=\frac{\binom{i+1}{j+1}_F}{F_{(m+p)}!\binom{i-j+m+p}{m+p}_F}\sum_{k=p}^{i-j+p}\frac{F_{(i-j+m+p)}!}{F_{(i-j+m+p-k)}!}R_{i-j+m+p-k}^{(m)}(x;F)R_k^{(p)}(y;F)\\&=\frac{\binom{i+1}{j+1}_F}{F_{(m+p)}!\binom{i-j+m+p}{m+p}_F}\sum_{k=p}^{i-j+m+p}\binom{i-j+m+p}{k}_F R_{i-j+m+p-k}^{(m)}(x;F)R_k^{(p)}(y;F)\\&=\frac{\binom{i+1}{j+1}_F}{F_{(m+p)}!\binom{i-j+m+p}{m+p}_F}\sum_{k=0}^{i-j+m+p}\binom{i-j+m+p}{k}R_{i-j+m+p-k}^{(m)}(x;F)R_k^{(p)}(y;F)\\&=\frac{\binom{i+1}{j+1}_F}{F_{(m+p)}!\binom{i-j+m+p}{m+p}_F}R_{i-j+m+p}^{(m+p)}(x+y;F),\end{aligned}$$

which proves the first equality of Theorem (3.2). The second equality can be obtained in a similar way. □

Corollary 3.1 *Let $(x_1, x_2, \cdots, x_k) \in \mathbb{R}^k$. For m_k natural numbers, the matrices $\mathscr{R}_n^{(m_j)}(x_j; F)$, $j = 1, 2, \cdots, k$ comply with the following product formula:*

$$\mathscr{R}_n^{(m_1+m_2+\cdots+m_k)}(x_1 + x_2 + \cdots + x_k; F) = \mathscr{R}_n^{(m_1)}(x_1; F)\mathscr{R}_n^{(m_2)}(x_2; F)\cdots\mathscr{R}_n^{(m_k)}(x_k; F), \tag{21}$$

particularly,

$$\mathscr{R}_n^{(km)}(kx; F) = \left[\mathscr{R}_n^{(m)}(x; F)\right]^k, \tag{22}$$

$$\mathscr{R}_n^{(k)}(kx; F) = [\mathscr{R}_n(x; F)]^k, \tag{23}$$

$$\mathscr{R}_n^{(k)}(F) = [\mathscr{R}_n(x; F)]^k. \tag{24}$$

Proof Assertion (21) can be obtain by applying induction on k.

To obtain (22), we take $m_1 = m_2 = \cdots = m_k = m$ and $x_1 = x_2 = \cdots = x_k = x$ in (21).

To prove (23), we take $m = 1$ in (22) and (24) is obtain by taking $x = 0$ in (23). □

Theorem 3.3 *The inverse matrix of the Fibo–Bernoulli matrix $\mathscr{R}_n^{(m)}(F)$ is given as follows:*

$$\mathscr{R}_n^{-1}(F) = \mathfrak{E}_n(F). \tag{25}$$

In particular,

$$\left(\mathscr{R}_n^{(k)}(F)\right)^{-1} = \mathfrak{E}_n^k(F).$$

Proof Let

$$\sum_{k=0}^{n} \frac{2}{F_{k+1}F_{n-k+1}} \binom{n}{k}_F R_{n-k+1}(F) = \delta_{n,0},$$

where $\delta_{n,0}$ is the Kronecker delta (see [17, p. 107]).

In order to prove (25), we show that $\mathscr{R}_n\mathfrak{E}_n = \mathscr{I}_n$, where $\mathscr{I}_n$ is the identity matrix of order n.

$$\begin{aligned}
(\mathscr{R}_n(F)\mathfrak{E}_n(F))_{ij} &= \sum_{k=j}^{i} \frac{\binom{i+1}{k+1}_F}{F_{(i-k+1)}} R_{i-k+1}(F) \frac{2}{F_{(k-j+1)}} \binom{k+1}{j+1}_F \\
&= \sum_{k=j}^{i} \binom{i+1}{j+1}_F \binom{i-j}{k-j}_F \frac{2R_{i-k+1}(F)}{F_{(i-k+1)}\,F_{(k-j+1)}} \\
&= \binom{i+1}{j+1}_F \sum_{k=0}^{i-j} \binom{i-j}{k-j}_F \frac{2R_{i-k+1}(F)}{F_{(i-k+1)}\,F_{(k-j+1)}} \\
&= \binom{i+1}{j+1}_F \sum_{k=0}^{i-j} \binom{i-j}{k}_F \frac{2R_{i-k-j+1}(F)}{F_{(i-k-j+1)}\,F_{(k+1)}} \\
&= \binom{i+1}{j+1}_F \sum_{k=0}^{i-j} \binom{i-j}{k}_F \frac{2R_{i-k-j+1}(F)}{F_{(k+1)}\,F_{(i-j-k+1)}} \\
&= \binom{i+1}{j+1}_F \delta_{i-j,0}.
\end{aligned}$$

This completes the demonstration. □

Now, we give the inverse matrix of the Fibo–Pascal polynomial matrix $\mathscr{P}_n(x; F)$ by the following definition:

Definition 3.4 The inverse of Fibo–Pascal polynomial matrix $\mathscr{P}_n(x; F)$ is an $(n+1)\times(n+1)$ matrix $\mathscr{P}_n^{-1}(x; F) = [\widetilde{p}_{ij}(x; F)]$; $i, j = 0, 1, 2, \ldots, n$ given as follows:

$$\widetilde{p}_{ij} = \begin{cases} 2R_{i-j+1}(F)\binom{i+1}{j+1}_F x^{i-j}, & i \geq j, \\ \\ 0, & \text{otherwise}. \end{cases}$$

Theorem 3.4 *The inverse Fibo–Bernoulli polynomial matrix $\mathscr{R}_n^{-1}(x; F)$ can be expressed as follows:*

$$\mathscr{R}_n^{-1}(x; F) = \mathscr{R}_n^{-1}(F)\mathscr{P}_n^{-1}(x; F) = \mathfrak{E}_n(F)\mathscr{P}_n^{-1}(x; F).$$

Proof In view of (20) we have

$$\mathscr{R}_n^{-1}(x; F) = \mathscr{R}_n^{-1}(F)\mathscr{P}_n^{-1}(x; F),$$

which on applying (25) gives

$$\mathscr{R}_n^{-1}(x; F) = \mathscr{R}_n^{-1}(F)\mathscr{P}_n^{-1}(x; F) = \mathfrak{E}_n(F)\mathscr{P}_n^{-1}(x; F).$$

This completes the demonstration. □

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