Bollettino dell'Unione Matematica Italiana
https://doi.org/10.1007/s40574-024-00423-w

The homotopy types of $SU(n)$-gauge groups over $\mathbb{C}P^3$

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Received: 16 April 2024 / Accepted: 16 July 2024
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Abstract
Let m and n be two positive integers such that $m \leq n$ and $n \geq 3$. In this article, by the unstable K-theory method, we will study the homotopy types of gauge groups of the principal $SU(n)$-bundles over $\mathbb{C}P^3$. Let $\mathcal{G}_{l,k}(\mathbb{C}P^3)$ be the gauge groups of the principal $SU(n)$-bundles over $\mathbb{C}P^3$, we will partially classify the homotopy types of $\mathcal{G}_{0,k}(\mathbb{C}P^3)$ by showing that if there is a homotopy equivalence $\mathcal{G}_{0,k}(\mathbb{C}P^3) \simeq \mathcal{G}_{0,k'}(\mathbb{C}P^3)$ then we have $(\frac{1}{2}(n-1)n(n+1)(n+2), k) = (\frac{1}{2}(n-1)n(n+1)(n+2), k')$, when n is odd and $(\frac{1}{4}(n-1)n(n+1)(n+2), k) = (\frac{1}{4}(n-1)n(n+1)(n+2), k')$, when n is even.

Keywords Gauge group · Homotopy type · Special unitary group · $\mathbb{C}P^3$

Mathematics Subject Classification Primary 54C35; Secondary 55P15

1 Introduction

Let G be a topological group and let M be a topological space. Let $P \to M$ be a principal G-bundle over M. The gauge group of this principal G-bundle, denote by $\mathcal{G}(P)$, is the topological group of automorphisms of P, where an automorphism of P is a G-equivariant self map of P covering the identity map of M. The main problem is to classify the homotopy types of $\mathcal{G}(P)$ as P ranges over all principal G-bundles over M for fixed G and M.

When G is a simple, simply-connected compact Lie group and M is a simply-connected closed four-manifold, then there is a one-to-one correspondence between the set of isomorphism classes of principal G-bundles over M and the homotopy set $[M, BG] \cong \mathbb{Z}$. Thus there are countably many equivalence classes of principal G-bundles over M. Each has a gauge group, so there are potentially countably many distinct gauge groups. While there are countably many inequivalent principal G-bundles, Crabb and Sutherland in [3] showed that their gauge groups have only finitely many distinct homotopy types. Let $P_k \to M$ represent the equivalence class of principal G-bundle whose second Chern class is k and $\mathcal{G}_k(M)$ be the gauge group of this principal G-bundle. In recent years there has been considerable interest

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Published online: 26 July 2024

in determining the precise number of homotopy types of these gauge groups and explicit classification results have been obtained. Let (a, b) be the their greatest common divisor of two integers a and b. When M is a spin 4-manifold, Theriault [14] showed that there is a homotopy equivalence

$$\mathcal{G}_k(M) \simeq \mathcal{G}_k(S^4) \times \prod_{i=1}^{t} \Omega^2 G,$$

where t is the second Betti number of M. Thus the homotopy type of $\mathcal{G}_k(M)$ depends on the special case $\mathcal{G}_k(S^4)$. Many cases of homotopy types of $\mathcal{G}_k(S^4)$ have been studied. When M is a non-spin 4-manifold, So [11] showed that there is a homotopy equivalence

$$\mathcal{G}_k(M) \simeq \mathcal{G}_k(\mathbb{C}P^2) \times \prod_{i=1}^{t-1} \Omega^2 G.$$

Thus the homotopy type of $\mathcal{G}_k(M)$ depends on the special case $\mathcal{G}_k(\mathbb{C}P^2)$. Only a few of the homotopy types of gauge groups over simply-connected non-spin four-manifolds have been studied, which we mention some results in the following.

- $U(n)$-gauge groups [2];
- for $G = SU(2)$, $\mathcal{G}_k(M) \simeq \mathcal{G}_{k'}(M)$ if and only if $(6, k) = (6, k')$ [7];
- if $G = SU(3)$ then an integral homotopy equivalence $\mathcal{G}_k(M) \simeq \mathcal{G}_{k'}(M)$ implies that $(12, k) = (12k')$, while $(12, k) = (12k')$ implies that there is a homotopy equivalence $\mathcal{G}_k(M) \simeq \mathcal{G}_{k'}(M)$ after localizing rationally or at any prime [13];
- for $G = Sp(2)$, if $\mathcal{G}_k(M) \simeq \mathcal{G}_{k'}(M)$ then $(20, k) = (20, k')$, and conversely, if $(20, k) = (20, k')$ then $\mathcal{G}_k(M) \simeq \mathcal{G}_{k'}(M)$ when localized rationally or at any prime [12];
- for $G = Sp(n)$, if there is a homotopy equivalence $\mathcal{G}_k(\mathbb{C}P^2) \simeq \mathcal{G}_{k'}(\mathbb{C}P^2)$ then we have $(4n(2n + 1), k) = (4n(2n + 1), k')$ [8].

So in [10] studies the homotopy types of $SU(n)$-gauge groups over non-spin 4-manifolds and shows that if $\mathcal{G}_k(\mathbb{C}P^2)$ is homotopy equivalent to $\mathcal{G}_{k'}(\mathbb{C}P^2)$, then $(\frac{1}{2}(n-1)n(n+1), k) = (\frac{1}{2}(n-1)n(n+1), k')$, if n is odd and $((n-1)n(n+1), k) = ((n-1)n(n+1), k')$, if n is even.

In this article, we will study the homotopy types of $SU(n)$-gauge groups over $\mathbb{C}P^3$ for $n > 2$. This is the first time $\mathbb{C}P^3$ gauge groups have been studied. Note that there is a one-to-one correspondence between the set of isomorphism classes of principal $SU(n)$-bundles over $\mathbb{C}P^3$ and the homotopy set $[\mathbb{C}P^3, BSU(n)] \cong \mathbb{Z} \oplus \mathbb{Z}$. One copy of $\mathbb{Z}$ corresponds to multiples of the map

$$\varepsilon_1 \colon \mathbb{C}P^3 \xrightarrow{pinch} S^6 \xrightarrow{\epsilon_1} BSU(n),$$

where ϵ_1 generates $\pi_6(BSU(n)) \cong \mathbb{Z}$. The other copy of $\mathbb{Z}$ corresponds to multiples of the map

$$\varepsilon_2 \colon \mathbb{C}P^3 \to \mathbb{C}P^3/\mathbb{C}P^1 \simeq S^4 \vee S^6 \xrightarrow{pinch} S^4 \xrightarrow{\epsilon_2} BSU(n),$$

where ϵ_2 generates $\pi_4(BSU(n)) \cong \mathbb{Z}$. Therefore the gauge groups are doubly-indexed, with $\mathcal{G}_{l,k}(\mathbb{C}P^3)$ corresponding to the principal $SU(n)$-bundle determined by the map $l\varepsilon_1 + k\varepsilon_2$. Since the classification results for $\mathcal{G}_{l,k}(\mathbb{C}P^3)$ with $l \neq 0$ are more complex, we will not study the homotopy types of $\mathcal{G}_{l,k}(\mathbb{C}P^3)$ and only consider the case $\mathcal{G}_{0,k}(\mathbb{C}P^3)$. We will partially classify the homotopy types of $\mathcal{G}_{0,k}(\mathbb{C}P^3)$ by using unstable K-theory to give a lower bound for the number of homotopy types. We will prove the following theorem.

Theorem 1.1 *Let $n > 2$, if $\mathcal{G}_{0,k}(\mathbb{C}P^3)$ is homotopy equivalent to $\mathcal{G}_{0,k'}(\mathbb{C}P^3)$ then we have*

$$\begin{cases} \left(\frac{1}{2}(n-1)n(n+1)(n+2), k\right) = \left(\frac{1}{2}(n-1)n(n+1)(n+2), k'\right) \text{ if } n \text{ is odd}, \\ \left(\frac{1}{4}(n-1)n(n+1)(n+2), k\right) = \left(\frac{1}{4}(n-1)n(n+1)(n+2), k'\right) \text{ if } n \text{ is even}. \end{cases}$$

2 Preliminaries

Let $BSU(n)$ and $B\mathcal{G}_{0,k}(\mathbb{C}P^3)$ be the classifying spaces of $SU(n)$ and $\mathcal{G}_{0,k}(\mathbb{C}P^3)$ respectively. Also, let $Map_{0,k}(\mathbb{C}P^3, BSU(n))$ and $Map^*_{0,k}(\mathbb{C}P^3, BSU(n))$ respectively be the components of the freely continuous and pointed continuous maps between $\mathbb{C}P^3$ and $BSU(n)$ containing the map ε_2. Observe that there is a fibration

$$Map^*_{0,k}(\mathbb{C}P^3, BSU(n)) \to Map_{0,k}(\mathbb{C}P^3, BSU(n)) \overset{ev}{\to} BSU(n),$$

where ev evaluates a map at the basepoint of $\mathbb{C}P^3$. By [1, 3], there is a homotopy equivalence

$$B\mathcal{G}_{0,k}(\mathbb{C}P^3) \simeq Map_{0,k}(\mathbb{C}P^3, BSU(n)).$$

The evaluation fibration therefore determines a homotopy fibration sequence

$$\mathcal{G}_{0,k}(\mathbb{C}P^3) \to SU(n) \overset{\alpha_k}{\to} Map^*_{0,k}(\mathbb{C}P^3, BSU(n)) \to B\mathcal{G}_{0,k}(\mathbb{C}P^3) \overset{ev}{\longrightarrow} BSU(n), \quad (2.1)$$

where $\alpha_k : SU(n) \to Map^*_{0,k}(\mathbb{C}P^3, BSU(n))$ is the boundary map.

In this article, we use the method in [10]. This article is organized as follows. In Sects. 3 and 4, respectively, in cases where $n-m$ is an even integer and $n-m$ is an odd integer, we first study the group $[\mathbb{C}P^m \wedge A, SU(n+1)]$, where A is the quotient $\mathbb{C}P^{n-m+2}/\mathbb{C}P^{n-m}$. Then we study the subgroup of $[\mathbb{C}P^m \wedge A, SU(n)]$ which is then used in Sect. 5 to show that if $\mathcal{G}_{0,k}(\mathbb{C}P^3) \simeq \mathcal{G}_{0,k'}(\mathbb{C}P^3)$ then $(\frac{1}{2}(n-1)n(n+1)(n+2), k) = (\frac{1}{2}(n-1)n(n+1)(n+2), k')$, when n is odd and $n \geq 3$ and $(\frac{1}{4}(n-1)n(n+1)(n+2), k) = (\frac{1}{4}(n-1)n(n+1)(n+2), k')$, when n is even and $n \geq 4$. In Sect. 5, we will prove Theorem 1.1.

3 The group $[\mathbb{C}P^m \wedge A, SU(n+1)]$ when $n-m$ is even

Let A be the quotient $\mathbb{C}P^{n-m+2}/\mathbb{C}P^{n-m}$. That is,

$$A = \begin{cases} \Sigma^{2n-2m}\mathbb{C}P^2 \simeq S^{2n-2m+2} \cup e^{2n-2m+4} & \text{if } n-m \text{ is even}, \\ S^{2n-2m+2} \vee S^{2n-2m+4} & \text{if } n-m \text{ is odd}. \end{cases}$$

Put $X = \mathbb{C}P^m \wedge A$. In this section, we first in case that $n-m$ is an even integer and $n \geq 3$ will study the group $[X, U(n+1)]$ and then obtain the order of group $[X, U(n)]$.

Denote the symmetric space $U(\infty)/U(n+1)$ by W_{n+1}. Recall that as an algebra

$$H^*(U(\infty); \mathbb{Z}) = \bigwedge(x_1, x_3, \ldots),$$
$$H^*(BU(\infty); \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \ldots],$$
$$H^*(U(n+1); \mathbb{Z}) = \bigwedge(x_1, x_3, \ldots, x_{2n+1}),$$

where c_i is the i-th universal Chern class and $x_{2i+1} = \sigma c_i$, σ is the cohomology suspension and x_{2i+1} has degree $2i+1$. Consider the projection $\pi: U(\infty) \to W_{n+1}$. As an algebra we have that the cohomology of W_{n+1} is given by

$$H^*(W_{n+1}; \mathbb{Z}) = \bigwedge(\bar{x}_{2n+3}, \bar{x}_{2n+5}, \ldots),$$

where $\pi^*(\bar{x}_{2i+1}) = x_{2i+1}$. Consider the following fibre sequence

$$\Omega U(\infty) \xrightarrow{\Omega\pi} \Omega W_{n+1} \xrightarrow{\delta} U(n+1) \xrightarrow{j} U(\infty) \xrightarrow{\pi} W_{n+1}. \tag{3.1}$$

Applying the functor $[X, -]$ to fibration (3.1), there is an exact sequence as follows

$$[X, \Omega U(\infty)] \xrightarrow{(\Omega\pi)_*} [X, \Omega W_{n+1}] \xrightarrow{\delta_*} [X, U(n+1)] \xrightarrow{j_*} [X, U(\infty)] \xrightarrow{\pi_*} [X, W_{n+1}].$$

Since W_{n+1} is $(2n+2)$-connected, for $i \leq 2n+2$ we have $\pi_i(W_{n+1}) \cong 0$. By the homotopy sequence of the fibration (3.1), we have $\pi_{2n+3}(W_{n+1}) \cong \mathbb{Z}$ and also

$$\pi_{2n+4}(W_{n+1}) \cong \begin{cases} 0 & \text{if } n \text{ is even,} \\ \mathbb{Z}_2 & \text{if } n \text{ is odd,} \end{cases} \qquad \pi_{2n+5}(W_{n+1}) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is even,} \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } n \text{ is odd.} \end{cases}$$

Since ΣX is a CW-complex consisting only of odd dimensional cells, therefore we have

$$[X, U(\infty)] \cong [\Sigma X, BU(\infty)] \cong \tilde{K}^0(\Sigma X) \cong 0.$$

Thus we get the following exact sequence

$$\tilde{K}^0(X) \xrightarrow{(\Omega\pi)_*} [X, \Omega W_{n+1}] \xrightarrow{\delta_*} [X, U(n+1)] \to 0.$$

Therefore we have the following lemma.

Lemma 3.1 $[X, U(n+1)] \cong Coker(\Omega\pi)_* \cong [X, \Omega W_{n+1}]/Im\Omega\pi_*$. □

We need to obtain the $Im\Omega\pi_*$. Define a homomorphism

$$\lambda\colon [X, \Omega W_{n+1}] \to H^{2n+2}(X) \oplus H^{2n+4}(X),$$

by $\lambda(\alpha) = (\alpha^*(a_{2n+2}), \alpha^*(a_{2n+4}))$, where $\alpha \in [X, \Omega W_{n+1}]$, a_{2n+2} and a_{2n+4} are generators of $H^{2n+2}(\Omega W_{n+1}) \cong \mathbb{Z}$ and $H^{2n+4}(\Omega W_{n+1}) \cong \mathbb{Z}$ respectively. Note that for $i = n, n+1$, $a_{2i+2} = \sigma(\bar{x}_{2i+3}) \in H^{2i+2}(\Omega W_{n+1})$. Since the cohomology class $\bar{x}_{2i+3}$ represents a map $\bar{x}_{2i+3}\colon W_{n+1} \to K(\mathbb{Z}, 2i+3)$ then a_{2i+2} is represented by a loop map $\Omega\bar{x}_{2i+3}\colon \Omega W_{n+1} \to \Omega K(\mathbb{Z}, 2i+3) \cong K(\mathbb{Z}, 2i+2)$. Taking the product of such maps for $i = n, n+1$, we obtain a map

$$a = a_{2n+2} \times a_{2n+4}\colon \Omega W_{n+1} \to K(\mathbb{Z}, 2n+2) \times K(\mathbb{Z}, 2n+4).$$

Now the map λ is given by the following composition

$$a_*\colon [X, \Omega W_{n+1}] \to H^{2n+2}(X) \oplus H^{2n+4}(X).$$

In the following lemma we show that the homomorphism λ is monomorphism.

Lemma 3.2 *The map λ is monic.*

Proof First, we need show to show the group $[X, \Omega W_{n+1}]$ is a free abelian group. We recall $A = \Sigma^{2n-2m}\mathbb{C}P^2 = S^{2n-2m+2} \cup e^{2n-2m+4}$. Consider the following cofibration sequence

$$S^{2m-1} \to \mathbb{C}P^{m-1} \to \mathbb{C}P^m \to S^{2m}. \tag{3.2}$$

Apply $[\Sigma^{2n-2m}\mathbb{C}P^2 \wedge -, \Omega W_{n+1}]$ to the cofibration (3.2),we get the following exact sequence

$$[\Sigma^{2n-2m+1}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-1}, \Omega W_{n+1}] \to [\Sigma^{2n}\mathbb{C}P^2, \Omega W_{n+1}] \to [\mathbb{C}P^m \wedge A, \Omega W_{n+1}]$$
$$\to [\Sigma^{2n-2m}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-1}, \Omega W_{n+1}] \to [\Sigma^{2n-1}\mathbb{C}P^2, \Omega W_{n+1}].$$

We show that the terms $[\Sigma^{2n-2m+1}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-1}, \Omega W_{n+1}]$ and $[\Sigma^{2n-1}\mathbb{C}P^2, \Omega W_{n+1}]$ are zero. Consider the following cofibration sequences

$$S^{2m-3} \to \mathbb{C}P^{m-2} \to \mathbb{C}P^{m-1} \to S^{2m-2}, \tag{3.3}$$
$$S^3 \to S^2 \to \mathbb{C}P^2 \to S^4. \tag{3.4}$$

Now apply $[\Sigma^{2n-2m+1}\mathbb{C}P^2 \wedge -, \Omega W_{n+1}]$ to the cofibration (3.3), we get the following exact sequence

$$[\Sigma^{2n-1}\mathbb{C}P^2, \Omega W_{n+1}] \to [\Sigma^{2n-2m+1}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-1}, \Omega W_{n+1}]$$
$$\to [\Sigma^{2n-2m+1}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-2}, \Omega W_{n+1}]$$
$$\to [\Sigma^{2n-2}\mathbb{C}P^2, \Omega W_{n+1}].$$

By apply $[\Sigma^{2n-1}-, \Omega W_{n+1}]$ to the cofibration (3.4), we get the following exact sequence

$$\pi_{2n+2}(\Omega W_{n+1}) \to \pi_{2n+3}(\Omega W_{n+1}) \to [\Sigma^{2n-1}\mathbb{C}P^2, \Omega W_{n+1}] \to \pi_{2n+1}(\Omega W_{n+1}).$$

When n is even then we get $[\Sigma^{2n-1}\mathbb{C}P^2, \Omega W_{n+1}]$ is zero. When n is odd then we get the following exact sequence

$$\pi_{2n+2}(\Omega W_{n+1}) \cong \mathbb{Z} \xrightarrow{f} \pi_{2n+3}(\Omega W_{n+1}) \cong \mathbb{Z}_2 \to [\Sigma^{2n-1}\mathbb{C}P^2, \Omega W_{n+1}] \to 0.$$

Since the map f sends $f_1\colon S^{2n+3} \to W_{n+1}$ to $f_2\colon S^{2n+4} \xrightarrow{\Sigma^{2n+1}\eta} S^{2n+3} \xrightarrow{f_1} W_{n+1}$, so the map f is surjective. Thus we get $[\Sigma^{2n-1}\mathbb{C}P^2, \Omega W_{n+1}]$ is isomorphic to zero.

Again apply $[\Sigma^{2n-2m+1} - \wedge \mathbb{C}P^{m-2}, \Omega W_{n+1}]$ to the cofibration (3.4), we get the following exact sequence

$$[\Sigma^{2n-2m+5}\mathbb{C}P^{m-2}, \Omega W_{n+1}] \to [\Sigma^{2n-2m+1}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-2}, \Omega W_{n+1}]$$
$$\to [\Sigma^{2n-2m+3}\mathbb{C}P^{m-2}, \Omega W_{n+1}]$$
$$\to [\Sigma^{2n-2m+4}\mathbb{C}P^2, \Omega W_{n+1}],$$

Since ΩW_{n+1} is $(2n+1)$-connected, we conclude that the terms $[\Sigma^{2n-2m+5}\mathbb{C}P^{m-2}, \Omega W_{n+1}]$ and $[\Sigma^{2n-2m+3}\mathbb{C}P^{m-2}, \Omega W_{n+1}]$ are zero. Therefore $[\Sigma^{2n-2m+1}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-2}, \Omega W_{n+1}]$ is isomorphic to zero. Therefore $[\Sigma^{2n-2m+1}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-1}, \Omega W_{n+1}]$ is isomorphic to zero. Thus there is an exact sequence

$$0 \to [\Sigma^{2n}\mathbb{C}P^2, \Omega W_{n+1}] \to [\mathbb{C}P^m \wedge A, \Omega W_{n+1}]$$
$$\to [\Sigma^{2n-2m}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-1}, \Omega W_{n+1}] \to 0.$$

We show the group $[\Sigma^{2n-2m}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-1}, \Omega W_{n+1}]$ is a free abelian group isomorphic to $\mathbb{Z}$. Again, apply $[\Sigma^{2n-2m}\mathbb{C}P^2 \wedge -, \Omega W_{n+1}]$ to the cofibration (3.3), we get the following exact sequence

$$\begin{aligned}[\Sigma^{2n-2m+1}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-2}, \Omega W_{n+1}] &\to [\Sigma^{2n-2}\mathbb{C}P^2, \Omega W_{n+1}] \\ &\to [\Sigma^{2n-2m}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-1}, \Omega W_{n+1}] \\ &\to [\Sigma^{2n-2m}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-2}, \Omega W_{n+1}].\end{aligned}$$

Note that the first term $[\Sigma^{2n-2m+1}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-2}, \Omega W_{n+1}]$ is zero, it is due to the connectivity of ΩW_{n+1}. Similarly we have that the last term $[\Sigma^{2n-2m}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-2}, \Omega W_{n+1}]$ is also zero. By apply $[\Sigma^{2n-2}-, \Omega W_{n+1}]$ to the cofibration (3.4), we can conclude that $[\Sigma^{2n-2}\mathbb{C}P^2, \Omega W_{n+1}] \cong \pi_{2n+2}(\Omega W_{n+1}) \cong \mathbb{Z}$. Therefore we obtain $[\Sigma^{2n-2m}\mathbb{C}P^2 \wedge \mathbb{C}P^{m-1}, \Omega W_{n+1}]$ is isomorphic to $\mathbb{Z}$. Also by [9], we know that $[\Sigma^{2n}\mathbb{C}P^2, \Omega W_{n+1}] \cong \mathbb{Z}\oplus\mathbb{Z}$. Therefore we obtain the exact sequence

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to [\mathbb{C}P^m \wedge A, \Omega W_{n+1}] \to \mathbb{Z} \to 0,$$

thus by exactness we conclude that there is a splitting that gives $[\mathbb{C}P^m \wedge A, \Omega W_{n+1}]$ is a free abelian group isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Now, since the maps $(a_{2n+2})^*\colon H^{2n+2}(K(\mathbb{Z}, 2n+2)) \to H^{2n+2}(\Omega W_{n+1})$ and $(a_{2n+4})^*\colon H^{2n+4}(K(\mathbb{Z}, 2n+4)) \to H^{2n+4}(\Omega W_{n+1})$ are isomorphism, the map $a^*\colon H^j(K(\mathbb{Z}, 2n+2) \times K(\mathbb{Z}, 2n+4)) \to H^j(\Omega W_{n+1})$ is also isomorphism for $j = 2n+2$ and $2n+4$. Since $[X, \Omega W_{n+1}]$ is a free abelian group then the map λ is monomorphism. □

Recall that $H^*(\mathbb{C}P^m) = \mathbb{Z}[t]/(t^{m+1})$, where $|t| = 2$ and $K(\mathbb{C}P^m) = \mathbb{Z}[x]/(x^{m+1})$. Let ζ_n be a generator of $\tilde{K}^0(S^{2n})$. Note that $\tilde{K}^0(X = \mathbb{C}P^m \wedge \Sigma^{2n-2m}\mathbb{C}P^2)$ is a free abelian group generated by $\theta_{i,j} = \zeta_{n-m} \otimes x^i \otimes x^j$, where $1 \le i \le m$ and $1 \le j \le 2$, with the following Chern characters

$$\begin{aligned}ch_{n+1}(\theta_{1,1}) &= ch_{n-m}(\zeta_{n-m})(ch_m(x) \otimes ch_1(x) + ch_{m-1}(x) \otimes ch_2(x)) \\ &= \sigma^{2n-2m}\left(\frac{1}{m!}t^m \otimes t + \frac{1}{(m-1)!}t^{m-1} \otimes \frac{1}{2}t^2\right),\end{aligned}$$

similarly

$$ch_{n+2}(\theta_{1,1}) = \sigma^{2n-2m}\frac{1}{m!}t^m \otimes \frac{1}{2}t^2,$$

$$ch_{n+1}(\theta_{1,2}) = \sigma^{2n-2m}\frac{1}{(m-1)!}t^{m-1} \otimes t^2, \qquad ch_{n+2}(\theta_{1,2}) = \sigma^{2n-2m}\frac{1}{m!}t^m \otimes t^2,$$

$$\vdots$$

$$ch_{n+1}(\theta_{m,1}) = \sigma^{2n-2m}A_1 t^m \otimes t, \qquad ch_{n+2}(\theta_{m,1}) = \sigma^{2n-2m}A_1 t^m \otimes \frac{1}{2}t^2,$$

$$ch_{n+1}(\theta_{m,2}) = 0, \qquad ch_{n+2}(\theta_{m,2}) = \sigma^{2n-2m}A_1 t^m \otimes t^2,$$

where

$$ch_m(x^m) = A_1 t^m = ch_1 x \sum_{\substack{i_1+\cdots+i_{m-1}=m-1,\\ 0\le i_1\le i_2\le\cdots\le i_{m-1}}} ch_{i_1}x^{i_1}\cdots ch_{i_{m-1}}x^{i_{m-1}}$$

$$+ch_2x^2 \sum_{\substack{i_1+\cdots+i_k=m-2,k=[\frac{m-2}{2}],\\ 2\le i_1\le i_2\le\cdots\le i_k}} ch_{i_1}x^{i_1}\cdots ch_{i_k}x^{i_k}$$
$$+ch_3x^3 \sum_{\substack{i_1+\cdots+i_k=m-3,k=[\frac{m-3}{3}],\\ 3\le i_1\le i_2\le\cdots\le i_k}} ch_{i_1}x^{i_1}\cdots ch_{i_k}x^{i_k}+\cdots$$
$$+ch_kx^k \sum_{i_1=m-k,k=[\frac{m}{2}]} ch_{i_1}x^{i_1}.$$

We will prove the following proposition.

Proposition 3.3 *Im* $\lambda\circ(\Omega\pi)_*$ *is generated by* $\alpha_{i,j}$, *for* $1\le i\le m$ *and* $1\le j\le 2$, *where*

$$\alpha_{1,1}=\frac{1}{2\cdot(m-1)!}(n+1)!\left(\frac{2}{m},1,\frac{n+2}{m}\right),$$
$$\alpha_{1,2}=\frac{1}{(m-1)!}(n+1)!\left(0,1,\frac{n+2}{m}\right),$$
$$\vdots$$
$$\alpha_{m,1}=\frac{1}{2}(n+1)!A_1(2,0,n+2),$$
$$\alpha_{m,1}=(n+2)!A_1(0,0,1).$$

Proof According to the definition of the map λ, we have

$$\lambda\circ(\Omega\pi)_*(\theta_{1,1})=((\Omega\pi\circ\theta_{1,1})^*(a_{2n+2}),(\Omega\pi\circ\theta_{1,1})^*(a_{2n+4})).$$

The calculation of the first component is as follows

$$(\Omega\pi\circ\theta_{1,1})^*(a_{2n+2})=a_{2n+2}\circ\Omega\pi(\theta_{1,1})=(n+1)!ch_{n+1}(\theta_{1,1})$$
$$=(n+1)!\left(\frac{1}{m!}t^m\otimes t+\frac{1}{(m-1)!}t^{m-1}\otimes\frac{1}{2}t^2\right)\sigma^{2n-2m},$$

and calculation the second component is as follows

$$(\Omega\pi\circ\theta_{1,1})^*(a_{2n+4})=a_{2n+4}\circ\Omega\pi(\theta_{1,1})=(n+2)!ch_{n+2}(\theta_{1,1})$$
$$=(n+2)!\left(\frac{1}{m!}t^m\otimes\frac{1}{2}t^2\right)\sigma^{2n-2m}.$$

Therefore we have

$$\lambda\circ(\Omega\pi)_*(\theta_{1,1})=\left(\frac{1}{m!}(n+1)!,\frac{1}{2\cdot(m-1)!}(n+1)!,\frac{1}{2\cdot m!}(n+2)!\right)$$
$$=\frac{1}{2\cdot(m-1)!}(n+1)!\left(\frac{2}{m},1,\frac{n+2}{m}\right).$$

Similarly we can show

$$\lambda\circ(\Omega\pi)_*(\theta_{1,2})=\left(0,\frac{1}{(m-1)!}(n+1)!,\frac{1}{m!}(n+2)!\right)$$
$$=\frac{1}{(m-1)!}(n+1)!\left(0,1,\frac{n+2}{m}\right),$$

$$\vdots$$

$$\lambda \circ (\Omega\pi)_*(\theta_{m,1}) = \left((n+1)!A_1, 0, \frac{1}{2}(n+2)!A_1\right) = \frac{1}{2}(n+1)!A_1(2, 0, n+2),$$

$$\lambda \circ (\Omega\pi)_*(\theta_{m,2}) = (0, 0, (n+2)!A_1) = (n+2)!A_1(0, 0, 1).$$

□

Now consider the map $\alpha_{k*}\colon [\Sigma A, SU(n)] \to [\Sigma A, Map_0^*(\mathbb{C}P^m, BSU(n))]$. Note that the group $[\Sigma A, SU(n)]$ is isomorphic to $\tilde{K}^1(\Sigma A) \cong \tilde{K}^0(\Sigma^2 A) \cong \mathbb{Z}\oplus\mathbb{Z}$ and is a free abelian group generated by $\xi_i = \zeta_{n-m+1} \otimes x^i$ for $i = 1, 2$. Let $\varepsilon_{m,n}\colon S^{2m-1} \to SU(n)$ represents the generator of $\pi_{2m-1}(SU(n)) \cong \mathbb{Z}$ and l_i for $i = 1, 2$, be the adjoint of the composition

$$\mathbb{C}P^m \wedge \Sigma A \xrightarrow{q\wedge \mathbb{1}} \Sigma S^{2m-1} \wedge \Sigma A \xrightarrow{\Sigma\varepsilon_{m,n}\wedge\xi_i} \Sigma SU(n) \wedge SU(n) \xrightarrow{[ev,ev]} BSU(n),$$

where $[ev, ev]$ is the Whitehead product. Let $j\colon SU(n) \to SU(n+1)$ is the canonical inclusion and H_1 be the subgroup of $[X, U(n+1)]$ generated by $j \circ l_1$ and $j \circ l_2$. We study the group H_1. First, we have the following proposition.

Proposition 3.4 *There are lifts $\tilde{\xi}_{i,k}$ of $j \circ l_i$ for $i = 1, 2$, respectively,*

$$\begin{array}{ccc} & & \Omega W_{n+1} \\ & \overset{\tilde{\xi}_{i,k}}{\nearrow} & \downarrow \\ \mathbb{C}P^m \wedge A & \xrightarrow{j\circ l_i} & SU(n+1) \end{array}$$

such that $(\tilde{\xi}_{i,k})^*(a_{2i+2}) = (m-1)!kt^m \otimes \Sigma^{-1}(\xi_i)^*(x_{2i-2m+3})$, where Σ is the cohomology suspension isomorphism.

Proof Hamanaka and Kono in [4, 5] showed that there is a lift $\gamma\colon \Sigma SU(n+1)\wedge SU(n+1) \to W_{n+1}$ of $[ev, ev]$ such that $\gamma^*(\bar{x}_{2i+3}) = \sum_{j+k=i} \Sigma x_{2j+1} \otimes x_{2k+1}$. Let $\tilde{\gamma}$ be the following composition

$$\tilde{\gamma}\colon \mathbb{C}P^m \wedge \Sigma A \xrightarrow{q\wedge\mathbb{1}} \Sigma S^{2m-1} \wedge \Sigma A \xrightarrow{\Sigma j\circ k\varepsilon_{m,n}\wedge j\circ\xi_i} \Sigma SU(n+1) \wedge SU(n+1) \xrightarrow{\gamma} W_{n+1}.$$

We have

$$\begin{aligned} \tilde{\gamma}^*(\bar{x}_{2i+3}) &= (q\wedge\mathbb{1})^*(\Sigma j\circ k\varepsilon_{m,n} \wedge j\circ\xi_i)^*\gamma^*(\bar{x}_{2i+3}) \\ &= (q\wedge\mathbb{1})^*(\Sigma j\circ k\varepsilon_{m,n} \wedge j\circ\xi_i)^*\left(\sum_{j+k=i} \Sigma x_{2j+1}\otimes x_{2k+1}\right) \\ &= (q\wedge\mathbb{1})^*((m-1)!\Sigma k u_{2m-1} \otimes (j\circ\xi_i)^*(x_{2i-2m+3})) \\ &= (m-1)!kt^m \otimes (\xi_i)^*(x_{2i-2m+3}), \end{aligned}$$

where u_{2m-1} is the generator of $H^{2m-1}(S^{2m-1})$. Let the map $S\colon \Sigma\mathbb{C}P^m\wedge A \longrightarrow \mathbb{C}P^m\wedge\Sigma A$ be the swapping map and the map $ad\colon [\Sigma\mathbb{C}P^m \wedge A, W_{n+1}] \longrightarrow [\mathbb{C}P^m \wedge A, \Omega W_{n+1}]$ be the adjunction. We take $\tilde{\xi}_{i,k}\colon \mathbb{C}P^m \wedge A \longrightarrow \Omega W_{n+1}$ to be the adjoint of the following composition

$$\Sigma\mathbb{C}P^m \wedge A \xrightarrow{S} \mathbb{C}P^m \wedge \Sigma A \xrightarrow{\tilde{\gamma}} W_{n+1},$$

that is $\tilde{\xi}_{i,k} : ad(\tilde{\gamma} \circ S)$, then $\tilde{\xi}_{i,k}$ is a lift of $i \circ l_i$, for $i = 1, 2$. We get

$$\begin{aligned}(\tilde{\gamma} \circ S)^*(\bar{x}_{2i+3}) = S^* \circ \tilde{\gamma}^*(\bar{x}_{2i+3}) = S^*((m-1)!kt^m \otimes (\xi_i)^*(x_{2i-2m+3})) \\ = (m-1)!\Sigma kt^m \otimes \Sigma^{-1}(\xi_i)^*(x_{2i-2m+3}),\end{aligned}$$

thus we have $(\tilde{\xi}_{i,k})^*(a_{2i+2}) = (m-1)!kt^m \otimes \Sigma^{-1}(\xi_i)^*(x_{2i-2m+3})$. □

Now let $H_1{}'$ be the subgroup generated by $\tilde{\xi}_{1,k}$ and $\tilde{\xi}_{2,k}$. By Lemma 3.1, H_1 is isomorphic to $H_1{}'/(Im(\Omega\pi)_* \cap H_1{}')$. We have

$$\begin{aligned}&c_{n-m+2}(\xi_1) = (n-m+1)!\sigma^{2n-2m+2}t, \quad &&c_{n-m+3}(\xi_1) = \frac{1}{2}(n-m+2)!\sigma^{2n-2m+2}t^2,\\ &c_{n-m+2}(\xi_2) = 0, &&c_{n-m+3}(\xi_2) = (n-m+2)!\sigma^{2n-2m+2}t^2.\end{aligned}$$

According to the map of λ, we have $\lambda(\tilde{\xi}_{1,k}) = ((\tilde{\xi}_{1,k})^*(a_{2n+2}), (\tilde{\xi}_{1,k})^*(a_{2n+4}))$. Note that $x_{2n-2m+3} = \sigma(c_{n-m+2})$ and $x_{2n-2m+5} = \sigma(c_{n-m+3})$. The calculation of the first component is as follows

$$(\tilde{\xi}_{1,k})^*(a_{2n+2}) = (m-1)!kt^m \otimes \Sigma^{-2}c_{n-m+2}(\xi_1) = (m-1)!kt^m \otimes (n-m+1)!\sigma^{2n-2m}t,$$

and computing the second component is as follows

$$\begin{aligned}(\tilde{\xi}_{1,k})^*(a_{2n+4}) &= (m-1)!kt^m \otimes \Sigma^{-2}c_{n-m+3}(\xi_1)\\ &= (m-1)!kt^m \otimes \frac{1}{2}(n-m+2)!\sigma^{2n-2m}t^2.\end{aligned}$$

Therefore $\lambda(\tilde{\xi}_{1,k}) = k((m-1)!(n-m+1)!, 0, \frac{1}{2}(m-1)!(n-m+2)!)$. Similarly we can show that $\lambda(\tilde{\xi}_{2,k}) = k(0, 0, (m-1)!(n-m+2)!)$. Therefore $H_1{}'$ is generated by α and α', where

$$\begin{aligned}\alpha &= \frac{1}{2}k(m-1)!(n-m+1)!(2, 0, n-m+2),\\ \alpha' &= k(m-1)!(n-m+2)!(0, 0, 1).\end{aligned}$$

Let t be the generator of $H^2(\mathbb{C}P^m)$, also $u_{2n-2m+2}$ and $u_{2n-2m+4}$ are generators of $H^{2n-2m+2}(A)$ and $H^{2n-2m+4}(A)$, respectively. We denote an element $at^m u_{2n-2m+2} + bt^{m-1}u_{2n-2m+4} + ct^m\zeta_{2n-2m+4}$ belong to $H^{2n+2}(X) \oplus H^{2n+4}(X)$by (a, b, c). Let $B = \{(a, b, c) | a + (m-1)b \equiv 0 \mod 2\}$. Recall $(2n+5)$-skeleton of ΩW_{n+1} is $S^{2n+4} \vee S^{2n+2}$. Let $(a, b, c) \in Im\lambda$, then there exists $f \in [X, \Omega W_{n+1}]$ such that

$$f^*(a_{2n+2}) = at^m u_{2n-2m+2} + bt^{m-1}u_{2n-2m+4}, \quad f^*(a_{2n+4}) = ct^m u_{2n-2m+4}. \quad (3.5)$$

We have $Sq^2(t^{m-1}) = (m-1)t^m$, $Sq^2(u_{2n-2m+2}) = u_{2n-2m+4}$ and $Sq^2(a_{2n+2}) = 0$. Now apply Sq^2 to (3.5), we get $a + (m-1)b \equiv 0 \mod 2$. Thus we have the following lemma.

Lemma 3.5 $Im\lambda \subseteq \{(a, b, c) | a + (m-1)b \equiv 0 \ \ mod \ \ 2\}$. □

In the following, we bring an application.

- $SU(n)$-gauge groups over $\mathbb{C}P^3$ where n is an odd integer and $n \geq 3$

In the previous calculations, we now take $m = 3$. First, we need the following lemma.

Lemma 3.6 $Im\lambda = \{(a, b, c) | a + 2b \equiv 0 \ \ mod \ \ 2\}$.

Proof Apply $[\Sigma^{2n}-, \Omega W_{n+1}]$ to cofibration $S^3 \xrightarrow{\eta} S^2 \xrightarrow{i} \mathbb{C}P^2 \xrightarrow{q} S^4$ to obtain the exact sequence

$$\pi_{2n+5}(W_{n+1}) \xrightarrow{q^*} [\Sigma^{2n}\mathbb{C}P^2, \Omega W_{n+1}] \xrightarrow{i^*} \pi_{2n+3}(W_{n+1}) \xrightarrow{\eta^*} \pi_{2n+4}(W_{n+1}),$$

where η, i and q are Hopf map, inclusion map and the quotient map, respectively, and the maps η^*, i^* and q^* are induced maps. We know that $\pi_{2n+3}(W_{n+1}) \cong \mathbb{Z}\{t_1\}$ and also $\pi_{2n+4}(W_{n+1}) \cong \mathbb{Z}_2\{t_2\}$, where $t_2\colon S^{2n+4} \xrightarrow{\eta} S^{2n+3} \xrightarrow{t_1} W_{n+1}$. Since η^* sends t_1 to t_2 so η^* is a surjection map. Thus by exactness we can conclude that $[\Sigma^{2n}\mathbb{C}P^2, \Omega W_{n+1}]$ has a $\mathbb{Z}$-summand with its generator t_3 that the map i^* sends t_3 to $2t_1$.

Now, let $B = \{(a, b, c)|a + 2b \equiv 0 \mod 2\}$. By Lemma 3.5, we have $Im\lambda \subseteq B$. Put $u = (0, 0, 1)$, $v = (0, 1, 1)$ and $w = (2, 0, 0)$. For the converse case, we show that u, v and w are in $Im\lambda$. Consider the following maps

$$\phi_1\colon \mathbb{C}P^3 \wedge A \xrightarrow{q} S^6 \wedge S^{2n-2} \hookrightarrow \Omega W_{n+1},$$
$$\phi_2\colon \mathbb{C}P^3 \wedge A \xrightarrow{q} \mathbb{C}P^3/\mathbb{C}P^1 \wedge S^{2n-2} \simeq S^{2n+4} \vee S^{2n+2} \hookrightarrow \Omega W_{n+1},$$
$$\phi_3\colon \mathbb{C}P^3 \wedge A \xrightarrow{q_1} S^6 \wedge A \simeq \Sigma^{2n}\mathbb{C}P^2 \xrightarrow{t_3} \Omega W_{n+1},$$

where q and q_1 are quotient maps and $\mathbb{C}P^3/\mathbb{C}P^1 \simeq S^6 \vee S^4$. We have $\lambda(\phi_1) = u$, $\lambda(\phi_2) = v$ and $\lambda(\phi_3) = w$, respectively. Thus Im $(\lambda) = B$. □

Put $\beta = \{u, v, w\}$. We know that $u, v, w \in Im\lambda$ and generators of Im λ, therefore β is a basis for Im λ. Let $p = (n + 1)n(n - 1)$. We have the following theorem.

Theorem 3.7 *$[X, U(n + 1)]$ is isomorphic to $\mathbb{Z}_{\frac{1}{4}(n+1)!} \oplus \mathbb{Z}_{\frac{1}{2}(n+2)!} \oplus \mathbb{Z}_{\frac{1}{2}(n+1)!}$.*

Proof By Proposition 3.3, Im $\lambda \circ (\Omega\pi)_*$ is generated by $\alpha_{i,j}$, where $1 \leq i \leq 3$ and $1 \leq j \leq 2$. Note that under basis β, Im $\lambda \circ (\Omega\pi)_*$ is generated by $\alpha_{i,j}$, where

$$\alpha_{1,1} = (n-2)!\left(\frac{1}{12}(n-1)p, \frac{1}{4}p, \frac{1}{12}p\right), \quad \alpha_{1,2} = (n-2)!\left(\frac{1}{6}(n-1)p, \frac{1}{2}p, 0\right),$$
$$\alpha_{2,1} = (n-2)!\left(\frac{1}{4}np, \frac{1}{2}p, \frac{1}{4}p\right), \quad \alpha_{2,2} = (n-2)!\left(\frac{1}{2}np, p, 0\right),$$
$$\alpha_{3,1} = (n-2)!((n+2)p, 0, p), \quad \alpha_{3,2} = (n-2)!(2(n+2)p, 0, 0).$$

We represent the coordinate of Im $\lambda \circ (\Omega\pi)_*$ by the following matrix

$$M = (n-2)!\begin{bmatrix} \frac{1}{12}(n-1)p & \frac{1}{4}p & \frac{1}{12}p \\ \frac{1}{6}(n-1)p & \frac{1}{2}p & 0 \\ \frac{1}{4}np & \frac{1}{2}p & \frac{1}{4}p \\ \frac{1}{2}np & p & 0 \\ (n+2)p & 0 & p \\ 2(n+2)p & 0 & 0 \end{bmatrix}$$

that is, Im $\lambda \circ (\Omega\pi)_*$ is generated by the row vectors of matrix M. By using the Smith normal form, there exist invertible 6×6 and 3×3-matrices M' and M'' such that

$$M' \cdot M \cdot M'' = (n-2)!\begin{bmatrix} \frac{1}{4}p & 0 & 0 \\ 0 & \frac{1}{2}(n+2)p & 0 \\ 0 & 0 & \frac{1}{2}p \end{bmatrix}$$

where $M' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -6 & 1 & 2 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ -24 & 0 & 12 & 0 & -1 & 0 \\ -6 & 3 & 2 & -1 & 0 & 0 \\ -24 & 12 & 0 & 0 & 2 & -1 \end{bmatrix}$ and $M'' = \begin{bmatrix} 0 & 3 & 0 \\ 1 & -(n-1) & -1 \\ 0 & 0 & 3 \end{bmatrix}$. Therefore we can conclude that

$$[X, U(n+1)] \cong \mathbb{Z}_{\frac{1}{4}(n+1)!} \oplus \mathbb{Z}_{\frac{1}{2}(n+2)!} \oplus \mathbb{Z}_{\frac{1}{2}(n+1)!}.$$

□

We write $|G|$ for the order of a group G. We will prove the following proposition.

Proposition 3.8 $|H_1| = \frac{\frac{1}{2}(n+2)(n+1)n}{\left(\frac{1}{2}(n+2)(n+1)n,k\right)} \cdot \frac{\frac{1}{36}p}{\left(\frac{1}{36}p,k\right)}$.

Proof We know that the subgroup $H_1{}'$ is generated by α and α', where

$$\alpha = k(2(n-2)!, 0, (n-1)!) = k(n-2)!(2, 0, n-1),$$
$$\alpha' = k(0, 0, 2(n-1)!) = k(n-2)!(0, 0, 2(n-1)).$$

Note that under basis β, the subgroup $H_1{}'$ is generated by $\alpha = k(n-2)!(n-1, 0, 1)$ and $\alpha' = k(n-2)!(2(n-1), 0, 0)$. We represent the coordinate of $H_1{}'$ by the following matrix

$$M_{H_1{}'} = k(n-2)! \begin{bmatrix} n-1 & 0 & 1 \\ 2(n-1) & 0 & 0 \end{bmatrix},$$

that is, $H_1{}'$ is generated by the row vectors of matrix $M_{H_1{}'}$. The new coordinate of $H_1{}'$

$$M_{H_1{}'} \cdot M'' = k(n-2)! \begin{bmatrix} 0 & 3(n-1) & 3 \\ 0 & 6(n-1) & 0 \end{bmatrix}.$$

Let $r = (n-2)!$, then we have

$$\begin{bmatrix} \frac{1}{3} & 0 \\ -6 & 3 \end{bmatrix} \cdot kr \begin{bmatrix} 0 & 3(n-1) & 3 \\ 0 & 6(n-1) & 0 \end{bmatrix} = kr \begin{bmatrix} 0 & n-1 & 1 \\ 0 & 0 & -18 \end{bmatrix}.$$

Put $\rho = (0, (n-1)kr, kr)$ and $\rho' = (0, 0, -18kr)$. Then we have

$$H_1{}' = \{x\rho + y\rho' \in [X, U(n+1)] | x, y \in \mathbb{Z}\}.$$

If $x\rho + y\rho'$ and $x'\rho + y'\rho'$ are the same modulo $\mathrm{Im}\, \lambda \circ (\Omega\pi)_*$ then we have

$$\begin{cases} (n-1)xkr \equiv (n-1)x'kr \mod \frac{1}{2}(n+2)p, \\ 18ykr \equiv 18y'kr \qquad\qquad \mod \frac{1}{2}p. \end{cases}$$

These conditions are equivalent to

$$\begin{cases} xk \equiv x'k \mod \frac{1}{2}(n+2)(n+1)n, \\ yk \equiv y'k \mod \frac{1}{36}p. \end{cases}$$

This implies that there are $\frac{\frac{1}{2}(n+2)(n+1)n}{(\frac{1}{2}(n+2)(n+1)n,k)}$ distinct value of x and $\frac{\frac{1}{36}p}{(\frac{1}{36}p,k)}$ distinct value of y, so we have

$$|H_1| = \frac{\frac{1}{2}(n+2)(n+1)n}{\left(\frac{1}{2}(n+2)(n+1)n,k\right)} \cdot \frac{\frac{1}{36}p}{\left(\frac{1}{36}p,k\right)}.$$

□

4 The group $[\mathbb{C}P^m \wedge A, SU(n+1)]$ when $n-m$ is odd

In this section, we in case that $n-m$ is an odd integer and $n \geq 3$ will study the group $[X, U(n+1)]$ and then obtain the order of group $[X, U(n)]$. Recall the homomorphism λ defined before in case one. To better distinguish the two cases we now relabel the homomorphism as λ'. That is, $\lambda' \colon [X, \Omega W_{n+1}] \to H^{2n+2}(X) \oplus H^{2n+4}(X)$ is defined by $\lambda'(\alpha) = (\alpha^*(a_{2n+2}), \alpha^*(a_{2n+4}))$. We have the following lemma.

Lemma 4.1 *The map λ' is monic.*

Proof Recall $A = S^{2n-2m+2} \vee S^{2n-2m+4}$ and $X = \mathbb{C}P^m \wedge A$. We show the group $[X, \Omega W_{n+1}]$ is a free abelian group. We have the following isomorphism

$$\begin{aligned}[X, \Omega W_{n+1}] &= [\mathbb{C}P^m \wedge (S^{2n-2m+2} \vee S^{2n-2m+4}), \Omega W_{n+1}] \\ &\cong [\Sigma^{2n-2m+2}\mathbb{C}P^m, \Omega W_{n+1}] \oplus [\Sigma^{2n-2m+4}\mathbb{C}P^m, \Omega W_{n+1}].\end{aligned}$$

Apply $[\Sigma^{2n-2m+2}-, \Omega W_{n+1}]$ to the cofibration (3.2), we get the following exact sequence

$$\begin{aligned}[\Sigma^{2n-2m+3}\mathbb{C}P^{m-1}, \Omega W_{n+1}] \to \pi_{2n+2}(\Omega W_{n+1}) \to [\Sigma^{2n-2m+2}\mathbb{C}P^m, \Omega W_{n+1}] \\ \to [\Sigma^{2n-2m+2}\mathbb{C}P^{m-1}, \Omega W_{n+1}].\end{aligned}$$

Since ΩW_{n+1} is $(2n+1)$-connected, we obtain that the first term $[\Sigma^{2n-2m+3}\mathbb{C}P^{m-1}, \Omega W_{n+1}]$ and the last term $[\Sigma^{2n-2m+2}\mathbb{C}P^{m-1}, \Omega W_{n+1}]$ are zero. Thus $[\Sigma^{2n-2m+2}\mathbb{C}P^m, \Omega W_{n+1}]$ is isomorphic to $\pi_{2n+2}(\Omega W_{n+1}) \cong \pi_{2n+3}(W_{n+1}) \cong \mathbb{Z}$. We prove that $[\Sigma^{2n-2m+4}\mathbb{C}P^m, \Omega W_{n+1}]$ is also a free abelian group. For this, again apply $[\Sigma^{2n-2m+4}-, \Omega W_{n+1}]$ to the cofibration (3.2), we get the exact sequence

$$\begin{aligned}[\Sigma^{2n-2m+5}\mathbb{C}P^{m-1}, \Omega W_{n+1}] \to \pi_{2n+4}(\Omega W_{n+1}) \to [\Sigma^{2n-2m+4}\mathbb{C}P^m, \Omega W_{n+1}] \\ \to [\Sigma^{2n-2m+4}\mathbb{C}P^{m-1}, \Omega W_{n+1}] \to \pi_{2n+3}(\Omega W_{n+1}).\end{aligned}$$

Apply $[\Sigma^{2n-2m+4}-, \Omega W_{n+1}]$ and $[\Sigma^{2n-2m+5}-, \Omega W_{n+1}]$ to the cofibration (3.3), we get the following exact sequences

$$\begin{aligned}[\Sigma^{2n-2m+5}\mathbb{C}P^{m-2}, \Omega W_{n+1}] \to \pi_{2n+2}(\Omega W_{n+1}) \to [\Sigma^{2n-2m+4}\mathbb{C}P^{m-1}, \Omega W_{n+1}] \\ \to [\Sigma^{2n-2m+4}\mathbb{C}P^{m-2}, \Omega W_{n+1}], \quad (4.1)\end{aligned}$$

$$\begin{aligned}[\Sigma^{2n-2m+6}\mathbb{C}P^{m-2}, \Omega W_{n+1}] \to \pi_{2n+3}(\Omega W_{n+1}) \to [\Sigma^{2n-2m+5}\mathbb{C}P^{m-1}, \Omega W_{n+1}] \\ \to [\Sigma^{2n-2m+5}\mathbb{C}P^{m-2}, \Omega W_{n+1}], \quad (4.2)\end{aligned}$$

respectively. Consider the exact sequence (4.1). Since ΩW_{n+1} is $(2n+1)$-connected then the first term and the last term are zero, thus $[\Sigma^{2n-2m+4}\mathbb{C}P^{m-1}, \Omega W_{n+1}]$ is isomorphic to $\pi_{2n+2}(\Omega W_{n+1}) \cong \mathbb{Z}$. Now, consider the exact sequence (4.2). We know that when n is even then $\pi_{2n+3}(\Omega W_{n+1})$ is zero, so the group $[\Sigma^{2n-2m+5}\mathbb{C}P^{m-1}, \Omega W_{n+1}]$ is isomorphic to zero, where by the exact sequence (4.1) we have that the group $[\Sigma^{2n-2m+5}\mathbb{C}P^{m-2}, \Omega W_{n+1}]$ is zero. When n is odd then we prove that the group $[\Sigma^{2n-2m+5}\mathbb{C}P^{m-1}, \Omega W_{n+1}]$ is isomorphic to $\mathbb{Z}_2$. Since n is odd, ΩW_{n+1} has $(2n+5)$-skeleton equal to $S^{2n+2} \vee S^{2n+4}$, so any map $\Sigma^{2n-2m+5}\mathbb{C}P^{m-1} \to \Omega W_{n+1}$ factors as

$$\Sigma^{2n-2m+5}\mathbb{C}P^{m-1} \xrightarrow{q} S^{2n+3} \xrightarrow{l} S^{2n+2} \hookrightarrow \Omega W_{n+1},$$

where q is the pinch map to the top cell and l is some map. Taking l to be the class of order 2 show that $[\Sigma^{2n-2m+5}\mathbb{C}P^{m-1}, \Omega W_{n+1}] \cong \mathbb{Z}_2$. Thus, in cases where n is even and n is odd, we get the following exact sequences

$$0 \to \mathbb{Z} \to [\Sigma^{2n-2m+4}\mathbb{C}P^{m}, \Omega W_{n+1}] \to \mathbb{Z} \to 0,$$
$$\mathbb{Z}_2 \xrightarrow{s_1} \mathbb{Z} \oplus \mathbb{Z}_2 \to [\Sigma^{2n-2m+4}\mathbb{C}P^{m}, \Omega W_{n+1}] \to \mathbb{Z} \to \mathbb{Z}_2,$$

respectively. We show that the map s_1 is injective. For this, it needs to be shown that the composite

$$S^{2n+4} \xrightarrow{s'} \Sigma^{2n-2m+5}\mathbb{C}P^{m-1} \xrightarrow{s''} \Omega W_{n+1}$$

is nontrivial, where s' is the suspension of the attaching map $S^{2m-1} \to \mathbb{C}P^{m-1}$ with cofibre $\mathbb{C}P^m$, and s'' generates $[\Sigma^{2n-2m+5}\mathbb{C}P^{m-1}, \Omega W_{n+1}]$. Note that by the connectivity of ΩW_{n+1}, the map s'' factors as the composite

$$\Sigma^{2n-2m+5}\mathbb{C}P^{m-1} \xrightarrow{q} S^{2n+3} \xrightarrow{c'} \Omega W_{n+1}$$

where q is the pinch map to the top cell and c' is $S^{2n+3} \xrightarrow{\eta} S^{2n+2} \hookrightarrow \Omega W_{n+1}$. On the other hand, the composite $S^{2n+4} \xrightarrow{s'} \Sigma^{2n-2m+5}\mathbb{C}P^{m-1} \xrightarrow{q} S^{2n+3}$ is homotopic to η since n is odd. Therefore $s'' \circ s'$ is homotopic to $S^{2n+4} \xrightarrow{\eta^2} S^{2n+2} \hookrightarrow \Omega W_{n+1}$, which is nontrivial. Thus in both cases, by exactness we obtain $[\Sigma^{2n-2m+4}\mathbb{C}P^{m}, \Omega W_{n+1}]$ is a free abelian group isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Therefore we can conclude that the group $[X, \Omega W_{n+1}]$ is a free abelian group that is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. □

Note that $\tilde{K}^0(X = \mathbb{C}P^m \wedge (S^{2n-2m+2} \vee S^{2n-2m+4}))$ is a free abelian group generated by $\theta_{i,j} = \zeta_{n-m+i} \otimes x^j$, where $1 \leq i \leq 2$ and $1 \leq j \leq m$, with the following Chern characters

$$\begin{aligned}
ch_{n+1}(\theta_{1,1}) &= \sigma^{2n-2m+2}\frac{1}{m!}t^m, & ch_{n+1}(\theta_{2,1}) &= \sigma^{2n-2m+4}\frac{1}{(m-1)!}t^{m-1},\\
ch_{n+1}(\theta_{1,2}) &= \sigma^{2n-2m+2}B_1 t^m, & ch_{n+1}(\theta_{2,2}) &= \sigma^{2n-2m+4}C_1 t^{m-1},\\
&\vdots & &\\
ch_{n+1}(\theta_{1,m}) &= \sigma^{2n-2m+2}A_1 t^m, & ch_{n+1}(\theta_{2,m}) &= 0,
\end{aligned}$$

and also

$$\begin{aligned}
ch_{n+2}(\theta_{1,1}) &= 0, & ch_{n+2}(\theta_{2,1}) &= \sigma^{2n-2m+4}\frac{1}{m!}t^m,\\
ch_{n+2}(\theta_{1,2}) &= 0, & ch_{n+2}(\theta_{2,2}) &= \sigma^{2n-2m+4}B_1 t^m,\\
&\vdots & &\\
ch_{n+2}(\theta_{1,m}) &= 0, & ch_{n+2}(\theta_{2,m}) &= \sigma^{2n-2m+4}A_1 t^m
\end{aligned}$$

where

$$ch_m(x^2) = B_1 t^m = \sum_{\substack{i+j=m,\\ 1\le i\le [\frac{m}{2}]}} ch_i x ch_j x = \sum_{k=1}^{[\frac{m}{2}]} \frac{1}{k!(m-k)!} t^m,$$

and $ch_{m-1}(x^2) = C_1 t^{m-1}$. We have the following proposition.

Proposition 4.2 *Im* $\lambda' \circ (\Omega\pi)_*$ *is generated by* ${\alpha'}_{i,j}$, *for* $1 \le i \le 2$ *and* $1 \le j \le m$, *where*

$$\begin{aligned}
{\alpha'}_{1,1} &= \frac{1}{m!}(n+1)!(1,0,0), & {\alpha'}_{2,1} &= \frac{1}{(m-1)!}(n+1)!\left(0, 1, \frac{n+2}{m}\right),\\
{\alpha'}_{1,2} &= B_1(n+1)!(1,0,0), & {\alpha'}_{2,2} &= (n+1)!(0, C_1, (n+2)B_1),\\
&\vdots\\
{\alpha'}_{1,m} &= A_1(n+1)!(1,0,0), & {\alpha'}_{2,m} &= A_1(n+2)!(0,0,1).
\end{aligned}$$

Proof Similar to the proof of Proposition 3.3, we get

$$\begin{aligned}
\lambda' \circ (\Omega\pi)_*(\theta_{1,1}) &= \left(\frac{1}{m!}(n+1)!, 0, 0\right),\\
\lambda' \circ (\Omega\pi)_*(\theta_{2,1}) &= \left(0, \frac{1}{(m-1)!}(n+1)!, \frac{1}{m!}(n+2)!\right),\\
\lambda' \circ (\Omega\pi)_*(\theta_{1,2}) &= (B_1(n+1)!, 0, 0),\\
\lambda' \circ (\Omega\pi)_*(\theta_{2,2}) &= (0, C_1(n+1)!, B_1(n+2)!),\\
&\vdots\\
\lambda \circ (\Omega\pi)_*(\theta_{1,m}) &= (A_1(n+1)!, 0, 0),\\
\lambda \circ (\Omega\pi)_*(\theta_{2,m}) &= (0, 0, A_1(n+2)!).
\end{aligned}$$

□

Let H_2 be the subgroup of $[X, U(n+1)]$ generated by $j \circ l_1$ and $j \circ l_2$. By proof of Proposition 3.4, there are lifts $\tilde{\xi}_{i,k}$ of $j \circ l_i$ for $i = 1, 2$, respectively, such that

$$(\tilde{\xi}_{i,k})^*(a_{2i+2}) = (m-1)!kt^m \otimes \Sigma^{-1}(\xi_i)^*(x_{2i-2m+3}).$$

Now let ${H_2}'$ be the subgroup generated by $\tilde{\xi}_{1,k}$ and $\tilde{\xi}_{2,k}$. By Lemma 3.1, we know that the subgroup H_2 is isomorphic to ${H_2}'/(Im(\Omega\pi)_* \cap {H_2}')$. We have

$$\begin{aligned}
c_{n-m+2}(\xi_1) &= (n-m+1)!\sigma^{2n-2m+4}, & c_{n-m+3}(\xi_1) &= 0,\\
c_{n-m+2}(\xi_2) &= 0, & c_{n-m+3}(\xi_2) &= (n-m+2)!\sigma^{2n-2m+6}.
\end{aligned}$$

According to the map of λ', we have $\lambda'(\tilde{\xi}_{1,k}) = ((\tilde{\xi}_{1,k})^*(a_{2n+2}), (\tilde{\xi}_{1,k})^*(a_{2n+4}))$. The calculation of the first and second components are as follows

$$\begin{aligned}
(\tilde{\xi}_{1,k})^*(a_{2n+2}) &= (m-1)!kt^m \otimes \Sigma^{-2} c_{n-m+2}(\xi_1)\\
&= (m-1)!kt^m \otimes (n-m+1)!\sigma^{2n-2m+2},\\
(\tilde{\xi}_{1,k})^*(a_{2n+4}) &= (m-1)!kt^m \otimes \Sigma^{-2} c_{n-m+3}(\xi_1) = 0.
\end{aligned}$$

Therefore $\lambda'(\tilde{\xi}_{1,k}) = k((m-1)!(n-m+1)!, 0, 0)$. Similarly we can show that

$$\lambda(\tilde{\xi}_{2,k}) = k(0, 0, (m-1)!(n-m+2)!).$$

Therefore $H_2{}'$ is generated by α and α', where

$$\alpha = k((m-1)!(n-m+1)!, 0, 0) = k(m-1)!(n-m+1)!(1, 0, 0),$$
$$\alpha' = k(0, 0, (m-1)!(n-m+2)!) = k(m-1)!(n-m+2)!(0, 0, 1).$$

Let $B' = \{(a, b, c)|(m-1)b \equiv c \mod 2\}$. We know that $(2n+5)$-skeleton of ΩW_{n+1} is $\Sigma^{2n}\mathbb{C}P^2 \simeq S^{2n+2} \cup e^{2n+4}$. Let $(a, b, c) \in Im\lambda'$, then there exists $g \in [X, \Omega W_{n+1}]$ such that

$$g^*(a_{2n+2}) = at^m\zeta_{2n-2m+2} + bt^{m-1}\zeta_{2n-2m+4}, \quad g^*(a_{2n+4}) = ct^m\zeta_{2n-2m+4}. \qquad (4.3)$$

Now apply Sq^2 to (4.3). Since $Sq^2(t^{m-1}) = (m-1)t^m$, $Sq^2(\zeta_{2n-4}) = 0$ and $Sq^2(a_{2n+2}) = a_{2n+4}$, we get $(m-1)b \equiv c \mod 2$. Thus we have the following lemma.

Lemma 4.3 $Im\lambda' \subseteq \{(a, b, c)|(m-1)b \equiv c \quad mod \quad 2\}$. □

In the following, we bring an application.

- $SU(n)$-gauge groups over $\mathbb{C}P^3$ where n is an even integer and $n \geq 4$

Now, we take $m = 3$. We need the following lemma.

Lemma 4.4 $Im\lambda' = \{(a, b, c)|2b \equiv c \quad mod \quad 2\}$.

Proof Let $B' = \{(a, b, c)|2b \equiv c \mod 2\}$. By Lemma 4.3, we have $Im\lambda' \subseteq B'$. Put $u' = (1, 0, 0)$, $v' = (0, 1, 0)$ and $w' = (0, 0, 2)$. For the converse case, we show that u', v' and w' are in Im λ'. Consider the following maps

$$\phi_1 : \mathbb{C}P^3 \wedge A \xrightarrow{q_1} S^6 \wedge A \xrightarrow{p_1} S^6 \wedge S^{2n-4} \hookrightarrow \Omega W_{n+1},$$
$$\phi_2 : \mathbb{C}P^3 \wedge A \xrightarrow{q_1} \mathbb{C}P^3/\mathbb{C}P^1 \wedge A \xrightarrow{p_1} S^4 \wedge A \xrightarrow{p_2} S^4 \wedge S^{2n-2} \hookrightarrow \Omega W_{n+1},$$
$$\phi_3 : \mathbb{C}P^3 \wedge A \xrightarrow{q_1} S^6 \wedge A \xrightarrow{p_2} S^6 \wedge S^{2n-2} \xrightarrow{\theta'} \Omega W_{n+1},$$

where p_1 and p_2 are pinch maps, q_1 is quotient map and θ' is the generator of $\pi_{2n+5}(W_{n+1})$. We have $\lambda'(\phi_1) = u'$, $\lambda'(\phi_2) = v'$ and $\lambda'(\phi_3) = w'$, respectively. Thus $Im(\lambda') = B'$. □

Put $\beta' = \{u', v', w'\}$. Since $u', v', w' \in Im\lambda'$ and generators of Im λ', therefore β' is a basis for Im λ'. Recall $p = (n+1)n(n-1)$. We have the following theorem.

Theorem 4.5 *$[X, U(n+1)]$ is isomorphic to* $\mathbb{Z}_{\frac{1}{6}(n+1)!} \oplus \mathbb{Z}_{\frac{1}{2}(n+1)!} \oplus \mathbb{Z}_{\frac{1}{4}(n+2)!}$.

Proof By Proposition 4.2, Im $\lambda' \circ (\Omega\pi)_*$ is generated by $\alpha'_{i,j}$ for $1 \leq i \leq 2$ and $1 \leq j \leq 3$. Note that under basis β', Im $\lambda' \circ (\Omega\pi)_*$ is generated by

$$\alpha'_{1,1} = (n-2)!\left(\frac{1}{6}p, 0, 0\right), \qquad \alpha'_{2,1} = (n-2)!\left(0, \frac{1}{2}p, \frac{1}{12}(n+2)p\right),$$
$$\alpha'_{1,2} = (n-2)!\left(\frac{1}{2}p, 0, 0\right), \qquad \alpha'_{2,2} = (n-2)!\left(0, p, \frac{1}{4}(n+2)p\right),$$
$$\alpha'_{1,3} = (n-2)!(2p, 0, 0), \qquad \alpha'_{2,3} = (n-2)!(0, 0, (n+2)p).$$

We represent the coordinate of Im $\lambda' \circ (\Omega\pi)_*$ by the following matrix

$$N = (n-2)! \begin{bmatrix} \frac{1}{6}p & 0 & 0 \\ 0 & \frac{1}{2}p & \frac{1}{12}(n+2)p \\ \frac{1}{2}p & 0 & 0 \\ 0 & p & \frac{1}{4}(n+2)p \\ 2p & 0 & 0 \\ 0 & 0 & (n+2)p \end{bmatrix}$$

Again, by using the Smith normal form, there exist invertible 6×6 and 3×3-matrices N' and N'' such that

$$N' \cdot N \cdot N'' = (n-2)! \begin{bmatrix} \frac{1}{6}p & 0 & 0 \\ 0 & \frac{1}{2}p & 0 \\ 0 & 0 & \frac{1}{4}(n+2)p \end{bmatrix}$$

where $N' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 3 & 0 & -1 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & -1 & 0 \\ 0 & -24 & 0 & 12 & 0 & -1 \end{bmatrix}$ and $N'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2}(n+2) \\ 0 & 0 & 3 \end{bmatrix}$. Therefore we can conclude that

$$[X, U(n+1)] \cong \mathbb{Z}_{\frac{1}{6}(n+1)!} \oplus \mathbb{Z}_{\frac{1}{2}(n+1)!} \oplus \mathbb{Z}_{\frac{1}{4}(n+2)!}.$$

□

We will prove the following proposition.

Proposition 4.6 $|H_2| = \frac{\frac{1}{4}(n+2)(n+1)n}{\left(\frac{1}{4}(n+2)(n+1)n,k\right)} \cdot \frac{\frac{1}{36}p}{\left(\frac{1}{36}p,k\right)}$.

Proof We know that the subgroup $H_2{}'$ is generated by α and α', where

$$\alpha = k(2(n-2)!, 0, 0) = k(n-2)!(2, 0, 0),$$
$$\alpha' = k(0, 0, 2(n-1)!) = k(n-2)!(0, 0, 2(n-1)).$$

Now under basis β', the subgroup $H_2{}'$ is generated by $\alpha = k(n-2)!(2, 0, 0)$ and $\alpha' = k(n-2)!(0, 0, n-1)$. We represent the coordinate of $H_2{}'$ by the following matrix

$$N_{H_2{}'} = k(n-2)! \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & n-1 \end{bmatrix}.$$

The new coordinate of $H_2{}'$ is as follow

$$N_{H_2{}'} \cdot N'' = k(n-2)! \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 3(n-1) \end{bmatrix}.$$

Recall $r = (n-2)!$. Then we have

$$\begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \cdot kr \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 3(n-1) \end{bmatrix} = kr \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & n-1 \end{bmatrix}.$$

Similar to the discussion in the proof of Proposition 3.7, we can conclude

$$|H_2| = \frac{\frac{1}{4}(n+2)(n+1)n}{\left(\frac{1}{4}(n+2)(n+1)n, k\right)} \cdot \frac{\frac{1}{36}p}{\left(\frac{1}{36}p, k\right)}.$$

□

The two cases are now being treated simultaneously.
Consider the map of $j_*\colon [X, SU(n)] \to [X, U(n+1)]$. We put

$$O_1 = \frac{\frac{1}{2}(n+2)(n+1)n}{\left(\frac{1}{2}(n+2)(n+1)n, k\right)} \cdot \frac{\frac{1}{36}p}{\left(\frac{1}{36}p, k\right)}, \quad O_2 = \frac{\frac{1}{4}(n+2)(n+1)n}{\left(\frac{1}{4}(n+2)(n+1)n, k\right)} \cdot \frac{\frac{1}{36}p}{\left(\frac{1}{36}p, k\right)}$$

Let P be the subgroup of $[X, SU(n)]$ generated by l_1 and l_2. We have the following lemma.

Lemma 4.7 *The following hold:*

$$|P| = \begin{cases} O_1 & \text{if } n \text{ is odd,} \\ O_2 & \text{if } n \text{ is even.} \end{cases}$$

Proof By definition of P and H_1, we have $j_*(P) = H_1$. When n is odd then the statement follows from Proposition 3.8 and when n is even then the statement follows from Proposition 4.6. □

5 Proof of Theorem 1.1

Apply the functor $[\Sigma A, -]$ to the fibration (2.1) to obtain the following exact sequence

$$[\Sigma A, \mathcal{G}_{0,k}(\mathbb{C}P^3)] \xrightarrow{(\Omega ev)_*} [\Sigma A, SU(n)] \xrightarrow{(\alpha_k)_*} [\Sigma A, Map^*_{0,k}(\mathbb{C}P^3, BSU(n))]$$
$$\to [\Sigma A, B\mathcal{G}_{0,k}(\mathbb{C}P^3)] \longrightarrow [\Sigma A, BSU(n)],$$

where $[\Sigma A, BSU(n)] \cong \tilde{K}^0(\Sigma A) \cong 0$. By adjunction, we have

$$[\Sigma A, Map^*_{0,k}(\mathbb{C}P^3, BSU(n))] \cong [\Sigma A \wedge \mathbb{C}P^3, BSU(n)].$$

The exact sequence becomes

$$[\Sigma A, \mathcal{G}_{0,k}(\mathbb{C}P^3)] \xrightarrow{(\Omega ev)_*} \tilde{K}^0(\Sigma^2 A) \xrightarrow{(\alpha_k)_*} [X, SU(n))] \to [\Sigma A, B\mathcal{G}_{0,k}(\mathbb{C}P^3)] \to 0.$$

Thus we get $[\Sigma A, B\mathcal{G}_{0,k}(\mathbb{C}P^3)] \cong Coker(\alpha_k)_*$. By definitions of α_k and P, the image of $(\alpha_k)_*$ is P. Let n be odd. If T is the order of $[X, SU(n)]$ then by exactness we have

$$T = |Im(\alpha_k)_*| \cdot |Coker(\alpha_k)_*| = |P| \cdot |Coker(\alpha_k)_*| = O_1 \cdot |Coker(\alpha_k)_*|.$$

Therefore $|Coker(\alpha_k)_*| = \frac{T}{O_1}$. Now suppose that $\mathcal{G}_{0,k}(\mathbb{C}P^3) \simeq \mathcal{G}_{0,k'}(\mathbb{C}P^3)$. Then there is an isomorphism of groups $[\Sigma A, B\mathcal{G}_{0,k}(\mathbb{C}P^3)] \cong [\Sigma A, B\mathcal{G}_{0,k'}(\mathbb{C}P^3)]$. Thus $|Coker(\alpha_k)_*| = |Coker(\alpha_{k'})_*|$. That is, $\frac{T}{O_1} = \frac{T}{O_1{}'}$, where

$$O_1{}' = \frac{\frac{1}{2}(n+2)(n+1)n}{\left(\frac{1}{2}(n+2)(n+1)n, k'\right)} \cdot \frac{\frac{1}{36}p}{\left(\frac{1}{36}p, k'\right)}.$$

Therefore we can conclude that if $\mathcal{G}_{0,k}(\mathbb{C}P^3) \simeq \mathcal{G}_{0,k'}(\mathbb{C}P^3)$ then

$$\left(\frac{1}{2}(n-1)n(n+1)(n+2), k\right) = \left(\frac{1}{2}(n-1)n(n+1)(n+2), k'\right).$$

If n is even, similarly we can conclude that if $\mathcal{G}_{0,k}(\mathbb{C}P^3) \simeq \mathcal{G}_{0,k'}(\mathbb{C}P^3)$ then

$$\left(\frac{1}{4}(n-1)n(n+1)(n+2), k\right) = \left(\frac{1}{4}(n-1)n(n+1)(n+2), k'\right). \qquad \square$$

Data availability Not applicable.

Declarations

Conflict of interest The author declares that I have no conflict of interest.

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