



# Some remarks on Vainikko integral operators in BV type spaces

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Ricordando Mimmo, collega e amico, con affetto, stima e gratitudine

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## Abstract

In this paper we study Vainikko integral operators which are similar to so-called cordial integral operators and contain the classical Hardy operator, the Schur operator, and the Hilbert transform as special cases. For such operators we obtain norm estimates and equalities, mainly in BV type spaces in the sense of Jordan, Wiener, Riesz, and Waterman. Several examples are also discussed.

**Keywords** Cordial integral operator · Vainikko integral operator · BV type space · Norm estimate

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## 1 Introduction

Domenico Candeloro (called “Mimmo” by his friends) was one of the leading specialists in the theory, methods, and applications of integral operators. He has given many important contributions to this field, with a particular emphasis on exotic measures, non-standard integrals, and multivalued maps. In the list of references at the end we mention only the more recent papers [2–25] he wrote, in part with coauthors, in the last 20 years.

In this paper we study a class of integral operators in a much simpler setting, using only single-valued scalar functions and integrals defined by the classical Lebesgue measure on the real line. In spite of their simplicity, we are convinced that our results on mapping properties of such integral operators would have been appreciated by Mimmo. In our discussion we will give particular attention to spaces of functions of bounded variation, a topic that is also very much *en vogue* in the Analysis School of the University of Perugia which owes so much to Mimmo’s scientific activity.

## 2 Cordial integral operators

Given a nonnegative  $L_1$  function  $\varphi : (0, 1) \rightarrow \mathbb{R}$ , in [27] the author defines an associated Volterra integral operator  $V_\varphi$  by

$$(V_\varphi x)(t) = \frac{1}{t} \int_0^t \varphi\left(\frac{s}{t}\right) x(s) ds = \int_0^1 \varphi(\tau) x(t\tau) d\tau. \quad (1)$$

Such operators are called *cordial*, the generating function  $\varphi$  the *core* of  $V_\varphi$ . In [27] and [28] the author gives necessary and sufficient conditions for  $V_\varphi$  to be bounded in the spaces  $C$ ,  $C^m$  and  $L_\infty$ . In the recent paper [29] he develops a parallel theory for Lebesgue spaces and proves that  $V_\varphi$  is bounded in  $L_p$  ( $1 \leq p \leq \infty$ ) iff

$$\int_0^1 s^{-1/p} |\varphi(s)| ds < \infty.$$

Moreover, he shows that in this case the norm  $\|V_\varphi\|_{L_p \rightarrow L_p}$  coincides with this integral. The spectrum and essential spectrum of the operator are also calculated.

Cordial integral operators have a series of remarkable properties. For example, one may show that the operator (1) has the continuum of eigenfunctions  $u_r(t) = t^r$  ( $0 \leq r < \infty$ ) in the space  $C$ , and so it cannot be compact. Moreover, the eigenvalues  $\lambda$  satisfy  $\|\lambda I - V_\varphi\|_{C \rightarrow C} = |\lambda| + \|\varphi\|_{L_1}$ ; in particular,  $\|V_\varphi\|_{C \rightarrow C} = \|\varphi\|_{L_1}$ .

Interestingly, a certain converse is also true: if the general Volterra operator

$$(Vx)(t) = \int_0^t k(t, s)x(s) ds$$

has the continuum of eigenfunctions  $u_r(t) = t^r$  ( $0 \leq r < \infty$ ), and its kernel function  $k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  satisfies some regularity requirements, then  $V = V_\varphi$  for  $\varphi := k(1, \cdot) \in L_1$ .

### 3 Vainikko integral operators in Lebesgue spaces

In this section we slightly modify the definition (1) to cover a wider range of examples. Given an  $L_1$  function  $\varphi : (0, \infty) \rightarrow \mathbb{R}$ , let us call the operator  $V_\varphi$  defined by

$$(V_\varphi x)(t) = \frac{1}{t} \int_0^\infty \varphi\left(\frac{s}{t}\right) x(s) ds = \int_0^\infty \varphi(\tau)x(t\tau) d\tau \tag{2}$$

*Vainikko integral operator* in the sequel. So the only difference between cordial and Vainikko operators is that we extend the integration in (2) over the semiaxis  $(0, \infty)$ . Prominent examples for such operators are the *Hardy operator*

$$(Hx)(t) = \frac{1}{t} \int_0^t x(s) ds \tag{3}$$

which has the form (2) for the choice  $\varphi(s) := \chi_{(0,1]}(s)$ , the *Schur operator*

$$(Sx)(t) = \int_0^\infty \frac{x(s)}{\max\{s, t\}} ds \tag{4}$$

which has the form (2) for the choice  $\varphi(s) := 1/\max\{1, s\}$ , and the (strongly singular) *Hilbert transform*

$$(Tx)(t) = \int_0^\infty \frac{x(s)}{s+t} ds \tag{5}$$

which has the form (2) for the choice  $\varphi(s) := 1/(1+s)$ .

Our first result gives a two-sided estimate for the norm of a Vainikko operator in the Lebesgue space  $L_p[0, \infty)$ . To this end, we will use the shortcut

$$\varphi_\theta(s) := s^\theta \varphi(s) \quad (s > 0), \tag{6}$$

where  $|\theta| \leq 1$ .

**Theorem 3.1** *For  $1 < p < \infty$ , suppose that the Vainikko integral operator  $V_\varphi$  defined by (2) maps  $L_p[0, \infty)$  into itself. Let  $\varphi : [0, \infty) \rightarrow \mathbb{R}$  be measurable with the property that*

$$\|\varphi_{-1/p}\|_{L_1} < \infty.$$

*Then  $V_\varphi$  is bounded and satisfies the estimate*

$$\left| \int_0^\infty s^{-1/p} \varphi(s) ds \right| \leq \|V_\varphi\|_{L_p \rightarrow L_p} \leq \int_0^\infty s^{-1/p} |\varphi(s)| ds. \tag{7}$$

*Consequently, in case  $\varphi \geq 0$  or  $\varphi \leq 0$ , the norm  $\|V_\varphi\|_{L_p \rightarrow L_p}$  of  $V_\varphi$  coincides with the right-hand side of (7).*

**Proof** Although the proof may be found in [29] for  $L_p[0, 1]$ , we give here another proof which requires slightly different arguments, since  $L_p(I)$  is not included in  $L_q(I)$  for  $q \leq p$  in case of an unbounded interval  $I$ . To estimate the norm  $\|V_\varphi\|_{L_p \rightarrow L_p}$ , we use the fact that the bilinear form

$$\langle y, z \rangle := \int_0^\infty y(t)z(t) dt \quad (y \in L_p, z \in L_{p'})$$

establishes a *duality* between  $L_p$  and  $L_{p'}$  for  $p' := p/(p-1)$ , in the sense that

$$\|y\|_{L_p} = \sup \{ \langle y, z \rangle : \|z\|_{L_{p'}} \leq 1 \}, \quad \|z\|_{L_{p'}} = \sup \{ \langle y, z \rangle : \|y\|_{L_p} \leq 1 \}.$$

For  $x \in L_p$  and  $y := V_\varphi x$  we get, by Fubini’s theorem

$$\begin{aligned} \langle y, z \rangle &= \int_0^\infty \left( \frac{1}{t} \int_0^\infty \varphi \left( \frac{s}{t} \right) x(s) ds \right) z(t) dt = \int_Q \frac{1}{t} \varphi \left( \frac{s}{t} \right) x(s) z(t) dt ds \\ &= \int_Q \left\{ \left( \frac{s}{t} \right)^{(p-1)/p^2} \left[ \frac{1}{t} \varphi \left( \frac{s}{t} \right) \right]^{1/p} x(s) \right\} \\ &\quad \times \left\{ \left( \frac{t}{s} \right)^{(p-1)/p^2} \left[ \frac{1}{t} \varphi \left( \frac{s}{t} \right) \right]^{1/p'} z(t) \right\} dt ds, \end{aligned}$$

where  $Q := (0, \infty) \times (0, \infty)$ . Therefore Hölder’s inequality implies

$$\begin{aligned} |\langle V_\varphi x, z \rangle| &\leq \left\{ \int_Q \left( \frac{s}{t} \right)^{1-1/p} \left| \varphi \left( \frac{s}{t} \right) \right| |x(s)|^p \frac{dt}{t} ds \right\}^{1/p} \\ &\quad \times \left\{ \int_Q \left( \frac{t}{s} \right)^{1/p} \left| \varphi \left( \frac{s}{t} \right) \right| |z(t)|^{p'} \frac{dt}{t} ds \right\}^{1/p'}. \end{aligned} \tag{8}$$

Using the change of variables  $s := t\tau$  we get

$$\int_0^\infty \left( \frac{s}{t} \right)^{1-1/p} \left| \varphi \left( \frac{s}{t} \right) \right| \frac{dt}{t} = \int_0^\infty \tau^{-1/p} |\varphi(\tau)| d\tau,$$

and applying again Fubini’s theorem we conclude that the first integral in (8) is

$$\int_Q \left( \frac{s}{t} \right)^{1-1/p} \left| \varphi \left( \frac{s}{t} \right) \right| |x(s)|^p \frac{dt}{t} ds = \left( \int_0^\infty \tau^{-1/p} |\varphi(\tau)| d\tau \right) \left( \int_0^\infty |x(s)|^p ds \right).$$

A similar calculation gives

$$\int_Q \left( \frac{t}{s} \right)^{1/p} \left| \varphi \left( \frac{s}{t} \right) \right| |z(t)|^{p'} \frac{dt}{t} ds = \left( \int_0^\infty \tau^{-1/p} |\varphi(\tau)| d\tau \right) \left( \int_0^\infty |z(t)|^{p'} dt \right)$$

for the second integral in (8). Combining these equalities we end up with

$$|\langle V_\varphi x, z \rangle| \leq \|x\|_{L_p} \|z\|_{L_{p'}} \left( \int_0^\infty \tau^{-1/p} |\varphi(\tau)| d\tau \right)^{1/p} \left( \int_0^\infty \tau^{-1/p} |\varphi(\tau)| d\tau \right)^{1/p'}$$

which in view of  $1/p + 1/p' = 1$  proves the claim. □

To illustrate Theorem 3.1 let us go back to our examples mentioned above.

**Example 3.1** For  $\varphi(s) := \chi_{(0,1]}(s)$  we have

$$\| \varphi_{-1/p} \|_{L_1} = \int_0^\infty s^{-1/p} \varphi(s) ds = \int_0^1 s^{-1/p} ds = \frac{p}{p-1} \tag{9}$$

which gives the precise norm  $\|H\|_{L_p \rightarrow L_p}$  of the Hardy operator (3) in  $L_p$ ; in particular,  $\|H\|_{L_2 \rightarrow L_2} = 2$ . Similarly, for  $\varphi(s) := 1/\max\{1, s\}$  we have

$$\| \varphi_{-1/p} \|_{L_1} = \int_0^\infty s^{-1/p} \varphi(s) ds = \int_0^1 s^{-1/p} ds + \int_1^\infty s^{-1-1/p} ds = \frac{p^2}{p-1} \tag{10}$$

which gives the precise norm  $\|S\|_{L_p \rightarrow L_p}$  of the Schur operator (4) in  $L_p$ ; in particular,  $\|S\|_{L_2 \rightarrow L_2} = 4$ . Finally, for  $\varphi(s) := 1/(1 + s)$  we have

$$\|\varphi_{-1/p}\|_{L_1} = \int_0^\infty s^{-1/p} \varphi(s) ds = \int_0^\infty \frac{s^{-1/p}}{1 + s} ds = \frac{\pi}{\sin(\pi/p)} \tag{11}$$

which gives the precise norm  $\|T\|_{L_p \rightarrow L_p}$  of the Hilbert transform (5) in  $L_p$ ; in particular,  $\|T\|_{L_2 \rightarrow L_2} = \pi$ .

The question arises whether or not it is possible to extend Theorem 3.1 to the extreme cases  $p = 1$  or  $p = \infty$ . Here, the norm equalities in Theorem 3.1 illustrate the difference between the operator (3), on the one hand, and the operators (4) and (5), on the other. Since the last expression in (9) tends to 1 as  $p \rightarrow \infty$ , say, one might hope that the Hardy operator also maps  $L_\infty$  into itself with  $\|H\|_{L_\infty \rightarrow L_\infty} = 1$ ; this may be in fact verified by a simple calculation. On the other hand, since the last expression in (10) or (11) tends to  $\infty$  as  $p \rightarrow \infty$ , one might suspect that the Schur operator and the Hilbert transform do not map  $L_\infty$  into itself. Indeed, for  $x(t) \equiv 1$  we get

$$(Sx)(t) = \int_0^\infty \frac{ds}{\max\{s, t\}} = 1 + \int_t^\infty \frac{ds}{s}, \quad (Tx)(t) = \int_0^\infty \frac{ds}{s + t} = \int_t^\infty \frac{ds}{s},$$

and so  $Sx \notin L_\infty$  and  $Tx \notin L_\infty$ .

### 4 Vainikko integral operators in Wiener spaces

In view of the importance of integral operators in spaces of functions of bounded (classical or generalized) variation, it seems reasonable to study the operator (2) in the space  $BV$  and its various generalizations. This is the purpose of this section. In contrast to  $L_p$ -spaces, however,  $BV$ -type spaces have a reasonable norm only for functions on compact intervals, but not on the semiaxis  $[0, \infty)$ . Since our main emphasis is on norm estimates in this paper, in what follows we consider functions  $x : [0, 1] \rightarrow \mathbb{R}$ . In this case for norm estimates we will use the fact that

$$\|\varphi_\theta\|_{L_1} \leq \|\varphi\|_{L_1}, \quad |(V_\varphi x)(0)| = \left| \int_0^1 \varphi(\tau)x(0) d\tau \right| \leq \|\varphi\|_{L_1} |x(0)|. \tag{12}$$

Given  $p \in [1, \infty)$  and a partition  $P := \{t_0, t_1, \dots, t_{m-1}, t_m\}$  of  $[0, 1]$ , we denote by

$$Var_p(x, P; [0, 1]) := \sum_{j=1}^m |x(t_j) - x(t_{j-1})|^p$$

the Wiener  $p$ -variation of  $x : [0, 1] \rightarrow \mathbb{R}$  w.r.t.  $P$ , and by

$$Var_p(x) = Var_p(x; [0, 1]) := \sup_P Var_p(x, P; [0, 1])$$

its total Wiener  $p$ -variation on  $[0, 1]$ . In case  $Var_p(x; [0, 1]) < \infty$  we write  $x \in BV_p[0, 1]$ . The set  $BV_p[0, 1]$  equipped with the norm

$$\|x\|_{BV_p} := |x(0)| + Var_p(x; [0, 1])^{1/p}$$

is a Banach space [32]. A particular important special case is of course  $p = 1$ , where we obtain the classical *Jordan variation*

$$Var(x) = Var(x; [0, 1]) := \sup_P Var(x, P; [0, 1]) = \sup_P \sum_{j=1}^m |x(t_j) - x(t_{j-1})|$$

and the space  $BV_1[0, 1] = BV[0, 1]$  equipped with the norm  $\|x\|_{BV} := |x(0)| + Var(x; [0, 1])$ .

**Theorem 4.1** *In case  $\varphi \in L_p$  the operator  $V_\varphi$  is bounded in  $BV_p$  and satisfies the estimates*

$$\left| \int_0^1 \varphi(s) ds \right| \leq \|V_\varphi\|_{BV_p \rightarrow BV_p} \leq \|\varphi\|_{L_p}. \tag{13}$$

**Proof** Fix  $x \in BV_p$  and a partition  $P := \{t_0, t_1, \dots, t_{m-1}, t_m\}$  of  $[0, 1]$ . Then we get, by Jensen’s inequality,

$$\begin{aligned} \sum_{j=1}^m |(V_\varphi x)(t_j) - (V_\varphi x)(t_{j-1})|^p &= \sum_{j=1}^m \left| \int_0^1 \varphi(s) [x(st_j) - x(st_{j-1})] ds \right|^p \\ &\leq \int_0^1 |\varphi(s)|^p \sum_{j=1}^m |x(st_j) - x(st_{j-1})|^p ds \leq \|\varphi\|_{L_p}^p Var_p(x). \end{aligned}$$

Combining this estimate with (12) we conclude that  $\|V_\varphi x\|_{BV_p} \leq \|\varphi\|_{L_p} \|x\|_{BV_p}$ , which proves the upper estimate in (13).

For the proof of the lower estimate it suffices to take  $e(t) \equiv 1$  and to note that, for all  $p \geq 1$ ,  $\|e\|_{BV_p} = 1$ ,  $\|V_\varphi e\|_{BV_p} = \left| \int_0^1 \varphi(s) ds \right|$ . □

Since  $BV_1 = BV$ , we get the estimate

$$\left| \int_0^1 \varphi(s) ds \right| \leq \|V_\varphi\|_{BV \rightarrow BV} \leq \int_0^1 |\varphi(t)| dt \tag{14}$$

as a special case of (13). Consequently, in case  $\varphi \geq 0$  or  $\varphi \leq 0$  a.e. on  $[0, 1]$  the norm  $\|V_\varphi\|_{BV \rightarrow BV}$  coincides with the  $L_1$ -norm of  $\varphi$ . It is not clear whether or not this is also true if  $\varphi$  changes its sign on subsets of positive measure. However, the lower and upper bounds in the estimates (13) may drift apart the more “symmetric”  $\varphi$  changes sign and has large absolute values, as the following example suggests.

**Example 4.1** For  $c > 0$ , consider the bang-bang function  $\psi_c : [0, 1] \rightarrow \mathbb{R}$  defined by  $\psi_c(t) := c\chi_{[0,1/2]}(t) - c\chi_{(1/2,1]}(t)$ . Clearly,  $\|\psi_c\|_{L_p} = c$  for all  $p \in [1, \infty]$ , and  $\int_0^1 \psi_c(t) dt = 0$ . Moreover, it is not hard to see that  $\psi_c \in BV_p$  for all  $p \geq 1$  with

$$\|\psi_c\|_{BV_p} = |\psi_c(0)| + Var_p(\psi_c; [0, 1])^{1/p} = 3c. \tag{15}$$

So we may apply the operator  $V_{\psi_c}$  to  $\psi_c$  itself and obtain, by (1),

$$(V_{\psi_c} \psi_c)(t) = \int_0^1 \psi_c(\tau) \psi_c(t\tau) d\tau = \begin{cases} 0 & \text{for } 0 \leq t \leq 1/2, \\ c^2(2 - 1/t) & \text{for } 1/2 < t \leq 1. \end{cases}$$

This implies that

$$\|V_{\psi_c} \psi_c\|_{BV_p} = |V_{\psi_c} \psi_c(0)| + Var_p(V_{\psi_c} \psi_c; [0, 1])^{1/p} = c^2. \tag{16}$$

Thus, a comparison of (15) and (16) shows that  $\|V_{\psi_c}\|_{BV_p \rightarrow BV_p} \geq c/3$ . We conclude that the lower bound in (13) is 0, while the upper bound is  $c$  which may become arbitrarily large.

### 5 Vainikko integral operators in Riesz spaces

Another generalization of the space  $BV$  is due to Riesz [26]. Given  $p \in [1, \infty)$  and a partition  $P := \{t_0, t_1, \dots, t_{m-1}, t_m\}$  of  $[0, 1]$  as before, we denote by

$$RVar_p(x, P; [0, 1]) := \sum_{j=1}^m \frac{|x(t_j) - x(t_{j-1})|^p}{(t_j - t_{j-1})^{p-1}}$$

the Riesz  $p$ -variation of a function  $x$  w.r.t.  $P$ . The total Riesz  $p$ -variation of  $x : [0, 1] \rightarrow \mathbb{R}$  on  $[0, 1]$  is then given by

$$RVar_p(x) = RVar_p(x; [0, 1]) := \sup_P RVar_p(x, P; [0, 1]).$$

In case  $RVar_p(x; [0, 1]) < \infty$  we write  $x \in RBV_p[0, 1]$ . The set  $RBV_p[0, 1]$  equipped with the norm

$$\|x\|_{RBV_p} := |x(0)| + RVar_p(x; [0, 1])^{1/p}$$

is a Banach space [26]. In spite of their similarity, the spaces  $BV_p$  and  $RBV_p$  have quite different properties. First of all, the scale of spaces  $BV_p$  is increasing in  $p$ , while the scale of spaces  $RBV_p$  is decreasing in  $p$ . Moreover, every function in the Riesz space  $RBV_p$  is continuous for  $p > 1$ , but  $RBV_1 = BV$  contains of course many discontinuous functions. However, the most interesting property of Riesz spaces is that, for  $1 < p < \infty$ , from  $x \in RBV_p$  it follows that  $x$  is absolutely continuous with  $x' \in L_p$  and  $RVar_p(x) = \|x'\|_{L_p}^p$ , and vice versa [1]. This means that Riesz discovered Sobolev spaces, at least in the scalar case, 20 years before Sobolev. This fact allows us to use Theorem 3.1 for finding a condition for  $V_\varphi$  to map  $RBV_p$  into itself.

**Theorem 5.1** *In case  $\varphi \in L_1$  the operator  $V_\varphi$  is bounded in  $RBV_p$  and satisfies the estimates*

$$\left| \int_0^1 \varphi(s) ds \right| \leq \|V_\varphi\|_{RBV_p \rightarrow RBV_p} \leq \|\varphi\|_{L_1}. \tag{17}$$

Consequently, in case  $\varphi \geq 0$  or  $\varphi \leq 0$ , the norm  $\|V_\varphi\|_{RBV_p \rightarrow RBV_p}$  of  $V_\varphi$  coincides with the right-hand side of (17).

**Proof** The case  $p = 1$  is covered by (14), so let  $1 < p < \infty$ . In this case,  $x$  is absolutely continuous and  $x' \in L_p$ . Moreover, using the second integral in (2) we see that

$$(V_\varphi x)'(t) = \int_0^1 \varphi(\tau) \tau x'(t\tau) d\tau = (V_{\varphi_1} x')(t), \tag{18}$$

where we have used the notation (6). From  $\varphi \in L_1$  it follows that also  $\varphi_{1-1/p} \in L_1$ ; so Vainikko's result implies that  $V_\varphi$  maps  $RBV_p$  into itself and, by (18),

$$\begin{aligned} RVar_p(V_\varphi x) &= \|V_{\varphi_1} x'\|_{L_p}^p \leq \|V_{\varphi_1}\|_{L_p \rightarrow L_p}^p \|x'\|_{L_p}^p \leq \|\varphi_{1-1/p}\|_{L_1}^p RVar_p(x) \\ &\leq \|\varphi\|_{L_1}^p RVar_p(x). \end{aligned}$$

So together with (12) this yields  $\|V_\varphi x\|_{RBV_p} \leq |(V_\varphi x)(0)| + RVar_p(V_\varphi x)^{1/p} \leq \|\varphi\|_{L_1} \|x\|_{RBV_p}$ .

The lower estimate in (17) is proved exactly as before. □

Observe that  $RBV_\infty = Lip$ , the linear space of all Lipschitz continuous maps with norm  $\|x\|_{Lip} = |x(0)| + lip(x; [0, 1])$ , where  $lip(x; [0, 1]) := \sup \left\{ \frac{|x(s)-x(t)|}{|s-t|} : 0 \leq s, t \leq 1, s \neq t \right\}$ . So we get in addition as a fringe benefit from Theorem 5.1 the following

**Theorem 5.2** *The operator  $V_\varphi$  is bounded in  $Lip$  iff  $\varphi_1 \in L_1$ . In this case we have*

$$\left| \int_0^1 \varphi(s) ds \right| \leq \|V_\varphi\|_{Lip \rightarrow Lip} \leq \|\varphi\|_{L_1}.$$

Consequently, in case  $\varphi \geq 0$  or  $\varphi \leq 0$ , the norm  $\|V_\varphi\|_{Lip \rightarrow Lip}$  of  $V_\varphi$  coincides with  $\|\varphi\|_{L_1}$ .

Observe that this result for the space  $Lip$  may be easily proved directly. In fact, from  $|x(s) - x(t)| \leq L|s - t|$  it follows that  $|(V_\varphi x)(s) - (V_\varphi x)(t)| \leq \int_0^1 \varphi(\tau) |x(\tau s) - x(\tau t)| d\tau \leq L \|\varphi_1\|_{L_1} |s - t|$ , hence  $lip(V_\varphi x) \leq \|\varphi_1\|_{L_1} lip(x) \leq \|\varphi\|_{L_1} lip(x)$ .

Let us illustrate our results by means of our “test animals”, the Hardy operator (3), the Schur operator (4), and the Hilbert transform (5).

For  $\varphi(s) := \chi_{(0,1]}(s)$  we have  $\|\varphi\|_{L_p}^p = \int_0^\infty \chi_{(0,1]}(s)^p ds \equiv 1$  and  $\|\varphi_{1-1/p}\|_{L_1} = \int_0^\infty s^{1-1/p} \chi_{(0,1]}(s) ds = \int_0^1 s^{1-1/p} ds < \infty$  for any  $p \geq 1$ . So the Hardy operator is bounded in  $BV_p$  for  $1 \leq p < \infty$  and in  $RBV_p$  for  $1 \leq p \leq \infty$ .

On the other hand, for  $\varphi(s) := 1/\max\{1, s\}$  we have  $\|\varphi\|_{L_p}^p = \int_0^1 1 ds + \int_1^\infty s^{-p} ds < \infty$  for  $p > 1$ , and  $\|\varphi_{1-1/p}\|_{L_1} = \int_0^1 s^{1-1/p} ds + \int_1^\infty s^{-1/p} ds = \infty$ . So the Schur operator maps the space  $BV_p$  for  $p > 1$  into itself, but none of the spaces  $RBV_p$ .

Finally, for  $\varphi(s) := 1/(1 + s)$  we have  $\|\varphi\|_{L_p}^p = \int_0^\infty \frac{ds}{(1+s)^p} = \int_1^\infty \frac{ds}{s^p} < \infty$  for  $p > 1$ , and  $\|\varphi_{1-1/p}\|_{L_1} = \int_0^\infty \frac{s^{1-1/p}}{1+s} ds = \infty$  for  $p > 1$ . So for  $p > 1$  the Hilbert transform maps the space  $BV_p$  into itself, but not the space  $RBV_p$ .

We close this section with another operator which depends on a real parameter  $\alpha$  and is, in contrast to the Hilbert transform, weakly singular.

**Example 5.1** For  $\alpha > 0$ , consider the *Liouville operator*

$$(L_\alpha x)(t) = \frac{1}{t^\alpha} \int_0^t s^{\alpha-1} x(s) ds. \tag{19}$$

This operator has the form (2) for the choice  $\varphi(s) := s^{\alpha-1} \chi_{(0,1]}(s)$ . Since  $\|\varphi_{-1/p}\|_{L_1} = \int_0^1 s^{-1/p} s^{\alpha-1} ds = \frac{1}{\alpha-1/p}$ , the operator (19) maps  $L_p$  for  $p\alpha > 1$  into itself. Moreover, since  $\|\varphi\|_{L_p}^p = \int_0^1 s^{(\alpha-1)p} ds$  and  $\|\varphi_{1-1/p}\|_{L_1} = \int_0^1 s^{\alpha-1/p} ds$ , the Liouville operator is bounded in  $BV_p$  for  $\alpha > 1 - 1/p$ , i.e.,  $p < 1/(1 - \alpha)$ , and in  $RBV_p$  for every  $\alpha > 0$ .

### 6 Vainikko integral operators in Waterman spaces

Finally, let us recall yet another  $BV$ -type space which was introduced by Waterman [30] and has very interesting applications.

Let  $\Lambda := (\lambda_k)_k$  be a positive decreasing sequence satisfying  $\lim_{k \rightarrow \infty} \lambda_k = 0$  and  $\sum_{k=1}^\infty \lambda_k = \infty$ . Recall that the Waterman variation of a function  $x : [0, 1] \rightarrow \mathbb{R}$  w.r.t. a collection  $S = \{[a_k, b_k] : k = 1, 2, \dots, n\}$  (with  $n$  variable) of pairwise non-overlapping



intervals  $[a_k, b_k] \subset [0, 1]$  is defined by

$$Var_{\Lambda}(x, S; [0, 1]) := \sum_{k=1}^n \lambda_k |x(b_k) - x(a_k)|,$$

and the total *Waterman variation* of  $x$  on  $[0, 1]$  by

$$Var_{\Lambda}(x; [0, 1]) := \sup_S Var_{\Lambda}(x, S; [0, 1]).$$

In case  $Var_{\Lambda}(x; [a, b]) < \infty$  we write  $x \in \Lambda BV[0, 1]$ . The set  $\Lambda BV[0, 1]$  equipped with the norm

$$\|x\|_{\Lambda BV} := |x(0)| + Var_{\Lambda}(x; [0, 1])$$

is a Banach space [30]. A typical example is of course  $\Lambda_q := (k^{-q})_k$  for  $0 < q \leq 1$ ; this has important applications to Fourier series. Thus, in [31] it was shown that, for  $f \in \Lambda_q BV$ , the Fourier series of  $f$  is everywhere  $(C, \beta)$ -bounded for  $\beta = q - 1$ , and  $(C, \alpha)$ -summable for  $\alpha > q - 1$ . Moreover, these estimates for  $\alpha$  and  $\beta$  are sharp. The starting point for the study of Waterman spaces was the choice  $q = 1$ ; in this case the elements of the space  $\Lambda_1 BV =: HBV$  are called functions of bounded *harmonic variation*.

**Theorem 6.1** *In case  $\varphi \in L_1$  the operator  $V_{\varphi}$  is bounded in  $\Lambda BV$  and satisfies the estimates*

$$\left| \int_0^1 \varphi(s) ds \right| \leq \|V_{\varphi}\|_{\Lambda BV \rightarrow \Lambda BV} \leq \|\varphi\|_{L_1}. \tag{20}$$

Consequently, in case  $\varphi \geq 0$  or  $\varphi \leq 0$ , the norm  $\|V_{\varphi}\|_{Lip \rightarrow Lip}$  of  $V_{\varphi}$  coincides with  $\|\varphi\|_{L_1}$ .

**Proof** Fix  $x \in \Lambda BV$  and a collection  $S = \{[a_k, b_k] : k = 1, 2, \dots, n\}$  of non-overlapping intervals  $[a_k, b_k] \subset [0, 1]$ . Then

$$\begin{aligned} \sum_{k=1}^n \lambda_k |(V_{\varphi}x)(b_k) - (V_{\varphi}x)(a_k)| &= \sum_{k=1}^n \lambda_k \left| \int_0^1 \varphi(s) [x(sb_k) - x(sa_k)] ds \right| \\ &\leq \int_0^1 |\varphi(s)| \sum_{k=1}^n \lambda_k |x(sb_k) - x(sa_k)| ds \leq \|\varphi\|_{L_1} Var_{\Lambda}(x). \end{aligned}$$

Adding this estimate to (12) we see that  $\|V_{\varphi}x\|_{\Lambda BV} \leq \|\varphi\|_{L_1} \|x\|_{\Lambda BV}$ , which proves the upper estimate in (20). For the proof of the lower estimate we take the same function  $e(t) \equiv 1$  as in the proof of Theorem 4.1. □

To conclude, let us summarize our norm estimates for the most important example, the Hardy operator (3), in the following

**Example 6.1** We already know that  $\|H\|_{L_p \rightarrow L_p} = p/(p - 1)$  for  $1 < p < \infty$  and  $\|H\|_{L_{\infty} \rightarrow L_{\infty}} = 1$ . Since  $\int_0^1 \varphi(s) ds = 1$  and  $\|\varphi\|_{L_p} = 1$  for  $\varphi(s) = \chi_{(0,1]}(s)$ , from Theorems 4.1, 5.1 and 6.1 we conclude that  $\|H\|_{BV_p \rightarrow BV_p} = \|H\|_{RBV_p \rightarrow RBV_p} = \|H\|_{\Lambda BV \rightarrow \Lambda BV} = 1$ .

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**Compliance with ethical standards**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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