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Essential dimension of inifinitesimal commutative unipotent group schemes

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Abstract
We propose a generalization of Ledet's conjecture, which predicts the essential dimension of cyclic p-groups in characteristic p, for finite commutative unipotent group schemes. And we present some evidence for this conjecture and discuss some consequences.

Keywords Group schemes · Torsors · Essential dimension

Mathematics Subject Classification 14L15 · 14L30 · 14F20

1 Introduction

The notion of essential dimension of a finite group over a field k was introduced by Buhler and Reichstein [2]. It was later extended to various contexts. First Reichstein generalized it to linear algebraic groups [11] in characteristic zero; afterwards Merkurjev gave a general definition for functors from the category of extension fields of the base field k to the category of sets [1]. In particular one can consider the essential dimension of group schemes over a field (see Definition 1.1).

If G is a flat group scheme of locally finite presentation over a scheme S, a G-torsor over X is an S-scheme Y with a left G-action by X-automorphisms and a faithfully flat and locally of finite presentation morphism $Y \to X$ over S such that the map $G \times_S Y \to Y \times_X Y$ given by $(g, y) \mapsto (gy, y)$ is an isomorphism. We recall that isomorphism classes of G-torsors over X are classified by the pointed set $\mathrm{H}^1(X, G)$ if G is affine [8, III, Theorem 4.3] or G an abelian scheme and X is regular [10, Proposition XIII 2.6]. We will restrict to these two cases. If G is commutative, then $\mathrm{H}^1(X, G)$ is a group, and coincides with the cohomology group of G in the fppf topology.

Definition 1.1 Let G be a group scheme of finite type over a field k. Let $k \subseteq K$ be an extension field and $[\xi] \in \mathrm{H}^1(\mathrm{Spec}(K), G)$ the class of a G-torsor ξ. Then the essential dimension of ξ over k, which we denote by $\mathrm{ed}_k\, \xi$, is the smallest non-negative integer n such that

(i) there exists a subfield L of K containing k, with $\mathrm{tr\,deg}(L/k) = n$,

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(ii) such that $[\xi]$ is in the image of the morphism

$$\mathrm{H}^1(\mathrm{Spec}(L), G) \longrightarrow \mathrm{H}^1(\mathrm{Spec}(K), G).$$

The essential dimension of G over k, which we denote by $\mathrm{ed}_k\, G$, is the supremum of $\mathrm{ed}_k\, \xi$, where K/k ranges through all the extension of K, and ξ ranges through all the G-torsors over $\mathrm{Spec}(K)$.

We study essential dimension of finite commutative unipotent group scheme over a field k of positive characteristic p. We recall the following conjecture due to Ledet [6].

Conjecture 1.2 The essential dimension of the cyclic group of order p^n over k is n.

We remark that it is known that that the essential dimension should be at most n. So the difficult part is the other inequality. We stress that this conjecture is known only for $n \leq 2$ [7, Propositions 5 and 7].

Here we propose a generalization of this conjecture. For any commutative group scheme G over k one can define a morphism $V : G^{(p)} \to G$, where $G^{(p)}$ is the fiber product of the morphism $G \to \mathrm{Spec}(k)$ and the absolute Frobenius $\mathrm{Spec}(k) \to \mathrm{Spec}(k)$. This morphism is called Verschiebung. See [5, IV,§3, n^o 4] for the definition. We remark that it can be defined also as the dual of the relative Frobenius $\mathrm{F} : G^\vee \to G^{\vee(p)}$ where $G^\vee$ is the Cartier dual of G.

Definition 1.3 Let G be a commutative unipotent group scheme over k. We call *V-exponent* for G the minimal integer $n \geq 1$ such that $V^n = 0$. We note it by $n_V(G)$.

This number exists since G is unipotent.

Conjecture 1.4 Let k be a field of positive characteristic and let G be a finite unipotent commutative group scheme. Then $\mathrm{ed}_k\, G \geq n_V(G)$.

In fact it is easy to see that Conjecture 1.2 is equivalent to Conjecture 1.4 in the case of finite commutative étale group schemes (Lemma 2.1).

We present some evidences for this conjecture and discuss some consequences. For instance we prove that the above conjecture is true if G is annihilated by the relative Frobenius (see Proposition 2.3). Moreover we will consider the case of V-exponent equal 2 : as remarked above, the conjecture is known in the étale case while we will precise what remains to do in the infinitesimal case. Finally we prove in Proposition 4.1, under the assumption that the conjecture is true, that the essential dimension of a nontrivial abelian variety over a field of positive characteristic is $+\infty$. We recall that the same statement is true over number fields [4, Theorem 2]. While over an algebraically closed field of characteristic zero it is two times the dimension of the abelian variety [3, Theorem 1.2].

2 Conjecture 1.4 vs Ledet's conjecture

In the following k is a field of positive characteristic p. The following Lemma proves that in fact Ledet's Conjecture is just a particular case of this conjecture.

Lemma 2.1 *Ledet's conjecture is equivalent to Conjecture 1.4 restricted to finite unipotent commutative étale group schemes.*

Proof We firstly prove that Conjecture 1.4 implies Ledet's conjecture. In fact the Verschiebung of $\mathbb{Z}/p^n\mathbb{Z}$ is just multiplication by p^n. So, since p^{n-1} is not trivial over $\mathbb{Z}/p^n\mathbb{Z}$, then by Conjecture 1.4 we have that $\mathrm{ed}_k\, \mathbb{Z}/p^n\mathbb{Z} \geq n$. On the other hand since $\mathbb{Z}/p^n\mathbb{Z}$ is contained in the special group of Witt vectors of length n, which has dimension n, then $\mathrm{ed}_k\, \mathbb{Z}/p^n\mathbb{Z} \leq n$.

Conversely, let us suppose Ledet's conjecture is true. Let G be a finite commutative unipotent *étale* group scheme over a field k of positive characteristic. Since essential dimension does not increase if we pass from k to its algebraic closure $\overline{k}$ [1, Prop 1.5] we can assume k algebraically closed. So we have that G is the product of cyclic p-groups. Let us take a direct summand of maximal order p^n. Then n is such that $V^n = 0$ but $V^{n-1} \neq 0$. By [1, Theorem 6.19] we have that $\mathrm{ed}_k\, G \geq \mathrm{ed}_k\, \mathbb{Z}/p^n\mathbb{Z} = n$, where the last equality follows from Ledet's Conjecture. So we are done. □

Remark 2.2 In fact we have proved slightly more: Ledet's conjecture for a fixed n is equivalent to Conjecture 1.4 restricted to finite unipotent commutative étale group schemes with V-exponent equal to n.

Some cases of Conjecture 1.4 can be proved using results of [12]. For instance we have the following proposition which treats the case orthogonal to the étale case.

Proposition 2.3 *The conjecture 1.4 is true for finite unipotent commutative group schemes of height* 1 *(i.e. annihilated by Frobenius).*

Proof In [12, Theorem 1.2] it has been proved that for a finite group scheme the essential dimension is greater than or equal the dimension of its Lie Algebra. In the case of the proposition the order of the group scheme is p^n, where n is the dimension of the Lie Algebra.

Since, by the lemma below, we have that $V^n = 0$ the conjecture is proven in this case. □

Lemma 2.4 *Let k be a field of characteristic p. The operator V^n is trivial over any unipotent commutative group scheme G of order p^n.*

Proof We consider V^n as morphism $G^{(p^n)} \to G$. Since G is unipotent the kernel of V is not trivial, so in particular the image of V has order strictly less than n. Iterating the argument and applying it to the subgroup image, we have that the image of V^{i+1} is strictly contained in the image of V^i, for any $i \geq 0$. So after n-iteration the image is trivial. □

3 *V*-exponent equal to two

Here some easy considerations about the conjecture.

Lemma 3.1 *Let k be a field of positive characteristic and let G_1 and G_2 be two finite commutative unipotent group schemes over k.*

(i) Let $f: G_1 \to G_2$ be an epimorphism (resp. monomorphism) of group schemes with $n_V(G_1) = n_V(G_2)$. If the Conjecture 1.4 is true for G_2 (resp. G_1) it is true for G_1 (resp. G_2).

(ii) If the Conjecture 1.4 is true for G_1 and G_2 then it is true for $G_1 \times G_2$.

(iii) It is sufficient to prove the Conjecture 1.4 under the following assumptions

(1) k is algebraically closed;

(2) G is contained in $W_{n,k}$, Witt vectors (of length n) group scheme, where $n = n_V(G)$.

(iv) It is sufficient to prove the conjecture for étale finite group schemes and infinitesimal group schemes.

Remark 3.2 It is easy to prove that if f is an epimorphism (resp. monomorphism) then one always has $n_V(G_1) \geq n_V(G_2)$ (resp. $n_V(G_1) \leq n_V(G_2)$).

Proof (i) Let us suppose f is an epimorphism. Then

$$0 \longrightarrow \ker f \longrightarrow G_1 \longrightarrow G_2 \longrightarrow 0$$

is exact. Then, since $\ker f$ is unipotent commutative, for any extension K of k we have that $H^2(\mathrm{Spec}(K), \ker f) = 0$ [12, Lemma 3.3]. So $H^1(\mathrm{Spec}(K), G_1) \to H^1(\mathrm{Spec}(K), G_2)$ is surjective. This implies, by [1, Lemma 1.9], that $\mathrm{ed}_k\, G_1 \geq \mathrm{ed}_k\, G_2$. So we have

$$\mathrm{ed}_k\, G_1 \geq \mathrm{ed}_k\, G_2 \geq n_V(G_2) = n_V(G_1)$$

and we are done.
If f is a monomorphism it is even easier. In fact we have $\mathrm{ed}_K\, G_1 \leq \mathrm{ed}_K\, G_2$ by [1, Theorem 6.19] and so we can conclude as above, switching G_1 with G_2.

(ii) It is sufficient to remark that the V-exponent of $G_1 \times G_2$ is the maximum between the V-exponent of G_1 and that one of G_2. So the result comes from (i) using as morphism the projection over the group scheme with greater V-exponent.

(iii) The essential dimension does not increase after a base change by [1, Proposition 1.5]. So we assume k algebraically closed. Let $n = n_V(G)$. Since $V_G^n = 0$, by [5, V,§1, Proposition 2.5], there exists r such that G is contained in $W_{n,k}^r$. For any $i = 1, \ldots, r$ let p_i the projection of $W_{n,k}^r$ on the i^{th} components and let $G_i = p_i(G)$. Since V^{n-1} is different from zero and the Verschiebung is compatible with morphisms there exists an i_0 such that V^{n-1} is different from 0 over G_{i_0}. Moreover we have an epimorphism $G \to G_{i_0}$. So it follows from part (i) that if the conjecture is true for G_{i_0} it is true for G.

(iv) Over an algebraically closed field a finite commutative group scheme is the direct product of its étale part and its connected (hence infinitesimal) part. So the statement follows from (ii). □

We now treat the case of finite unipotent group schemes with $n_V(G) = 2$. The case with trivial Verschiebung is immediate since nontrivial finite group schemes have positive essential dimension.

Proposition 3.3 *The conjecture is true for group schemes G with $n_V(G) = 2$ if and only if it is true for the group scheme* $\ker(V - \mathrm{F}^m) : W_{2,k} \to W_{2,k}$, *for $m \geq 1$.*

Proof Clearly we have only to prove the *if part*. Using Lemma 3.1(iv), we have to prove the conjecture only for infinitesimal group schemes since the étale case, which we can reduce to $G = \mathbb{Z}/p^2\mathbb{Z}$ by Remark 2.2, is known [1, Proposition 7.10]. In particular the Frobenius is not injective. By the above Lemma we can suppose that k is algebraically closed and we can suppose that G is contained in $W_{2,k}$. We have to prove that the essential dimension is at least 2. Let us consider the exact sequence

$$0 \longrightarrow \ker \mathrm{F} \longrightarrow G \longrightarrow \mathrm{F}(G) \longrightarrow 0.$$

If $V(\ker \mathrm{F}) \neq 0$ then we are done by Lemma 3.1(i) and Proposition 2.3. If $V(\mathrm{F}(G)) \neq 0$ we are reduced to prove the conjecture for $\mathrm{F}(G)$, again by Lemma 3.1(i). If m is the smallest integer such that $\mathrm{F}^m(G) = 0$ then iterating the argument (at most $m - 1$ times) we finally have two possibilities: we have to prove the conjecture for a group scheme with $V(\ker \mathrm{F}) =$

$V(\mathrm{F}(G)) = 0$ or for a group annihilated by F (this happens if we have to iterate exactly $m-1$ times). The second case is already known by Proposition 2.3. Therefore we can suppose we are in the first case. Therefore $\ker \mathrm{F}$ and $\mathrm{F}(G)$ are contained in $\mathbb{G}_{a,k}$. This means that $\ker \mathrm{F} = \alpha_{p,k}$ and $\mathrm{F}(G) = \alpha_{p^m,k}$ for some m. So we can suppose that we have an exact sequence

$$0 \longrightarrow \alpha_{p,k} \longrightarrow G \longrightarrow \alpha_{p^m,k} \longrightarrow 0.$$

Now we have that V and F^m induce a morphism from G to $\alpha_{p,k}$. Moreover

$$\mathrm{Hom}_k(G, \alpha_{p,k}) = \mathrm{Hom}_k(G/\,\mathrm{F}(G^{(1/p)}), \alpha_{p,k}).$$

But G has order p^{m+1} and $\mathrm{F}(G^{(1/p)})$ has order p^m, so $G/\,\mathrm{F}(G^{(1/p)}) \simeq \alpha_{p,k}$. Then

$$\mathrm{Hom}_k(G, \alpha_{p,k}) = \mathrm{Hom}_k(\alpha_{p,k}, \alpha_{p,k}) = k.$$

This implies that

$$V = a\,\mathrm{F}^m$$

for some $a \in k$. Therefore G is contained in $\ker(V - a\,\mathrm{F}^m) : W_{2,k} \to W_{2,k}$. Since these two group schemes have the same order, as it is easy to check, they are equal.

Finally we remark that since k is algebraically closed it is straightforward to prove that, if we fix m, the group schemes $\ker(V - a\,\mathrm{F}^m)$ are pairwise isomorphic.

□

If $m = 1$ and k is algebraically closed the group scheme in the Proposition is nothing else that the p-torsion group scheme of a supersingular elliptic curve.

Finally we give a consequence of the conjecture for group schemes of essential dimension one.

Proposition 3.4 *Let us suppose that conjecture 1.4 is true for group schemes with V-exponent* 2.

(i) If k is algebraically closed, a finite commutative unipotent group scheme has essential dimension 1 *if and only if it is isomorphic to $\alpha_{p^m,k} \times (\mathbb{Z}/p\mathbb{Z})^r$ for some $m, r > 0$.*

(ii) An infinitesimal commutative unipotent group scheme over a perfect field has essential dimension 1 *if and only if it is isomorphic to $\alpha_{p^m,k}$ for some $m > 0$.*

Proof We now prove (i). Let k be algebraically closed. One has just to prove the *only if* part since $\alpha_{p^m,k} \times (\mathbb{Z}/p\mathbb{Z})^r$ is contained in $\mathbb{G}_{a,k}$ if k is algebraically closed . First of all we remark that if the essential dimension is 1 then the Verschiebung is trivial. In fact if $n_G(V) > 1$ then $V^{n_G(V)-2}(G^{(p^{n_G(V)-2})})$ is a subgroup scheme of G with V-exponent 2. Therefore if the conjecture is true for group schemes with V-exponent 2 then G would have essential dimension strictly greater than 1. So $V = 0$. Then by [5, IV §3, Corollaire 6.9] we have that G is isomorphic to $\prod_{i=1}^{l} \alpha_{p^{n_i},k} \times (\mathbb{Z}/p\mathbb{Z})^r$ for some $l, n_i, r > 0$. Since the essential dimension of G is one then, by [12, Theorem 1.2], we have $\dim \mathrm{Lie}(G) \le 1$. So G is isomorphic to $\alpha_{p^m,k} \times (\mathbb{Z}/p\mathbb{Z})^r$ for some $m, r > 0$

Using (i), to prove (ii) we have just to show that $\alpha_{p^m,k}$ has no nontrivial twisted forms. This follows from the fact that $\mathrm{Aut}_{\bar{k}}(\alpha_{p^n,\bar{k}})$ is isomorphic to $\mathbb{G}_{m,\bar{k}} \times \mathbb{G}_{a,\bar{k}}^{m-1}$. Since twisted forms are classified by the first cohomology of this group then there is only one twisted form. □

Remark 3.5 For étale group schemes the result is unconditional since the conjecture is true ([7, Proposition 5 and 7]).

4 Abelian varieties in positive characteristic

Finally we give a consequence of the conjecture for the essential dimension of abelian varietis in positive characteristic.

Proposition 4.1 *Let A be a nontrivial abelian variety over a field k of characteristic $p > 0$. If the Conjecture 1.4 is true then*

$$\mathrm{ed}_k\, A = +\infty.$$

Proof As usual we can suppose that k is algebraically closed. For any positive integer n we call $A[p^n]$ the group scheme of p^n-torsion of A. Since $A[p^n] \subseteq A$, by [3, Principle 2.9][1], we have that

$$\mathrm{ed}_k\, A[p^n] \leq \mathrm{ed}_k\, A + \dim A. \tag{4.1}$$

Now by [9, pag. 147] we have that there exists an integer $r \geq 0$ such that

$$A[p^n] \simeq (\mathbb{Z}/p^n\mathbb{Z} \times \mu_{p^n})^r \times G_n^0$$

where G_n^0 is unipotent and infinitesimal. If $r = 0$ then we remark that V^{n-1} is not trivial over $G_n^0 = A[p^n]$ otherwise $A[p^n] = A[p^{n-1}]$. If $r > 0$ then $V^{n-1}((\mathbb{Z}/p^n\mathbb{Z})^r) \neq 0$. So, if we call U_n the unipotent part of $A[p^n]$ we have that V^{n-1} is not trivial over U_n. So by conjecture 1.4, and [1, Theorem 6.19], we have that

$$\mathrm{ed}_k\, A[p^n] \geq \mathrm{ed}\, U_n \geq n.$$

which, together with (4.1), gives $\mathrm{ed}_k\, A = +\infty$. □

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[1] We remark that the proof of [1, Theorem 6.19], which we used before and which is the standard reference for this result, works only for affine group schemes since it uses versal torsors. We remark that in [1] an *algebraic group* is intended to be affine.

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