

Weak solutions for a thermoelectric problem with power-type boundary effects

Luisa Consiglieri¹ 

Received: 21 August 2017 / Accepted: 10 January 2018 / Published online: 18 January 2018
© Unione Matematica Italiana 2018

Abstract This paper deals with thermoelectric problems including the Peltier and Seebeck effects. The coupled elliptic and doubly quasilinear parabolic equations for the electric and heat currents are stated, respectively, under power-type boundary conditions that describe the thermal radiative effects. To verify the existence of weak solutions to this coupled problem (Theorem 1), analytical investigations for abstract multi-quasilinear elliptic-parabolic systems with nonsmooth data are presented (Theorems 2 and 3). They are essentially approximated solutions based on the Rothe method. It consists on introducing time discretized problems, establishing their existence, and then passing to the limit as the time step goes to zero. The proof of the existence of time discretized solutions relies on fixed point and compactness arguments. In this study, we establish quantitative estimates to clarify the smallness conditions.

Keywords Doubly quasilinear parabolic equation · Quantitative estimates · Time discretization · Thermoelectric system

Mathematics Subject Classification 35K51 · 35R05 · 35J62 · 35Q79

1 Introduction

The study of the heat equation with constant coefficients is a simplification from both mathematical and engineering points of view. From the real world point of view, constant coefficients are not appropriate because the density and the thermal conductivity both depend on the temperature itself, and often also on the space variable. The concern of discontinuous

Dedicated to my colleague and friend Eduardo Corregedor Borges Pires.

✉ Luisa Consiglieri
lconsiglieri@gmail.com
<http://sites.google.com/site/luisaconsiglieri>

¹ Pateo Central, 1600-256 Lisboa, Portugal

leading coefficient is being a long matter of study in the mathematical literature, as long as the works [22,25]. The complete concern is achieved by the doubly quasilinear parabolic equation [1,3,5]. It is well known that the determination of estimates is the crucial key in the theory of partial differential equations (PDE), which involve the so called universal bounds. These bounds are only qualitative and they do not have any practical use on the real world applications because their abstract form. In their majority, the expression of the universal bounds would be truly cumbersome if the proof of estimates was remade step by step, or even impossible if the contradiction argument is applied. Also regularity estimates have being a subject of study in the last decades [11,12,15], but these ones only occur by admitting data smoothness. With this in mind, our main objective is to find quantitative estimates, i.e. their involved constants have an explicit expression, that are useful on the real applications. In particular, the quantitative estimates clarify the smallness conditions on the data when a fixed point argument is used.

For the two-dimensional space situation, a first attempt on the finding smallness conditions that assure the existence and regularity results for some thermoelectric problems is presented in [9,10], where some domain dependent constants were kept abstract. Indeed, the central result, which is only 2D valid, is provided by some higher regularity. This particular technique makes the smallness conditions quite bizarre. Here, we establish more elegant smallness conditions and they are extended to the n -dimensional space situation, by finding weak solutions. The present model also extends the thermal effects, of the previous works [9,10], to the unsteady state.

Existence of solutions for parabolic-elliptic systems with nonlinear no-flux boundary conditions is not a new idea if taking constant coefficients into account [4]. Application of elliptic PDE system in divergence form with Dirichlet boundary conditions in doubly-connected domain of the plane are given in [7] to the problem of electrical heating of a conductor whose thermal and electrical conductivities depend on the temperature and to the flow of a viscous fluid in a porous medium, taking into account the Soret and Dufour effects. In [8], the authors deal with a traditional RLC circuit in which a thermistor has been inserted, representing the microwave heating process with temperature-induced modulations on the electric field. In particular, the existence of a solution to a coupled system of three differential equations (an ODE, an elliptic equation and a nonlinear parabolic PDE) and appropriate initial and boundary conditions is proved. A one-dimensional thermal analysis for the performance of thermoelectric cooler is conducted in [13] under the influence of the Thomson effect, the Joule heating, the Fourier heat conduction, and the radiation and convection heat transfer. Simulation studies have been performed to investigate the thermal balance affected by anode shorting in an aluminum reduction cell [6].

The method of discretization in time, whose basic idea (coming from the implicit Euler formula) was investigated by Rothe, is a very well-known effective technique for both theoretical and numerical analysis, [14,17,24] and [21,26], respectively (see also the pioneering work [1] of Alt and Luckhaus).

This paper is organized as follows. The thermoelectric (TE) model is introduced in Sect. 2. After discussing the physical model, the main result with respect to this model is formulated with a detailed description of the relevant constants. In Sect. 3, one abstract model related to the problem under consideration is introduced to simplify the proofs of the existence results of time-discretized solutions (Sect. 3), and their corresponding steady-state solutions (Sect. 4). Indeed, the analysis of the problem is structured via two different approaches to exemplify alternative assumptions on the data smallness, namely the existence results of time-discretized solutions (Sects. 5.1 and 5.2), and their corresponding steady-state solutions (Sects. 4.1 and 4.2).

2 The thermoelectric model

Let $[0, T] \subset \mathbb{R}$ be the time interval with $T > 0$ being an arbitrary (but preassigned) time. Let Ω be a bounded domain (that is, connected open set) in \mathbb{R}^n ($n \geq 2$). Its boundary is constituted by two disjoint open $(n - 1)$ -dimensional sets $\partial\Omega = \overline{\Gamma_N} \cup \overline{\Gamma}$. We consider Γ_N over which the Neumann boundary condition is taken into account, and Γ over which the radiative effects may occur. Each one, Γ_N and Γ , may be alternatively of zero $(n - 1)$ -Lebesgue measure. Set $Q_T = \Omega \times]0, T[$ and $\Sigma_T = \Gamma \times]0, T[$.

The electrical current density \mathbf{j} and the energy flux density $\mathbf{J} = \mathbf{q} + \phi\mathbf{j}$, with \mathbf{q} being the heat flux vector, are given by the constitutive relations (see [9] and the references therein)

$$\mathbf{q} = -k(\cdot, \theta)\nabla\theta - \Pi(\cdot, \theta)\sigma(\cdot, \theta)\nabla\phi; \tag{1}$$

$$\mathbf{j} = -\alpha_S(\cdot, \theta)\sigma(\cdot, \theta)\nabla\theta - \sigma(\cdot, \theta)\nabla\phi. \tag{2}$$

Here, θ denotes the absolute temperature, ϕ is the electric potential, α_S represents the Seebeck coefficient, and the Peltier coefficient $\Pi(\theta) = \theta\alpha_S(\theta)$ is due to the first Kelvin relation. The electrical conductivity σ , and the thermal conductivity $k = k_T + \Pi\alpha_S\sigma$, with k_T denotes the purely conductive contribution, are, respectively, the known positive coefficients of Ohm and Fourier laws.

The Seebeck coefficient α_S has a constant sign corresponding to the behaviour of the charge carriers as it occurs in the Hall effect under the existence of a magnetic field. With positive sign ($\alpha_S > 0$), there are as examples: the alkali metals Li, Rb and Cs [2, p. 17], and the noble metals Ag and Au [2, p. 49, 192] or [20, p. 71]. With negative sign ($\alpha_S < 0$), there are as examples: the alkali metals Na and K [20, p. 97], the transition metals Fe and Ni [2, p. 215], and the semiconductor Pb [2, p. 48]. We refer to [10, p. 3], and the references therein, for more examples and their increase and decrease behaviors.

Although heat generation starts instantaneously when the current begins to flow, it takes time before the heat transfer process is initiated to allow the transient conditions to disappear. Thus, the electrical current density \mathbf{j} and the energy flux density \mathbf{J} satisfy

$$\begin{cases} \nabla \cdot \mathbf{j} = 0 & \text{in } \Omega \\ -\mathbf{j} \cdot \mathbf{n} = g & \text{on } \Gamma_N \\ \mathbf{j} \cdot \mathbf{n} = 0 & \text{on } \Gamma \end{cases} \tag{3}$$

$$\begin{cases} \rho(\cdot, \theta)c_v(\cdot, \theta)\partial_t\theta - \nabla \cdot \mathbf{J} = 0 & \text{in } Q_T \\ \mathbf{J} \cdot \mathbf{n} = 0 & \text{on } \Gamma_N \times]0, T[\\ -\mathbf{J} \cdot \mathbf{n} = \gamma(\cdot, \theta)|\theta|^{\ell-2}\theta - h & \text{on } \Sigma_T, \end{cases} \tag{4}$$

for $\ell \geq 2$. Here, ρ denotes the density, c_v denotes the heat capacity (at constant volume), \mathbf{n} is the unit outward normal to the boundary $\partial\Omega$, and g denotes the surface current source,

The boundary operators, γ and h , are temperature dependent functions that express, respectively, the radiative convection depending on the wavelength, and the external heat sources. For $\ell = 5$, the Stefan–Boltzmann radiation law says that $\gamma(T) = \sigma_{SB}\epsilon(T)$ and $h(T) = \sigma_{SB}\alpha(T)\theta_e^{\ell-1}$, where $\sigma_{SB} = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$ is the Stefan-Boltzmann constant for blackbodies, and θ_e denotes an external temperature. The parameters, the emissivity ϵ and the absorptivity α , both depend on the space variable and the temperature function θ . If $\ell = 2$, the boundary condition corresponds to the Newton law of cooling with heat transfer coefficient $\gamma = h/\theta_e^{\ell-1}$.

In the framework of Sobolev and Lebesgue functional spaces, we use the following spaces of test functions:

$$\begin{aligned}
 V &= \left\{ \begin{aligned} V(\Omega) &= \{v \in H^1(\Omega) : \int_{\Omega} v dx = 0\} \\ V(\partial\Omega) &= \{v \in H^1(\Omega) : \int_{\partial\Omega} v ds = 0\} \end{aligned} \right\} \\
 V_{\ell}(\Omega) &= \{v \in H^1(\Omega) : v|_{\Gamma} \in L^{\ell}(\Gamma)\}; \\
 V_{\ell}(Q_T) &= \{v \in L^2(0, T; H^1(\Omega)) : v|_{\Sigma_T} \in L^{\ell}(\Sigma_T)\},
 \end{aligned}$$

with their usual norms, $\ell > 1$. Notice that $V_{\ell}(\Omega) \equiv H^1(\Omega)$ if $\ell \leq 2_*$, where 2_* is the critical trace exponent, i.e. $2_* = 2(n - 1)/(n - 2)$ if $n > 2$ and $2_* > 1$ is arbitrary if $n = 2$.

In the presence of the previous considerations, it is expected that the temperature-potential pair is neither regular nor even bounded. The thermoelectric problem is formulated as follows. (TE) Find the temperature-potential pair (θ, ϕ) such that it verifies the variational problem:

$$\begin{aligned}
 &\int_0^T \langle \rho(\cdot, \theta)c_v(\cdot, \theta)\partial_t\theta, v \rangle dt + \int_{Q_T} k(\cdot, \theta)\nabla\theta \cdot \nabla v dx dt \\
 &+ \int_{Q_T} \sigma(\cdot, \theta) (T_{\mathcal{M}}(\phi)\alpha_S(\cdot, \theta)\nabla\theta + (\Pi(\cdot, \theta) + T_{\mathcal{M}}(\phi))\nabla\phi) \cdot \nabla v dx dt \\
 &+ \int_{\Sigma_T} \gamma(\cdot, \theta)|\theta|^{\ell-2}\theta v ds dt = \int_{\Sigma_T} h(\cdot, \theta)v ds dt; \tag{5}
 \end{aligned}$$

$$\int_{\Omega} \sigma(\cdot, \theta)\nabla\phi \cdot \nabla w dx + \int_{\Omega} \sigma(\cdot, \theta)\alpha_S(\cdot, \theta)\nabla\theta \cdot \nabla w dx = \int_{\Gamma_N} g w ds, \quad \text{a.e. in }]0, T[, \tag{6}$$

for every $v \in V_{\ell}(Q_T)$ and $w \in V$, where $\langle \cdot, \cdot \rangle$ accounts for the duality product, and $T_{\mathcal{M}}$ is the \mathcal{M} -truncation function defined by $T_{\mathcal{M}}(z) = \max(-\mathcal{M}, \min(\mathcal{M}, z))$.

We assume the following.

(H1) The density and the heat capacity $\rho, c_v : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions, i.e. measurable with respect to $x \in \Omega$ and continuous with respect to $e \in \mathbb{R}$. Furthermore, they verify

$$\exists b^{\#}, b_{\#} > 0 : b_{\#} \leq \rho(x, e)c_v(x, e) \leq b^{\#}, \quad \text{for a.e. } x \in \Omega, \quad \forall e \in \mathbb{R}. \tag{7}$$

(H2) The thermal and electrical conductivities $k, \sigma : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions. Furthermore, they verify

$$\exists k^{\#}, k_{\#} > 0 : k_{\#} \leq k(x, e) \leq k^{\#}; \tag{8}$$

$$\exists \sigma^{\#}, \sigma_{\#} > 0 : \sigma_{\#} \leq \sigma(x, e) \leq \sigma^{\#} \quad \text{for a.e. } x \in \Omega, \quad \forall e \in \mathbb{R}. \tag{9}$$

(H3) The Seebeck and Peltier coefficients $\alpha_S, \Pi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions such that

$$\exists \alpha^{\#} > 0 : |\alpha_S(x, e)| \leq \alpha^{\#}; \tag{10}$$

$$\exists \Pi^{\#} > 0 : |\Pi(x, e)| \leq \Pi^{\#}, \quad \text{for a.e. } x \in \Omega, \quad \forall e \in \mathbb{R}. \tag{11}$$

(H4) The boundary function h belongs to $L^{\ell'}(\Sigma_T)$.

(H5) The boundary function g belongs to $L^2(\Gamma_N)$.

(H6) The boundary operator γ is a Carathéodory function from $\Sigma_T \times \mathbb{R}$ into \mathbb{R} such that

$$\exists \gamma_{\#}, \gamma^{\#} > 0 : \gamma_{\#} \leq \gamma(x, t, e) \leq \gamma^{\#}; \quad \text{for a.e. } (x, t) \in \Sigma_T, \quad \forall e \in \mathbb{R}. \tag{12}$$

Moreover, γ is strongly monotone:

$$(\gamma(u)|u|^{\ell-2}u - \gamma(v)|v|^{\ell-2}v)(u - v) \geq \gamma_{\#}|u - v|^{\ell}.$$

Let us state our main existence theorem.

Theorem 1 *Let (H1)–(H6) be fulfilled. The thermoelectric problem (TE) admits a solution $(\theta, \phi) \in V_\ell(Q_T) \times L^2(0, T; V)$, for \mathcal{M} being such that*

$$\mathcal{M}\alpha^\# \sigma^\# < k_\#, \tag{13}$$

and one of the following hypothesis is assured:

1. there holds

$$4(k_\# - \mathcal{M}\alpha^\# \sigma^\#)\sigma_\# > (\sigma^\#)^2(\Pi^\# + \mathcal{M} + \alpha^\#)^2; \tag{14}$$

2. there holds

$$4(k_\# - \mathcal{M}\alpha^\# \sigma^\#) > \sigma^\#(\Pi^\# + \mathcal{M} + \alpha^\#)^2; \tag{15}$$

3. there holds

$$k_\# > \sigma^\# \alpha^\# (2\Pi^\# + 3\mathcal{M}). \tag{16}$$

3 Existence of approximated solutions

The thermoelectric problem provides the abstract initial boundary value problem

$$b(\theta)\partial_t\theta - \nabla \cdot (a(\theta, \phi)\nabla\theta) = \nabla \cdot (\sigma(\theta)F(\theta, \phi)\nabla\phi); \tag{17}$$

$$-\nabla \cdot (\sigma(\theta)\nabla\phi) = \nabla \cdot (\sigma(\theta)\alpha_S(\theta)\nabla\theta) \text{ in } Q_T; \tag{18}$$

$$(a(\theta, \phi)\nabla\theta + \sigma(\theta)F(\theta, \phi)\nabla\phi) \cdot \mathbf{n} = (h - \gamma(\theta)|\theta|^{\ell-2}\theta)\chi_\Gamma; \tag{19}$$

$$(\sigma(\theta)\nabla\phi + \sigma(\theta)\alpha_S(\theta)\nabla\theta) \cdot \mathbf{n} = g\chi_{\Gamma_N} \text{ on } \partial\Omega \times]0, T[. \tag{20}$$

This abstract problem is formulated in the form that the coefficients are correlated with the leading coefficient σ . We emphasize that this interrelation must be clear.

Let us assume the hypothesis set.

(H) The operators a, F and b, σ, α_S are Carathéodory functions from $\Omega \times \mathbb{R}^2$ and $\Omega \times \mathbb{R}$, respectively, into \mathbb{R} , which enjoy the following properties. There exist positive constants $F^\#, a_\#, \alpha^\#, b_\#, b^\#$ such that

$$|F(x, e, d)| \leq F^\#; \tag{21}$$

$$a_\# \leq a(x, e, d) \leq a^\#; \tag{22}$$

$$b_\# \leq b(x, e) \leq b^\# \text{ for a.e. } x \in \Omega, \quad \forall e, d \in \mathbb{R}, \tag{23}$$

and $\sigma_\#, \sigma^\#, \alpha^\#$ verifying (9) and (10), respectively.

Definition 1 We say that (θ, ϕ) is a weak solution to (17)–(20) if it solves the variational problem

$$\begin{aligned} & \int_0^T \langle b(\cdot, \theta)\partial_t\theta, v \rangle dt + \int_{Q_T} a(\cdot, \theta, \phi)\nabla\theta \cdot \nabla v dx dt + \int_{\Sigma_T} \gamma(\cdot, \theta)|\theta|^{\ell-2}\theta v ds dt \\ & = - \int_{Q_T} \sigma(\theta)F(\cdot, \theta, \phi)\nabla\phi \cdot \nabla v dx dt + \int_{\Sigma_T} h(\cdot, \theta)v ds dt; \end{aligned} \tag{24}$$

$$\begin{aligned} & \int_\Omega \sigma(\cdot, \theta)\nabla\phi \cdot \nabla w dx + \int_\Omega \sigma(\cdot, \theta)\alpha_S(\cdot, \theta)\nabla\theta \cdot \nabla w dx \\ & = \int_{\Gamma_N} g w ds, \text{ a.e. in }]0, T[, \end{aligned} \tag{25}$$

for every $v \in V_\ell(Q_T)$ and $w \in V$.

We define an auxiliary operator. Denote by B the operator from $H^1(\Omega)$ into $L^2(\Omega)$ defined by

$$B(v) = \int_0^v b(\cdot, z)dz, \tag{26}$$

for all $v \in H^1(\Omega)$.

Different approaches in the finding of solutions according to Definition 1 provide different smallness conditions (27), (28) or (31). We emphasize that the difference between these smallness conditions has its importance in the real world applications.

Theorem 2 *Let (H) and (H4)–(H6) be fulfilled. If there exists $\varepsilon > 0$ such that one of the following relations holds, that is, either*

$$a_\# > \varepsilon\sigma^\#(F^\# + \alpha^\#)/2 \quad \text{and} \quad \varepsilon\sigma_\# > \sigma^\#(F^\# + \alpha^\#)/2, \tag{27}$$

or

$$a_\# > \varepsilon\sqrt{\sigma^\#}(F^\# + \alpha^\#)/2 \quad \text{and} \quad \varepsilon > \sqrt{\sigma^\#}(F^\# + \alpha^\#)/2, \tag{28}$$

then the variational problem (24), (25) admits a sequence of approximate solutions $\{(\theta_M, \phi_M)\}_{M \in \mathbb{N}}$ in the sense established in Sect. 5.1.

The proof of Theorem 2 relies on the limit solution to the recurrent sequence of time-discretized problems

$$\begin{aligned} & \frac{1}{\tau} \int_\Omega B(\theta^m)vdx + \int_\Omega a(\theta^m, \phi^m)\nabla\theta^m \cdot \nabla vdx + \int_\Gamma \gamma(\theta^m)|\theta^m|^{\ell-2}\theta^m vds \\ & + \int_\Omega \sigma(\theta^m)F(\theta^m, \phi^m)\nabla\phi^m \cdot \nabla vdx = \frac{1}{\tau} \int_\Omega B(\theta^{m-1})vdx + \int_\Gamma h_m vds; \end{aligned} \tag{29}$$

$$\int_\Omega \sigma(\theta^m)\nabla\phi^m \cdot \nabla wdx + \int_\Omega \sigma(\theta^m)\alpha_S(\theta^m)\nabla\theta^m \cdot \nabla wdx = \int_{\Gamma_N} g wds, \tag{30}$$

where τ is the so called time step, B is defined in (26), $m \in \mathbb{N}$ and h_m is conveniently chosen in Sect. 5 (the time discretization technique). We call ϕ^m the corresponding solution to the time independent temperature θ^m .

Theorem 3 *Let (H) and (H4)–(H6) be fulfilled. If there holds*

$$a_\# > 2\sigma^\#\alpha^\#F^\#, \tag{31}$$

then the variational problem (24), (25) admits a sequence of approximate solutions $\{(\theta_M, \phi_M)\}_{M \in \mathbb{N}}$ in the sense established in Sect. 5.2.

The proof of Theorem 3 relies on the limit solution to the recurrent sequence of time-discretized problems

$$\begin{aligned} & \frac{1}{\tau} \int_\Omega B(\theta^m)vdx + \int_\Omega a(\theta^m, \phi^m)\nabla\theta^m \cdot \nabla vdx + \int_\Gamma \gamma(\theta^m)|\theta^m|^{\ell-2}\theta^m vds \\ & + \int_\Omega \sigma(\theta^m)F(\theta^m, \phi^m)\nabla\phi^m \cdot \nabla vdx = \frac{1}{\tau} \int_\Omega B(\theta^{m-1})vdx + \int_\Gamma h_m vds; \end{aligned} \tag{32}$$

$$\begin{aligned} & \int_\Omega \sigma(\theta^{m-1})\nabla\phi^m \cdot \nabla wdx = - \int_\Omega \sigma(\theta^{m-1})\alpha_S(\theta^{m-1})\nabla\theta^{m-1} \cdot \nabla wdx \\ & + \int_{\Gamma_N} g wds, \end{aligned} \tag{33}$$

where τ is the time step, B is defined in (26), $m \in \mathbb{N}$ and h_m is conveniently chosen in Sect. 5 (the time discretization technique). We call ϕ^m the corresponding solution to the time independent temperature θ^{m-1} .

4 Steady-state solvability

In this section, we prove the existence of solutions to the recurrent sequence of time-discretized problems (29), (30) and (32), (33) in Sects. 4.1 and 4.2, respectively. Since $m \in \mathbb{N}$ is fixed and $\theta^{m-1} \in V_\ell(\Omega)$ is given, for the sake of simplicity, we set $f = B(\theta^{m-1})$ and $H = h_m$, and we omit the index to the unknown pair, i.e. we simply write (θ, ϕ) .

Denoting by K_2 the continuity constant of the trace embedding $H^1(\Omega) \hookrightarrow L^2(\Gamma)$, with $2_* = 2(n-1)/(n-2)$ if $n > 2$, and any $2_* > 2$ if $n = 2$, and by P_2 the Poincaré constant correspondent to the space exponent 2, the constant $K_2(P_2 + 1)$ obeys

$$\|v\|_{2,\Gamma} \leq K_2 (\|v\|_{2,\Omega} + \|\nabla v\|_{2,\Omega}) \leq K_2(P_2 + 1)\|\nabla v\|_{2,\Omega}, \quad \forall v \in H^1(\Omega). \quad (34)$$

Let us introduce [1, 16]

$$\Psi(s) := B(s)s - \int_0^s B(r)dr = \int_0^s (B(s) - B(r))dr.$$

We state the main properties of the auxiliary operators B and Ψ , the ones that we will use later. For completeness sake, we sketch the proof of the property (35).

Lemma 1 *There holds*

$$\int_{\Omega} (B(u) - B(v))u dx \geq \int_{\Omega} \Psi(u) dx - \int_{\Omega} \Psi(v) dx. \quad (35)$$

In particular, if the Assumption (23) is fulfilled then there holds

$$\int_{\Omega} \Psi(u) dx \leq \int_{\Omega} B(u)u dx \leq b^\# \|u\|_{2,\Omega}^2.$$

Under the Assumption (23) the operator B verifies

$$(B(u) - B(v), u - v) \geq b_\# \|u - v\|_{2,\Omega}^2.$$

Proof Let us write the decomposition

$$(B(u) - B(v))u = B(u)u - B(v)v - B(v)(u - v).$$

Thanks to the mean value theorem for definite integrals, there exists c between u and v such that

$$\int_v^u B(r)dr = B(c)(u - v).$$

Since $-B$ is a decreasing function, we obtain

$$\int_{\Omega} (B(u) - B(v))u dx \geq \int_{\Omega} (B(u)u - B(v)v) dx - \int_{\Omega} \int_v^u B(r)dr dx,$$

which concludes the proof by definition of Ψ .

Finally, we recall the following remarkable lemma [1, Lemma 1.9].

Lemma 2 *Suppose u_m weakly converge to u in $L^p(0, T; W^{1,p}(\Omega))$, $p > 1$, and the estimates*

$$\int_{\Omega} \Psi(u_m(t))dx \leq C \text{ for } 0 < t < T,$$

and for $z > 0$

$$\int_0^{T-z} \int_{\Omega} (B(u_m(t+z)) - B(u_m(t)))(u_m(t+z) - u_m(t))dxdt \leq Cz, \tag{36}$$

hold with C being positive constants. Then, $B(u_m) \rightarrow B(u)$ in $L^1(Q_T)$ and $\Psi(u_m) \rightarrow \Psi(u)$ almost everywhere in Q_T .

4.1 Fixed point argument (solvability to (29) and (30))

Let $\ell \geq 2$, and define an operator \mathcal{T} from $\mathbf{V}_\ell = V_\ell(\Omega) \times V$ into itself such that $(\theta, \phi) = \mathcal{T}(\mathbf{u})$ is the unique solution of Proposition 1.

Proposition 1 *Let $\mathbf{u} = (u_1, u_2) \in \mathbf{V}_\ell$, and $u = u_1$. Then, there exists a unique solution $(\theta, \phi) \in \mathbf{V}_\ell$ to the Neumann-power-type elliptic problem*

$$\begin{aligned} & \frac{1}{\tau} \int_{\Omega} b(u)\theta v dx + \int_{\Omega} a(\mathbf{u})\nabla\theta \cdot \nabla v dx + \int_{\Omega} \sigma(u)F(\mathbf{u})\nabla\phi \cdot \nabla v dx \\ & + \int_{\Gamma} \gamma(u)|\theta|^{\ell-2}\theta v ds = \frac{1}{\tau} \int_{\Omega} f v dx + \int_{\Gamma} H v ds; \end{aligned} \tag{37}$$

$$\int_{\Omega} \sigma(u)\nabla\phi \cdot \nabla w dx + \int_{\Omega} \sigma(u)\alpha_S(u)\nabla\theta \cdot \nabla w dx = \int_{\Gamma_N} g w ds, \tag{38}$$

for all $v \in V_\ell(\Omega)$ and $w \in V$. In addition, the following estimate

$$\begin{aligned} & \frac{b_{\#}}{2\tau} \|\theta\|_{2,\Omega}^2 + (L_1)_{\#} \|\nabla\theta\|_{2,\Omega}^2 + \frac{(L_2)_{\#}}{2} \|\nabla\phi\|_{2,\Omega}^2 + \frac{\gamma_{\#}}{\ell'} \|\theta\|_{\ell,\Gamma}^{\ell} \leq \frac{1}{2\tau b_{\#}} \|f\|_{2,\Omega}^2 \\ & + \frac{1}{\ell' \gamma_{\#}^{1/(\ell-1)}} \|H\|_{\ell',\Gamma}^{\ell'} + \frac{(K_2)^2(P_2 + 1)^2}{2(L_2)_{\#}} \|g\|_{2,\Gamma_N}^2 := \mathcal{R}(\|f\|_{2,\Omega}^2, \|H\|_{\ell',\Gamma}^{\ell'}) \end{aligned} \tag{39}$$

holds true, if provided by one of the following definitions

$$\begin{cases} (L_1)_{\#} = a_{\#} - \varepsilon\sigma^{\#}(F^{\#} + \alpha^{\#})/2 \\ (L_2)_{\#} = \sigma_{\#} - \sigma^{\#}(F^{\#} + \alpha^{\#})/(2\varepsilon) \end{cases} \tag{40}$$

$$\begin{cases} (L_1)_{\#} = a_{\#} - \varepsilon\sqrt{\sigma^{\#}}(F^{\#} + \alpha^{\#})/2 \\ (L_2)_{\#} = \sigma_{\#}(1 - \sqrt{\sigma^{\#}}(F^{\#} + \alpha^{\#})/(2\varepsilon)). \end{cases} \tag{41}$$

Proof The existence of a solution to the variational system (37), (38) relies on the direct application of the Browder–Minty Theorem [18]. Indeed, the form $\mathcal{F} : \mathbf{V}_\ell \rightarrow \mathbb{R}$ defined by

$$\mathcal{F}(v, w) = \frac{1}{\tau} \int_{\Omega} f v dx + \int_{\Gamma} H v ds + \int_{\Gamma_N} g w ds$$

is continuous and linear, and the form $\mathcal{L} : \mathbf{V}_\ell \times \mathbf{V}_\ell \rightarrow \mathbb{R}$ defined by

$$\mathcal{L}((\theta, \phi), (v, w)) = \frac{1}{\tau} \int_{\Omega} b(u)\theta v dx + \int_{\Omega} \left(\mathbf{L}(\mathbf{u})\nabla \begin{bmatrix} \theta \\ \phi \end{bmatrix} \right) \cdot \nabla \begin{bmatrix} v \\ w \end{bmatrix} dx,$$

is continuous and bilinear, with L being the (2×2) -matrix

$$L(\mathbf{u}) = \begin{bmatrix} a(\mathbf{u}) & \sigma(u)F(\mathbf{u}) \\ \sigma(u)\alpha_S(u) & \sigma(u) \end{bmatrix}.$$

Moreover, \mathcal{L} is coercive:

$$\sum_{i,j=1}^2 \sum_{l=1}^n (L_{i,j}(\mathbf{u})\xi_{j,l})\xi_{l,i} \geq (L_1)_\# |\xi_1|^2 + (L_2)_\# |\xi_2|^2, \tag{42}$$

with $(L_1)_\#$ and $(L_2)_\#$ being the positive constants defined in (40) or (41), taking the assumptions (27) and (28) into account. The difference of the definitions is consequence of the different application of the Young inequality $2AB \leq \varepsilon A^2 + B^2/\varepsilon$ ($\varepsilon, A, B > 0$), see Remark 1. Namely, with

1. $A = |\xi_1|$ and $B = |\xi_2|$, for (40). That is,

$$\sum_{l=1}^n (\sigma(u)F(\mathbf{u})\xi_{2,l}\xi_{l,1} + \sigma(u)\alpha_S(u)\xi_{1,l}\xi_{l,2}) \leq \sigma^\#(F^\# + \alpha^\#) \left(\frac{\varepsilon}{2} A^2 + \frac{1}{2\varepsilon} B^2 \right).$$

2. $A = |\xi_1|$ and $B = \sqrt{\sigma(u)}|\xi_2|$, for (41). That is,

$$\sum_{l=1}^n (\sigma(u)F(\mathbf{u})\xi_{2,l}\xi_{l,1} + \sigma(u)\alpha_S(u)\xi_{1,l}\xi_{l,2}) \leq \sqrt{\sigma^\#}(F^\# + \alpha^\#) \left(\frac{\varepsilon}{2} A^2 + \frac{1}{2\varepsilon} B^2 \right).$$

Finally, observing that the function $e \in \mathbb{R} \mapsto \gamma(u)|e|^{\ell-2}e$ is monotonically increasing, we conclude the existence of the required solution.

In order to obtain (39), we take $v = \theta$ and $w = \phi$ as test functions in (37) and (38), respectively. Summing the obtained relations, and applying (23) and (12), the coercivity (42) of L , and the Hölder inequality, we find

$$\begin{aligned} & \frac{b_\#}{\tau} \|\theta\|_{2,\Omega}^2 + (L_1)_\# \|\nabla\theta\|_{2,\Omega}^2 + (L_2)_\# \|\nabla\phi\|_{2,\Omega}^2 + \gamma_\# \|\theta\|_{\ell,\Gamma}^\ell \\ & \leq \frac{1}{\tau} \|f\|_{2,\Omega} \|\theta\|_{2,\Omega} + \|H\|_{\ell',\Gamma} \|\theta\|_{\ell,\Gamma} + \|g\|_{2,\Gamma_N} \|\phi\|_{2,\Gamma_N}. \end{aligned} \tag{43}$$

We successively apply (34) and the Young inequality to obtain

$$\begin{aligned} & \|H\|_{\ell',\Gamma} \|\theta\|_{\ell,\Gamma} + \|g\|_{2,\Gamma_N} \|\phi\|_{2,\Gamma_N} \\ & \leq \frac{1}{\ell'\gamma_\#^{1/(\ell-1)}} \|H\|_{\ell',\Gamma}^{\ell'} + \frac{\gamma_\#}{\ell} \|\theta\|_{\ell,\Gamma}^\ell + \frac{K_2^2(P_2 + 1)^2}{2(L_2)_\#} \|g\|_{2,\Gamma_N}^2 + \frac{(L_2)_\#}{2} \|\nabla\phi\|_{2,\Omega}^2. \end{aligned} \tag{44}$$

Inserting (44) into (43), we deduce (39).

Remark 1 Even $\varepsilon > 0$ may be an arbitrary (but fixed) number, we may differently define $(L_1)_\#$ and $(L_2)_\#$. Indeed, the Young inequality $2AB \leq \varepsilon A^2 + B^2/\varepsilon$ ($\varepsilon, A, B > 0$) may be applied to obtain

$$\begin{aligned} & \sum_{l=1}^n (\sigma(u)F(\mathbf{u})\xi_{2,l}\xi_{l,1} + \sigma(u)\alpha_S(u)\xi_{1,l}\xi_{l,2}) \leq \sigma^\# \left(F^\# \left(\frac{\varepsilon_1}{2} |\xi_1|^2 + \frac{1}{2\varepsilon_1} |\xi_2|^2 \right) \right. \\ & \left. + \alpha^\# \left(\frac{\varepsilon_2}{2} |\xi_1|^2 + \frac{1}{2\varepsilon_2} |\xi_2|^2 \right) \right). \end{aligned}$$

Next, let us determine whose radius make possible that the operator \mathcal{T} maps a closed ball into itself.

Proposition 2 For $R = \max\{R_1, R_2\}$ with R_1 and R_2 being defined in (45) and (46), respectively, the operator \mathcal{T} verifies $\mathcal{T}(K) \subset K$, with

$$K = \{(v, w) \in \mathbf{V}_\ell : \|\nabla w\|_{2,\Omega} + \|\nabla v\|_{2,\Omega} + \|v\|_{\ell,\Gamma} \leq R\}.$$

Proof Let $\mathbf{u} \in \mathbf{V}_\ell$, $u = u_1$ and (θ, ϕ) be the unique solution of Proposition 1, i.e. $(\theta, \phi) = \mathcal{T}(\mathbf{u})$. In order to prove that $(\theta, \phi) \in K$ we consider two different cases: (1) if $\|\theta\|_{\ell,\Gamma} \leq 1$; and (2) if $\|\theta\|_{\ell,\Gamma} > 1$,

1. if $\|\theta\|_{\ell,\Gamma} \leq 1$, then there holds

$$\|\nabla\phi\|_{2,\Omega} + \|\nabla\theta\|_{2,\Omega} + \|\theta\|_{\ell,\Gamma} \leq \sqrt{2}(\|\nabla\phi\|_{2,\Omega}^2 + \|\nabla\theta\|_{2,\Omega}^2)^{1/2} + 1,$$

by applying the elementary inequality $(a + b)^2 \leq 2(a^2 + b^2)$ for every $a, b \geq 0$. By using (39), we may take

$$R_1 = \left(\frac{2\mathcal{R}}{\min\{(L_1)_\#, (L_2)_\#/2\}} \right)^{1/2} + 1. \tag{45}$$

2. if $\|\theta\|_{\ell,\Gamma} > 1$, then using $\ell \geq 2$ there holds

$$\|\nabla\phi\|_{2,\Omega} + \|\nabla\theta\|_{2,\Omega} + \|\theta\|_{\ell,\Gamma} \leq \sqrt{2}(2(\|\nabla\phi\|_{2,\Omega}^2 + \|\nabla\theta\|_{2,\Omega}^2) + \|\theta\|_{\ell,\Gamma}^\ell)^{1/2},$$

by applying the elementary inequality $(a + b)^2 \leq 2(a^2 + b^2)$ for every $a, b \geq 0$. By using (39), we may take

$$R_2^2 = \left(\frac{2}{\min\{(L_1)_\#, (L_2)_\#/2\}} + \frac{\ell'}{\gamma_\#} \right) \mathcal{R}. \tag{46}$$

Then, the proof is complete by taking R such that is the maximum of R_1 and R_2 defined in (45) and (46), respectively.

Proposition 3 The operator \mathcal{T} is continuous.

Proof Let $\{\mathbf{u}^m\}_{m \in \mathbb{N}}$ be a sequence which weakly converges to $\mathbf{u} = (u, u_2)$ in \mathbf{V}_ℓ , and $(\theta_m, \phi_m) = \mathcal{T}(\mathbf{u}^m)$ for each $m \in \mathbb{N}$. Proposition 1 guarantees that (θ_m, ϕ_m) solves, for each $m \in \mathbb{N}$, the variational system (37)_m, (38)_m, with \mathbf{u} replaced by \mathbf{u}^m . The uniform boundedness ensured by Proposition 2 guarantees the existence of a limit $(\theta, \phi) \in \mathbf{V}_\ell$, for at least a subsequence of (θ_m, ϕ_m) still denoted by (θ_m, ϕ_m) , such that

$$\theta_m \rightharpoonup \theta \text{ in } V_\ell(\Omega) \text{ and } \phi_m \rightharpoonup \phi \text{ in } V.$$

The Rellich–Kondrachov theorem guarantees the strong convergences

$$\begin{aligned} u^m &\rightarrow u \text{ and } u_2^m \rightarrow u_2 \text{ in } L^2(\Omega); \\ \theta_m &\rightarrow \theta \text{ and } \phi_m \rightarrow \phi \text{ in } L^2(\Omega); \\ u^m &\rightarrow u \text{ and } \theta_m \rightarrow \theta \text{ in } L^2(\Gamma). \end{aligned}$$

To show that $(\theta, \phi) = \mathcal{T}(\mathbf{u})$, it remains to pass to the limit in the system (37)_m, (38)_m as m tends to infinity.

Applying the Krasnoselski theorem to the Nemytskii operators b, a, σ , we have

$$\begin{aligned} b(u^m)v &\rightarrow b(u)v && \text{in } L^2(\Omega); \\ a(\mathbf{u}^m)\nabla v &\rightarrow a(\mathbf{u})\nabla v && \text{in } \mathbf{L}^2(\Omega); \\ \sigma(u^m)\nabla v &\rightarrow \sigma(u)\nabla v && \text{in } \mathbf{L}^2(\Omega), \end{aligned}$$

for all $v \in H^1(\Omega)$, making use of the Lebesgue dominated convergence theorem and the assumptions (9), (22) and (23). Also the terms $\sigma(u^m)F(\mathbf{u}^m)\nabla v$ and $\sigma(u^m)\alpha_S(u^m)\nabla v$ pass to the limit making recourse to the assumptions (21) and (10), respectively.

Similarly, the boundary term $\gamma(u^m)v$ converges to $\gamma(u)v$ in $L^{\ell'}(\Gamma)$, for all $v \in L^{\ell'}(\Gamma)$, due to (12). Observe that θ_m strongly converges to θ in $L^p(\Gamma)$, for all $1 < p < \ell$. Then, the nonlinear boundary term $\gamma(u^m)|\theta^m|^{\ell-2}\theta^m$ weakly passes to the limit, as m tends to infinity, to $\gamma(u)\Lambda$ in $L^{\ell'}(\Gamma)$. Therefore, we pass to the limit the variational system (37)_m, (38)_m, as m tends to infinity, to conclude that ϕ is the required limit solution, *i.e.* it solves the limit equality (38), while θ verifies

$$\begin{aligned} \frac{1}{\tau} \int_{\Omega} b(u)\theta v dx + \int_{\Omega} a(\mathbf{u})\nabla\theta \cdot \nabla v dx + \int_{\Omega} \sigma(u)F(\mathbf{u})\nabla\phi \cdot \nabla v dx \\ + \int_{\Gamma} \gamma(u)\Lambda v ds = \frac{1}{\tau} \int_{\Omega} f v dx + \int_{\Gamma} H v ds. \end{aligned} \tag{47}$$

It remains to identify the limit Λ by using the Minty trick as follows. The argument is slightly different from the classical one (see [18]).

Making recourse to the lower bound (12) of γ and the monotone property of the function $v \mapsto |v|^{\ell-2}v$, we have

$$0 \leq \gamma_{\#}2^{2-\ell}|\theta_m - v|^{\ell} \leq \gamma(u^m)(|\theta_m|^{\ell-2}\theta_m - |v|^{\ell-2}v)(\theta_m - v).$$

Thanks to the coercivity coefficients (40) or (41), the monotone property of the boundary term, and the Hölder and Young inequalities, let us consider

$$\begin{aligned} \int_{\Omega} a(\mathbf{u}^m)|\nabla(\theta_m - v)|^2 dx + \int_{\Omega} \sigma(u^m)|\nabla(\phi_m - \phi)|^2 dx \\ + \int_{\Omega} \sigma(u^m)F(\mathbf{u}^m)\nabla(\phi_m - \phi) \cdot \nabla(\theta_m - v) dx \\ + \int_{\Omega} \sigma(u^m)\alpha_S(u^m)\nabla(\theta_m - v) \cdot \nabla(\phi_m - \phi) dx \\ + \int_{\Gamma} \gamma(u^m) \left(|\theta_m|^{\ell-2}\theta_m - |v|^{\ell-2}v \right) (\theta_m - v) ds \\ \geq (L_1)_{\#} \int_{\Omega} |\nabla(\theta_m - v)|^2 dx + (L_2)_{\#} \int_{\Omega} |\nabla(\phi_m - \phi)|^2 dx \geq 0. \end{aligned} \tag{48}$$

Let us define

$$\begin{aligned} \mathcal{J}_m := \int_{\Omega} (a(\mathbf{u}^m)|\nabla\theta_m|^2 + \sigma(u^m)F(\mathbf{u}^m)\nabla\phi_m \cdot \nabla\theta_m) dx \\ + \int_{\Omega} (\sigma(u^m)|\nabla\phi_m|^2 + \sigma(u^m)\alpha_S(u^m)\nabla\theta_m \cdot \nabla\phi_m) dx + \int_{\Gamma} \gamma(u^m)|\theta_m|^{\ell} ds. \end{aligned}$$

On the one hand, we deduce

$$\begin{aligned} \liminf_{m \rightarrow \infty} \mathcal{J}_m &\geq \int_{\Gamma} \gamma(u) \Lambda v ds + \int_{\Gamma} \gamma(u) |v|^{\ell-2} v (\theta - v) ds \\ &\quad + \int_{\Omega} a(\mathbf{u}) \nabla \theta \cdot \nabla v dx + \int_{\Omega} a(\mathbf{u}) \nabla v \cdot \nabla (\theta - v) dx \\ &\quad + \int_{\Omega} \sigma(u) |\nabla \phi|^2 dx + \int_{\Omega} \sigma(u) F(\mathbf{u}) \nabla \phi \cdot \nabla \theta dx + \int_{\Omega} \sigma(u) \alpha_S(u) \nabla \theta \cdot \nabla \phi dx. \end{aligned}$$

On the other hand, taking in (37)_m the test function $v = \theta^m$, in (47) the test function $v = \theta$, in (38)_m the test function $w = \phi_m$, and in (38) the test function $w = \phi$, we deduce

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathcal{J}_m &= \frac{1}{\tau} \int_{\Omega} f \theta dx + \int_{\Gamma} H \theta ds - \frac{1}{\tau} \int_{\Omega} b(u) \theta dx + \int_{\Gamma_N} g \phi ds \\ &= \int_{\Omega} a(\mathbf{u}) |\nabla \theta|^2 dx + \int_{\Omega} \sigma(u) F(\mathbf{u}) \nabla \phi \cdot \nabla \theta dx + \int_{\Gamma} \gamma(u) \Lambda \theta ds \\ &\quad + \int_{\Omega} \sigma(u) |\nabla \phi|^2 dx + \int_{\Omega} \sigma(u) \alpha_S(u) \nabla \theta \cdot \nabla \phi dx. \end{aligned}$$

Gathering the above two relations, we find

$$\int_{\Omega} a(\mathbf{u}) |\nabla (\theta - v)|^2 dx + \int_{\Gamma} \gamma(u) (\Lambda - |v|^{\ell-2} v) (\theta - v) ds \geq 0.$$

We continue the argument by taking $v = \theta - \delta \varphi$, with $\varphi \in \mathcal{D}(\Gamma)$. After dividing by $\delta > 0$, and finally letting $\delta \rightarrow 0^+$ we arrive to

$$\int_{\Gamma} \gamma(u) (\Lambda - |\theta|^{\ell-2} \theta) \varphi ds \geq 0, \quad \forall \varphi \in \mathcal{D}(\Gamma),$$

which implies that $\Lambda = |\theta|^{\ell-2} \theta$.

Thus, we are in the condition of concluding that (θ, ϕ) is the required limit solution, *i.e.* it solves the limit system (37, 38).

Thanks to Propositions 1, 2 and 3, there exists at least one fixed point of \mathcal{T} , that is $(\theta, \phi) = \mathcal{T}(\theta, \phi)$, which concludes the solvability to (29) and (30).

4.2 Fixed point argument (solvability to (32) and (33))

Let $\ell \geq 2$, and define an operator \mathcal{T} from $V_{\ell}(\Omega)$ into itself such that $\theta = \mathcal{T}(u)$ is the unique solution of Proposition 5.

Denote by \mathcal{F} the well defined continuous operator such that $\mathcal{F}(u) = \phi$. The existence of a unique weak auxiliary solution ϕ to the variational equality (38) is standard and it can be stated as follows.

Proposition 4 *Let $u \in H^1(\Omega)$. Under the assumptions (9), (10) and (H5), the Neumann problem*

$$\int_{\Omega} \sigma(u) \nabla \phi \cdot \nabla w dx = \int_{\Omega} \sigma(u) \alpha_S(u) \nabla u \cdot \nabla w dx + \int_{\Gamma_N} g w ds, \quad \forall w \in V, \quad (49)$$

admits a unique solution $\phi \in V$. Moreover, the estimate

$$\|\sqrt{\sigma(u)} \nabla \phi\|_{2,\Omega} \leq \sqrt{\sigma^{\#} \alpha^{\#}} \|\nabla u\|_{2,\Omega} + \frac{K_2(P_2 + 1)}{\sqrt{\sigma^{\#}}} \|g\|_{2,\Gamma_N} \quad (50)$$

holds true.

Proof Let us establish the quantitative estimate (50). We take $w = \phi$ as a test function in (49), and we compute by applying the Hölder inequality and (34)

$$\|\sqrt{\sigma(u)}\nabla\phi\|_{2,\Omega}^2 \leq \left(\alpha^\# \|\sqrt{\sigma(u)}\nabla u\|_{2,\Omega} + \frac{K_2(P_2 + 1)}{\sqrt{\sigma^\#}} \|g\|_{2,\Gamma_N} \right) \|\sqrt{\sigma(u)}\nabla\phi\|_{2,\Omega}.$$

Then, (50) arises.

Proposition 5 *Let $\mathbf{u} = (u, \phi) \in (H^1(\Omega))^2$. Under the assumptions (9), (12) and (21)–(23), there exists a unique solution $\theta \in V_\ell(\Omega)$ to the power-type elliptic problem*

$$\begin{aligned} & \frac{1}{\tau} \int_\Omega b(u)\theta v dx + \int_\Omega a(\mathbf{u})\nabla\theta \cdot \nabla v dx + \int_\Omega \sigma(u)F(\mathbf{u})\nabla\phi \cdot \nabla v dx \\ & + \int_\Gamma \gamma(u)|\theta|^{\ell-2}\theta v ds = \frac{1}{\tau} \int_\Omega f v dx + \int_\Gamma H v ds, \end{aligned} \tag{51}$$

for all $v \in V_\ell(\Omega)$. If $\phi \in V$ satisfies (50), then the following estimate

$$\begin{aligned} & \frac{b_\#}{2\tau} \|\theta\|_{2,\Omega}^2 + \frac{a_\#}{2} \|\nabla\theta\|_{2,\Omega}^2 + \frac{\gamma_\#}{\ell'} \|\theta\|_{\ell,\Gamma}^\ell \leq \frac{1}{2\tau b_\#} \|f\|_{2,\Omega}^2 + \frac{1}{\ell' \gamma_\#^{1/(\ell-1)}} \|H\|_{\ell',\Gamma}^{\ell'} \\ & + \frac{(F^\#)^2 \sigma^\#}{a_\#} \left(\sigma^\# (\alpha^\#)^2 \|\nabla u\|_{2,\Omega}^2 + \frac{K_2^2(P_2 + 1)^2}{\sigma_\#} \|g\|_{2,\Gamma_N}^2 \right), \end{aligned} \tag{52}$$

holds true.

Proof Taking $v = \theta$ as a test function in (51), and applying (23), (22), (12), (21), and the Hölder inequality, we find

$$\begin{aligned} & \frac{b_\#}{\tau} \|\theta\|_{2,\Omega}^2 + a_\# \|\nabla\theta\|_{2,\Omega}^2 + \gamma_\# \|\theta\|_{\ell,\Gamma}^\ell \\ & \leq \frac{1}{\tau} \|f\|_{2,\Omega} \|\theta\|_{2,\Omega} + F^\# \|\sqrt{\sigma(u)}\nabla\phi\|_{2,\Omega} \|\sqrt{\sigma(u)}\nabla\theta\|_{2,\Omega} + \|H\|_{\ell',\Gamma} \|\theta\|_{\ell,\Gamma}. \end{aligned}$$

Applying (50) and the Young inequality, we compute

$$\begin{aligned} & \|\sqrt{\sigma(u)}\nabla\phi\|_{2,\Omega} \|\sqrt{\sigma(u)}\nabla\theta\|_{2,\Omega} \leq \frac{a_\#}{2} \|\nabla\theta\|_{2,\Omega}^2 \\ & + \frac{\sigma^\#}{a_\#} \left(\sigma^\# (\alpha^\#)^2 \|\nabla u\|_{2,\Omega}^2 + \frac{K_2^2(P_2 + 1)^2}{\sigma_\#} \|g\|_{2,\Gamma_N}^2 \right). \end{aligned}$$

Then, arguing as in (44), we deduce (52).

Next, let us determine whose radius make possible that the operator \mathcal{T} maps a closed ball into itself.

Proposition 6 *Let (31) be fulfilled. For $\tau \leq a_\#/b_\#$ and $R > 0$ being defined as in (53), the operator \mathcal{T} verifies $\mathcal{T}(B_R) \subset B_R$, with B_R denoting the open ball of $H^1(\Omega)$ with radius R .*

Proof Let $u \in H^1(\Omega)$ and $\theta = \mathcal{T}(u)$ be the unique solution according to Proposition 5. Considering (52) and $\min \{b_\#/\tau, a_\#\} = a_\#$, the proof is complete by defining R such that

$$R \left(\sqrt{a_\#} - 2 \frac{F^\# \sigma^\# \alpha^\#}{\sqrt{a_\#}} \right) = \left(\frac{1}{\tau b_\#} \|f\|_{2,\Omega}^2 + \frac{2}{\ell' \gamma_\#^{1/(\ell-1)}} \|H\|_{\ell',\Gamma}^{\ell'} \right)^{1/2} + 2F^\# K_2(P_2 + 1) \sqrt{\frac{\sigma^\#}{a_\# \sigma_\#}} \|g\|_{2,\Gamma_N}, \tag{53}$$

taking the Assumption (31) be account.

Proposition 7 *Let $\{u_m\}_{m \in \mathbb{N}}$ be a sequence which weakly converges to u in $H^1(\Omega)$, then the solution (θ_m, ϕ_m) , according to Propositions 4 and 5, weakly converges in $V_\ell(\Omega) \times H^1(\Omega)$, and its limit is a solution according to Propositions 4 and 5*

Proof Let (θ_m, ϕ_m) be the solution according to Propositions 4 and 5 and corresponding to u_m for each $m \in \mathbb{N}$. The estimates (50) and (52) guarantee that the sequence (θ_m, ϕ_m) is uniformly bounded in $V_\ell(\Omega) \times H^1(\Omega)$. Thus, we can extract a subsequence of (θ_m, ϕ_m) still denoted by (θ_m, ϕ_m) , weakly convergent to (θ, ϕ) in $V_\ell(\Omega) \times H^1(\Omega)$. Similar arguments in the proof of Proposition 3 the weak limit (θ, ϕ) solves the variational system consisting of (49) and (51), which concludes the proof of Proposition 7.

Thanks to Propositions 4, 5, 6 and 7, there exists at least one fixed point of

$$\mathcal{T} : u \mapsto (u, \mathcal{F}(u)) \mapsto \theta,$$

that is $\theta = \mathcal{T}(\theta)$ and $\phi = \mathcal{F}(\theta)$, which concludes the solvability to (32) and (33).

5 Time discretization technique

In this section, we apply the method of discretization in time [17, 23, 24].

We decompose the time interval $I = [0, T]$ into M subintervals $I_{m,M}$ of size τ such that $M = T/\tau \in \mathbb{N}$, i.e. $I_{m,M} = [(m - 1)T/M, mT/M]$ for $m \in \{1, \dots, M\}$. We set $t_{m,M} = mT/M$. Thus, the problem (24) is approximated by the following recurrent sequence of time-discretized problems

$$\frac{1}{\tau} \int_\Omega B(\theta^m) v dx + \int_\Omega a(\theta^m, \phi^m) \nabla \theta^m \cdot \nabla v dx + \int_\Gamma \gamma(\theta^m) |\theta^m|^{\ell-2} \theta^m v ds + \int_\Omega \sigma(\theta^m) F(\theta^m, \phi^m) \nabla \phi^m \cdot \nabla v dx = \frac{1}{\tau} \int_\Omega B(\theta^{m-1}) v dx + \int_\Gamma h(t_{m,M}) v ds, \tag{54}$$

for all $v \in V_\ell(\Omega)$, and the problem (25) is approximated by either (30) or (33) for all $w \in V$, corresponding to the two different approaches. The existence of weak solutions pair $(\theta^m, \phi^m) \in V_\ell(\Omega) \times V$ to the above systems of elliptic problems is established in Sect. 4 with $H = h(t_{m,M})$.

Since $\theta^0 \in L^2(\Omega)$ is known, we determine (θ^1, ϕ^1) as the unique solution of the Neumann-power-type elliptic problems (29), (30) or (32), (33), and we inductively proceed.

Denote by $\{\theta_M\}_{M \in \mathbb{N}}$, $\{\phi_M\}_{M \in \mathbb{N}}$ and $\{Z_M\}_{M \in \mathbb{N}}$ the sequences of the (piecewise constant in time) functions, $\theta_M : [0, T] \rightarrow V_\ell(\Omega)$, $\phi_M : [0, T] \rightarrow V$ and $Z_M : [0, T] \rightarrow L^2(\Omega)$,

defined by, respectively, a.e. in Ω

$$\theta_M(t) := \begin{cases} \theta^0 & \text{if } t = 0 \\ \theta^m & \text{if } t \in]t_{m-1,M}, t_{m,M}] \end{cases} \tag{55}$$

$$\phi_M(t) := \phi^m \quad \text{for all } t \in]t_{m-1,M}, t_{m,M}], \tag{56}$$

in accordance with one of the two variational formulations (30) and (33), while

$$Z_M(t) := \begin{cases} B(\theta^0) & \text{if } t = 0 \\ Z^m & \text{if } t \in]t_{m-1,M}, t_{m,M}] \end{cases} \quad \text{in } \Omega, \tag{57}$$

with the discrete derivative with respect to t at the time $t = t_{m,M}$:

$$Z^m := \frac{B(\theta^m) - B(\theta^{m-1})}{\tau}.$$

While θ_M is the Rothe function obtained from θ^m by piecewise constant interpolation with respect to time t , the Rothe function, obtained from θ^m by piecewise linear interpolation with respect to time t , Θ_M is

$$\Theta_M(\cdot, t) = \theta^{m-1} + (t - t_{m-1,M}) \frac{\theta^m - \theta^{m-1}}{\tau}.$$

For our purposes, we introduce the following definition.

Definition 2 We say that $\{\tilde{B}_M = \tilde{B}(\theta_M)\}_{M \in \mathbb{N}}$ is the Rothe sequence (affine on each time interval) if

$$\tilde{B}(\cdot, \theta_M(t)) = B(\cdot, \theta^{m-1}) + \frac{t - t_{m-1,M}}{\tau} (B(\cdot, \theta^m) - B(\cdot, \theta^{m-1}))$$

in Ω , for all $t \in I_{m,M}$, and for all $m \in \{1, \dots, M\}$.

Denoting $h_M(t) = h(t_{m,M})$ for $t \in]t_{m-1,M}, t_{m,M}]$ and $m \in \{1, \dots, M\}$, the triple (θ_M, ϕ_M, Z_M) solve

$$\begin{aligned} & \int_{Q_T} Z_M v dx dt + \int_{Q_T} a(\theta_M, \phi_M) \nabla \theta_M \cdot \nabla v dx dt + \int_{\Sigma_T} \gamma(\theta_M) |\theta_M|^{\ell-2} \theta_M v ds dt \\ & + \int_{Q_T} \sigma(\theta_M) F(\theta_M, \phi_M) \nabla \phi_M \cdot \nabla v dx dt = \int_{\Sigma_T} h_M v ds dt. \end{aligned} \tag{58}$$

5.1 Proof of Theorem 2

Here, ϕ_M solves

$$\int_{\Omega} \sigma(\theta_M) \nabla \phi_M \cdot \nabla w dx + \int_{\Omega} \sigma(\theta_M) \alpha_S(\theta_M) \nabla \theta_M \cdot \nabla w dx = \int_{\Gamma_N} g w ds, \tag{59}$$

for $w \in V$ and a.e. in $]0, T[$.

We begin by establishing the uniform estimates to θ_M and ϕ_M .

Proposition 8 *Let θ_M and ϕ_M be the (piecewise constant in time) functions defined in (55) and (56). Then the following estimate holds:*

$$\begin{aligned} & \max_{1 \leq m \leq M} \int_{\Omega} \Psi(\theta^m) dx + (L_1)_{\#} \|\nabla \theta_M\|_{2, Q_T}^2 + \frac{(L_2)_{\#}}{2} \|\nabla \phi_M\|_{2, Q_T}^2 \\ & + \frac{\gamma_{\#}}{\ell'} \|\theta_M\|_{\ell', \Sigma_T}^{\ell} \leq b_{\#} \|\theta^0\|_{2, \Omega}^2 + \frac{1}{\ell' \gamma_{\#}} \|h\|_{\ell', \Sigma_T}^{\ell'} + T \frac{K_2^2 (P_2 + 1)^2}{2(L_2)_{\#}} \|g\|_{2, \Gamma_N}^2. \end{aligned} \tag{60}$$

Proof Let $m \in \{1, \dots, M\}$ be arbitrary. Choosing $v = \theta^m \in V_\ell(\Omega)$ and $w = \phi^m \in V$ as test functions in (29) and (30), we sum the obtained relations, and arguing as in (43) and (44), we have

$$\begin{aligned} & \frac{1}{\tau} \int_{\Omega} (B(\theta^m) - B(\theta^{m-1}))\theta^m dx + (L_1)_{\#} \|\nabla\theta^m\|_{2,\Omega}^2 + \frac{(L_2)_{\#}}{2} \|\nabla\phi^m\|_{2,\Omega}^2 \\ & + \int_{\Gamma} \gamma(\theta^m)|\theta^m|^\ell ds \leq \mathcal{R}(0, \|h(t_{m,M})\|_{\ell',\Gamma}^{\ell'}) + \frac{1}{\ell} \int_{\Gamma} \gamma(\theta^m)|\theta^m|^\ell ds, \end{aligned} \tag{61}$$

with \mathcal{R} being the increasing continuous function defined in (39). By (35), we have

$$\sum_{i=1}^m \int_{\Omega} (B(\theta^i) - B(\theta^{i-1}))\theta^i dx \geq \int_{\Omega} (\Psi(\theta^m) - \Psi(\theta^0)) dx.$$

Therefore, summing over $i = 1, \dots, m$ into (61), multiplying by τ , and inserting the previous inequality, we obtain

$$\begin{aligned} & \int_{\Omega} \Psi(\theta^m) dx + \tau \sum_{i=1}^m \left((L_1)_{\#} \|\nabla\theta^i\|_{2,\Omega}^2 + \frac{1}{\ell'} \int_{\Gamma} \gamma(\theta^i)|\theta^i|^\ell ds + \frac{(L_2)_{\#}}{2} \|\nabla\phi^i\|_{2,\Omega}^2 \right) \\ & \leq \int_{\Omega} \Psi(\theta^0) dx + \tau \sum_{i=1}^m \mathcal{R}(0, \|h(t_{i,M})\|_{\ell',\Gamma}^{\ell'}). \end{aligned} \tag{62}$$

Therefore, we find the uniform estimate (60) by taking the maximum over $m \in \{1, \dots, M\}$ in the previous estimate and applying Lemma 1 provided by (23).

A direct application of Proposition 8 ensures the following proposition.

Proposition 9 *There exist $\theta, \phi : Q_T \rightarrow \mathbb{R}$ and subsequences of (θ_M, ϕ_M) , still labelled by (θ_M, ϕ_M) , such that*

$$\theta_M \rightharpoonup \theta \text{ in } V_\ell(Q_T); \tag{63}$$

$$\phi_M \rightharpoonup \phi \text{ in } L^2(0, T; V), \tag{64}$$

as M tends to infinity. Moreover, there exists $Z : Q_T \rightarrow \mathbb{R}$ such that

$$\partial_t \tilde{B}(\theta_M) \rightharpoonup Z \text{ in } L^{\ell'}(0, T; (V_\ell(\Omega))'),$$

as M tends to infinity.

Proof Considering the uniform estimates to θ_M and ϕ_M that are established in Proposition 8, we extract subsequences, still denoted by θ_M and ϕ_M , weakly convergent in $V_\ell(Q_T)$ and $L^2(0, T; V)$, respectively, to θ and ϕ .

Let $p = \max\{\ell, 2\} = \ell$. Note that $L^p(0, T; V_\ell(\Omega)) \hookrightarrow V_\ell(Q_T)$. By definition of norm, we find

$$\|\partial_t \tilde{B}(\theta_M)\|_{L^{p'}(0,T;(V_\ell(\Omega))')} = \sum_{m=1}^M \int_{(m-1)\tau}^{m\tau} \sup_{\substack{v \in L^p(0,T;V_\ell(\Omega)) \\ \|v\| \leq 1}} \langle Z^m, v \rangle dt \leq C,$$

with $C > 0$ being a constant independent on M , by estimating in (54) the term involving Z^m by means of the rest terms using the uniform estimates established in (62). Hence, we can extract a subsequence, still denoted by $\partial_t \tilde{B}(\theta_M)$, weakly convergent to Z in $L^{p'}(0, T; (V_\ell(\Omega))')$.

In the following proposition, we state the strong convergence of $B(\theta_M)$ and of θ_M .

Proposition 10 *Under (9)–(12) and (21)–(22), the solution θ^m of (54) satisfies*

$$\begin{aligned} b_{\#} \|\theta^m - \theta^{m-1}\|_{2,\Omega}^2 &\leq \int_{\Omega} (B(\theta^m) - B(\theta^{m-1}))(\theta^m - \theta^{m-1}) dx \\ &\leq C(\|\theta^m - \theta^{m-1}\|_{\ell,\Gamma} \tau^{1/\ell} + \|\nabla(\theta^m - \theta^{m-1})\|_{2,\Omega} \sqrt{\tau}), \end{aligned} \tag{65}$$

with C being a positive constant. Moreover, for a subsequence, there hold

$$B(\theta_M) \rightarrow B(\theta) \text{ in } L^1(Q_T); \tag{66}$$

$$\theta_M \rightarrow \theta \text{ a.e. in } Q_T, \tag{67}$$

as M tends to infinity.

Proof Let $k \in \mathbb{N}$. Let us sum up (54) for $m = j + 1, \dots, j + k$ and multiply by τ , obtaining

$$\int_{\Omega} (B(\theta^{j+k}) - B(\theta^j)) v dx \leq \mathcal{I}_{\Gamma}^j + \mathcal{I}_{\Omega}^j,$$

where

$$\begin{aligned} \mathcal{I}_{\Gamma}^j &:= \tau \sum_{m=j+1}^{j+k} \int_{\Gamma} |(\gamma(\theta^m)|\theta^m|^{\ell-2}\theta^m - h(t_m, M))v| dx \\ &\leq \int_{j\tau}^{(j+k)\tau} \|\gamma(\theta_M)|\theta_M|^{\ell-2}\theta_M - h\|_{\ell',\Gamma} \|v\|_{\ell,\Gamma} dt; \\ \mathcal{I}_{\Omega}^j &:= \tau \sum_{m=j+1}^{j+k} \int_{\Omega} |(\alpha(\theta^m, \phi^m)\nabla\theta^m + F(\theta^m, \phi^m)\nabla\phi^m) \cdot \nabla v| dx \\ &\leq \int_{j\tau}^{(j+k)\tau} \|\alpha(\theta_M, \phi_M)\nabla\theta_M + F(\theta_M, \phi_M)\nabla\phi_M\|_{2,\Omega} \|\nabla v\|_{2,\Omega} dt. \end{aligned}$$

Here, we used the Hölder inequality and the definition of θ_M and of ϕ_M .

Let us compute \mathcal{I}_{Γ}^j and \mathcal{I}_{Ω}^j by applying the estimate (60). Using the Assumption (12) and after the Hölder inequality, we deduce

$$\mathcal{I}_{\Gamma}^j \leq \|v\|_{\ell,\Gamma} \int_{j\tau}^{(j+k)\tau} \left(\gamma^{\#} \|\theta_M\|_{\ell,\Gamma}^{\ell-1} + \|h\|_{\ell',\Gamma} \right) dt \leq \|v\|_{\ell,\Gamma} C(k\tau)^{1/\ell}. \tag{68}$$

Using the Assumptions (9), (21) and (22), and after the Hölder inequality, we deduce

$$\mathcal{I}_{\Omega}^j \leq \|\nabla v\|_{2,\Omega} \int_{j\tau}^{(j+k)\tau} \left(a^{\#} \|\nabla\theta_M\|_{2,\Omega} + \sigma^{\#} F^{\#} \|\nabla\phi_M\|_{2,\Omega} \right) dt \leq \|\nabla v\|_{2,\Omega} C\sqrt{k\tau}. \tag{69}$$

Hence, we find

$$\int_{\Omega} (B(\theta^{j+k}) - B(\theta^j)) v dx \leq \|v\|_{\ell,\Gamma} C(k\tau)^{1/\ell} + \|\nabla v\|_{2,\Omega} C\sqrt{k\tau}. \tag{70}$$

In particular, the estimate (65) follows by taking $j = m - 1, k = 1$ and $v = \theta^m - \theta^{m-1}$, and applying Lemma 1.

To prove the convergences, we will apply Lemma 2. Considering the weak convergence of θ_M established in Proposition 9 and the estimate (60), in order to apply Lemma 2 it remains to prove that the condition (36) is fulfilled.

Let $0 < z < T$ be arbitrary. Since the objective is to find convergences, it suffices to take $M > T/z$, which means $\tau < z$. Thus, there exists $k \in \mathbb{N}$ such that $k\tau < z \leq (k + 1)\tau$. Moreover, we may choose $M > k + 1$ deducing

$$\begin{aligned} & \int_0^{T-z} \int_{\Omega} (B(\theta_M(t+z)) - B(\theta_M(t)))(\theta_M(t+z) - \theta_M(t)) dx dt \\ & \leq \sum_{j=1}^{M-k} \int_{(j-1)\tau}^{(j+k)\tau} \int_{\Omega} (B(\theta^{j+k}) - B(\theta^j))(\theta^{j+k} - \theta^j) dx. \end{aligned}$$

Taking $v = \theta^{j+k} - \theta^j$ in (70) and then summing up for $j = 1, \dots, M - k$, we find

$$\begin{aligned} & \int_0^{T-z} \int_{\Omega} (B(\theta_M(t+z)) - B(\theta_M(t)))(\theta_M(t+z) - \theta_M(t)) dx dt \\ & \leq \sum_{j=1}^{M-k} \int_{(j-1)\tau}^{(j+k)\tau} \left(\|\theta^{j+k} - \theta^j\|_{\ell, \Gamma} C(k\tau)^{1/\ell} + \|\nabla(\theta^{j+k} - \theta^j)\|_{2, \Omega} C\sqrt{k\tau} \right) dt. \end{aligned}$$

Arguing as in (68) and (69), we conclude

$$\begin{aligned} & \int_0^{T-z} \int_{\Omega} (B(\theta_M(t+z)) - B(\theta_M(t)))(\theta_M(t+z) - \theta_M(t)) dx dt \\ & \leq C \left((k\tau)^{1/\ell} (k\tau + \tau)^{1/\ell'} + (k\tau)^{1/2} (k\tau + \tau)^{1/2} \right) = C \left(2^{1/\ell'} + 2^{1/2} \right) z. \end{aligned}$$

Thus, all hypothesis of Lemma 2 are fulfilled. Therefore, Lemma 2 assures that $B(\theta_M)$ strongly converges to $B(\theta)$ in $L^1(Q_T)$. Consequently, up to a subsequence, $B(\theta_M)$ converges to $B(\theta)$ a.e. in Q_T . Since B is strictly monotone, θ_M converges to θ a.e. in Q_T (see, for instance, [19]).

Now we are able to identify the limit Z .

Proposition 11 *The limit Z satisfies*

$$Z = \partial_t(B(\theta)) \text{ in } L^{\ell'}(0, T; (V_{\ell}(\Omega))').$$

Proof For a fixed t , there exists $m \in \{1, \dots, M\}$ such that $t \in]t_{m-1, M}, t_{m, M}]$. From Definition 57 we have

$$\begin{aligned} \int_0^t Z_M(\varsigma) d\varsigma &= \sum_{j=1}^{m-1} \int_{(j-1)\tau}^{j\tau} \frac{B(\theta^j)(\varsigma) - B(\theta^{j-1})(\varsigma)}{\tau} d\varsigma \\ &+ \int_{(m-1)\tau}^t \frac{B(\theta^m)(\varsigma) - B(\theta^{m-1})(\varsigma)}{\tau} d\varsigma \\ &= B(\theta^{m-1}) - B(\theta^0) + \frac{t - (m-1)\tau}{\tau} (B(\theta^m) - B(\theta^{m-1})) = \tilde{B}(\theta_M(t)) - B(\theta^0) \end{aligned}$$

in Ω . By the Riesz theorem, the above bounded linear functional is (uniquely) representable by the element $\tilde{B}(\theta_M(t)) - B(\theta^0)$ from $L^2(\Omega)$. Using the corresponding definitions we

compute

$$\begin{aligned} \int_0^T \|\tilde{B}(\theta_M(t)) - B(\theta_M)\|_{2,\Omega}^2 dt &= \sum_{m=1}^M \|Z^m\|_{2,\Omega}^2 \int_{(m-1)\tau}^{m\tau} (t - m\tau)^2 dt \\ &= \frac{\tau^3}{3} \sum_{m=1}^M \|Z^m\|_{2,\Omega}^2 = \frac{\tau}{3} \sum_{m=1}^M \|B(\theta^m) - B(\theta^{m-1})\|_{2,\Omega}^2 \\ &\leq (b^\#)^2 \frac{\tau}{3} \sum_{m=1}^M \|\theta^m - \theta^{m-1}\|_{2,\Omega}^2. \end{aligned}$$

Applying (65) and Proposition 8 we have

$$\int_0^T \|\tilde{B}(\theta_M(t)) - B(\theta_M)\|_{2,\Omega}^2 dt \leq C (\tau^{1/\ell+1/\ell'} + \tau^{1/2+1/2}) = C\tau,$$

and consequently $\tilde{B}(\theta_M)$ converges to $B(\theta)$ in $L^1(Q_T)$, considering (66). By the uniqueness of limit, we deduce

$$\int_0^t Z(\zeta) d\zeta = B(\theta) - B(\theta^0),$$

which concludes the proof.

We emphasize that the above convergences are sufficient to identify the limit ϕ as stated in the following proposition, but they are not sufficient to identify the temperature θ as a solution, because on the one hand the apparent nonlinearity of the coefficients destroy the weak convergence, on the other hand, the weak-weak convergence does not imply weak convergence.

Corollary 1 *Let (θ, ϕ) be in accordance with Propositions 9 and 10, then they verify (25).*

Proof Let (θ_M, ϕ_M) solve (58), (59). Applying Propositions 9 and 10, and the Krasnoselski theorem to the Nemytskii operators σ and α_S , we have

$$\sigma(\theta_M)\nabla\phi_M \rightharpoonup \sigma(\theta)\nabla\phi \text{ in } \mathbf{L}^2(Q_T); \tag{71}$$

$$\sigma(\theta_M)\alpha_S(\theta_M)\nabla\theta_M \rightharpoonup \sigma(\theta)\alpha_S(\theta)\nabla\theta \text{ in } \mathbf{L}^2(Q_T) \text{ as } M \rightarrow +\infty. \tag{72}$$

Thus, we may pass to the limit in (59) as M tends to infinity, concluding that (θ, ϕ) verifies (25).

5.2 Proof of Theorem 3

This proof follows *mutatis mutandis* the structure of the proof of Theorem 2 (cf. Sects. 5.1). We only sketch its main steps.

1. The uniform estimates to θ_M and ϕ_M are as follows. The quantitative estimate (60) reads

$$\begin{aligned} &\max_{1 \leq m \leq M} \int_{\Omega} \Psi(\theta^m) dx + L_\# \|\nabla\theta_M\|_{2,Q_T}^2 + \frac{\gamma_\#}{\ell'} \|\theta_M\|_{\ell,\Sigma_T}^\ell \\ &\leq b^\# \|\theta^0\|_{2,\Omega}^2 + \frac{1}{\ell' \gamma_\#^{1/(\ell-1)}} \|h\|_{\ell',\Sigma_T}^{\ell'} + T \frac{K_2^2 (P_2 + 1)^2}{2(L_2)_\#} \|g\|_{2,\Gamma_N}^2, \end{aligned}$$

by using the argument to estimate (52). Namely, (61) reads

$$\begin{aligned} & \frac{1}{\tau} \int_{\Omega} (B(\theta^m) - B(\theta^{m-1}))\theta^m \, dx + L_{\#} \|\nabla\theta^m\|_{2,\Omega}^2 + \int_{\Gamma} \gamma(\theta^m)|\theta^m|^\ell \, ds \\ & \leq \mathcal{R}(0, \|h(t_{m,M})\|_{\ell',\Gamma}^\ell) + \frac{1}{\ell} \int_{\Gamma} \gamma(\theta^m)|\theta^m|^\ell \, ds, \end{aligned}$$

where \mathcal{R} is the increasing continuous function defined in (39), with $(L_2)_{\#} = a_{\#}\sigma_{\#}/(2(F_{\#})^2\sigma_{\#})$. In addition, summing the quantitative estimate

$$\sqrt{\sigma_{\#}} \|\nabla\phi^m\|_{2,\Omega} \leq \sqrt{\sigma_{\#}}\alpha_{\#} \|\nabla\theta^{m-1}\|_{2,\Omega} + \frac{K_2(P_2 + 1)}{\sqrt{\sigma_{\#}}} \|g\|_{2,\Gamma_N},$$

it results in

$$\|\nabla\phi_M\|_{2,Q_T} \leq \alpha_{\#} \|\nabla\theta_M\|_{2,Q_T} + T \frac{K_2(P_2 + 1)}{\sigma_{\#}} \|g\|_{2,\Gamma_N}.$$

2. For subsequences of θ_M and ϕ_M , the weak convergences hold according to Propositions 9 and 10, which guarantee the required result.

6 Existence of solutions to the TE problem

The objective is the passage to the limit in the abstract boundary value problems introduced in Sect. 3 as the time step goes to zero ($M \rightarrow +\infty$), with the coefficients being defined by

$$\begin{aligned} b(\cdot, v) &= \rho(\cdot, v)c_v(\cdot, v); \\ a(\cdot, v, w) &= k(\cdot, v) + T_{\mathcal{M}}(w)\alpha_S(\cdot, v)\sigma(\cdot, v); \\ F(\cdot, v, w) &= \Pi(\cdot, v) + T_{\mathcal{M}}(w), \end{aligned}$$

where $T_{\mathcal{M}}$ is the \mathcal{M} -truncation function defined by $T_{\mathcal{M}}(z) = \max(-\mathcal{M}, \min(\mathcal{M}, z))$. By the definition of truncated functions, we choose

$$\begin{aligned} a_{\#} &= k_{\#} - \mathcal{M}\alpha_{\#}^{\sigma_{\#}}; \\ F_{\#} &= \Pi_{\#} + \mathcal{M}. \end{aligned}$$

taking (13) into account. Under these choices, the assumptions (14), (15) and (16) imply (27), (28) and (31), respectively.

Let us focus the present proof in accordance with the approximated solutions that are established in Theorem 2. Analogous argument is valid for the approximated solutions that are established in Theorem 3.

Let us redefine the electrical current density as

$$\mathbf{j}(\theta, \phi) = \sigma(\theta)\nabla\phi + \sigma(\theta)\alpha_S(\theta)\nabla\theta.$$

Analogously for $\mathbf{j}_M = \mathbf{j}(\theta_M, \phi_M)$ or simply \mathbf{j}_M and \mathbf{j} whenever the meaning is not ambiguous.

Let (θ_M, ϕ_M) solve (58), (59), which may be rewritten as

$$\begin{aligned} & \int_0^T \int_{\Omega} Z_M v dx dt + \int_{Q_T} k(\theta_M) \nabla \theta_M \cdot \nabla v dx dt + \int_{\Sigma_T} \gamma(\theta_M) |\theta_M|^{\ell-2} \theta_M v ds dt \\ & + \int_{Q_T} \sigma(\theta_M) \Pi(\theta_M) \nabla \phi_M \cdot \nabla v dx dt + \int_{Q_T} \phi_M \mathbf{j}(\theta_M, \phi_M) \cdot \nabla v dx dt \\ & = \int_{\Sigma_T} h_M v ds dt; \end{aligned} \tag{73}$$

$$\int_{\Omega} \mathbf{j}(\theta_M, \phi_M) \cdot \nabla w dx = \int_{\Gamma_N} g w ds, \tag{74}$$

for every $v \in V_{\ell}(Q_T)$ and $w \in V$. There exist $\theta, \phi : Q_T \rightarrow \mathbb{R}$ and subsequences of (θ_M, ϕ_M) , still labelled by (θ_M, ϕ_M) , weakly convergent in accordance with Proposition 9. By Corollary 1, the pair (θ, ϕ) verifies the electric equality

$$\int_{\Omega} \mathbf{j}(\theta, \phi) \cdot \nabla w dx = \int_{\Gamma_N} g w ds, \tag{75}$$

for every $w \in V$.

We emphasize that the weak convergence of \mathbf{j}_M to \mathbf{j} in $L^2(Q_T)$, taking (71) and (72) into account, is not sufficient to pass to the limit the term $\phi_M \mathbf{j}(\theta_M, \phi_M)$. Moreover, the non smoothness of the coefficients destroy the possibility of obtaining strong convergences of $\nabla \theta_M$ and of ϕ_M .

Thanks to Proposition 10, we have a.e. pointwise convergence for a subsequence of θ_M , which we still denote by θ_M . Considering the assumptions (8)–(11), the Nemytskii operators are continuous due to the Krasnoselski theorem, and applying the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} & k(\theta_M) \nabla v \rightarrow k(\theta) \nabla v \quad \text{in } L^2(Q_T); \\ & \sigma(\theta_M) \alpha_S(\theta_M) \nabla v \rightarrow \sigma(\theta) \alpha_S(\theta) \nabla v \quad \text{in } L^2(Q_T); \\ & \sigma(\theta_M) \Pi(\theta_M) \nabla v \rightarrow \sigma(\theta) \Pi(\theta) \nabla v \quad \text{in } L^2(Q_T), \end{aligned}$$

Applying (60) to the following estimates

$$\begin{aligned} & \|T_{\mathcal{M}}(\phi_M) \nabla \theta_M\|_{2, Q_T} \leq \mathcal{M} \|\nabla \theta_M\|_{2, Q_T}; \\ & \|\sigma(\theta_M) T_{\mathcal{M}}(\phi_M) \nabla \phi_M\|_{2, Q_T} \leq \sigma^{\#} \mathcal{M} \|\nabla \phi_M\|_{2, Q_T}; \\ & \|\gamma(\theta_M) |\theta_M|^{\ell-2} \theta_M\|_{\ell', \Sigma_T} \leq \gamma^{\#} \|\theta_M\|_{\ell, \Sigma_T}, \end{aligned}$$

there exist $\Lambda_1, \Lambda_2 \in L^2(Q_T)$ and $\Lambda_3 \in L^{\ell'}(\Sigma_T)$ such that

$$\begin{aligned} & T_{\mathcal{M}}(\phi_M) \nabla \theta_M \rightharpoonup \Lambda_1 \quad \text{in } L^2(Q_T); \\ & \sigma(\theta_M) T_{\mathcal{M}}(\phi_M) \nabla \phi_M \rightharpoonup \Lambda_2 \quad \text{in } L^2(Q_T); \\ & \gamma(\theta_M) |\theta_M|^{\ell-2} \theta_M \rightharpoonup \Lambda_3 \quad \text{in } L^{\ell'}(\Sigma_T). \end{aligned}$$

Thus, we may pass to the limit in (73) as M tends to infinity, concluding that (θ, ϕ) verifies

$$\begin{aligned} & \int_0^T \langle Z, v \rangle dt + \int_{Q_T} k(\theta) \nabla \theta \cdot \nabla v dx dt \\ & + \int_{Q_T} \sigma(\theta) \Pi(\theta) \nabla \phi \cdot \nabla v dx dt + \int_{Q_T} \sigma(\theta) \alpha_S(\theta) \Lambda_1 \cdot \nabla v dx dt \\ & + \int_{Q_T} \Lambda_2 \cdot \nabla v dx dt + \int_{\Sigma_T} \Lambda_3 v ds dt = \int_{\Sigma_T} h v ds dt, \quad \forall v \in V_\ell(Q_T). \end{aligned} \tag{76}$$

To identify the temperature θ as a solution, we need to identify $\Lambda_1, \Lambda_2 \in \mathbf{L}^2(Q_T)$ and $\Lambda_3 \in L^\ell(\Sigma_T)$.

To prove that $\Lambda_1 = T_{\mathcal{M}}(\phi) \nabla \theta$, let us consider the Green formula

$$\int_{Q_T} T_{\mathcal{M}}(\phi_M) \nabla \theta_M \cdot \mathbf{v} dx dt = - \int_{Q_T[|\phi_M| < \mathcal{M}]} \theta_M \nabla \phi_M \cdot \mathbf{v} dx dt,$$

for every $\mathbf{v} \in \mathbf{L}^p(0, T; \mathbf{W}^{1,p}(\Omega))$ such that $\nabla \cdot \mathbf{v} = 0$ in Q_T and $\mathbf{v} \cdot \mathbf{n} = 0$ in $\partial\Omega \times]0, T[$. Next we choose the exponent $p > 1$ to ensure the meaning of the involved terms. By $\theta_M \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ and $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ with 2^* being the critical Sobolev exponent, *i.e.* $2^* = 2n/(n - 2)$ if $n > 2$ and any $2^* > 1$ if $n = 2$, making recourse to the interpolation with exponents being

$$\frac{\beta}{2} = \frac{1 - \beta}{2} + \frac{\beta}{2^*} = \frac{1}{q}, \quad (0 < \beta < 1),$$

then θ_M converges to θ in $L^q(Q_T)$ for every $q < 2(n + 2)/n$. In particular, we take $p > n + 2$ such that

$$\frac{1}{2} = \frac{1}{p} + \frac{1}{q} > \frac{1}{p} + \frac{n}{2(n + 2)}.$$

Consequently, we have that $\theta_M \mathbf{v}$ converges to $\theta \mathbf{v}$ in $\mathbf{L}^2(Q_T)$. Therefore, the uniqueness of the weak limit implies that $\Lambda_1 = T_{\mathcal{M}}(\phi) \nabla \theta$. In particular, we find

$$\int_{Q_T} a(\theta_M, \phi_M) \nabla \theta_M \cdot \nabla v dx dt \rightarrow \int_{Q_T} a(\theta, \phi) \nabla \theta \cdot \nabla v dx dt,$$

and consequently (76) reads

$$\begin{aligned} & \int_0^T \langle Z, v \rangle dt + \int_{Q_T} a(\theta, \phi) \nabla \theta \cdot \nabla v dx dt + \int_{Q_T} \sigma(\theta) \Pi(\theta) \nabla \phi \cdot \nabla v dx dt \\ & + \int_{Q_T} \Lambda_2 \cdot \nabla v dx dt + \int_{\Sigma_T} \Lambda_3 v ds dt = \int_{\Sigma_T} h v ds dt, \quad \forall v \in V_\ell(Q_T). \end{aligned} \tag{77}$$

Now, we are in the conditions to identify the limits Λ_2 and Λ_3 by making recourse to the Minty argument as follows. We rephrase (48) as

$$\begin{aligned} & \mathcal{J}_M - \mathcal{J}_1^M - \mathcal{J}_2^M - \mathcal{J}_3^M - \mathcal{J}_4^M - \mathcal{J}_5^M \\ & + \int_{\Sigma_T} \left(\gamma(\theta_M) |\theta_M|^{\ell-2} \theta_M - \gamma(v) |v|^{\ell-2} v \right) (\theta_M - v) ds dt \\ & \geq (L_1)_\# \int_{Q_T} |\nabla(\theta_M - v)|^2 dx dt + (L_2)_\# \int_{Q_T} |\nabla(\phi_M - \phi)|^2 dx dt \geq 0, \end{aligned}$$

where

$$\begin{aligned}\mathcal{J}_M &:= \int_{Q_T} (a(\theta_M, \phi_M)|\nabla\theta_M|^2 + \sigma(\theta_M)F(\theta_M, \phi_M)\nabla\phi_M \cdot \nabla\theta_M)dxdt \\ &\quad + \int_{Q_T} (\sigma(\theta_M)|\nabla\phi_M|^2 + \sigma(\theta_M)\alpha_S(\theta_M)\nabla\theta_M \cdot \nabla\phi_M)dxdt \\ &\quad + \int_{\Sigma_T} \gamma(\theta_M)|\theta_M|^\ell dsdt; \\ \mathcal{J}_1^M &:= 2 \int_{Q_T} a(\theta_M, \phi_M)\nabla\theta_M \cdot \nabla v dxdt - \int_{Q_T} a(\theta_M, \phi_M)|\nabla v|^2 dxdt; \\ \mathcal{J}_2^M &:= 2 \int_{Q_T} \sigma(\theta_M)\nabla\phi_M \cdot \nabla\phi dxdt - \int_{Q_T} \sigma(\theta_M)|\nabla\phi|^2 dxdt; \\ \mathcal{J}_3^M &:= \int_{Q_T} \sigma(\theta_M)\Pi(\theta_M)(\nabla\phi_M \cdot \nabla v + \nabla\phi \cdot \nabla(\theta_M - v))dxdt; \\ \mathcal{J}_4^M &:= \int_{Q_T} \sigma(\theta_M)T_{\mathcal{M}}(\phi_M)(\nabla\phi_M \cdot \nabla v + \nabla\phi \cdot \nabla(\theta_M - v))dxdt; \\ \mathcal{J}_5^M &:= \int_{Q_T} \sigma(\theta_M)\alpha_S(\theta_M)(\nabla\theta_M \cdot \nabla\phi + \nabla v \cdot \nabla(\phi_M - \phi))dxdt.\end{aligned}$$

Considering the convergences

$$\begin{aligned}\mathcal{J}_1^M &\rightarrow 2 \int_{Q_T} a(\theta, \phi)\nabla\theta \cdot \nabla v dxdt - \int_{Q_T} a(\theta, \phi)|\nabla v|^2 dxdt := \mathcal{J}_1; \\ \mathcal{J}_2^M &\rightarrow \int_{Q_T} \sigma(\theta)|\nabla\phi|^2 dxdt := \mathcal{J}_2; \\ \mathcal{J}_3^M &\rightarrow \int_{Q_T} \sigma(\theta)\Pi(\theta)\nabla\phi \cdot \nabla\theta dxdt := \mathcal{J}_3; \\ \mathcal{J}_4^M &\rightarrow \int_{Q_T} \Lambda_2 \cdot \nabla v dxdt + \int_{Q_T} \sigma(\theta)T_{\mathcal{M}}(\phi)\nabla\phi \cdot \nabla(\theta - v) dxdt := \mathcal{J}_4; \\ \mathcal{J}_5^M &\rightarrow \int_{Q_T} \sigma(\theta)\alpha_S(\theta)\nabla\theta \cdot \nabla\phi dxdt := \mathcal{J}_5,\end{aligned}$$

we deduce

$$\begin{aligned}\liminf_{M \rightarrow \infty} \mathcal{J}_M &\geq \int_{\Sigma_T} \Lambda_3 v dsdt + \int_{\Sigma_T} \gamma(v)|v|^{\ell-2}v(\theta - v) dsdt \\ &\quad + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 + \mathcal{J}_5.\end{aligned}$$

We continue the Minty argument by taking in (73) and (74), respectively, the test function $v = \theta_M$ and the test function $w = \phi_M$, in (77) the test function $v = \theta$, and in (75) the test function $w = \phi$, we deduce

$$\begin{aligned}\limsup_{M \rightarrow \infty} \mathcal{J}_M &\leq - \int_0^T \langle Z, \theta \rangle dt + \int_{\Sigma_T} h\theta dsdt + \int_0^T \int_{\Gamma_N} g\phi dsdt \\ &= \int_{Q_T} a(\theta, \phi)|\nabla\theta|^2 dxdt + \int_{Q_T} \sigma(\theta)\Pi(\theta)\nabla\phi \cdot \nabla\theta dxdt \\ &\quad + \int_{Q_T} \Lambda_2 \cdot \nabla\theta dxdt + \int_{\Sigma_T} \Lambda_3\theta dsdt + \int_{Q_T} \mathbf{j}(\theta, \phi) \cdot \nabla\phi dxdt.\end{aligned}$$

Gathering the above two relations, we find

$$\int_{Q_T} a(\theta, \phi) |\nabla(\theta - v)|^2 dxdt + \int_{Q_T} (\Lambda_2 - \sigma(\theta) T_{\mathcal{M}}(\phi) \nabla \phi) \cdot \nabla(\theta - v) dxdt + \int_{\Sigma_T} (\Lambda_3 - \gamma(v) |v|^{\ell-2} v) (\theta - v) dsdt \geq 0. \quad (78)$$

Next, taking $v = \theta - \delta\varphi$, with $\varphi \in \mathcal{D}(Q_T)$, and after dividing by $\delta > 0$, we arrive to

$$\delta \int_{Q_T} a(\theta, \phi) |\nabla\varphi|^2 dxdt + \int_{Q_T} (\Lambda_2 - \sigma(\theta) T_{\mathcal{M}}(\phi) \nabla \phi) \cdot \nabla\varphi dxdt \geq 0.$$

Finally letting $\delta \rightarrow 0^+$ then we obtain $\Lambda_2 = \sigma(\theta) T_{\mathcal{M}}(\phi) \nabla \phi$.

We conclude the Minty argument by taking $v = \theta - \delta\varphi$, with $\varphi \in \mathcal{D}(\Sigma_T)$, in (78). After dividing by $\delta > 0$, and finally letting $\delta \rightarrow 0^+$ we arrive to

$$\int_{\Sigma_T} (\Lambda_3 - \gamma(\theta) |\theta|^{\ell-2} \theta) \varphi dsdt \geq 0, \quad \forall \varphi \in \mathcal{D}(\Sigma_T),$$

which implies that $\Lambda_3 = \gamma(\theta) |\theta|^{\ell-2} \theta$.

Therefore, the weak formulation (5) yields concluding the proof of Theorem 1.

Compliance with ethical standards

Conflict of interest The author states that there is no conflict of interest.

References

- Alt, H.W., Luckhaus, S.: Quasilinear elliptic-parabolic differential equations. *Math. Z.* **183**, 311–341 (1983)
- Banard, R.D.: *Thermoelectricity in Metals and Alloys*. Taylor and Francis Ltd., London (1972)
- Benilan, P., Wittbold, P.: On mild and weak solutions of elliptic-parabolic problems. *Adv. Differ. Equ.* **1**, 1053–1073 (1996)
- Biler, P.: Existence and asymptotics of solutions for a parabolic-elliptic system with nonlinear no-flux boundary conditions. *Nonlinear Anal.* **19**(12), 1121–1136 (1992)
- Caffarelli, L.A., Stefaneli, U.: A counterexample to $C^{2,1}$ regularity for parabolic fully nonlinear equations. *Commun. Partial Differ. Equ.* **33**(7), 1216–1234 (2008)
- Cheung, C.-Y., Menictas, C., Bao, J., Skyllas-Kazacos, M., Welch, B.J.: Spatial temperature profiles in an aluminum reduction cell under different anode current distributions. *AIChE J.* **59**(5), 1544–1556 (2013)
- Cimatti, G.: Invariance of flows in doubly-connected domains with the same modulus. *Bollettino dell'Unione Matematica Italiana* **7**(3), 217–226 (2014)
- Colli, P., Scarpa, L.: Existence of solutions for a model of microwave heating. *Discrete Contin. Dyn. Syst.* **36**(6), 3011–3034 (2016)
- Consiglieri, L.: Radiative effects for some bidimensional thermoelectric problems. *Adv. Nonlinear Anal.* **5**(4), 347–366 (2016)
- Consiglieri, L.: *Quantitative Estimates on Boundary Value Problems. Smallness Conditions to Thermoelectric and Thermoelectrochemical Problems*. LAP LAMBERT Academic Publishing, Saarbrücken (2017)
- Disser, K., Kaiser, H.-C., Rehberg, J.: Optimal Sobolev regularity for linear second-order divergence elliptic operators occurring in real-world problems. *SIAM J. Math. Anal.* **47**(3), 1719–1746 (2015)
- Haller-Dintelmann, R., Jonsson, A., Knees, D., Rehberg, J.: Elliptic and parabolic regularity for second-order divergence operators with mixed boundary conditions. *Math. Methods Appl. Sci.* **39**(17), 5007–5026 (2016)
- Huang, M.-J., Yen, R.-H., Wang, A.-B.: The influence of the Thomson effect on the performance of a thermoelectric cooler. *Int. J. Heat Mass Transf.* **48**, 413–418 (2005)

14. Igbida, J., El Hachimi, A., Jamea, A., Asmaa, A.: Doubly non-linear elliptic-parabolic equations by Rothe's method. *Br. J. Math. Comput. Sci.* **4**(22), 3224–3235 (2014)
15. Ivanov, A.V.: Second-order quasilinear degenerate and nonuniformly elliptic and parabolic equations. *Proc. Steklov Inst. Math.* **160**, 1–288 (1984)
16. Jäger, W., Kačur, J.: Solution of doubly nonlinear and degenerate parabolic problems by relaxation schemes. *RAIRO-Modélisation mathématique et analyse numérique* **29**(5), 605–627 (1995)
17. Kačur, J.: Solution to strongly nonlinear parabolic problems by a linear approximation scheme. *IMA J. Numer. Anal.* **19**, 119–145 (1999)
18. Leray, J., Lions, J.L.: Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty–Browder. *Bull. Soc. Math. Fr.* **93**, 97–107 (1965)
19. Lions, J.L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod et Gauthier-Villars, Paris (1969)
20. MacDonald, D.K.C.: *Thermoelectricity: An Introduction to the Principles*. Wiley, New York (1962)
21. Mikula, K.: Numerical solution of nonlinear diffusion with finite extinction phenomenon. *Acta Math. Univ. Comen.* **64**(2), 173–184 (1995)
22. jr Morrey, C.B.: Second order elliptic equations in several variables and Hölder continuity. *Math. Z.* **72**, 146–164 (1959)
23. Pluschke, V., Weber, F.: The local solution of a parabolic-elliptic equation with a nonlinear Neumann boundary condition. *Comment. Math. Univ. Carolin.* **40**(1), 13–38 (1999)
24. Rektorys, K.: *The Method of Discretization in Time and Partial Differential Equations*. D. Reidel, Dordrecht (1982)
25. Stampacchia, G.: Problemi al contorno ellittici con dati discontinui dotati di soluzioni hölderiane. *Ann. Math. Pura Appl.* **51**, 1–32 (1960)
26. Yuan, G.-W., Hang, X.-D.: Acceleration methods of nonlinear iteration for nonlinear parabolic equations. *J. Comput. Math.* **24**(3), 412–424 (2006)