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Some kinetic models for a market economy

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Abstract We review the main results on some basic kinetic models for wealth distribution in a simple market economy, with interaction rules involving random variables to take into account effects due to market risks. Then, we investigate in more detail long time behavior of a model which includes the taxation phenomenon and the redistribution of collected wealth according to proper criterions. Finally, we propose a new class of kinetic equations in which agent's trading propensity varies according to the personal amount of wealth.

Keywords Boltzmann and Fokker–Planck equations · Kinetic approach to economic sciences · Continuous trading limit

Mathematics Subject Classification 91B60 · 82C40 · 35B40

1 Introduction

Kinetic theory was originally proposed in nineteenth century by Maxwell and Boltzmann as a tool to describe the evolution of a rarefied gas, by means of methods typical of statistical mechanics. It is an intermediate (mesoscopic) approach between the unfeasible microscopic description of single particles, and the investigation only of main macroscopic fields of the gas, typical of fluid-dynamics. The most famous model is the Boltzmann equation, an integro-differential equation for the "distribution function" $f(\mathbf{x}, \mathbf{v}, t)$, representing the density of molecules at time t in the phase space (where $\mathbf{x}$ stands for the position and $\mathbf{v}$ for the molecular velocity, which is the kinetic variable, peculiar of this theory). In classical kinetic approach collisions are assumed to be elastic, namely with conservation of global momentum and kinetic energy in each encounter, and these constraints allow to determine uniquely post-collision velocities $(\mathbf{v}', \mathbf{w}')$ in terms of the ingoing $(\mathbf{v}, \mathbf{w})$ and of the "impact parameters" of the collision. Suitable moments (with respect to $\mathbf{v}$) of the distribution function provide major

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macroscopic fields, therefore Boltzmann approach allows to recover hydrodynamic Euler or Navier-Stokes equations in proper asymptotic limits. All details of classical kinetic theory may be found in several books, among which we mention [5,18].

Boltzmann's analysis has been later adapted to describe problems with non-conservative interactions (as reactive mixtures of polyatomic gases or plasmas, granular materials, biological propagation phenomena), and even to model interacting multi-agent systems. The parallelism between gases and crowds at the basis of kinetic socio-economic models is the following: a gas is constituted by an enormous number of particles, changing their molecular velocities through binary collisions; in analogous way, a crowd may be modelled as a huge amount of agents who modify their goods (wealth, opinion, ...) by means of binary interactions. It should be emphasized however that there are substantial differences between the collision mechanism of particles and human interactions: in social sciences, usual mechanical conservations are lacking, and random effects may play a crucial role, therefore they have to be suitably introduced in interaction rules. Moreover, in classical Boltzmann theory the velocity variable ranges over the whole space $\mathbb{R}^3$, while in social sciences the variable v must fulfill some proper bounds: in a market economy, v stands for the amount of wealth, and it should be $v \geq 0$ if debts are not allowed; in the opinion formation, the opinion v may vary between two extremal points, namely on a closed interval. This fact implies the presence of suitable discontinuities, represented by unit step functions in the collision operator, able to cancel non-admissible interactions (which would give rise to post-collision variables (v', w') out of the allowed range). The mathematical study of such models is therefore more complicated, but in several cases a kinetic approach to socio-economic problems is able to identify some universal behaviors and asymptotic profiles, which turn out to depend only on some basic features of the interaction rules, neglecting all other details. The major results about kinetic equations for multi-agent systems have been recently summarized in the book [27].

In this paper, we focus the attention on kinetic models for simple market economies. A good model should reproduce the fact, pointed out by the Italian economist Pareto [29], that for $v \to +\infty$ the wealth distribution $f(v, t)$ has the so-called "Pareto tails", namely it is $f(v, t) \sim v^{-(\alpha+1)}$ and α is called "Pareto index". Pareto observed that this distribution is somehow universal, in the sense that different populations yield very similar curves, varying little in space and time [6]. Following numerous later studies, today it is known that the bulk of wealth distribution fits log-normal ($f(v, t) \sim \frac{1}{v}\, \mathrm{e}^{-(\log v - M)^2}$) or Gamma distributions ($f(v, t) \sim v^{\alpha-1}\, \mathrm{e}^{-\beta v}$), reasonably well; the first one is preferred by economists [25], while the second type is mainly used by statisticians and physicists [17,37]. Anyway, with different parameters in the exchange processes modelling multi-agents systems, wealth distribution may turn from one form to the other [31]. There is much more consensus on the tail of the distribution, which turns out to be well described by an inverse power law, with Pareto index in capitalist countries usually such that $1.3 < \alpha < 3$. Studies based on real empirical data may be found for instance in [19,38] for the USA, in [2] for Japan, in [10] for Italy.

Kinetic models, even if simple and not really realistic from an economic point of view, allow to capture some key factors in socio-economic interactions, and to prove that very different societies converge to similar forms of unequal distributions, and that such economic inequalities might be reduced or increased by modifying some basic parameters, such as the saving habits or the taxation rates [1]. Necessary and sufficient assumptions on trading rules in kinetic models guaranteeing the formation of steady distributions (or of self-similar solutions) with Pareto tails have been proved in [23]. Nevertheless, the explicit determination of the limit distribution of the kinetic equation is very difficult, and it usually requires suitable numerical methods, or particular asymptotic limits leading to simpler kinetic models (typically, Fokker–

Planck-type equations). This kinetic approach has received even critical comments [20], but more recently, also because of the failure of main economic research lines to correctly anticipate and analyze the economic crisis, some of the previously critical authors have shown interest in market dynamics based on statistical mechanics [21].

The present article will be organized as follows. In Sect. 2 we present some mesoscopic economic equations, introducing notations, major macroscopic observables, and main results concerning a basic deterministic model and a more recent approach which takes into account also risks of the market by means of suitable random variables. Then, in Sect. 3 we investigate a model in which an external entity (for instance, the state) introduces a small taxation in all binary interactions, and redistributes the collected wealth to the people according to proper rules that, for varying parameters, may favor a specific class of individuals (the poor people, or the rich agents, or the middle class). In Sect. 4 we propose a new class of kinetic models in which the trading propensity parameter is not a constant, as usual in existing literature on the matter, but depends on the individual amount of wealth; the distribution behavior in suitable asymptotic limits is discussed and compared with the corresponding results available for simpler models. Some concluding remarks and hints for future works are presented in Sect. 5.

2 Some basic kinetic equations for a market economy

In this section we present some basic features of kinetic models for the economy, and we discuss in more detail a simple deterministic model (in Sect. 2.1) and a more recent model accounting for market risks (in Sect. 2.2).

We are interested in describing the evolution of wealth distribution $f(v,t)$ in a simple market economy in which agents interact through binary trades. In this frame, the quantity $f(v,t)\,dv$ represents the number of agents who at time t have wealth in $(v, v+dv)$. As usual in the kinetic approach, macroscopic information may be extracted from the moments of the distribution with respect to the kinetic variable v (representing individuals' wealth), namely

$$M_s(t) = \int_0^{+\infty} v^s\, f(v,t)\,dv. \tag{1}$$

Indeed, M_0 is nothing but the total number of individuals (constant, usually normalized as $M_0 \equiv 1$), M_1 provides the global amount of wealth (coinciding with the mean wealth if $M_0 = 1$), while M_2 is related to the variance of the distribution function. It is well known [9,14] that in a capitalist country wealth distribution usually follows a "Pareto law", namely $f(v,t) \sim v^{-(\alpha+1)}$ for $v \to +\infty$, with the exponent $\alpha \geq 1$. The "Pareto index" α, representing also the sup $\{s > 0 : M_s < +\infty\}$, describes the size of the rich people class, in the sense that the smaller α is, the more of the total wealth is owned by a small group of individuals. If a distribution $f(v,t)$ has all of its moments M_s finite, as it may happen in societies with an almost uniformly distributed wealth, it is said to possess "slim tails".

When two agents with wealths (v, w) undertake in a trade, they change their amount of wealth according to proper rules $(v', w') = h(v, w)$, that may be usually cast as

$$\begin{cases} v' = p_{11}\, v + p_{12}\, w \\ w' = p_{21}\, v + p_{22}\, w \end{cases} \tag{2}$$

The interaction coefficients p_{ij} are non-negative (in order to guarantee that post-interaction wealths (v', w') are non-negative), and are assumed to have fixed laws, independent of wealths and time, so that the amount of wealth that each agent transfers to the other is proportional to the respective wealth.

From now on we assume that $f(v,t)$ is a probability density $f \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ with $f(0,t) = 0$ (therefore individuals have strictly positive wealth) and with $M_{2+\delta} < +\infty$ for all $t \geq 0$, for some $\delta > 0$. The Boltzmann-type equation for the distribution $f(v,t)$ may be derived by standard methods of kinetic theory [5]. The variation in the time interval $(t, t+dt)$ of the number of individuals with wealth in $(v, v+dv)$ is provided by

$$f(v, t+dt)dv - f(v,t)dv = Q(f,f)(v,t)\,dv\,dt, \tag{3}$$

where the interaction operator $Q(f,f)(v,t)$ measures the contribution due to the trading interactions. This operator may be cast as the difference between a "gain term" $Q^+(f,f)$ and a "loss term" $Q^-(f,f)$. The loss term counts the agents having wealth in $(v, v+dv)$ at time t and different wealth (due to trade exchanges) $v' \notin (v, v+dv)$ at time $t+dt$:

$$Q^-(f,f)\,dv\,dt = dt\,f(v,t)\,dv \int_0^{+\infty} f(w,t)\,dw.$$

On the other hand, $Q^+(f,f)$ accounts for the individuals with wealth $v_\circ \notin (v, v+dv)$ at time t, but in the desired interval at time $t+dt$:

$$Q^+(f,f)\,dv\,dt = dt\,dv \int_0^{+\infty} \frac{1}{|J|} f(v_\circ,t)\,f(w_\circ,t)\,dw$$

where $(v_\circ, w_\circ) = h^{-1}(v,w)$ and J is the Jacobian of the linear transformation $h(v,w)$ defined in (2). In this way, the new variables $(v_\circ, w_\circ)$ denote the pre-interaction wealths having (v,w) as post-interaction output (i.e. $h(v_\circ, w_\circ) = (v,w)$), while the variables (v', w') defined in (2) denote the output of a trade with ingoing wealths (v,w) (i.e. $h(v,w) = (v',w')$).

The kinetic evolution equation for $f(v,t)$ follows from (3) dividing by $dv\,dt$ and passing to the limit $dt \to 0$:

$$\frac{\partial f(v,t)}{\partial t} = Q(f,f)(v,t).$$

The most useful tool in the sequel will be the weak form of the Boltzmann equation, that may be cast as

$$\frac{d}{dt}\left(\int_0^{+\infty} \varphi(v)\,f(v,t)\,dv\right) = \int_0^{+\infty}\int_0^{+\infty} \Big[\varphi(v') - \varphi(v)\Big] f(v,t) f(w,t)\,dv\,dw \tag{4}$$

where $\varphi(v) \in C^\infty(\mathbb{R}^+)$ stands for a generic test function. Notice that on the right hand side a suitable change of variables ($(v_\circ, w_\circ) \to (v,w)$) has allowed to get rid of the Jacobian.

One may immediately note that total number of individuals M_0 (corresponding to $\varphi \equiv 1$) is constant, therefore it may be assumed equal to 1 without loss of generality, while the evolution of higher moments depends on the nature of the chosen coefficients p_{ij}. In the following subsections we briefly present a simple deterministic model in which p_{ij} are fixed constants, and a more realistic model in which coefficients p_{ij} involve random variables taking into account market risks. A more detailed description of existing kinetic models for a market economy and social sciences may be found in Refs. [8,27].

2.1 A deterministic market model

One of the simplest descriptions of a binary wealth exchange dates back to Chakraborti and Chakrabarti [7], who proposed a trade rule of the form (2) depending only on a constant parameter γ:

$$\begin{cases} v' = (1-\gamma)\, v + \gamma\, w \\ w' = \gamma\, v + (1-\gamma)\, w \end{cases} \tag{5}$$

The coefficient γ represents the "trading propensity", since it measures the fraction of the own wealth that each agent transfers to the other; since each individual would like to preserve the major part of his wealth, it is reasonable to assume $0 < \gamma < 1/2$. The model (5) is pointwise conservative, namely the total wealth is preserved in each trade ($v' + w' = v + w$), and consequently, as it may be obviously deduced from (4) setting $\varphi(v) = v$, the mean wealth is preserved in time: $M_1(t) = M_1$.

For this simple model, it is possible to find an explicit steady state resorting to the Laplace transform

$$\tilde{f}(s,t) = \int_0^{+\infty} \mathrm{e}^{-sv} f(v,t)\, dv.$$

Indeed, using the weak form of the kinetic equation (4) with the weight function $\varphi(v) = \mathrm{e}^{-sv}$ we get

$$\frac{d}{dt}(\tilde{f}(s,t)) = \tilde{f}((1-\gamma)s,t)\tilde{f}(\gamma s,t) - \tilde{f}(s,t)$$

so that each stationary (time-independent) state $\tilde{f}_\infty(s)$ must fulfill the equation

$$\tilde{f}_\infty(s) = \tilde{f}_\infty((1-\gamma)s)\, \tilde{f}_\infty(\gamma s)$$

with the constraints $\tilde{f}_\infty(0) = 1$ and $\tilde{f}'_\infty(0) = -M_1$. It is easy to check that the unique solution is $\tilde{f}_\infty(s) = \mathrm{e}^{-M_1 s}$, and consequently the inverse Laplace transform provides the steady state $f_\infty(v) = \delta(v - M_1)$, namely a distribution with all agents having the same amount of wealth. The evolution of the variance of the distribution function, provided by (4) with $\varphi(v) = (v - M_1)^2$, reads as

$$\frac{d}{dt}\left(\int_0^{+\infty} (v - M_1)^2 f(v,t)\, dv\right) = -2\gamma(1-\gamma)\int_0^{+\infty} (v - M_1)^2 f(v,t)\, dv,$$

proving that the variance tends to zero at the exponential rate $2\gamma(1-\gamma)$, therefore we may conclude that in this basic kinetic model the wealth distribution really evolves towards the steady state consisting in a Dirac delta concentrated at the mean wealth. Of course this model describes an ideal (not realistic) closed society in which each individual is ready to share his wealth and to reduce economic inequalities, and for this reason new models, taking into account market risks or other external phenomena, have been proposed and investigated.

2.2 A model with random market risks

In 2005, the authors Cordier et al. proposed a kinetic model [12], usually referred to as CPT model, in which coefficients p_{ij} appearing in the interaction rule (2) also contain random variables, in order to take into account risks and other non deterministic effects in the market. More precisely, trading rule in CPT reads as

$$\begin{cases} v' = (1-\gamma+\mu)\, v + \gamma\, w \\ w' = \gamma\, v + (1-\gamma+\eta)\, w \end{cases} \tag{6}$$

where μ, η are independent and identically distributed random variables, with probability distribution $\Theta(\cdot)$ assumed to be symmetric ($\Theta(\mu) = \Theta(-\mu)$), with zero mean and variance σ^2:

$$\langle \mu \rangle = \int_\Omega \mu\, \Theta(\mu)\, d\mu = 0, \quad \langle \mu^2 \rangle = \int_\Omega \mu^2\, \Theta(\mu)\, d\mu = \sigma^2,$$

where the domain Ω has to be properly determined. Indeed, the idea of this model is that individual wealth changes not only because of deterministic trades, but also thanks to proper investments, that of course have some risks (causing a loss of wealth), but allow also the possibility to further increase (in a non-deterministic way) the personal amount of wealth. Parameters μ and η may thus take positive and even negative values, provided that the post-interaction wealths (v', w') turn out to be both positive. In real problems, individuals have self-thinking capability and they try to enter in a trade only if they feel that this trade could be a favourable investment; therefore, interactions in which non deterministic effects (provided by μ and η) are both positive are quite frequent. Moreover, correlations between random market effects occurring on the two interacting individuals should not be neglected, therefore the independence of the two random variables μ and η is somehow restrictive. For this reason, a model in which μ and η assume the same value has been presented in [35], but a general model taking into account of stochastic dependency between the two random risks is still lacking.

A simple way to guarantee non-negative post-trade wealths consists in assuming random variables with compact support $\Omega = (-1+\gamma, 1-\gamma)$. In this case, denoting with $\langle\cdot\rangle$ the mean of a given function $k(\cdot,\cdot)$ with respect to random variables:

$$\langle k(\mu,\eta) \rangle = \int_{-1+\gamma}^{1-\gamma} \int_{-1+\gamma}^{1-\gamma} k(\mu,\eta)\, \Theta(\mu)\, \Theta(\eta)\, d\mu\, d\eta,$$

it can be easily checked that the model (6), even if no more pointwise conservative (since $v' + w' \neq v + w$), is "conservative in the mean", in the sense that $\langle v' + w' \rangle = v + w$. This implies that the mean wealth is preserved in time: $M_1(t) = M_1(0) = M_1^0$, and this is confirmed also by inserting the test function $\varphi(v) = v$ into the weak form of the Boltzmann equation, that in the present non-deterministic case may be cast as

$$\frac{d}{dt}\left(\int_0^{+\infty} \varphi(v)\, f(v,t)\, dv\right) = \left\langle \int_0^{+\infty} \int_0^{+\infty} \Big[\varphi(v') - \varphi(v)\Big] f(v,t) f(w,t)\, dv\, dw \right\rangle. \quad (7)$$

The evolution of the second moment M_2 (corresponding to the option $\varphi(v) = v^2$), taking into account that

$$(v')^2 - v^2 = \big(\gamma^2 + \mu^2 - 2\gamma + 2\mu - 2\gamma\mu\big)v^2 + \gamma^2 w^2 + 2\gamma\big(1 - \gamma + \mu\big)vw,$$

reads, after some simple computations that we skip here for brevity, as

$$\frac{dM_2(t)}{dt} = -\big[2\,\gamma(1-\gamma) - \sigma^2\big]M_2(t) + 2\gamma(1-\gamma)(M_1^0)^2. \quad (8)$$

Consequently, if the variance of random variables, measuring non-deterministic effects, is large enough, the second moment M_2 increases in time, exponentially if $\sigma^2 > 2\,\gamma(1-\gamma)$, and linearly if the equality holds. Thus, the wealth distribution for infinite time is characterized by a Pareto tail with index less than two, quite in agreement with empirical observations. On the other hand, if the variance of random variables is lower than the threshold $2\,\gamma(1-\gamma)$, the evolution of $M_2(t)$ is explicitly provided by

$$M_2(t) = M_2^0 \exp\Big[-\Big(2\gamma(1-\gamma) - \sigma^2\Big)t\Big] + \frac{2\gamma(1-\gamma)(M_1^0)^2}{2\gamma(1-\gamma) - \sigma^2}\Big\{1 - \exp\Big[-\Big(2\gamma(1-\gamma) - \sigma^2\Big)t\Big]\Big\}$$

and

$$\lim_{t\to+\infty} M_2(t) = \frac{2\gamma(1-\gamma)(M_1^0)^2}{2\gamma(1-\gamma)-\sigma^2} > 0,$$

therefore even in this case with few random effects, the steady state is no more a Dirac delta with all people sharing the same amount of wealth, typical of the simple basic model described in the previous subsection. Estimates on the possible formation of Pareto tails may be performed resorting to the following function involving powers of the coefficients p_{ij} $(i, j = 1, 2)$ of the interaction rule:

$$\mathcal{S}(s) = \frac{1}{2}\left(\sum_{i=1}^{2}\left\langle (p_{i1})^s + (p_{i2})^s \right\rangle\right) - 1.$$

Indeed, for "conservative in the mean" models, $\mathcal{S}(s)$ is convex in $s > 0$, with $\mathcal{S}(0) = 1$ and $\mathcal{S}(1) = 0$, and it has been proved in [23] that if $S(\alpha) = 0$ for some $\alpha > 1$ then the steady state has a Pareto tail of index α, otherwise it has slim tails; therefore the macroscopic features of the asymptotic profiles are completely determined by the interaction coefficients in (6). In [23] tails formation has been analytically studied for simple random variables, for instance when μ and η take only the two values $\pm k$ with probability 1/2 each, or when the probability distribution $\Theta(\cdot)$ has an exponential decay; for more complicated options, the behavior of $\mathcal{S}(s)$ may be numerically investigated.

For general symmetric probability distributions $\Theta(\cdot)$, with unbounded support, one has to insert suitable unit step functions into the kernel of the Boltzmann collision operator, in order to only retain interactions with non-negative post-trade wealths. In this case, the weak form of the kinetic equation reads as

$$\begin{aligned}\frac{d}{dt}\left(\int_0^{+\infty}\varphi(v)\,f(v,t)\,dv\right) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\Big[\varphi(v')-\varphi(v)\Big]\\ &\times\Psi(v'\ge 0)\Psi(w'\ge 0)\Theta(\mu)\Theta(\eta)\,f(v,t)\,f(w,t)\,dv\,dw\,d\mu\,d\eta,\end{aligned} \tag{9}$$

where $\Psi(B)$ is the indicator function of the set B. For the evolution of mean wealth we have

$$\frac{dM_1(t)}{dt} = \iiiint \mu\, v\,\Psi(v'\ge 0)\Psi(w'\ge 0)\Theta(\mu)\Theta(\eta)\,f(v,t)\,f(w,t)\,dv\,dw\,d\mu\,d\eta \tag{10}$$

(the integral of the other term of $v' - v$ vanishes for parity arguments). Noticing that, since $\Theta(\cdot)$ is symmetric,

$$\int \Theta(\eta)\Psi\left(\eta \ge -(1-\gamma) - \gamma\frac{v}{w}\right)d\eta \ge \int \Theta(\eta)\Psi(\eta\ge 0)d\eta = \frac{1}{2}$$

and

$$\begin{aligned}\int \mu\,\Theta(\mu)\Psi\left(\mu \ge -(1-\gamma)-\gamma\frac{w}{v}\right)d\mu \;&=\; \int \mu\,\Theta(\mu)\Psi\left(\mu\ge 1-\gamma+\gamma\frac{w}{v}\right)d\mu\\ &\ge\; \Psi(w\le v)\int \mu\,\Theta(\mu)\Psi(\mu\ge 1)d\mu\\ &:= A\,\Psi(w\le v),\end{aligned}$$

one has the estimate

$$\frac{dM_1(t)}{dt} \geq \frac{A}{2} \iint v\,\Psi(w \leq v)\, f(v)\, f(w)\, dv\, dw$$
$$= \frac{A}{4} \iint \Big[v\,\Psi(w \leq v)\; + w\,\Psi(v \leq w) \Big] f(v)\, f(w)\, dv\, dw \geq \frac{A}{4}\, M_1(t).$$

In conclusion, the mean wealth increases in time at least exponentially, with $M_1(t) \geq M_1^0 \exp(\frac{A}{4}\,t)$; more precisely, by similar arguments it can be also proved [12] that it does not increase more than exponentially in time. Of course these proofs are expected to be much more involved in case of not independent pairs of random variables. The global increasing of total (and mean) wealth is in complete agreement with historical observations showing that global wealth, even if not strictly monotone (especially in economic crisis periods), has an increasing trend over long times [33]. From a mathematical point of view, such a property is due to the fact that in (9) the collision kernel is no more symmetric, and it keeps all interactions giving rise to gains while it cancels all trades producing "too much" loss (in the sense that one of the individuals would remain with a negative wealth). Indeed, agents are usually willing to be involved in a trade only if they have reasonable possibilities to improve their personal conditions, learning also from their previous mistakes [26], and this kinetic model is somehow able to reproduce the effects of this way of thinking.

3 A kinetic model with taxation and redistribution

In this section, we review and investigate a kinetic model which tries to take into account the taxation and the redistribution of the collected wealth. The problem of finding an "optimal taxation" has gained interest since almost one century ago, with the advent of New Welfare in 1930s [36]: it seems morally obvious that taxation rate should be progressive versus income, but this is not true for economists, whose role consists in finding the most efficient strategy, irrespective of population opinions. A review on studies about relations between politics and economy may be found in [16]. The standard theory of optimal taxation looks for a tax system which maximizes a social welfare function subject to a set of constraints (some examples may be found in [24,30]). However, it has been recently asserted that a "flat taxation", with the same rate at every income level, could be close to optimal [22]. On the other hand, other authors recommend to rise taxes for very high earning and to supply subsidies to low income families [13]. In this respect, in [4] we have introduced and investigated a kinetic model for wealth distribution which includes taxation with a fixed rate in each trading process, and also redistribution of the collected money among the population according to a given criterion, which could be uniform (reproducing thus the case of "flat taxation"), or even favourable for poor or rich people. The aim of this section is to discuss in more detail the kinds of redistribution described by the model for varying parameters, and to show that, at least in some proper asymptotic limit, the redistribution is able to modify the Pareto index of the steady wealth distribution.

The crucial idea of this model is to introduce a simple taxation mechanism at the level of each single trade, in order to generate a portion of the total wealth of the society that will be totally redistributed to agents, maintaining thus the total wealth constant. The mechanism of redistribution will be sufficiently flexible to be able to redistribute to agents a constant amount of wealth independently of the wealth itself, or to redistribute proportionally (or inversely proportionally) to their wealth. As concerns the interaction rule, let us assume that the transaction coefficients p_{11}, p_{22} in (2) are bounded from below: $\min\{p_{11}, p_{22}\} > \delta$, for a given small constant $\delta > 0$; then, for any positive constant $\varepsilon \leq \delta$, the trade

$$\begin{cases} v'_\varepsilon = (p_{11} - \varepsilon)v + p_{12}\,w \\ w'_\varepsilon = p_{21}\,v + (p_{22} - \varepsilon)w \end{cases} \tag{11}$$

is such that both v'_ε and w'_ε are non-negative, but conservation of wealth is generally lost. We assume that parameters p_{ij} contain random variables such that $\langle p_{11}\rangle + \langle p_{21}\rangle = \langle p_{12}\rangle + \langle p_{22}\rangle = 1$; consequently, total wealth decreases even "in the mean":

$$\langle v'_\varepsilon + w'_\varepsilon\rangle = (1-\varepsilon)(v+w). \tag{12}$$

Moreover, since the role of the two interacting individuals is symmetric, we take $\langle p_{11}\rangle = \langle p_{22}\rangle$ and $\langle p_{12}\rangle = \langle p_{21}\rangle$ and, owing to the weak Boltzmann equation (7) with $\varphi(v) = v$, since $v'_\varepsilon - v = p_{11}v + p_{12}\,w - (1+\varepsilon)v$ we notice that $\frac{dM_1(t)}{dt} = -\,\varepsilon\, M_1(t)$, hence the mean wealth is exponentially decaying in time as $M_1(t) = M_1^0\, \exp(-\,\varepsilon\, t)$.

The percentage of mean wealth that comes out by taxation, can be restituted to the agents, resorting to a proper redistribution operator $\mathcal{R}(f)$, in such a way that the total wealth is left unchanged. Let us assume that the amount of collected wealth $q = \varepsilon\, M_1(t)$ is entirely and instantaneously redistributed to the population; the operator $\mathcal{R}$ has thus to fulfill the constraints

$$\int_0^{+\infty} \mathcal{R}(f)(v,t)\,dv = 0, \quad \int_0^{+\infty} v\,\mathcal{R}(f)(v,t)\,dv = q, \tag{13}$$

and, if $\psi(v)$ denotes the fraction of the available wealth q to be given to each agent with wealth v, it should be

$$\int_0^{+\infty} \psi(v)\, f(v,t)\,dv = 1. \tag{14}$$

We consider a balance equation for agents with wealth in $(v, v+dv)$ due to redistribution. Each individual having wealth v at time t, will have wealth $v^* = v + \psi(v)\,q\,dt$ at time $t+dt$. Therefore

$$\frac{\partial f(v,t)}{\partial t}\,dv\,dt = f(v_*,t)\,dv_* - f(v,t)\,dv$$

where v_* is the pre-redistribution wealth having a corresponding output v at time $t+dt$:

$$v_* = v - \psi(v_*)\,q\,dt = v - \psi(v)\,q\,dt + O(dt^2), \quad dv_* = dv - \psi'(v)\,q\,dv\,dt + O(dt^2).$$

Neglecting $O(dt^2)$ terms we find

$$\frac{\partial f(v,t)}{\partial t} = \mathcal{R}(f)(v,t) = -\,q\,\frac{\partial}{\partial v}\Big[\psi(v)\, f(v,t)\Big]. \tag{15}$$

This operator fulfills the first constraint in (13) if $f(0,t) = 0$, while it is not so easy to find reasonable functions $\psi(v)$ satisfying (14) and the second of (13). If we look for a linear redistribution function $\psi(v) = C\,v + D$ (case to which we will stick here for simplicity), we find that admissible options are

$$\psi(v) = \frac{1}{M_1(t)}\Big[-\,\chi\, v + (\chi+1)\,M_1(t)\Big], \tag{16}$$

where χ is a free parameter. Notice that positivity of $\psi(v)$ is not guaranteed, therefore for a part of the population redistribution could be negative (corresponding thus to a further taxation). More precisely, for $\chi < -1$ we have $\psi(v) < 0$ for $v < M_1(t)(1/\chi + 1)$, therefore the poorest agents supply additional resources to the richest ones. For $-1 < \chi < 0$, $\psi(v)$ is positive for each $v > 0$, but it remains an increasing function, therefore rich people are again favored. The particular choice $\chi = 0$ corresponds to a uniform redistribution. Finally, for positive values of the parameter χ, money is redistributed especially to agents with little wealth, while people with wealth $v > M_1(t)(1/\chi + 1)$ are taxed once more.

We consider now the whole kinetic model with taxation included in the interaction rules and with the additional redistribution operator defined in (15) with $\psi(v)$ provided by (16):

$$\frac{\partial f(v,t)}{\partial t} = Q_\varepsilon(f, f)(v,t) + \mathcal{R}(f)(v,t).$$

Trading rule is assumed given by (11), with the same coefficients p_{ij} considered in the CPT model (6), including market risks by means of suitable random variables, that we assume here independent and with compact support $(-1+\gamma+\varepsilon, 1-\gamma-\varepsilon)$. The weak form of the Boltzmann equation with taxation and redistribution reads as

$$\begin{aligned}\frac{d}{dt}\left(\int_0^{+\infty} \varphi(v)\, f(v,t)\, dv\right) = \left\langle \int_0^{+\infty}\int_0^{+\infty} \Big[\varphi(v'_\varepsilon) - \varphi(v)\Big] f(v,t) f(w,t)\, dv\, dw \right\rangle \\ - \varepsilon \int_0^{+\infty} \varphi'(v)\Big[\chi\, v - (\chi+1)\, M_1\Big] f(v,t)\, dv, \qquad (17)\end{aligned}$$

where the first term on the right hand side is analogous to the standard weak Boltzmann operator in (7), with $v'_\varepsilon = (1-\gamma+\mu-\varepsilon)v + \gamma\, w$, while last contribution is obtained by a simple integration by parts of $\mathcal{R}(f)$, bearing in mind that $f(0,t) = 0$. The redistribution operator has been built up in order to compensate the loss of wealth due to taxation, and for this reason in the global equation M_1 is constant (independent from t).

We investigate now situations in which most of the trades correspond to a very small exchange of money (therefore with the trading propensity γ small), but at the same time we want to keep trace at the macroscopic level of all phenomena affecting market rules (market risks, taxation, and redistribution). This kind of asymptotic analysis is referred to as "continuous trading limit", and it allows to prove that under proper assumptions on the interaction parameters, at suitably large times the kinetic Boltzmann-type equation may be approximated by a simpler Fokker–Planck equation, allowing an analytical investigation of long time behavior of wealth distribution [3,12]. We measure the variance of random variables and the taxation parameter in terms of γ as

$$\sigma^2 = \lambda\,\gamma, \quad \varepsilon = \kappa\,\gamma \qquad (18)$$

(with $\lambda > 0$ and $\kappa > 0$), and we investigate the limit $\gamma \to 0$. The first of (18) implies that in this limit even the variance of random variables tends to zero, so that risks are really low in most of binary interactions, but they are still present in the market and they will affect the kinetic equation we will recover in this asymptotic regime. A second order Taylor expansion of $\varphi(v)$ around v provides

$$\begin{aligned}\varphi(v'_\varepsilon) - \varphi(v) = \Big[\gamma(w-v) + \mu\, v - \varepsilon\, v\Big]\varphi'(v) + \frac{1}{2}\Big[\gamma(w-v) + \mu\, v - \varepsilon\, v\Big]^2 \varphi''(v) \\ + o((v'_\varepsilon - v)^2). \qquad (19)\end{aligned}$$

Setting $\tau = \gamma t$ as new time variable, and inserting (19) into (17), the weak form for the scaled distribution function $g(v, \tau)$ turns out to be

$$\frac{d}{d\tau}\left(\int_0^{+\infty} \varphi(v)\, g(v,\tau)\, dv\right) = \frac{1}{\gamma}\left\langle \int_0^{+\infty}\int_0^{+\infty} \left\{\left[\gamma(w-v) + \mu\, v - \varepsilon\, v\right]\varphi'(v) + \frac{1}{2}\left[\gamma(w-v) + \mu\, v - \varepsilon\, v\right]^2 \varphi''(v)\right\} g(v,\tau) g(w,\tau)\, dv\, dw \right\rangle - \frac{\varepsilon}{\gamma}\int_0^{+\infty} \varphi'(v)\left[\chi\, v - (\chi+1)\, M_1\right] g(v,\tau)\, dv, \tag{20}$$

where $o((v'_\varepsilon - v)^2)$ terms have been neglected. Passing now to the limit $\gamma \to 0$ one may note that Eq. (20) is nothing but the weak form of the Fokker–Planck equation

$$\frac{\partial g}{\partial \tau} = \frac{\partial}{\partial v}\left\{\frac{\lambda}{2}\, v\, \frac{\partial}{\partial v}(vg) + \left[\left(\frac{\lambda}{2} + \kappa(\chi+1) + 1\right) v - \left(\kappa(\chi+1)+1\right) M_1\right] g\right\}. \tag{21}$$

The evolution of wealth distribution is thus a balance between a diffusion operator and a drift term, which would concentrate distribution towards the mean value M_1. The formal derivation of such a Fokker–Planck equation can be made rigorous repeating the very technical computations of [3,12]. Different balances among the parameters γ, σ^2, ε may of course give purely diffusive, or purely drift equations.

In Eq. (21) the taxation/redistribution mechanism is represented solely by the positive parameter κ, limiting ratio of vanishingly small taxation rate and transaction coefficient, and by the constant χ describing the redistribution policy. The presence of random effects is accounted for by the parameter $\lambda > 0$. The steady state $g_\infty(v)$ has to fulfill

$$\lambda\, v\, g_\infty(v) + \frac{\lambda}{2}\, v^2 g'_\infty(v) + \left[\, (\kappa(\chi+1)+1)\, v - \left(\kappa(\chi+1)+1\right) M_1\right] g_\infty(v) = 0$$

from which, integrating over the kinetic variable v:

$$\int_0^{+\infty} \frac{g'_\infty(v)}{g_\infty(v)}\, dv = -\int_0^{+\infty}\left\{\frac{2}{\lambda\, v^2}\left[\, (\kappa(\chi+1)+1)\, v - \left(\kappa(\chi+1)+1\right) M_1\right] + \frac{2}{v}\right\} dv$$

and skipping intermediate details we get

$$g_\infty(v) = C\, v^{-[2+r]} \exp\left\{-\frac{r\, M_1}{v}\right\}, \tag{22}$$

where C may be explicitly chosen so that $\int_0^{+\infty} g_\infty(v)\, dv = 1$, and

$$r = \frac{2[\kappa(\chi+1)+1]}{\lambda}. \tag{23}$$

We have $g_\infty(0) = 0$, while for $v \to +\infty$ we get $g_\infty(v) = O\left(v^{-(2+r)}\right)$. Therefore the Pareto index is here $1+r$, or, in other words, the moment of order α exists finite if $\alpha < 1+r$. Notice that Pareto tails are strongly influenced by the random effects. Indeed, in almost absence of market risks (with λ small), α increases making fairer the distribution of wealth for large times, while in a market with high risks (λ big) wealth distribution becomes really unfair. Unlike other statistical models of wealth exchange between agents [11], this kinetic model is able to reproduce also situations appearing in feudal societies, or even in India [34], in which the Pareto index is unusually low (around 1), as discussed in [32].

The original CPT model of [12] is recovered for $\kappa = 0$. The effects of taxation and redistribution are accounted for and explicitly quantified by the additional exponent $-2\kappa(\chi + 1)/\lambda$ for the wealth variable, which always makes tails slimmer, and thus wealth distribution fairer. It is clear that this effect is the stronger the higher is the factor κ, i.e. the higher the taxation rate. It is also evident that, if κ and λ are fixed, tails become less and less thick as χ increases, and in all cases with $\chi > -1$ the percentage of agents at high wealth decreases, whether or not they are in turn favored (as it would occur when $\chi > 0$). No effects on the wealth distribution are present for $\chi = -1$, in which case the same index as for the CPT model is recovered, independently of κ, namely, no matter how high is taxation.

4 Kinetic economic models with variable trading propensity

In existing kinetic models summarized in the reviews [8,15,27], the parameters involved in the interaction rules of kinetic equations for market economies are the same for the whole population. Actually, in real markets the saving propensity and the risk perception vary according to the agent's amount of wealth. For instance, a rich individual could have a lower saving propensity, and could be willing to risk a considerable amount of his money, conscious that high risk could correspond to a high gain, and that even in case of high loss, the remaining amount of money would allow him to survive; but also situations in which rich agents try to preserve almost all of their own wealth are admissible. Even for poor people there are essentially two ways of trading: one could try to save as much as possible, or could try the opposite strategy, risking almost all of his wealth hoping to be lucky and to rapidly increase his wealth. For this reason, a long-time objective is to build up more general kinetic models able to investigate all these possibilities, with trading propensity γ in the rule (6) depending on the individual's wealth v, and external market risks for the two interacting individuals not independent (actually, the independency assumption for the random variables μ and η is very simplistic, even if it allows to reproduce realistic macroscopic effects). Even risk perception of each agent should be taken into account, leading to different choices for their own portfolio.

In this section, we try to do a step in this direction, proposing models in which trading propensity γ is not a fixed constant, but a function of wealth v. Then, we discuss two different choices for $\gamma(v)$, one increasing and the other decreasing versus v, and we will notice that the strategy of rich people determines the basic features of steady distributions.

We consider an interaction rule of the form

$$\begin{cases} v' = (1-\gamma(v))\,v + \gamma(w)\,w + \mu\,v \\ w' = \gamma(v)\,v + (1-\gamma(w))\,w + \eta\,w \end{cases} \tag{24}$$

where $\gamma(v)$ is a generic function depending on wealth v, with the bounds $0 < \gamma(v) \leq \gamma_0 < 1$. A sufficient condition in order to guarantee that post-trade wealths (v', w') are non-negative consists in assuming random variables μ, η with a bounded support $(-1+\gamma_0, 1-\gamma_0)$. For simplicity we stick here to this assumption, but in similar problems it has been rigorously proved [3,12] that the remainders neglected in the continuous trading limits actually vanish even with market risks with unbounded domain, provided that the initial wealth distribution has $M_{2+\delta} < +\infty$ for some $\delta > 0$ (the standard Gaussian distribution is of course included in this class), and that the distribution of random variables has $2+\alpha$ finite moments, with $\alpha > \delta$. The weak form of our Boltzmann equation is again provided by formula (7).

We rewrite the trading propensity as $\gamma(v) = \gamma_0\,\tilde{\gamma}(v)$, and we consider the same continuous trading limit investigated in the CPT model in [12], assuming that

$$\sigma^2 = \lambda\,\gamma_0 \tag{25}$$

and letting $\gamma_0 \to 0$. A Taylor expansion of the test function $\varphi(v)$ provides now

$$\varphi(v') - \varphi(v) = \Big[\gamma(w)\,w - \gamma(v)\,v + \mu\,v\Big]\varphi'(v) + \frac{1}{2}\Big[\gamma(w)\,w - \gamma(v)\,v + \mu\,v\Big]^2\varphi''(v) + o((v'-v)^2).$$

By inserting this expression into (7) and solving the integrals over the random variables (bearing in mind that $\langle\mu\rangle = 0$ and $\langle\mu^2\rangle = \sigma^2$) we get for the re-scaled distribution $g(v,\tau)$, with $\tau = \gamma_0\,t$, the equation

$$\frac{d}{d\tau}\left(\int_0^{+\infty}\varphi(v)\,g(v,\tau)\,dv\right) = \int_0^{+\infty}\!\!\int_0^{+\infty}\Big\{\Big[\tilde\gamma(w)\,w - \tilde\gamma(v)\,v\Big]\varphi'(v) + \frac{1}{2}\Big[\gamma_0\,\tilde\gamma^2(w)\,w^2 - 2\,\gamma_0\,\tilde\gamma(v)\,\tilde\gamma(w)\,v\,w + \gamma_0\,\tilde\gamma^2(v)\,v^2 + \frac{\sigma^2}{\gamma_0}\,v^2\Big]\varphi''(v)\Big\}\,g(v,\tau)g(w,\tau)\,dv\,dw. \tag{26}$$

With the assumption (25), in the asymptotic limit $\gamma_0 \to 0$ all but the last term multiplying $\varphi''(v)$ vanish, and bearing in mind that $\int_0^{+\infty} g(w,\tau)\,dw = 1$ one is left with

$$\frac{d}{d\tau}\left(\int_0^{+\infty}\varphi(v)\,g(v,\tau)\,dv\right) = \int_0^{+\infty}\Big\{\Big[\Big(\int_0^{+\infty}\tilde\gamma(w)\,w\,g(w,\tau)\,dw\Big) - \tilde\gamma(v)\,v\Big]\varphi'(v) + \frac{\lambda}{2}\,v^2\varphi''(v)\Big\}\,g(v,\tau)dv. \tag{27}$$

This is the weak form associated to the nonlinear Fokker–Planck-type equation

$$\frac{\partial g}{\partial\tau} = \frac{\partial}{\partial v}\left\{\frac{\lambda}{2}\,\frac{\partial}{\partial v}(v^2 g) + \left[\tilde\gamma(v)\,v - \left(\int_0^{+\infty}\tilde\gamma(w)\,w\,g(w,\tau)\,dw\right)\right]g\right\}. \tag{28}$$

Recalling (25), we note that the random variables μ, η affect this equation through the parameter λ, hence in the diffusion term which guarantees the spread of wealth among population, and prevents the trend towards a Dirac delta distribution, as it would occur in models neglecting market risks [7]. The steady state $g_\infty(v)$ of Eq. (28) is provided by the solution of the equation

$$\frac{\lambda}{2}\,\frac{\partial}{\partial v}(v^2 g_\infty) + \Big[\tilde\gamma(v)\,v - \mathcal{M}_\infty\Big]g_\infty = 0 \tag{29}$$

where

$$\mathcal{M}_\infty = \int_0^{+\infty}\tilde\gamma(w)\,w\,g_\infty(w)\,dw.$$

Skipping some intermediate steps (similar to the ones in the previous section), Eq. (29) provides

$$\log(g_\infty(v)) = -\int_0^{+\infty}\left[\frac{2}{v} + \frac{2}{\lambda}\,\frac{\tilde\gamma(v)}{v} + \frac{2}{\lambda}\,\frac{\mathcal{M}_\infty}{v^2}\right]dv = -2\,\log v + \frac{2}{\lambda}\,\frac{\mathcal{M}_\infty}{v} - \frac{2}{\lambda}\int_0^{+\infty}\frac{\tilde\gamma(v)}{v}\,dv. \tag{30}$$

In the case of constant trading propensity $\gamma = \gamma_0$ (namely $\tilde{\gamma}(v) = 1$), we have $\mathcal{M}_\infty = M_1$, and we correctly recover the stationary state found in [12],

$$g_\infty(v) = C\, v^{-\left(2+\frac{2}{\lambda}\right)} \exp\left(-\frac{2}{\lambda}\frac{M_1}{v}\right),$$

with Pareto index $\alpha = 1 + \frac{2}{\lambda}$. We note that for varying saving propensity, since by definition we have $\tilde{\gamma}(v) \le 1$, the value $\alpha = 1 + \frac{2}{\lambda}$ is always an upper bound for the Pareto index of the steady state. Moreover, $\mathcal{M}_\infty \le M_1 < +\infty$.

Let us consider at first a situation in which trading propensity is provided by

$$\gamma(v) = \gamma_0 \frac{v+1}{2\,v+1}, \tag{31}$$

describing thus a market in which the fraction of wealth that each individual is willing to put into a single trade decreases versus wealth; more precisely, for v ranging from 0 to $+\infty$, the function $\gamma(v)$ decreases from γ_0 to $\frac{1}{2}\gamma_0$. This option could describe a population in which a poor agent decides to invest in (hopefully good) trades almost all his remaining money, in order to try to improve his present condition, while, on the other hand, a rich individual tries to save the major part of his money, and to invest only a small part of his wealth. The function (31) implies

$$\int_0^{+\infty} \frac{\tilde{\gamma}(v)}{v}\, dv = \int_0^{+\infty} \left(\frac{1}{v} - \frac{1}{2\,v+1}\right) dv = \log v - \frac{1}{2}\log\left(v + \frac{1}{2}\right),$$

and consequently the steady state of the Fokker–Planck equation (28) may be cast as

$$g_\infty(v) = C\, v^{-\left(2+\frac{2}{\lambda}\right)} \left(v + \frac{1}{2}\right)^{\frac{1}{\lambda}} \exp\left(-\frac{2}{\lambda}\frac{\mathcal{M}_\infty}{v}\right),$$

which has a Pareto tail with index $\alpha = 1 + \frac{1}{\lambda}$, less than the one corresponding to constant trading parameter $\gamma(v) = \gamma_0$. By analogous arguments it may be proved that to the choice $\gamma(v) = \gamma_0\,(v+1)/(K\,v+1)$ (with $K > 1$) there corresponds a stationary distribution with Pareto index $\alpha = 1 + \frac{2}{\lambda K}$, describing the fact that when rich people decide to keep their huge amount of money for themselves, without investing it in further trades, then the wealth distribution becomes really unequal for long times, and even the second moment (the variance of the distribution) is not finite for $K > \frac{2}{\lambda}$.

Then, let us investigate an opposite situation, in which trading propensity $\gamma(v)$ increases versus wealth, for instance as

$$\gamma(v) = \gamma_0 \left(1 - \frac{1}{v+\beta}\right), \tag{32}$$

with $\beta > 1$, so that for $0 \le v < +\infty$ we have $\gamma(v)$ increasing from $\gamma_0\left(1 - \frac{1}{\beta}\right) > 0$ to γ_0. In this case, the steady state of the Fokker–Planck equation (28) corresponding to the asymptotic limit (25) turns out to be

$$g_\infty(v) = C\, v^{-\left(2+\frac{2(\beta-1)}{\lambda\beta}\right)} (v+\beta)^{-\frac{2}{\lambda\beta}} \exp\left(-\frac{2}{\lambda}\frac{\mathcal{M}_\infty}{v}\right),$$

therefore its Pareto index is $\alpha = 1 + \frac{2}{\lambda}$ as in case of constant $\gamma(v) = \gamma_0$. This analysis allows to deduce that the macroscopic properties of wealth distribution among the individuals is essentially determined by the trading strategy of the rich people, while fluctuations in the trading propensity of poor people do not influence long time behavior, since such agents can invest and redistribute to other individuals only a small amount of wealth.

5 Concluding remarks

In this paper we have at first provided a review of some basic kinetic models for economic problems, in which coefficients appearing in trading rules may be deterministic as in [7], or random parameters as in [12]. The presence of random unpredictable effects favors the presence of long time steady distributions with Pareto tails (consistent with quoted economic references), in which few very rich individuals possess the major part of total wealth. This process could be modified by an external entity (typically, the state) by means of a proper taxation policy, and of redistribution and investments of collected wealth according to appropriate criterions. A complete overview on the state of art about mathematical and numerical aspects of models for socio-economic sciences may be found in [6,8,27]. In the last part of the paper we have proposed new models in which trading propensity parameter depends on the individual wealth, and in a suitable asymptotic regime we have shown that formation of Pareto tails depends above all on the strategy adopted by rich people.

A strict interaction with economists community should be pursued, in view of a comparison between our kinetic equations and real data, and even of possible generalizations of the kinetic approach to current challenging economic problems. From the mathematical point of view, the investigation of models with non-constant parameters could be extended, since trading strategies, risk perception, and even optimal taxes should depend on the personal characteristics [22]; a first attempt in this direction may be found in [28], in which the evolution of wealth is influenced by the level of knowledge of single agents. Moreover, in order to help poor people to improve their condition, the possibility of taking on some debts should be allowed, but in this way of modelling the definition of proper interaction rules seems to be not trivial, since the nature of trades among individuals with positive wealth should be different from interactions involving people with debts. All these open problems are left to future work.

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