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On traces spaces connected with a class of intermediate weighted spaces

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Abstract Methods developed by Lions and Peetre (Pub Math de l'IHES 19:5–68, 1964) are used to extend results derived in Artola (Bolletino UMI (9) V:125–158, 2012) for traces of weighted spaces. The weights are required to belong to the Hardy class H(p) defined in Artola (Bolletino UMI (9) V:125–158, 2012) to ensure that a necessary convolution product remains valid in weighted spaces. The restriction, apparently new, is necessary for the present treatment.

1 Introduction

The paper establishes certain trace properties that extend those for weighted spaces studied in [5]. The approach involves a (θ, c, p) method originally developed (for unweighted spaces) by J. Peetre but generalised to include weighted spaces with weights belonging to the Hardy class $\mathcal{H}(p)$, $1 \leq p < +\infty$. An integral representation for traces of order j is shown to belong to a type of weighted spaces that correspond to the unweighted spaces introduced by Lions and Peetre [21] and called "espaces de moyenne" by these authors.

A similar problem solved in [21] deals with the weights $c(t) = t^\alpha$, with $\alpha + 1/p \in (0, 1)$ so that c belongs to $\mathcal{H}(p)$ and $1/c$ to $\mathcal{H}(p')$. See also: [1,13–15,17,18,20].

A convolution product is introduced which in terms of a normed vector space A and weight c is defined to be

$$\rho * \phi(t) = \int_0^t \rho(t-\tau)\phi(\tau)d\tau, \quad \rho \in L^1(0, +\infty; \mathbf{R}),\ c\phi \in L^p(0, +\infty; \mathbf{A}). \tag{1.1}$$

Associated with the convolution product is a Young's inequality of type

$$|c(\rho * \phi)|_{L^p(\mathbf{A})} \leq \kappa\, |\rho|_{L^1} \,.\, |c\phi|_{L^p(\mathbf{A})} \tag{1.2}$$

In memory of Jacques-Louis Lions, 1928–2001.

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where κ is a positive constant. Implications of the inequality described here may be regarded as a stability result and are believed to be new. In this respect, it is worth noting that the inequality, called "$\mathcal{P}$-*condition*" in [7], is obvious when the weight is *not increasing*.

When $c \in A(p)$, the Muckenhoupt class [23], the inequality has been proved in [7]. But it is also known (cf. [7]) that $A(p) \subset \mathcal{H}(p)$ for all p, which suggests that the inequality is valid under improved conditions. In fact, it is shown in Theorem 3.1 in Sect. 3 that a sufficient condition for (1.2) is $c \in \mathcal{H}(p)$. An implicit proof using the complex derivative $D^{i\eta}(\eta \in R)$ is presented in [6] while heuristic arguments are announced in [7]. Consequently, we are led to the problem, which as far as the author is aware is open and not completely resolved here, which may stated as

Is $\mathcal{H}(p)$ the widest class of weight c for which (1.2) *holds*?

Nevertheless a positive answer is presented in a particular case in Remark 3.4.

For convenience, notation and conclusions previously obtained mainly in [5–7] are recalled in Sect. 2.

Section 3 defines a trace of a weighted space that is an alternative to the definition studied in [5] but which leads to an improved definition of a class of intermediate weighted spaces.

Inequality (1.2) then can be used to embed our trace space into another type of weighted space.

Specifically inequality (1.2) combined with the procedure introduced in [21] is employed to show that elements of the trace space $\mathbf{T}_j^{(m)}$ are represented by integrals. Consequently $\mathbf{T}_j^{(m)}$ can be identified as an "espace de moyenne" in the sense of [21] appropriately modified to accommodate weighted spaces.

Section 4 interrupts the main discussion and introduces certain intermediates spaces and other relevant definitions required subsequently. Invariance under the change of variable $t \longrightarrow \lambda t,\ (\lambda > 0)$, or $t \longrightarrow 1/t$, with respect to the Haar measure dt/t on $\mathbf{R}^*$, serves to guide the choice of weights.

By reference to inequality (1.2) two equivalent definitions are formulated for a space Σ called here "*intermediate mean space*".

A complete account of such spaces, which may found in [8], is omitted. Nevertheless, key properties and their proofs are recalled partly to enable the space Σ to be represented as

$$\Sigma = \Sigma_\theta(p_0, \theta, \hat{c}_0, \mathbf{A}_0;\ p_1, \theta - 1, \hat{c}_1, \mathbf{A}_1)$$

for a suitable choice of θ and of $\hat{c}_i \in \mathcal{H}(p_i),\ i = 0, 1$.

Section 5, which resumes the main discussion, interprets the space $\mathbf{T}_j^{(m)}$ as a particular "intermediate weighted mean space" Σ_{θ_j} and contrasts this result with those of [21] which deals with weights t^α where α appears as a parameter. Dependence upon the weight requires clarification.

It remains to investigate the new spaces with respect to the usual parameters and weights. For this purpose the methods of Peetre [25–27] are extended in the final Sect. 6. The "quasi invariance" of all definitions of $a \in \Sigma$, with respect to the change of variable $t \longrightarrow 1/t$, together with a new (**Jw**- *or* **Kw**-) *method* used to prove, following [27], that the space $\Sigma = \Sigma_\theta$ depends here, only upon three parameters: $\theta \in (0, 1)$, a power p_θ, and a weight c_θ.

I wish to dedicate this work to the memory of Jacques-Louis Lions who was my thesis adviser[1] during 1964–1968 and who introduced me not only to weighted spaces [2,3] but also to interpolation theory [4].

[1] My thesis on Partial Differential Equations with delay was published in 1969 at the Annals of E.N.Sup.ULM Paris.

The publication of the notes completed during this earlier period was prevented by heavy demands on my time, due not only to growing research interests in applied mathematical subjects mainly unrelated to weighted spaces, but also to responsibility for leading the development of the Applied Mathematic Group at the University Bordeaux I. (Indeed my entire professional career has been spent at the University of Bordeaux I.) Throughout this burdensome administrative task, J. L. Lions provided unfailing support and encouragement.

Fortunately, it is now possible to prepare and complete the remaining (1964–1968) notes for publication. So far , this has led to the appearance of articles [7] in 1998, and [5,6], recently. The present paper is intended as a contribution toward the continuation of the series.

2 Definitions and background

If $\mathcal{X}, \mathcal{Y}$ are vectorial topological spaces, $\mathcal{X} \subset \mathcal{Y}$ means always algebraic inclusion with a continuous injective mapping and $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, *resp.* ($\mathcal{L}(\mathcal{X})$ *if* $\mathcal{X} = \mathcal{Y}$), denotes the space of linear continuous mappings from $\mathcal{X}$ *into* $\mathcal{Y}$.

Let $\mathbf{X}$ be a normed space with norm $|\,.\,|_{\mathbf{X}}$, and let $I = (a, b) \subset \mathbf{R}^+ = (0, +\infty)$, $L^p(I;\ \mathbf{X})$, (*resp.* $L^p(\mathbf{X})$ *if* $I = \mathbf{R}^+$) denotes the space of (class of) functions which are strongly measurable with respect to the Lebesgue measure and *p-integrable* $(1 \le p \le +\infty)$ *on* $I \subset \mathbf{R}^+$ *with values in* $\mathbf{X}$.

If $\mathbf{X}$ is a Banach space, then provided with the norm $u \longrightarrow |u|_p = \left(\int_I |u(x)|_{\mathbf{X}}^p dx\right)^{1/p}$, $(1 \le p < +\infty)$, $L^p(I; \mathbf{X})$ is a Banach space; Similarly for the usual modification when $p = +\infty$.

Finally if $\mathbf{X} = \mathbf{R}$ (*resp.* $\mathbf{C}$)) we denote by L^p the space $L^p(\mathbf{R}^+; \mathbf{R})$ (*resp.* $L^p(\mathbf{R}^+; \mathbf{C})$ and by $L^p_*(\mathbf{X})$ (resp. L^p_*) the space $L^p(\mathbf{X})$ *for the Haar measure* $\frac{dt}{t}$ *on* $\mathbf{R}^*$.

Let ω be a positive measurable function locally integrable on $I \subset \mathbf{R}^+ = (0, +\infty)$ with values in $\mathbf{R}^+$, we can define a measure ν such that $d\nu = \omega(t)dt$, where $\omega > 0$ is a density with respect to the Lebesgue measure. Such a density ω is also called a *weight* and we can define the *weighted space* $L^p_\omega(I; \mathbf{X})$, of (class of) functions u such that

$$\int_I |u(t)|_{\mathbf{X}}^p d\nu(t) < +\infty,$$

with usual modification when $p = +\infty$. Provided with the natural norm $L^P_\omega(I; \mathbf{X})$ is a Banach space.

In what follows we let $\omega(t) = c^p(t)$ and we assume that $c > 0$ satisfies

$$\forall T > 0, \quad \text{(i) } c \in L^p(0, T; \mathbf{R}^+), \quad \text{(ii) } c^{-1} \in L^{p'}(0, T; \mathbf{R}^+), \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.1}$$

When $\omega = c^p$, the condition $u \in L^p_\omega(I; \mathbf{X})$ is equivalent to $cu \in L^p(I; \mathbf{X})$ provided with the Lebesgue measure. Accordingly, we still refer to c *as a weight*. So, in what follows, we shall denote by $L^p_c(I; \mathbf{X})$ *the space of functions* u, *such that* $cu \in L^p(I; \mathbf{X})$.The letter ω is always reserved for the *density* $\omega = c^p$, where c *satisfies* (2.1).

Remark 2.1 1. Obviously the condition [(2.1), (ii)] for c is reasonable to satisfy by Hölder inequality:

$$\forall I \in \mathbf{R}^+, \quad I \neq (a, +\infty), \quad L^p_\omega(I; \mathbf{X}) \subset L^1(I; \mathbf{X})$$

with continuous injective mapping. On the other hand, the condition is necessary for $c \in \mathcal{H}(p)$ (see later).

2. Suppose $\mathbf{X}$ is reflexive and $\mathbf{X}'$ the dual (or antidual) of $\mathbf{X}$. Then the dual of $L^p_\omega(\mathbf{X})$ *is* $L^{p'}_{\omega'}(\mathbf{X}')$ with $\omega' = \omega^{1-p'} = c^{-p'}$.

As in [5], we are concerned with weights in the Hardy class $\mathcal{H}(p)$, that is weights c for which the Hardy operator $\mathcal{H} :: u \longrightarrow \frac{1}{t}\int_0^t u(\tau)d\tau$ is continuous from $L^p_c(\mathbf{X})$ into itself.

We recall[2] that $c \in \mathcal{H}(p),\ 1 \le p < \infty,$ *if and only if c satisfies the inequality*:

$$\underset{t>0}{Sup} \left(\int_t^{+\infty} \left[\frac{c(\tau)}{\tau}\right]^p d\tau\right)^{1/p} \left(\int_0^t \frac{d\tau}{[c(\tau)]^{p'}}\right)^{1/p'} < +\infty \tag{2.2}$$

with the usual modifications when $p = 1$ $(or\ p' = +\infty)$, where (2.2) is replaced by

$$\textit{there is a constant } K,\ (0 < K < +\infty) \textit{ such that } \forall t_0 > 0,\ \int_{t_0}^{+\infty} \frac{c(t)}{t} dt \le K c(t_0).$$

Remark 2.2 (i) It is of interest to notice that the condition

$$\forall t > 0, \quad \int_t^{+\infty} \left[\frac{c(\tau)}{\tau}\right]^p < +\infty, \tag{2.3}$$

is only a necessary condition for c to be in $\mathcal{H}(p)$.

(ii) The condition [(2.1), (ii)] is also necessary for $c \in \mathcal{H}(p)$, but [(2.1), (i)] is not necessary for (2.2).
Indeed if $c \in \mathcal{H}(p)$ and if ϕ is non-increasing, then $\phi c \in \mathcal{H}(p)$: for example the weight $c(t) = t^{-1/p}(p \ge 1)$ (which corresponds to the density $\omega(t) = \frac{1}{t}$ for the Haar measure in L^p_*) belongs to $\mathcal{H}(p)$ but $\int_0^t c^p(\tau)d\tau = +\infty$. Thus we could assume only $c \in L^p(\epsilon, t; \mathbf{R}^+)$ *for all* $(\epsilon, t), 0 < \epsilon < t$, in place of [(2.1) (i)] for c, but the last condition is needed for the existence of traces $\neq 0$.

Assume now that $\mathbf{X}$ is reflexive so that the dual (or antidual) operator $\mathcal{H}^*$ of $\mathcal{H}$ is defined by $\mathcal{H}^*(t) = \int_t^{+\infty} \frac{u(\tau)}{\tau} d\tau$ which is continuous from $L^{p'}_{1/c}(\mathbf{X}')$ into itself *if and only if* c satisfies (2.2) [5].

2.1 Spaces $\mathbf{W}^{(m)}$ and spaces of traces

Following [5], let $\mathbf{A}_0, \mathbf{A}_1$ be two Banach spaces continuously imbedded in a topological vector space $\mathcal{A}$ with

$$\mathbf{X} = \mathbf{A}_0 \cap \mathbf{A}_1 \textit{ equipped with the norm } |u|_{\mathbf{X}} = max\{|u|_{\mathbf{A}_0}, |u|_{\mathbf{A}_1}\} \tag{2.4}$$

$$\mathbf{Y} = \mathbf{A}_0 + \mathbf{A}_1 \textit{ equipped with the norm } |u|_{\mathbf{Y}} = \underset{u=a_0+a_1}{inf} (|a_0| + |a_1|). \tag{2.5}$$

Thus $\mathbf{X}, \mathbf{Y}$ are Banach spaces and $\mathbf{X} \subset \mathbf{A}_i \subset \mathbf{Y},\ (i = 0, 1)$. We assume that

$$\mathbf{A}_i, \quad (i = 0, 1) \textit{ is reflexive} \tag{2.6}$$

$$\mathbf{X} \textit{ is dense in } \mathbf{A}_i, \quad i = 0, 1. \tag{2.7}$$

For $i = 0, 1,$ *let* c_i *satisfy* (2.1), *and let* $p_i, 1 \le p_i \le +\infty$. *Consider the spaces*

$$\mathbf{X}_i = L^{p_i}_{c_i}(\mathbf{A}_i), \quad \textit{with norm denoted } N_i(.). \tag{2.8}$$

[2] See [16] and the bibliography therein.

and define for $m \geq 1$

$$\mathbf{W}^{(m)}(p_0, c_0, \mathbf{A}_0; p_1, c_1, \mathbf{A}_1) = \mathbf{W}^{(m)},$$

be the space of functions u, locally integrable on $\mathbf{R}^+$ with $u \in \mathbf{X}_0$, such that $D^m u \in \mathbf{X}_1$. The last condition must be understood as follows: u is *m-times* differentiable in the distribution sense with values in $\mathbf{Y}$ and $D^m u$ locally integrable, so that the product with c makes sense.

Equipped with the norm

$$u \longrightarrow \|u\|_{\mathbf{W}^{(m)}} = max\{N_0(u), N_1(D^m u)\} \tag{2.9}$$

$\mathbf{W}^{(m)}$ is a Banach space.

Let $\mathbf{W}_K^{(m)}(\mathbf{X})$ be the subspace of functions $u \in W^{(m)}$ with values in $\mathbf{X}$, with compact support in $[0, +\infty[$, then from [5] we have

Lemma 2.3 *If* $c_1 \in \mathcal{H}(p_1)$, *then* $\mathbf{W}_K^{(m)}(\mathbf{X})$ *is dense in* $\mathbf{W}^{(m)}$.

Indeed, since $D^m u$ is locally integrable with values in $\mathbf{Y}$, *then* $D^{m-1}u$ is absolutely continuous, hence continuous.

Then we can consider that u is $(m-1)$-times continuously differentiable on $\mathbf{R}^+$ with values in $\mathbf{Y}$ and $D^j u(t)$, $1 \leq j \leq m-1$ is well defined for $t \in (0, +\infty)$.

Therefore, when $lim_{t \longrightarrow +0} D^j u(t) = a_j \; in \; \mathbf{Y}$ exists, we shall say that $D^j u$ has *a trace of order* j, $D^j u(0) = a_j$ *at* $t = 0$.

We have proved in [5] that if *for* $j \in \{0, 1, \ldots, m-1\}$, $t^j c_0 \notin L^{p_0}(0, 1)$ *then the trace* $a_j = 0$. Consequently we can adapt a result of Poulsen [2,3,28] to obtain

Lemma 2.4 *Assume,*

$$t^j c_0 \in L^{p_0}(0, 1) \tag{2.10}$$

then a necessary and sufficient condition for the existence of a trace of order j *is*

$$\frac{t^{m-j-1}}{c_1} \in L^{p_1'}(0, 1). \tag{2.11}$$

Denote by $\mathbf{T}_j^{(m)}(p_0, c_0, \mathbf{A}_0; p_1, c_1, \mathbf{A}_1) = \mathbf{T}_j^{(m)}$ the space spanned in $\mathbf{Y}$ by $D^j u(0) = a_j$ when u spans $\mathbf{W}^{(m)}$. Equipped with the norm

$$\|a\|_{\mathbf{T}_j^{(m)}} \inf_{D^j u(0)=a} \|u\|_{\mathbf{W}^{(m)}} \tag{2.12}$$

one obtains a Banach space. *The spaces* $\mathbf{T}_j^{(m)}$ *are called spaces of traces.*

It follows that (see [5], Proposition 2.6), we have

Lemma 2.5 *Let* $u \in \mathbf{W}^{(m)}$ *with* $D^j u(0 = a_j)$ *then for* $1 \leq j \leq m-1$:

$$|a_j|_{\mathbf{T}_j^{(m)}} = \mathop{Inf}_{u} max\{N_0(u)^{1-\gamma_{j,m}}, N_1(D^m u)^{\gamma_{j,m}}\}, \quad \gamma_{j,m} = \frac{j + 1/p_0}{m + 1/p_0 - 1/p_1}. \tag{2.13}$$

From [5, Theorem 4.9] we can reduce the study of $\mathbf{T}_j^{(m)}$ for $j \in \{0, 1, \ldots, m-1\}$, only to $\mathbf{T}_0^1 = \mathbf{T}(p_0, c_0, \mathbf{A}_0; p_1, c_1, \mathbf{A}_1)$ and the condition on the weights in order to possess a trace $u(0)$ is

$$\forall T > 0, \quad c_0 \in L^{p_0}(0, T), \quad c_1^{-1} \in L^{p_1'}(0, T). \tag{2.14}$$

Then we have from [5]

Proposition 2.6 *Assume*, (2.10), (2.11) *hold for* $j \in \{0, 1, \ldots, m-1\}$ (*in fact* (2.14) *is sufficient*) *then*

$$\mathbf{T}_j^{(m)}(p_0, c_0, A_0; p_1, c_1, A_1) = \mathbf{T}_0^1(p_0, t^j c_0, A_0; p_1, t^{j-m+1} c_1, A_1) \tag{2.15}$$

with equivalent norms.

Now let $\mathbf{A}$ be a topological vector space such that

$$\mathbf{X} \subset \mathbf{A} \subset \mathbf{Y}, \tag{2.16}$$

we shall say that $\mathbf{A}$ *is an intermediate space* (*between* $\mathbf{A}_0$ *and* $\mathbf{A}_1$).

With this definition $\mathbf{A}_i$, $i = 0, 1$ is itself an intermediate space and we recall (see [15, p. 145]) that

$$\forall j, \quad 0 \le j \le m-1, \quad \mathbf{X} \subset \mathbf{T}_j^{(m)} \subset \mathbf{Y}. \tag{2.17}$$

Thus $\mathbf{T}_j^{(m)}$ are intermediate spaces with the following *interpolation property*:
let $(\mathbf{B}_0, \mathbf{B}_1, \mathcal{B})$ is a family of spaces with properties analogous to the family $(\mathbf{A}_0, \mathbf{A}_1, \mathcal{A})$. Assume $\mathbf{X} \subset \mathbf{A} \subset \mathbf{Y}$, $\mathbf{B}_0 \cap \mathbf{B}_1 \subset \mathbf{B} \subset \mathbf{B}_0 + \mathbf{B}_1$. Let π be a linear mapping from $\mathbf{Y}$ *into* $\mathbf{B}_0 + \mathbf{B}_1$ which restricted to $\mathbf{A}_i$ is linear and continuous from $\mathbf{A}_i$ *into* $\mathbf{B}_i$ $(i = 0, 1)$ (that is $\pi \in \mathcal{L}(\mathbf{A}_i, \mathbf{B}_i)$). Then the restriction of π to $\mathbf{A}$ belongs to $\mathcal{L}(\mathbf{A}, \mathbf{B})$ (see [5]).

3 Another representation of the traces in $\mathbf{W}^{(m)}$

Orientation *Let* $a_j \in \mathbf{T}_j^{(m)}$. *We want show in Sect.* 3.2 *that a function*, $\tilde{u} \in \mathbf{W}^{(m)}$, *can be found*, *eventually with compact support*, *such that*

$$a_j = \int_0^{+\infty} \tilde{u}(t) \frac{dt}{t}, \quad \text{with } t^j \tilde{u} \in L_{\pi_0}^{p_0}(\mathbf{A}_0),\ t^{j-m} \tilde{u} \in L_{\pi_1}^{p_1}(A_1). \tag{3.1}$$

This will enable us to introduce in Sect. 4 *new weighted spaces which extends those of* [21].

Since the main tool used in extension of the proofs involves convolution products with some weighted functions, it is of prime necessity to establish beforehand some essential results (see especially Theorem 3.1) which appear to be new.

3.1 A stability result for convolution product with weight

Theorem 3.1 *Let* $\mathbf{B}$ *a Banach space*, c *a weight satisfying* (2.1), *with* $c \in \mathcal{H}(p)$, $1 \le p < +\infty$, $\phi \in L^1(\mathbf{R}^+)$, $cu \in L^p(\mathbf{B})$ *then* $\phi * u \in L_c^p(\mathbf{B})$ (*where* $*$ *means the convolution*) *and there is a constant* $\kappa > 0$, *such that*

$$|\phi * u|_{L_c^p(\mathbf{B})} \le \kappa\, |\phi|_{L^1(\mathbf{R}^+)} \cdot |u|_{L_c^p(\mathbf{B})} \tag{3.2}$$

Remark 3.2 As mentioned in the introduction, the result is the $(\mathcal{P})$-*condition* of [7] proved, only for non-increasing weights (that is obvious) and for the weights $c \in A(p)$ (the class of Muckenhoupt see: [24]), where, following [7], it was proved by a method of Stein [30] that gives one estimate for u using the maximal theorem of Hardy–Littlewood. Thus we have a sufficient condition for $c \in A(p)$. But we know that strictly $A(p) \subset \mathcal{H}(p)$ and accordingly Theorem 3.1 gives the best result.

Moreover a direct procedure, independent of Stein's method is used.

Proof of Theorem 3.1 The proof is divided into three steps, repeating the outlines of certain proofs of [6] in order to correct some misprints.
(i) *First step:*

We introduce the operator $D^{i\eta} = Y_{-i\eta}*,\ \eta \in \mathbf{R}$ where[3]

$$Y_{-i\eta} = \frac{1}{\Gamma(-i\eta)} Pf\left[\frac{1}{x^{1+i\eta}}\right] \quad if\, \eta \neq 0,\ Y_0 = \delta$$

If $\Phi \in \mathcal{D}(\mathbf{R}^+)$ then

$$D^{i\eta}\Phi(t) = \frac{1}{\Gamma(-i\eta)} \underset{\epsilon \longrightarrow 0^+}{Lim} \left(\int_\epsilon^t \frac{\Phi(t-x)}{x^{1+i\eta}}dx - \frac{\Phi(t)\epsilon^{-i\eta}}{-i\eta}\right), \quad \eta \neq 0,\ D^0\Phi = \Phi$$

For convenience we set $D^{i\eta}\Phi = \hat{\Phi}$. It is of interest for what follows to apply $D^{i\eta}$ to the characteristic function $\chi_{]a,b[}$ on the interval $(a, b),\ 0 \leq a < b < +\infty$. We get

$$\hat{\chi}_{]a,b[}(t) = 0 \quad if\, t < a, = \frac{i}{\eta\Gamma(-i\eta)}\left[\frac{1}{(t-a)^{i\eta}}\right] \quad if\, a < t < b,$$
$$= \frac{i}{\eta\Gamma(-i\eta)}\left[\frac{1}{(t-a)^{i\eta}} - \frac{1}{(t-b)^{i\eta}}\right] \quad if\, t > b,$$

and we can check that

$$t > b \Longrightarrow \left|\hat{\chi}_{]a,b[}(t)\right| = \gamma(\eta)\frac{2}{|\eta|}\left|sin\left[\frac{\eta}{2}Log\left(1 + \frac{b-a}{t-b}\right)\right]\right|,$$
$$\gamma(\eta) = \frac{1}{|\Gamma(-i\eta)|} = \left(\frac{\eta sh\eta}{\pi}\right)^{1/2} \tag{3.3}$$

Then the first step is to prove[4] the

Theorem 3.3 *Let B a Banach space, and let c satisfy* (2.1) *and* $c \in \mathcal{H}(p)$. *Then for* $1 \leq p < +\infty$ *one has* $D^{i\eta} \in \mathcal{L}(L_c^p(\mathbf{B}))$.

Remark 3.4 (1) For unweighted spaces (i.e.: $c \equiv 1$) the result is known only for $1 < p < \infty$ with $\mathbf{B} = \mathbf{R}$ or $=\mathbf{C}$ (see: [22,30]). It is also true when B is a Hilbert or Banach spaces (see: [2–4,7]) but again for $1 < p < \infty$.

(2) The conditions on c are here sufficient conditions, nevertheless if $Logc$ is of finite order with respect to $Logt$ as $t \longrightarrow +0$, *or* $t \longrightarrow \infty$, then (using Bourbaki [11]), we can show that those conditions are also necessary.

Actually, for $\beta \in \mathbf{B}$, *if we want* $\hat{\chi}_{]a,b[} \otimes \beta \in L_c^p(\mathbf{B})$, then from (3.3), we see that the norm in $\mathbf{B}$ of the function is equivalent (up to a multiplicative constant) to $1/t$ as $t \longrightarrow \infty$, so that (2.3) must hold.

Now if the order of $Logc$ with respect to $Logt$ is -1 then (2.1) is not true. If the order is ∞, then (2.3) fails. So from a result of [11], the integral in (2.3) is equivalent (up to a multiplicative constant) to $t^{-p+1}c^p(t)$ and $\int_0^t c^{-p'}(\tau)d\tau \simeq (constant)tc^{-p'}(t)$ as $t \longrightarrow +\infty$ *or* $\longrightarrow +0$, we easily check that (2.2) is true.

In this case the condition (2.2) is necessary for Theorem 3.3 and one has an answer for the problem posed in the Sect. 1.

(3) When $c \equiv 1$, *the integral of* (2.3) is divergent, then $D^{i\eta}$ does not act in L^1.

[3] Pf = Finite Part at the sense of Laurent Schwartz [29].

[4] See Theorem 3.3 of [6].

(1) To prove Theorem 3.3 we need the

Lemma 3.5 *Let ϕ a locally measurable function with compact support $\subset (o, T)$ taking values in* $\mathbf{B}$. *Then there is a constant γ such that*

$$\forall T > 0, \quad \left|\hat{\phi}(2T)\right|_{\mathbf{B}} \leq \gamma \frac{1}{T} \int_0^T |\phi(\tau)|_{\mathbf{B}}\, d\tau. \tag{3.4}$$

To prove the lemma, consider a step function $\phi_k,\ supp(\phi_k) \subset (0, T)$, given by $\phi_k = \sum_{i=0}^{i=k-1} \beta_i \otimes \chi_{]a_i, a_{i+1}[}$, where $\beta_i \in \mathbf{B},\ a_0 \geq 0$ and $a_k = T$. From (3.3) we obtain for $t > T$

$$\left|\hat{\phi}_k(t)\right|_{\mathbf{B}} \leq \gamma(\eta) \sum_{i=0}^{i=k-1} \frac{a_{i+1} - a_i}{t - a_{i+1}} |\beta_i|_{\mathbf{B}}, \quad \forall t > T$$

(where we have used $|sinu| \leq |u|$, and $Log(1 + v) \leq v,\ 0 < v < 1$).

Now choosing $t = 2T$, since $t - a_{i+1} \geq T,\ o \leq i \leq k - 1$, one has for all $T > 0$

$$\left|\hat{\phi}_k(2T)\right|_{\mathbf{B}} \leq \gamma(\eta) \frac{1}{T} \sum_{i=0}^{i=k-1} (a_{i+1} - a_i)\, |\beta_i|_{\mathbf{B}} = \gamma(\eta) \frac{1}{T} \int_0^T |\phi_k(\tau)|_{\mathbf{B}}\, d\tau,$$

which is (3.4) and the lemma is proved for ϕ_k.

Now if $\phi \in L^1(0, T); \mathbf{B})$ with compact support contained in $(0, T)$, we may always find a step function ϕ_k with compact support in $(0, T)$ such that: $\phi_k \longrightarrow \phi$ *a.e. and in* $L^1(0, T; \mathbf{B})$ *norm as* $k \longrightarrow \infty$. Then, observing that the kernel of $\hat{\phi}_k(2T)$ is bounded because $2T - x \geq T,\ 0 \leq x \leq T$, we can pass to the limit by Lebesgue's theorem and Lemma 3.5 is proved.

(2) Now to complete the first step in the proof of Theorem 3.3, consider $\phi \in L^1_{loc}(\mathbf{R}^+)$ *and fix* $t > 0$.

Introduce, for $n \in \mathbf{N}$, the truncating sequence θ_n:

$$\begin{aligned} &\theta_n(\tau) = 1, \quad 0 \leq \tau \leq t - 1/n, \\ &\theta_n(\tau) = 2n(t - 1/2n - \tau), \quad t - 1/n \leq \tau \leq t - 1/2n, \\ &\theta_n(\tau) = 0, \quad \tau \geq t - 1/2n. \end{aligned}$$

then $\phi_n = \theta_n \phi$ has a compact support $\subset (0, t - 1/2n)$, so we may apply Lemma 3.5 and pass to the limit by Lebesgue's theorem as $n \longrightarrow +\infty$. One obtains

$$\left|\hat{\phi}(2t)\right|_{\mathbf{B}} \leq \gamma \mathcal{H}(|\phi|_{\mathbf{B}})(t) \quad \textit{for all } t > 0. \tag{3.5}$$

If we assume $\phi \in L^p_c(\mathbf{B})$, then from [(2.1) (ii)], one has $\phi \in L^1_{loc}(\mathbf{R}^+; \mathbf{B})$ and (3.5) holds. Noticing that if $f > 0$, we infer that $\mathcal{H}(f)(t) \leq 2\mathcal{H}(f)(2t)$ and on multiplying the two members of (3.5) by $c(2t)$, we integrate over $\mathbf{R}^+$ the power p to each side of (3.5) and because Hardy's operator belongs to $\mathcal{L}(L^p_c(\mathbf{B}))$ Theorem 3.3 is proved.

(*ii*) *Second step:*

Now we prove another result obtained in the same way:

Theorem 3.6 *Assume $c \in \mathcal{H}(p), 1 \leq p < +\infty,\ \phi \in L^1(\mathbf{R}^+),\ u \in L^p_c(\mathbf{B})$ and let $\hat{v} = Y_{-i\eta} * v$. Then $\phi * \hat{u} \in L^p_c(\mathbf{B})$ and there is a constant $\kappa_1 > 0$, such that*

$$\left|\phi * \hat{u}\right|_{L^p_c(\mathbf{B})} \leq \kappa_1 |\phi|_{L^1(\mathbf{R}^+)} \cdot |u|_{L^p_c(\mathbf{B})} \tag{3.6}$$

Proof We introduce for *fixed* $t > 0$ the truncating sequence θ_n and let $u_n = \theta_n u$. We may write $g_n = \phi * \hat{u}_n = \hat{\phi} * u_n$, so that

$$|g_n(2t)|_{\mathbf{B}} \leq \int_0^{t-1/2n} \left|\hat{\phi}(2t-\sigma)\right| |u_n(\sigma)|_B \, d\sigma.$$

from the definition of u_n.

Now from the proof of Lemma 3.5 we have $\left|\hat{\phi}(2s)\right| \leq \frac{\gamma}{s} \int_0^s |\phi(\xi)| \, d\xi$ and from the last inequality we deduce

$$\begin{aligned} |g_n(2t)|_{\mathbf{B}} &\leq 2\gamma \int_0^{t-1/2n} \left\{ \frac{|u_n(\sigma)|_{\mathbf{B}}}{2t-\sigma} \int_0^{t-\sigma/2} |\phi(\xi)| \, d\xi \right\} d\sigma \\ &\leq 2\gamma \, |\phi|_{L^1(\mathbf{R}^+)} \frac{1}{t+1/2n} \int_0^{t-1/2n} |u_n(\sigma)|_{\mathbf{B}} \, d\sigma, \end{aligned}$$

because $\sigma \in (0, t - 1/2n)$ and $t + 1/2n \leq 2t - \sigma \leq 2t$.

As at the end of the proof of Theorem 3.3, we can pass to the limit as $n \longrightarrow +\infty$, to get

$$\forall t > 0, \quad |g(2t)|_{\mathbf{B}} \leq 2\gamma \, |\phi|_{L^1(\mathbf{R}^+)} \, .\mathcal{H}(|u|_{\mathbf{B}})(t). \tag{3.7}$$

(*iii*) *Third step:*

Since $Y_{i\eta} * Y_{-i\eta} = \delta, \ \forall \eta \in \mathbf{R}$, *we may write* $u = Y_{i\eta} * \hat{u}$, *so that*

$$\forall \eta \in \mathbf{R}, \quad \phi * u = \phi * [Y_{i\eta} * \hat{u}],$$

and gathering the results of steps 1–2 Theorem 3.1 *is proved.*

On observing that the weight: $t^{-1/p}c(t) \in \mathcal{H}(p)$ *if* $c \in \mathcal{H}(p)$ and that $|cu|_{L^p_*(\mathbf{B})} = \left|t^{-1/p}cu\right|_{L^p(\mathbf{B})}$, we obtain:

Corollary 3.7 *Assume* $c \in \mathcal{H}(p), \ 1 \leq p < \infty, \ \phi \in L^1, \ cu \in L^p_*(\mathbf{B})$. *Then one has*

$$|c(\phi * u)|_{L^p_*(\mathbf{B})} \leq \kappa \, |\phi|_{L^1} \, . \, |cu|_{L^p_*(\mathbf{B})} \, . \tag{3.8}$$

Proof From (3.5) we obtain obviously a version of Theorem 3.3 stating that $cD^{i\eta} \in \mathcal{L}[L^p_*(\mathbf{B})]$ because $t^{-1/p}c \in \mathcal{H}(p)$ (like also $t^{-1/p}$), while from (3.6) a version of Theorem 3.6 for the same reason implies that there exists a constant $\kappa_1 > 0$, such that

$$\left|c(\phi * \hat{u})\right|_{L^p_*(\mathbf{B})} \leq \kappa_1 \, |\phi|_{L^1} \, . \, |cu|_{L^p_*(\mathbf{B})} \, .$$

The proof is completed as in the third step.

Remark 3.8 1. With a convenient extension to $\mathbf{R}$ of the functions only defined on R^+, for example by the relation $t = e^x$, we can deduce some variants of previous results in particular in the important case where ϕ has a compact support.

2. Generally if we start from $g(t) = (\phi * u)(t) = \int_{\mathbf{R}} \phi(t-s)u(s)ds$ and take $t = Log\tau, s = Log\sigma$, then on setting $\tilde{f}(\xi) = f(Logx)$, we have that g is expressed by $\tilde{g}(\tau) = \int_0^{+\infty} \tilde{\phi}(\tau/\sigma)\tilde{u}(\sigma)\frac{d\sigma}{\sigma}$; that is the convolution product on the multiplicative group $\mathbf{R}^*$ provided with Haar's measure $\frac{d\sigma}{\sigma.}$. Notice that $|u|_{L^p_c(\mathbf{B})} \simeq \left|t^{1/p}cu\right|_{L^p_*(\mathbf{B})}$ and Theorem 3.1 may be extended to a convolution as $\tilde{g}(\tau)$ with respect to Haar's measure (see also Corollary 3.7). In consequence we have

$$\left|t^{1/p}c\tilde{g}\right|_{L^p_*(\mathbf{B})} \leq \kappa \left|\tilde{\phi}\right|_{L^1_*} \left|t^{1/p}c\tilde{u}\right|_{L^p_*(\mathbf{B})} \, . \tag{3.9}$$

3.2 An integral représentation for the trace of order j

Henceforward, to simplify notation, we set $u^{(r)}$ *for* $D^r u$ and use the procedure of [21] (even though it involves adaptation to our structure), which is justified by Theorem 3.1.

First, we state,

Lemma 3.9 *Let* $a_j \in \mathbf{T}_j^{(m)}$ *and* $v \in \mathbf{W}^{(m)}$ *be such that* $v^{(j)}(0) = a_j$. *Then we can construct a function* $u \in \mathbf{W}^{(m)}$, *with compact support in* $\mathbf{R}^+$ *such that*

$$t^k D^k u \in L_{c_0}^{p_0}(\mathbf{A}_0), \quad k \in \{1, 2, \ldots, m \ldots\}, \quad \textit{with } u^{(j)}(0) = a_j. \tag{3.10}$$

Proof Define a function f *on* $\mathbf{R}$ by $f(\sigma) = v(e^\sigma)$, $\sigma \in \mathbf{R}$. Because

$$\left[\frac{d^r f}{d\sigma}\right]_{\sigma = logt} = \sum_{i=1}^{k} \gamma_{ik} t^i \frac{d^i v(t)}{dt},$$

when $\rho \in \mathcal{D}(\mathbf{R})$, $\int_R e^{-j\sigma} \rho(\sigma) d\sigma = 1$, we obtain

$$e^{-j\sigma}(f * \rho^{(j)})(\sigma) = e^{-j\sigma}(f * \rho)^{(j)}(\sigma) \longrightarrow \gamma_{jj} v^{(j)}(0) \quad \textit{in } \mathbf{Y} \textit{ as } \sigma \longrightarrow -\infty.$$

Since $v \in \mathbf{W}^{(m)}$ is equivalent to $\{t^{1/p_0} c_0 v \in L_*^{p_0}(\mathbf{A}_0),\ t^{1/p_1} c_1 v \in L_*^{p_1}(\mathbf{A}_1)\}$, and consequently

$$t^{1/p_0} c_0 f|_{\sigma = logt} \in L_*^{p_0}(\mathbf{A}_0) \ \simeq \ f|_{\sigma = logt} \in L_{c_0}^{p_0}(\mathbf{A}_0)$$

so that from Theorem 3.1, we deduce

$$(f * \rho)^{(k)}\,(logt) \in L_{c_0}^{p_0}(\mathbf{A}_0).$$

Because we want traces at $t = 0$, only functions in a neighborhood of $t = 0$ are required, so that choosing $\Phi \in \mathcal{D}(\mathbf{R}^+)$ with $\Phi^{(j)}(0) = 1$ we can take

$$u(t) = \Phi(t)(f * \rho)(logt)$$

which satisfies

$$u^{(j)}(0) = a_j, \quad t^k D^k u \in L_{c_0}^{p_0}(\mathbf{A}_0)$$

and (3.10) holds.

Remark 3.10 Lemma 2.3 implies that $\mathbf{W}_K^{(m)}(\mathbf{X})$ is dense in $\mathbf{W}^{(m)}$, so that there exists $v \in \mathbf{W}_K^{(m)}(\mathbf{X})$ with $v^{(j)}(0) = a_j$ *in* $\mathbf{Y}$. Then u as previously constructed can be chosen in $\mathbf{W}_K^{(m)}(\mathbf{X})$.

Secondly one has

Lemma 3.11 *Assume that* [(2.1), (ii)], (2.10), (2.11) *hold true and let u satisfy* (3.10). *Then*

$$u^{(j)}(0) = \gamma_j \int_0^{+\infty} t^{m-j-1} u^{(m)}(t) dt, \quad \gamma_j = \frac{(-1)^{(m-1)}}{(m-j-1)!}. \tag{3.11}$$

Proof Four steps are required
Step 1: $T > 0$, $\int_0^T t^{m-j-1} u^{(m)} dt < +\infty$.

From (2.11) we have $\frac{t^{m-j-1}}{c_1} \in L^{p_1'}(0T; \mathbf{R}^+)$, and $u^{(m)} \in L_{c_1}^{p_1}(\mathbf{A}_1)$. The result then follows from Hölder 's inequality.

Step 2: $\int_T^{+\infty} t^{m-j-1} u^{(m)}(t) dt < +\infty$.

Assume u has a compact support (say $[0, T_0]$) in $\overline{R}^+$. *From* (3.10) *with* $k = m$, Hölder's inequality gives

$$\int_T^{+\infty} t^{m-j-1} u^{(m)}(t)dt \leq \left|t^m u^{(m)}\right|_{\mathbf{A}_0} \left(\int_T^{T_0} \frac{dt}{[t^{j+1}c_0(t)]^{p_0'}}\right)^{1/p_0'}.$$

and similarly, we have

$$\int_T^{T_0} \frac{dt}{[t^{j+1}c_0(t)]^{p_0'}} \leq \frac{1}{[T^{j+1}]^{p_0'}} \int_0^{T_0} \frac{dt}{[c_0(t)]^{p_0'}},$$

and the result follows by [(2.10), (ii)].

Remark 3.12 The choice of u with compact support was partly to derive the last inequality. Otherwise it is necessary to assume that: $J = \int_T^{+\infty} \frac{dt}{[t^{j+1}c_0(t)]^{p_0'}} < +\infty$. This is done in [21] when $c_0(t) = t^\alpha$ and gives a condition on the weight: $1/p_0 + \alpha + j > 0$. Actually since $j \in \{0, 1, \ldots, m-1\}$ it is sufficient to assume $\forall T > 0,\ [tc_0]^{-1} \in L_{c0}^{p_0'}(T, +\infty; R^+)$ (because $J \leq \frac{1}{T^{jp_0'}} \int_T^{+\infty} \frac{dt}{[tc_0]^{p_0'}}$) which is a necessary condition for $1/c_0 \in \mathcal{H}(p_0')$. Then in this case it seems that the additional assumption

$$\frac{1}{c_0} \in \mathcal{H}(p_0') \tag{3.12}$$

is sufficient to establish step 2 when u does not have compact support.

Furthermore, on setting $\phi(t) = [tc_0]^{-1}$, taking into account [(2.1), (ii)], we recover a set of functions

$$\{\Phi\} = \{\phi;\ \phi \in L^{p_0'}(1, +\infty; \overline{\mathbf{R}}^+),\ t\phi \in L^{p_0'}(0, 1; \overline{\mathbf{R}}^+\} \tag{3.13}$$

introduced by Lions in [19] for a problem of interpolation that has only recently been solved. The reader is referred to [9,19].

Step 3:

Next we prove:

$$\int_0^{+\infty} t^{m-j-1} u^{(m)}(t)dt = -(m-j-1) \int_0^{+\infty} t^{m-j-2} u^{(m-1)}(t)dt. \tag{3.14}$$

For $\epsilon > 0$, we start from $I_\epsilon^t = \int_\epsilon^t \tau^{m-j-1} u^{(m)}(\tau)d\tau$ and after an integration by parts, obtain

$$[\tau^{m-j-1} u^{m-1}(\tau)]_\epsilon^t = -(m-i-1) \int_\epsilon^t \tau^{m-j-2} u^{(m-1)}(\tau)d\tau.$$

Since u has compact support, we deduce that

$$t^{m-j-1} u^{(m-1)}(t) \longrightarrow 0 \quad in\ \mathbf{Y}\ as\ t \longrightarrow \infty.$$

On the other hand, the identity

$$u^{(m-1)}(\epsilon) = u^{(m-1)}(1) - \int_\epsilon^1 u^{(m)}(\tau)dt$$

implies that

$$\left|u^{(m-1)}(\epsilon)\right|_{\mathbf{Y}} \leq \left|u^{(m-1)}(1)\right|_{\mathbf{Y}} + \left(\int_\epsilon^1 c_1^{p_1} \left|u^{(m)}(\tau)\right|_{\mathbf{Y}}^{p_1} d\tau\right)^{1/p_1} \left(\int_0^1 \frac{d\tau}{[c_1(\tau)]^{p_1'}}\right)^{1/p_1'}$$

and the right side is bounded because $D^m u \in \mathbf{X}_1$, and [(2.1), (ii)] holds. In consequence

$$\epsilon^{m-j-1}u^{(m-1)}(\epsilon)\longrightarrow 0 \quad in\ \mathbf{Y}\ as\ \epsilon\longrightarrow 0,$$

and (3.14) holds true.
Step 4:
Repeating the process leads to

$$\int_0^{+\infty} t^{m-j-1}u^{(m)}(t)dt =(-1)^{m-j-1}(m-j-1)!\int_0^{+\infty} u^{(j+1}(t)dt,$$

but $u^{(j)}(t)\longrightarrow 0\ in\ \mathbf{Y}\ as\ t\longrightarrow +\infty$. Consequently we get (3.11).

Resume of Sect. 3.2
If $a_j \in \mathbf{T}_j^{(m)}$ we have constructed a function $u^*(t)=t^{m-j}u^{(m)}(t)$, such that

$$a_j=\int_0^{+\infty} u^*(t)\frac{dt}{t}, \quad \{t^j u^* \in \mathbf{X}_0,\ t^{j-m}u^* \in \mathbf{X}_1\}. \tag{3.15}$$

The interpretation of this result requires some definitions and properties of some spaces.

4 Some intermediate weighted spaces

Here we extend to weighted spaces the spaces that in [21] called “Espaces de moyennes” while preserving some properties of invariance with respect to Haar's measure $\frac{dt}{t}$ subject to the change of the variable t into $t+T,\ \lambda t,\ or\ 1/t$.

4.1 A first definition

Assume, for the moment, that $i \in \{0,1\},\ c_i \in \mathcal{H}(p_i),\ 1 \le p_i \le +\infty,\ \xi_i$, with $\xi_0\xi_1 < 0$ and define the space

$$\mathbf{V}=\mathbf{V}(p_0,\xi_0,c_0,\mathbf{A}_0;\ p_1,\xi_1,c_1,\mathbf{A}_1)=\{v;\ t^{\xi_0}c_0 v \in L_*^{p_0}(\mathbf{A}_0),\ c_1 t^{\xi_1} v \in L_*^{p_1}(\mathbf{A}_1)\} \tag{4.1}$$

which being equipped with the natural norm is a Banach Space.
When $c_i \in \mathcal{H}(p_i),\ i=0,1$, it is of interest to set

$$\hat{c}_i(t)=t^{-1/p_i}c_i(t), \quad \hat{X}_i=L_{\hat{c}_i}^{p_i}(\mathbf{A}_i) \quad (and\ also\ \hat{c}_i'(t)=t^{-1/p_i'}c_i(t)). \tag{4.2}$$

By virtue of

$$v \in \hat{\mathbf{X}}_i \simeq c_i v \in L_*^{p_i}(\mathbf{A}_i),$$

one has

$$\mathbf{V}\equiv\hat{\mathbf{V}}=\{v;\ t^{\xi_0}v \in \hat{\mathbf{X}}_0,\ t^{\xi_1}v \in \hat{\mathbf{X}}_1\}, \quad with\ \hat{c}_i \in \mathcal{H}(p_i),\ i=\{0,1\}. \tag{4.3}$$

and we denote by $\hat{N}_i(v)$ the norm of $v \in \hat{X}_i = L_{\hat{c}_i}^{p_i}(\mathbf{A}_i)$.
We can check that

$$\int_0^{+\infty} v(t)\frac{dt}{t} \tag{4.4}$$

exists under some conditions on $(c_i,\xi_i),\ i=\{0,1\}$.

For example, assume $\xi_1 < 0$. Then Hölder's inequality

$$\int_0^1 |v(t)|_{\mathbf{Y}} \leq \left(\int_0^1 [t^{\xi_1}\hat{c}_1(t)\,|v(t)|_{\mathbf{Y}}^{p_1}\,dt\right)^{1/p_1} \left(\int_0^1 \frac{dt}{[t^{1+\xi_1}\hat{c}_1(t)]^{p_1'}}\right)^{1/p_1'} < +\infty$$

holds provided

$$[t^{1+\xi_1}\hat{c}_1]^{-1} \in L^{p_1'}(0,1;R^+), \tag{4.5}$$

that is assumed for $-\xi_1 \geq 1$, if $[\hat{c}_1]^{-1} \in L^{p_1'}(0,1)$ (that is [(2.1, (ii)] for $\hat{c}_1$).

On the other hand, we further employ

$$\int_1^{+\infty} |v(t)|_{\mathbf{Y}} \leq \left(\int_1^{+\infty} (t^{\xi_0}\hat{c}_0(t)\,|v(t)|_{\mathbf{Y}}\,dt\right)^{1/p_0} \left(\int_1^{+\infty} \frac{dt}{[t^{1+\xi_0}\hat{c}_0(t)]^{p_0'}}\right)^{1/p_0'} < +\infty$$

which is valid provided

$$[t^{1+\xi_0}\hat{c}_0]^{-1} \in L^{p_0'}(1,+\infty;\mathbf{R}^+), \tag{4.6}$$

assumed for $\xi_0 \geq 0$, if $[t\hat{c}_0]^{-1} \in L^{p_0'}(1,+\infty)$ which holds if $[\hat{c}_0]^{-1} \in \mathcal{H}(p_0')$ (see also Corollary 3.7) and vice versa if $\xi_0 < 0$.

Now, in what follows, to fix ideas we assume $\xi_1 < 0$.

Accordingly we assume (4.4) holds (when, (ξ_0, ξ_1) are chosen to satisfy (4.5)–(4.6)), and we consider a function v with values *a.e. in X*. Set

$$\Sigma = \Sigma(p_0, \xi_0, \hat{c}_0, \mathbf{A}_0; p_1, \xi_1, \hat{c}_1, \mathbf{A}_1),$$

which is the space spanned by $a = \int_0^{+\infty} v(t)\frac{dt}{t}$ in $\mathbf{Y}$ as v spans the space $\mathbf{V} \simeq \hat{\mathbf{V}}$.

Equipped with the norm

$$|a|_\Sigma = Inf_v \left\{max\left(\hat{N}_0(t^{\xi_0}v),\, \hat{N}_1(t^{\xi_1}v)\right)\right\} \tag{4.7}$$

(where Inf_v means the Inf taken on v *such that* $a = \int_0^{+\infty} v(t)\frac{dt}{t}$), Σ is a Banach space.

To understand better the properties of the spaces $(\mathbf{V}, \Sigma)$ we make the change of variable $t = e^x,\ x \in \mathbf{R}$ in $\mathbf{V}$.

On setting $\tilde{f}(x) = f(e^x)$ we have an isomorphism between the space $\mathbf{V}$ and the space

$$\tilde{\mathbf{V}} = \{v;\, e^{\xi_0 x}\tilde{v} \in L^{p_0}_{\tilde{c}_0}(\mathbf{R};\mathbf{A}_0),\, e^{\xi_1 x}\tilde{v} \in L^{p_1}_{\tilde{c}_1}(\mathbf{R};\mathbf{A}_1)\}. \tag{4.8}$$

We denote

$$\tilde{\Sigma} = \tilde{\Sigma}(p_0, \xi_0, \tilde{c}_0, \mathbf{A}_0; p_1, \xi_1, \tilde{c}_1, \mathbf{A}_1) \tag{4.9}$$

the associated space spanned by $a = \int_{-\infty}^{+\infty} \tilde{v}(x)dx\ in\ \mathbf{Y}$ when $\tilde{v}$ spans the space $\tilde{\mathbf{V}}$, which is naturally a Banach space equipped with the norm induced by (4.3) upon the change of variable $t \longrightarrow e^x$.

The space $\tilde{\Sigma}$ is analogous to those studied in [21] by Lions and Peetre and called "Espaces de Moyennes" and accordingly we call $\tilde{\Sigma}$ is a "weighted mean space".

4.2 A second definition

We continue to assume $\xi_0 > 0, \xi_1 < 0,\ \forall i \in \{0,1\},\ c_i \in \mathcal{H}(p_i),\ 1 \leq p_i \leq +\infty$, and we consider v_i measurable with values in $\mathbf{A}_i$ such that, the derivatives being taken in the sense of distributions in Y

$$\frac{\partial}{\partial x}(v_0(x) + v_1(x)) = 0, \quad a.e.\ in\ \mathbf{Y}$$

which implies

$$v_0(x) + v_1(x) = constant, \quad (a.e)\ in\ \mathbf{Y} = a \in \mathbf{Y}. \tag{4.10}$$

In what follows, we set $\tilde{N}_i(f) = |\tilde{c}_i f|_{L^{p_i}(\mathbf{R};\mathbf{A}_i)},\ i = (0, 1)$.

Consider the space (temporarily) denoted by $\tilde{\Sigma}_- = \tilde{\Sigma}_-(p_0, \xi_0, \tilde{c}_0, \mathbf{A}_0;\ p_1, \xi_1, \tilde{c}_1, \mathbf{A}_1)$ which is spanned by $a = v_0(x) + v_1(x)$ when the v_i spans the space

$$\tilde{\mathbf{V}}_- = \{e^{\xi_0 x}\tilde{v}_0 \in L^{p_0}_{\tilde{c}_0}(\mathbf{R}; \mathbf{A}_0),\ e^{\xi_1 x}\tilde{v}_1 \in L^{p_1}_{\tilde{c}_1}(\mathbf{R}; \mathbf{A}_1)\} \tag{4.11}$$

when equipped with the norm

$$|a|_{\tilde{\Sigma}_-} = \underset{v_0(x)+v_1(x)=a}{Inf}\ max[\tilde{N}_0(e^{\xi_0 x}\tilde{v}_0), \tilde{N}(e^{\xi_1 x}\tilde{v}_1)] \tag{4.12}$$

this space is a Banach space.

Note that we have also

$$a \in \tilde{\Sigma}_- = \{a = v_0(x) + v_1(x),\ such\ that\ t^{\xi_0} v_0 \in \hat{\mathbf{X}}_0, t^{\xi_1} v_1 \in \hat{\mathbf{X}}_1\}. \tag{4.13}$$

Remark 4.1 We can observe, thanks Theorem 3.1, that the spaces $\tilde{\Sigma}$ (*resp.* $\tilde{\Sigma}_-$) are not changed if we replace the conditions (4.8) for v, (resp. (4.11) for $v_i,\ i = (0, 1)$), by

$$\forall j \geq 1, \quad e^{\xi_i x} D^j \tilde{v} \in L^{p_i}_{\tilde{c}_i}(\mathbf{R}; \mathbf{A}_i), \quad (resp.\ e^{\xi_i x} D^j \tilde{v}_i \in L^{p_i}_{\tilde{c}_i}(\mathbf{R}; \mathbf{A}_i)), \quad i = (0, 1).$$

(Indeed we can generally replace v by the convolution $v * \rho$, where $\rho \in \mathcal{D}(R)$, *with* $\int_R \rho(x)dx = 1$, so that $D^j\tilde{v} = \tilde{v} * D^j\rho$ and because $D^j\rho \in L^1$, Theorem 3.1 leads to the result).

We claim

Theorem 4.2 *The following equalities hold*:

$$\begin{aligned}\tilde{\Sigma}_-(p_0, \xi_0, \tilde{c}_0, \mathbf{A}_0;\ p_1, \xi_1, \tilde{c}_1, \mathbf{A}_1) &= \tilde{\Sigma}(p_0, \xi_0, \tilde{c}_0, \mathbf{A}_0;\ p_1, \xi_1, \tilde{c}_1, \mathbf{A}_1)\\ &= \Sigma(p_0, \xi_0, \hat{c}_0, \mathbf{A}_0;\ p_1, \xi_1, \hat{c}_1, \mathbf{A}_1)\end{aligned} \tag{4.14}$$

with equivalent norms.

Proof Assume, $a \in \tilde{\Sigma}$; and note that $a = \int_R v(t)dt = 1 * v$, with v satisfying (4.8). Let χ_- be the characteristic function of the interval $]-\infty, 0[$ and χ_+ that of $]0, +\infty[$. Because $1 = \chi_- + \chi_+$ we can take $v_0 = \chi_- * v,\ v_1 = \chi_+ * v$, to give $v_0 + v_1 = a$, so that

$$e^{\xi_0 x} v_0 = (e^{\xi_0 x}\chi_-) * (e^{\xi_0 x} v), \quad e^{\xi_1 x} v_1 = (e^{\xi_1 x}\chi_+) * (e^{\xi_1 x} v),$$

where $e^{\xi_0 x}\chi_-$ and $e^{\xi_1 x}\chi_+$ belong to L^1 with the norms $\frac{1}{\xi_0}$ and $\frac{1}{|\xi_1|}$. On applying Theorem 3.1 we obtain

$$\tilde{\Sigma}(p_0, \xi_0, \tilde{c}_0, \mathbf{A}_0;\ p_1, \xi_1, \tilde{c}_1, \mathbf{A}_1) \subset \tilde{\Sigma}_-(p_0, \xi_0, \tilde{c}_0, \mathbf{A}_0;\ p_1, \xi_1, \tilde{c}_1, \mathbf{A}_1), \tag{4.15}$$

More precisely from Theorem 3.1, we infer that

$$\begin{aligned}|a|_{\tilde{\Sigma}_-} &\leq max(\tilde{N}_0(e^{\xi_0 x}\tilde{v}_0, \tilde{N}_1(e^{\xi_1 x}\tilde{v}_1)) \leq max\left(\frac{1}{\xi_0}\tilde{N}_0(e^{\xi_0 x}\tilde{v}), \frac{1}{|\xi_1|}\tilde{N}_1(e^{\xi_1 x}\tilde{v})\right)\\ &\leq max\left(\frac{1}{\xi_0}, \frac{1}{|\xi_1|}\right)|a|_{\tilde{\Sigma}}\end{aligned} \tag{4.16}$$

To prove the converse embedding, Remark 4.1 enables us to put $a = v_0 + v_1$, with $e^{\xi_i x} v_i' \in L^{p_i}_{\tilde{c}_i}(\mathbf{R}; \mathbf{A}_i)$, $i = \{0, 1\}$. As $v_0' + v_1' = 0$, we can let $v = v_0' = -v_1'$, to conclude that v *satisfies* (4.8). Moreover, we can write

$$1 * v = \chi_+ * v + \chi_- * v = \chi_+ * v_0' - \chi_- * v_1' = D\chi_+ * v_0 - D\chi_- * v_1$$

and since $D\chi_+ = -D\chi_\_ = \delta$, we obtain $\int_R v(x)dx = a$. Thus $a \in \tilde{\Sigma}(p_0, \xi_0, \tilde{c}_0, \mathbf{A}_0; p_1, \xi_1, \tilde{c}_1, \mathbf{A}_1)$ and the theorem is proved.

By the method of [5], we can easily demonstrate that $\mathbf{X}$ is dense in $\tilde{\Sigma}$ and use

$$\mathbf{X} \subset \tilde{\Sigma} \subset \mathbf{Y},$$

to deduce that $\tilde{\Sigma}$ is an intermediate space and one can check (see: Sect. 4.2 here after) that $\tilde{\Sigma}$ has the interpolation property (analogous to the spaces in [21]).

Nevertheless it is of interest to establish beforehand some others properties related with symmetry (this is obvious by the definition) or with the invariance of the integral $I(\tilde{v}) = \int_{-\infty}^{+\infty} \tilde{v}(x)dx$ with respect to the changes of variable $x \longrightarrow x + T$, or $x \longrightarrow \lambda x$, $\lambda \neq 0$.

Lemma 4.3 *Let* $a \in \tilde{\Sigma}$. *Then*

$$|a|_{\tilde{\Sigma}} = \underset{I(\tilde{v})=a}{Inf} \; [\tilde{N}_0(e^{\xi_0 x}\tilde{v})]^{1-\theta}[\tilde{N}_1(e^{\xi_1 x}\tilde{v})\,]^{\theta}, \tag{4.17}$$

where

$$\theta = \frac{\xi_0}{\xi_0 - \xi_1}. \tag{4.18}$$

Proof If $T \in R$, set $f^T(x) = f(x + T)$; then

$$\tilde{v}^T \in \tilde{\mathbf{V}}^T(p_0, e^{\xi_0(x+T)}\tilde{c}_0^T, \mathbf{A}_0; p_1, e^{\xi_1(x+T)}\tilde{c}_1^T, \mathbf{A}_1), \quad \int_{-\infty}^{+\infty} \tilde{v}(x + T)dx = a.$$

It can be checked that

$$\tilde{N}_i(e^{\xi_i(x+T)}\tilde{c}_i^T \tilde{v}^T) = e^{-\xi_i T}\tilde{N}_i(e^{\xi_i x}\tilde{c}_i \tilde{v}), \quad i = (0, 1)$$

and consequently

$$|a|_{\tilde{\Sigma}} \leq max\left(e^{-\xi_0 T}\tilde{N}_0(e^{\xi_0 x}\tilde{c}_0\tilde{v}), e^{-\xi_1 T}\tilde{N}_1(e^{\xi_1 x}\tilde{c}_1\tilde{v})\right).$$

Choosing T such that each term in the bracket, on the right in the last formula, takes the same value, we obtain (4.17), (4.18).

Remark 4.4 The second definition leads to

$$|a|_{\tilde{\Sigma}_-} = \underset{v_0(x)+v_1(x)=a}{Inf} \; max[\tilde{N}_0(e^{\xi_0 x}\tilde{v}_0)^{1-\theta}, \; \tilde{N}(e^{\xi_1 x}\tilde{v}_1)^{\theta}], \tag{4.19}$$

with θ given by (4.18).

Corollary 4.5 *There is a constant* $\kappa = \kappa(\xi_0, p_0, c_0; \xi_1, p_1, c_1)$ *such that*

$$|a|_{\tilde{\Sigma}} \leq \kappa \, |a|_{\mathbf{A}_0}^{1-\theta} \, |a|_{\mathbf{A}_1}^{\theta}, \tag{4.20}$$

for all $a \in \mathbf{X}$.

Proof of Corollary 4.5 Consider $\rho \in \mathcal{D}(\mathbf{R})$ such that $\int_R \rho(x)dx = 1$. Then taking $\tilde{v}(x) = \phi(x)a$ in (4.17) gives the result.

Naturally the result is valid for all other definitions of $\tilde{\Sigma}$ (for example $\tilde{\Sigma}_-$) with equivalent norms but with different constants κ.

Now on setting $f_\lambda(x) = f(\lambda x),\ \lambda > 0$, *we obtain a homogeneity result*:

Lemma 4.6 *One has*

$$\forall\lambda > 0, \quad \tilde{\Sigma}(p_0, \lambda\xi_0\tilde{c}_0, \mathbf{A}_0;\ p_1, \lambda\xi_1, \tilde{c}_1, \mathbf{A}_1) \simeq \tilde{\Sigma}(p_0,\ \xi_0, \tilde{c}_0, \mathbf{A}_0;\ p_1, \xi_1, \tilde{c}_1,\ \mathbf{A}_1) \quad (4.21)$$

with equivalent norms. *Moreover*

$$|a|_{\tilde{\Sigma}_\lambda} = \lambda^{1-1/p_\theta}\,|a|_{\tilde{\Sigma}} \quad (4.22)$$

and

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \textit{where } \theta \textit{ is given by } (4.18). \quad (4.23)$$

Proof It is obvious that the function $\tilde{v}_\lambda(x) = \lambda\tilde{v}(\lambda x)$ belongs to the space

$$\tilde{V}_\lambda(p_0, \lambda\xi_0, \tilde{c}_{0\lambda}, \mathbf{A}_0;\ p_1, \lambda\xi_1, \tilde{c}_{1\lambda}, \mathbf{A}_1)$$

and that

$$\int_{-\infty}^{+\infty} \tilde{v}_\lambda(x)dx = \int_{-\infty}^{+\infty} \tilde{v}(x)dx = a,$$

Consequently, we obtain

$$\forall\lambda > 0, \quad \tilde{\Sigma}_\lambda = \tilde{\Sigma}(p_0, \lambda\xi_0, \tilde{c}_{0\lambda}, \mathbf{A}_0;\ p_1, \lambda\xi_1, \tilde{c}_{1\lambda}\mathbf{A}_1) \ \simeq\ \tilde{\Sigma}(p_0, \xi_0, \tilde{c}_0, \mathbf{A}_0;\ p_1, \xi_1, \tilde{c}_1, \mathbf{A}_1).$$

More precisely, we have

$$\left|e^{\lambda\xi_i x}\tilde{c}_{i\lambda}\tilde{v}_\lambda\right|_{L^{p_i}(\mathbf{R};\mathbf{A}_i)} = \lambda^{1-1/p_i}\left|e^{\xi_i x}\tilde{c}_i\tilde{v}\right|_{L^{p_i}(\mathbf{R};\mathbf{A}_i)} \quad (4.24)$$

On taking into account (4.17)–(4.20) for $|a|_{\tilde{\Sigma}}$ and (4.22). The result of the Lemma 4.6 follows.

Remark 4.7 As in the case of unweighted space [21], the estimate (4.22) shows that the result seems to depend upon the parameters (θ, p_θ) but provided no information about dependence on the weights.

From Lemma 4.6, we deduce the main formula:

$$\begin{aligned}\tilde{\Sigma}(p_0, \xi_0, \tilde{c}_0, \mathbf{A}_0;\ p_1, \xi_1, \tilde{c}_1, \mathbf{A}_1) &= \tilde{\Sigma}_\theta \\ &= \tilde{\Sigma}(p_0, \theta, \tilde{c}_0, \mathbf{A}_0;\ p_1, 1-\theta, \tilde{c}_1, \mathbf{A}_1), \quad \theta \textit{ given by } (4.18),\end{aligned} \quad (4.25)$$

with equivalent norms.

4.3 Interpolation property

Consider a family of spaces $\{\mathbf{B}_0, \mathbf{B}_1, \mathcal{B}\}$ analogous to $\{\mathbf{A}_0, \mathbf{A}_1, \mathcal{A}\}$.Denote $\mathbf{Y}_a = \mathbf{Y},\ \mathbf{Y}_b = \mathbf{B}_0 + \mathbf{B}_1$ and let $\Sigma_a = \tilde{\Sigma}(p_0, \xi_0, \tilde{c}_0, \mathbf{A}_0;\ p_1, \xi_1, \tilde{c}_1, \mathbf{A}_1),\ \Sigma_b = \tilde{\Sigma}(p_0, \xi_0, \tilde{c}_0, \mathbf{B}_0;\ p_1, \xi_1, \tilde{c}_1, \mathbf{B}_1)$. We have

Theorem 4.8 *Let π be a linear mapping from $\mathbf{Y}_a$ into $\mathbf{Y}_b$ which restricted to $\mathbf{A}_i$ is linear and continuous from $\mathbf{A}_i$ into $\mathbf{B}_i$ (said $\pi \in \mathcal{L}(\mathbf{A}_i, \mathbf{B}_i)$) $(i = 0, 1)$. Then the restriction of π to $\mathbf{A}$ belongs to $\mathcal{L}(\mathbf{A}, \mathbf{B})$.*

Moreover, let ω_i the norm of $\mathcal{L}(\mathbf{A}_i, \mathbf{B}_i)$, ω that of $\mathcal{L}(\mathbf{A}, \mathbf{B})$, one has

$$\omega \leq \omega_0^{1-\theta} \omega_1^{\theta}, \tag{4.26}$$

where θ is given by (4.18).

Proof Let $a \in \mathbf{A}$. There exists a function $\tilde{v} \in \sim V$ with $\int_{\mathbf{R}} \tilde{v}(x)dx = a$, the last integral being convergent into $\mathbf{Y}_a$. From the assumptions upon π, we deduce that $\pi \in \mathcal{L}(\mathbf{Y}_a, \mathbf{Y}_b)$ and thus

$$\pi a = \int_{\mathbf{R}} \pi \tilde{v}(x)dx.$$

From Lemma 4.3, we obtain

$$|\pi a|_{\mathbf{B}} \leq \omega_0^{1-\theta} \omega_1^{\theta} [\tilde{N}_0(e^{\xi_0 x} \tilde{v})]^{1-\theta} [\tilde{N}_1(e^{\xi_1 x} \tilde{v})]^{\theta}.$$

Again with Lemma 4.3, we have

$$|\pi a|_{\mathbf{B}} \leq \omega_0^{1-\theta} \omega_1^{\theta} \, |a|_{\mathbf{A}}$$

and Theorem 4.8 is proved.

Remark 4.9 From Remark 4.4 Theorem 4.8 stay again valid if we replace $\tilde{\Sigma}$ by $\tilde{\Sigma}\_$.

5 Intermediate mean spaces and spaces to traces

Let us return to the definitions and observe that the space Σ is spanned by $a = \int_0^{+\infty} v(t) \frac{dt}{t}$, as v spans the space $\mathbf{V}_\theta = \{v \ : t^\theta v \in \hat{\mathbf{X}}_0;\ t^{\theta-1} v \in \hat{\mathbf{X}}_1\}$ with $\theta \in (0, 1)$ given by (4.18).

Accordingly, with the result obtained in the summary of Sect. 3.2, we conclude that the function

$$u^* = t^{m-j} u^{(m)} \in \mathbf{V}_j = \{v\ ;\ t^{1/p_0+j} \in \hat{\mathbf{X}}_0;\ t^{1/p_1+m-j} v \in \hat{\mathbf{X}}_1\}, \quad j \in \{0, 1, \ldots, m-1\}. \tag{5.1}$$

But $\int_0^{+\infty} u^*(t) \frac{dt}{t} = a_j \in \mathbf{T}^{(m)}$and thus we obtain the algebraic and topological inclusion

$$\mathbf{T}_j^{(m)} \subset \tilde{\Sigma}_j = \Sigma(p_0, 1/p_0 + j, \hat{c}_0, \mathbf{A}_0;\ p_1, 1/p_1 + j - m, \hat{c}_1, \mathbf{A}_1), \quad j \in \{0, 1, \ldots, m-1\}. \tag{5.2}$$

$$\left|a_j\right|_{\Sigma_{\theta_j}} \leq c_1 \left|a_j\right|_{\mathbf{T}_j^{(m)}}. \tag{5.3}$$

Furthermore from (4.23), we have

$$\mathbf{T}_j^{(m)} \subset \Sigma_{\theta_j} = \Sigma(p_0, \theta_j, \hat{c}_0, \mathbf{A}_0;\ p_1, \theta_j - 1, \hat{c}_1, \mathbf{A}_1), \quad j \in \{0, 1, \ldots, m-1\} \tag{5.4}$$

$$\left|a_j\right|_{\Sigma_{\theta_j}} \leq c_1 \left|a_j\right|_{\mathbf{T}_j^{(m)}}. \tag{5.5}$$

where θ_j (=γ_{jm}, (see (2.13)) is given by

$$\theta_j = \frac{1/p_0 + j}{1/p_0 - 1/p_1 + m}, \quad m \geq 1,\ j \in \{0, 1, \ldots, m-1\}. \tag{5.6}$$

Remark 5.1 As in [5] for $\mathbf{W}^{(m)}$ (see also [8]), the fact that the space $\mathbf{V}_K(\mathbf{X})$ of functions v, with values in $\mathbf{X}$, and compact support, for $v \in \mathbf{V}$, *is dense in* $\mathbf{V}$, can be used to obtain the previous results.

Next we want to establish the identity with equivalent norms between the space $\mathbf{T}_j^{(m)}$ *and* Σ_j (or Σ_{θ_j}). It then remains to prove a converse of the inclusion (5.2).

Assume that $a \in \Sigma_j$, then we can find a function $u_* \in V_j = \{u_*;\ t^{1/p_0+j}u_* \in \hat{X}_0, t^{1/p_1+j-m}u_* \in \hat{X}_1\}$ such that $a = \int_0^{+\infty} u_*(t)\frac{dt}{t}$. We require however a function $u \in W^{(m)}$ such that $D^j u(0) = a$.

From the direct proof of (3.15) we must have that u is *an integral of order* m of the function $ct^{j-m}u_*$, (c a constant) then because $t^{j-m}u_* \in X_1$ we have obviously:

$$D^m u \in X_1 \tag{5.7}$$

According to Laurent Schwartz [24] (see also [12]), an integral of order m of a distribution T with support restricted on the left, is defined by the convolution

$$I^m(T) = Y_m^+ * T,$$

where

$$Y_m^+(x) = \frac{x_+^{m-1}}{\Gamma(m)}.$$

When S has a support restricted on the right, then

$$I^m(S) = (-1)^m Y_m^- * S,$$

where

$$Y_m^-(x) = \frac{(-x)_+^{m-1}}{\Gamma(m)}.$$

Since we can choose u_* with compact support, both definitions are valid. Consequently, it is the derivative of order j for u which leads us to take

$$u(t) = \frac{(-1)^m c}{(m-1)!}\int_t^{+\infty} (\tau - t)^{m-1}\tau^{j-m}u_*(\tau)d\tau \tag{5.8}$$

so that

$$D^j u(t) = \frac{(-1)^{m-j} c}{(m-j-1)!}\int_t^{+\infty} (\tau - t)^{m-j-1}\tau^{j-m}u_*(\tau)d\tau. \tag{5.9}$$

Then on choosing c such that $\frac{(-1)^{m-j}c}{(m-j-1)!} = 1$, and tacking $t \longrightarrow +0$ we deduce from (5.9) that

$$u^{(j)}(0) = \int_0^{+\infty} u_*(\tau)\frac{d\tau}{\tau} = a, \tag{5.10}$$

which is the required result.

It remains to prove that

$$u \in X_0. \tag{5.11}$$

To this end, we observe that the integral in (5.8) is $I(t) = \int_t^{+\infty}(1-\frac{t}{\tau})^{m-1}\tau^j u_*(\tau)\frac{d\tau}{\tau}$, which is the convolution on R^* with respect to Haar's measure $\frac{d\tau}{\tau}$, of two functions with compact support one, $f(t) = (1-t)^{m-1}$ if $0 \le t \le 1$, $f(t) = 0\ if\ t > 1$, (which is in L^1_*) while the other is $t^j u_*$. We know that $t^{1/p_0+j}u_* \in \hat{X}_0$, $i.e.$ $t^j u_* \in X_0$. From Theorem 3.1 and Corollary 3.7, or Remark 3.8(2), as $c_0 t^j u_* \in L^{p_0}_*(A_0)$ then we can deduce that there is a constant $\mu > 0$, such that

$$|u|_{X_0} \le \mu \left|t^j u_*\right|_{X_0}.$$

which gives (5.11).

Thus we have proved that

$$\Sigma_j \simeq \Sigma_{\theta_j} \subset T_j^{(m)} \tag{5.12}$$

with equivalent norms. The conclusion is the converse of the embeddings (5.2)–(5.3), the topological part following from the inequalities.

The study of Sect. 5 can be summarised by

Theorem 5.2 *Assume $c_i \in \mathcal{H}(p_i)$, $i = (0,1)$, (2.10)–(2.11), and that $[c_0]^{-1} \in \mathcal{H}(p_0')$ holds. Then we obtain*

$$T_j^{(m)}(p_0, c_0, A_0; p_1, c_1, A_1) = \Sigma(p_0, 1/p_0 + j, \hat{c}_0, A_0; p_1, 1/p_1 + j - m, \hat{c}_1, A_1), \quad j \in \{0, 1, \ldots, m-1\} \tag{5.13}$$

with equivalent norms.

We also have

Corollary 5.3 *In particular, there holds*

$$\mathbf{T}_j^{(m)}(p_0, c_0, \mathbf{A}_0; p_1, c_1, \mathbf{A}_1) = \Sigma(p_0, \theta_0, \hat{c}_0, \mathbf{A}_0; p_1, \theta_j - 1, \hat{c}_1, \mathbf{A}_1) \tag{5.14}$$

where

$$\theta_j = \frac{j + 1/p_0}{1/p_0 - 1/p_1 + m}, \quad m \ge 1,\ j \in \{0, 1, \ldots, m-1\} \tag{5.15}$$

with equivalent norms.

If we take into account Proposition 2.6, we deduce from Theorem 5.2 that

$$\begin{aligned}\mathbf{T}_j^{(m)} &= \mathbf{T}_0^1(p_0, t^j c_0, \mathbf{A}_0; p_1, t^{j-m+1}c_1, \mathbf{A}_1)\\ &= \Sigma(p_0, 1/p_0 + j, \hat{c}_0, \mathbf{A}_0; p_1, 1/p_1 + j - m, \hat{c}_1, \mathbf{A}_1)\end{aligned} \tag{5.16}$$

and from Corollary 5.3

$$\Sigma(p_0, \theta_j, \hat{c}_0, \mathbf{A}_0; p_1, \theta_j - 1, \hat{c}_1, \mathbf{A}_1) = \mathbf{T}_0^1(p_0, \chi_0, \mathbf{A}_0; p_1, \chi_1, \mathbf{A}_1) \tag{5.17}$$

where

$$\chi_i(t) = t^{\theta_j} \hat{c}_i, \quad i = \{0, 1\}. \tag{5.18}$$

To compare with the trace spaces studied in [21], we have

Remark 5.4 In [21] the weights $c_i(t) = t^{\alpha_i}$, $\alpha_i + 1/p_i \in]0, 1[$ are considered and the exponents α_i appear in the expression for θ_j. Indeed, if we set $\eta_i = \alpha_i + 1 p_i$, then $\theta_j = \frac{\eta_0 + j}{\eta_0 - \eta_1 + m}$, and the exponents of the weights can be considered as parameters ξ_i. Here, the weights are different to these parameters.

Notice that the condition $\alpha_i + 1/p_i \in]0, 1[$ implies that the weights belong to $\mathcal{H}(p_i)$ with, in particular the Poulsen condition (2.11) given by $1/p_1 + \alpha_1 + j < m$ while the condition $1/p_0 + \alpha_0 + j > 0$ gives $t^{-(1+j+\alpha_0)} \in L^{p_0'}(1, +\infty; \mathbf{R}^+)$ which is implied by $t^{-\alpha_0} \in \mathcal{H}(p_0')$, that is true in [21] (see Theorem 3.1).

6 On dependence of the spaces upon parameters

In what follows, we want to prove that $\Sigma_\theta(p_0, \theta, \hat{c}_0, \mathbf{A}_0; p_1, \theta - 1, \hat{c}_1, \mathbf{A}_1)$ depends only on three parameters: θ, p_θ (see Corollary 4.5) and a weight π_θ.

As a preliminary, we note

Remark 6.1 Let $f_\#(t) = f(1/t)$, then $a = \int_0^{+\infty} v(t) \frac{dt}{t} = \int_0^{+\infty} v_\#(\tau) \frac{d\tau}{\tau}$, so that $\int_0^{+\infty} [\hat{c}_i(t) \, |v(t)|_{\mathbf{A}_i}]^{p_i} dt = \int_0^{+\infty} [c_\#(\tau) \, |v_\#(\tau)|_{\mathbf{A}_i}]^{p_i} \frac{d\tau}{t} = \int_0^{+\infty} [\tau^{-1/p_i} c_\#(\tau) \, |v_\#(\tau)|_{\mathbf{A}_i}]^{p_i} d\tau$ and the weight $\hat{c}_i(t) = t^{-1/p_i} c_i(t)$ is changed to $\hat{c}_{i\#}(\tau) = \tau^{-1/p_i} c_{i\,\#}(\tau)$, with respect to Lebesgue measure when t is changed to $1/t$.

This property, of "quasi invariance" when t *is changed to* $1/t$, led us to the definitions $\{(4.1), (4.2)\}$ for the space V. We obtain

$$\Sigma_\theta = \Sigma(p_0, \theta, \hat{c}_0, \mathbf{A}_0; p_1, \theta - 1, \hat{c}_1, \mathbf{A}_1) = \Sigma_{-\theta} = \Sigma(p_0, -\theta, \hat{c}_{0\#}, \mathbf{A}_0; p_1, 1 - \theta, \hat{c}_{1\#}, \mathbf{A}_1) \tag{6.1}$$

with equivalent norms.

Naturally since $\tilde{\Sigma}_- = \tilde{\Sigma} = \Sigma$, from $\{(4.19), (6.1)\}$, we have also

$$\Sigma_{-\theta} = \{a; a = v_0 + v_1 \in \mathbf{Y}, \underset{v_0 + v_1 = a}{Inf} \{ \left| t^{-\theta} c_{0\#} v_0 \right|_{L_*^{p_0}(\mathbf{A}_0)} + \left| t^{1-\theta} c_{1\#} v_1 \right|_{L_*^{p_1}(\mathbf{A}_1)} \} \tag{6.2}$$

with equivalent norms.

6.1 Some definitions

Now we want to present the technique used by J. Peetre in the case of unweighted spaces[5] to define adapted J and K methods in the particular case where $\forall i = (0, 1)$ $p_i = p$, $c_i = \pi$.

(1) J.W-method: Define a measurable function $u = u(t)$ *on* $\mathbf{R}^+$, taking values in $\mathbf{X}$, such that

$$a = I(u) = \int_0^{+\infty} u(t) \frac{dt}{t}, \ (in\ \mathbf{Y}), \ t^{-\theta} \pi_\# J(t, u(t) \in L_*^p, \ \pi \in \mathcal{H}(p), 1 \le p \le +\infty, \tag{6.3}$$

[5] See [22–24].

where the quantity

$$\forall t > 0, \quad J(t,a) = max(|a|_{\mathbf{A}_0}, t\,|a|_{\mathbf{A}_1}) \tag{6.4}$$

for a fixed t, is a norm on $\mathbf{X}$ equivalent to the norm $J(1,a)$ of $.\mathbf{X}$.

Let $(\mathbf{A}_0, \mathbf{A}_1)^J_{\theta,p,\pi}$, the space spanned by a equipped with the norm

$$|a|_{(\mathbf{A}_0,\mathbf{A}_1)^J_{\theta,p,\pi}} = \underset{I(u)}{Inf} \left| t^{-\theta} J(t,u(t)) \right|_{L^p_{\hat{\pi}_\#}}$$

(where $\hat{\pi}_\#(t) = t^{-1/p}\pi_\#(t)$) which is a Banach space.

When $p_i = p, c_i = c$, (6.1) implies

$$\Sigma_\theta(p,\theta,\hat{c},\mathbf{A}_0;\, p,\theta-1,\hat{c},\mathbf{A}_1) = (\mathbf{A}_0,\mathbf{A}_1)^J_{\theta,p,c} \tag{6.5}$$

with equivalence of norms.

(2) K.W-Method: Let v_i be two measurable functions with values in A_i $(i = 0, 1)$ and $a \in Y$ such that

$$a = v_0(t) + v_1(t), \quad t^{-\theta}\pi_\# K(t,a) \in L^p_*, \tag{6.6}$$

where $K(t,a)$ defined by

$$K(t,a) = \underset{a=a_0+a_1}{Inf} \left(|a_0|_{\mathbf{A}_0} + t\,|a_1|_{\mathbf{A}_1} \right), \tag{6.7}$$

is, for a fixed t, a norm on $\mathbf{Y}$ equivalent to the norm $K(1,a)$ of $\mathbf{Y}$.

Let $(\mathbf{A}_0, \mathbf{A})^K_{\theta,p,\pi}$ be the space spanned by $a = v_0(t) + v_1(t)$ a.e. in $\mathbf{Y}$ with

$$t^{-\theta}\pi_\# v_0 \in L^p_*(\mathbf{A}_0), \quad t^{1-\theta}\pi_\# v_1 \in L^p_*(\mathbf{A}_1),$$

and equipped with the norm

$$|a|_{(\mathbf{A}_0,\mathbf{A}_1)^K_{\theta,p,\pi}} = \underset{a=v_0(t)+v_1(t),a.e.}{inf} \left(\int_0^{+\infty} \left(t^{-\theta}\pi_\#(t)[|v_0(t)|_{\mathbf{A}_0} + t\,|v_1(t)|_{\mathbf{A}_1} \right)^p \frac{dt}{t} \right)^{1/p} \tag{6.8}$$

$(\mathbf{A}_0, \mathbf{A}_1)^K_{\theta,p,\pi}$ is a Banach space.

Now when $p_0 = p_1 = p,\ c_0 = c_1 = c$, we easily include from (6.2) that

$$\Sigma_\theta(p,\theta,\hat{c},\mathbf{A}_0;\, p,\theta-1,\hat{c},\mathbf{A}_1) = (\mathbf{A}_0,\mathbf{A}_1)^K_{\theta,p,c} \tag{6.9}$$

Henceforward, we write $\Sigma_\theta(p_0,\theta,\hat{c}_0,\mathbf{A}_0;\, p_1,\theta-1,\hat{c}_1,\mathbf{A}_1) = (\mathbf{Y}_0,\mathbf{A}_1)_{\theta,p_0,p_1,c_0,c_1}$ and state

Proposition 6.2 *For* $p_i = p, c_i = c, i \in \{0,1\}, \theta \in (0,1)$, *the following conditions are equivalent*:

1. $a \in (\mathbf{A}_0,\mathbf{A}_1)_{\theta,p,p,c,c}$ (*denoted* $(\mathbf{A}_0,\mathbf{A}_1)^*_{\theta,p,c}$)
2. $a \in (\mathbf{A}_0,\mathbf{A}_1)^J_{\theta,p,c}$,
3. $t^{-\theta}c_\# K(t,a) \in L^p_*$.

(*That means*: $(\mathbf{A}_0,\mathbf{A}_1)^*_{\theta,p,c} = (\mathbf{A}_0,\mathbf{A}_1)^J_{\theta,p,c} = (\mathbf{A}_0,\mathbf{A}_1)^K_{\theta,p,c}$ *with equivalent norms*).

Proof (1) $\Longleftrightarrow$ (2) and (1) $\Longleftrightarrow$ (3) are obvious from definitions.

Proposition 6.3 *We assume* $0 < \theta < 1$, $1 \leq p \leq q \leq +\infty$, and $c^{-1} \in \mathcal{H}(p')$ [*so that* $\forall T > 0$, $\int_0^T [c(t)]^p dt < +\infty$, (*see* (2.1)] *then,*

$$(\mathbf{A}_0, \mathbf{A}_1)^*_{\theta,p,\pi} \subset (\mathbf{A}_0, \mathbf{A}_1)^*_{\theta,q,\pi} \tag{6.10}$$

with continuous imbedding.

Proof From Proposition 6.2 we can define $(\mathbf{A}_0, \mathbf{A}_1)^*_{\theta,p,\pi}$ by the *K.W-method* and on noting for all $\tau \geq 0$, that[6]

$$t > \tau \Longrightarrow K(t, a) > K(\tau, a)$$

then

$$|a|^p \geq [K(\tau, a)]^p \int_\tau^{+\infty} [t^{-\theta} c_\#(t)]^p \frac{dt}{t}, \tag{6.11}$$

implying that

$$\forall \tau > 0, \quad c_\#(\tau)\tau^{-\theta} K(\tau, a) \leq \kappa \, |a|_{\theta,p,c}$$

Although, from (6.11) the integral must make sense, but there is a contradiction when the function $log(\tau^{-\theta} c_{\#(\tau)})$ is *of order* -1 (*resp.* ∞) *with respect to* $Log\tau$. Thus from Bourbaki [11], the integral in (6.11) is equivalent to $\tau^{-\theta p} c_\#^p(\tau)$ (up to a multiplicative constant). In consequence

$$\left|t^{-\theta} K(t, a)\right|_{L^\infty_{\hat{c}_\#}} \leq \gamma \, |a|_{\theta,p,c} \, , \tag{6.12}$$

and (6.10) follows for $q = \infty$, implying that

$$\forall t > 0, \, K(t, a) \leq \kappa t^\theta \, |a|_{\theta,p,c} \, . \tag{6.13}$$

Now if $1 \leq p \leq q < \infty$, on letting $h(t) = t^{-\theta} \hat{c}_\#(t) K(t, a)$ and using (6.12), we may check that

$$|a|^q_{\theta,q,c} = \int_0^{+\infty} [h(t)]^p [h(t)]^{q-p} dt \leq \gamma_1 \, |a|^q_{\theta,p,c} \, .$$

and the proof is complete.

6.2 Some lemmas

Lemma 6.4 (Inequality of Carlson's type) *Let* $\lambda \in (0, 1)$, $1 \leq p_i \leq +\infty$, $\pi_i \in \mathcal{H}(p_i)$, $i = 0, 1$, Φ *a positive function and assume*

$$t\phi_i \in L^{p_i'}(0, 1), \; \phi_i \in L^{p_i'}(1, +\infty) \quad \text{where } t\phi_i = t^{-1/p_i'} [t^{-(1-\lambda)} \pi_i(t)]^{-1}. \tag{6.14}$$

Then one has the inequality

$$\int_0^{+\infty} \Phi(t) \frac{dt}{t} \leq \gamma \left(\int_0^{+\infty} [t^{-\lambda} \pi_{0\#}(t) \Phi(t)]^{p_0} \frac{dt}{t} \right)^{\frac{1-\lambda}{p_0}} \left(\int_0^{+\infty} [t^{1-\lambda} \pi_{1\#}(t) \Phi(t)]^{p_1} \frac{dt}{t} \right)^{\frac{\lambda}{p_1}} \tag{6.15}$$

where γ *is a constant depending on* λ, π_0, π_1.

[6] See also [31] for unweighted spaces.

Remark 6.5 A similar inequality proved in [10] (with only "weights" constructed with general terms of the form t^α), is also given in [23,24] with $\pi_i \equiv 1, i = 0, 1$. Conditions (6.14) are always fulfilled in this case. The choice of the weight $\pi_i(1/t) = \pi_{i\#}(t)$ is useful in what follows.

Proof of Lemma 6.4 [7] We start from

$$\int_0^{+\infty} \Phi(t)\frac{dt}{t} = \int_0^{+\infty} \frac{\Phi(t)}{t(1+t)}dt + \int_0^{+\infty} \frac{\Phi(t)}{(1+t)}dt = I_0 + I_1$$

and using Hölder's inequality, one obtain

$$I_0 \leq \left|t^{-\lambda}\pi_{0\#}\Phi\right|_{L_*^{p_0}} (J_0)^{1/p_0'}, \quad where\ J_0 = \int_0^{+\infty} \left[\frac{t^{-1/p_0'}}{t^{-\lambda}\pi_0(1/t)(1+t)}\right]^{p_0'} dt.$$

Now taking $\tau = 1/t$ in J_0, we have

$$J_0 \leq \int_0^{+\infty} \frac{d\tau}{\tau[\tau^\lambda \pi_0(\tau)(1+1/\tau)]^{p_0'}} \leq \int_0^1 \tau\phi_0(\tau)d\tau + \int_1^{+\infty} \phi_0(\tau)d\tau = \gamma_0,$$
$$\phi_0(\tau) = \frac{1}{\tau(\tau^{-(1-\lambda)}\pi_0(\tau))},$$

which is (6.10) for $i = 0$.

On the other hand,

$$I_1 \leq \left|t^{1-\lambda}\pi_{1\#}\Phi\right|_{L_*^{p_1}} (J_1)^{1/p_1'} \quad where\ J_1 = \int_0^{+\infty} \left[\frac{t^{1-1/p_1'}}{t^{1-\lambda}\pi_1(1/t)(1+t)}\right]^{p_1'}$$
$$= \int_0^{+\infty} \frac{\tau^{p_1'}}{(\tau^\lambda \pi_1(\tau)(1+\tau))^{p_1'}} \frac{d\tau}{\tau}$$

and we can easily check that

$$J_1 \leq \int_0^1 (\tau\phi_1(\tau))^{p_1'} d\tau + \int_1^{+\infty} (\phi_1(\tau))d\tau = \gamma_1, \quad \phi_1(t) = \frac{1}{\tau(t^{-(1-\lambda)}\pi_1(\tau))}$$

which leads to

$$\int_0^{+\infty} \Phi(t)\frac{dt}{t} \leq \gamma_0 \left|t^{-\lambda}\pi_{0\#}\Phi\right|_{L_*^{p_0}} + \gamma_1 \left|t^{1-\lambda}\pi_{1\#}\Phi\right|_{L_*^{p_1}}.$$

The change of variable $t = k\tau,\ k > 0$, and a convenient choice of k yields (6.15).

Now we introduce the space $L_{\hat{c}}^{p,\alpha} = \{f;\ s^{-\alpha}cf \in L_*^p,\ \alpha \in R,\ c \in \mathcal{H}(p)\}$, which provided with the natural norm becomes a Banach space.

We state

Lemma 6.6 *Let* $\theta \in (0, 1),\ 1 \leq p_i \leq +\infty, c_i \in \mathcal{H}(p_i)$ *satisfy* (6.14), *for* $i = 0, 1$. *Then*

$$\Sigma(-\theta, p_0, c_{0\#}, L_{\hat{c}_0}^{p_0,\alpha_0}; 1-\theta, p_1, c_{1\#}, L_{\hat{c}_1}^{p_1,\alpha_1}) \simeq (L_{\hat{c}_0}^{p_0,\alpha_0}, L_{\hat{c}_1}^{p_1,\alpha_1})_{\theta,p_0,p_1,c_0,c_1} = L_{c\theta}^{p_\theta,\alpha}, \tag{6.16}$$

[7] For the convenience of the Reader we adapt the proof given in [10].

with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\ \alpha = (1-\theta)\alpha_0 + \theta\alpha_1,\ , c_\theta(t) = t^{-1/p_\theta}[c_0(t)]^{1-\theta}\ [c_1(t)]^\theta. \tag{6.17}$$

Proof of Lemma 6.6 (1) Assume

$$a(.) \in \Sigma(-\theta, p_0, \hat{c}_{0\#}, L^{p_0}_{\hat{c}_0}; 1-\theta, p_1, \hat{c}_{1\#}, L^{p_1}_{\hat{c}_1})$$

then there is a function $u(., t)$, $a(.) = \int_0^{+\infty} u(., t)\frac{dt}{t}$, with

$$t^{-\theta} c_{0\#} u(., t) \in L^{p_0}_*[L^{p_0}_{\hat{c}_0}], \quad t^{1-\theta} c_{1\#} u(., t) \in L^{p_1}_*[L^{p_1}_{\hat{c}_1}]. \tag{6.18}$$

which is equivalent to

$$t^{-\theta} s^{-\alpha_0} u(s, t) \in L^{p_0}_{\hat{c}_{0\#}}[L^{p_0}_{\hat{c}_0}], \quad t^{1-\theta} s^{-\alpha_1} u(s, t) \in L^{p_1}_{\hat{c}_{1\#}}[L^{p_1}_{\hat{c}_1}].$$

An application of inequality (6.15), gives

$$\begin{aligned} |a(s)| &\le \int_0^{+\infty} |u(s, t)|\,\frac{dt}{t} \\ &\le \gamma \left(\int_0^{+\infty} [t^{-\theta}\hat{c}_{0\#}(t)u(s, t)]^{p_0} dt\right)^{\frac{1-\theta}{p_0}} \left(\int_0^{+\infty} [t^{1-\theta}\hat{c}_{1\#}(t)u(s, t)]^{p_1} dt\right)^{\frac{\theta}{p_1}} \end{aligned}$$

from which may be deduced the inequality

$$\begin{aligned} \left|\hat{c}_\theta s^{-\alpha} a(s)\right|^{p_\theta} &\le \gamma \left(\int_0^{+\infty} [t^{-\theta}\hat{c}_{0\#}(t)s^{-\alpha_0}\hat{c}_0(s)u(s, t)]^{p_0} dt\right)^{\frac{p_\theta(1-\theta)}{p_0}} \\ &\quad \times \left(\int_0^{+\infty} [t^{1-\theta}\hat{c}_{1\#}(t)s^{-\alpha_1}\hat{c}_1(s)u(s, t)]^{p_1} dt\right)^{\frac{p_\theta\theta}{p_1}} \end{aligned}$$

But $\frac{p_\theta(1-\theta)}{p_0} + \frac{p_\theta\theta}{p_1} = 1$, and Hölder's inequality gives, $\int_0^{+\infty} [\hat{c}_\theta s^{-\alpha}\,|a(s)|]^{p_\theta} ds \le (\int_0^{+\infty}\int_0^{+\infty} [\alpha(s, t)]^{p_0}(\hat{c}_0(s))^{p_0} ds (\hat{c}_{0\#}(t))^{p_0} dt)^{\frac{p_\theta(1-\theta)}{p_0}} (\int_0^{+\infty}\int_0^{+\infty} [\beta(s, t)]^{p_1}(\hat{c}_1(s))^p_1 ds\ (\hat{c}_{1\#}(t))^{p_1} dt)^{\frac{p_\theta\theta}{p_1}}$ where

$$\alpha(s, t) = t^{-\theta} s^{-\alpha_0} u(s, t), \quad \beta(s, t) = t^{1-\theta} s^{-\alpha_1} u(s, t)$$

and

$$s^{-\alpha} a(s) \in L^{p_\theta}_{\hat{c}_\theta}, \quad \hat{c}_\theta = s^{-1/p_\theta} c_0^{1-\theta} c_1^\theta.$$

We have therefore proved that

$$(L^{p_0,\alpha_0}_{\hat{c}_0}, L^{p_1,\alpha_1}_{\hat{c}_1})_{\theta, p_0, p_1, c_0, c_1} \subset L^{p_\theta,\alpha}_{\hat{c}_\theta} \tag{6.19}$$

with continuous injection.

(2) To prove the converse of (6.19) assume $a(s) \in L^{p_\theta,\alpha}_{c_\theta}$. We must show that there is a function $u(s, t)$ satisfying (6.18) such that $a(s) = \int_0^{+\infty} u(s, t)\frac{dt}{t}$.

For this purpose,we adopt a strategy by J. Peetre in [24] and consider a function Φ, such that $\int_0^{+\infty} \Phi(t)\frac{dt}{t} = 1$. We define

$$u(s, t) = \Phi(\chi(s)t)a(s)$$

where $\chi(s)$ is a function defined later. Then we have

$$\int_0^{+\infty} u(s,t)\frac{dt}{t} = a(s)$$

Moreover, we note that

$$\begin{aligned} K_0 &= \int\int_{R^{+2}} |t^{-\theta}s^{-\alpha_0}u(s,t)|^{p_0}(\hat{c}_0(s))^{p_0}ds(\hat{c}_{1\#}(t))^{p_0}dt \\ &= \int_0^{+\infty}\left(\int_0^{+\infty}[t^{-\theta}\Phi(\chi(s)t)]^{p_0}(c_{0\#}(\chi(s)t))^{p_0}\frac{dt}{t}\right)s^{-\alpha_0 p_0}\,|a(s)|^{p_0}\,(\hat{c}_0(s))^{p_0}ds \\ &= \int_0^{+\infty}[t^{-\theta}\Phi(t)]^{p_0}(\hat{c}_{0\#}(t))^{p_0}dt\int_0^{+\infty}x(s)^{\theta p_0}s^{-\alpha_0 p_0}\,|a(s)|^{p_0}(\hat{c}_0(s))^{p_0}ds. \end{aligned}$$

and similarly obtain

$$\begin{aligned} K_1 &= \int\int_{R^{+2}} \left|t^{1-\theta}s^{-\alpha_1}u(s,t)\right|^{p_1}(\hat{c}_1(s))^{p_1}ds(\hat{c}_{1\#}(t))^{p_1}dt \\ &= \int_0^{+\infty}[t^{1-\theta}\Phi(t)]^{p_1}(\hat{c}_{1\#}(t))^{p_1}dt\int_0^{+\infty}\chi(s)^{-(1-\theta)p_1}s^{-\alpha_1 p_1}\,|a(s)|^{p_1}\,(\hat{c}_1(s))^{p_1}ds \end{aligned}$$

Now following [24, pp. 254–255] we set $\chi(s) = f([c_0,c_1](s))s^{-y}\,|a(s)|^{-x}$ and determine x, y, and f such that

$$[f(s)]^{\theta p_0}[c_0(s)]^{p_0} = [f(s)]^{-(1-\theta)p_1}[c_1(s)]^{p_1} = [c_\theta(s)]^{p_\theta}$$

to obtain

$$\chi(s)^{\theta p_0}\left|c_0(s)s^{-\alpha_0}a(s)\right|^{p_0} \equiv \chi(s)^{-(1-\theta)p_1}|c_1(s)s^{-\alpha_1}a(s)|^{p_1} \equiv (c_\theta(s)s^{-\alpha}\,|a(s)|)^{p_\theta}. \quad (6.20)$$

In consequence, we have

$$\begin{aligned} K_0 &= \int_0^{+\infty}[\hat{c}_{0\#}(t)t^{-\theta}\Phi(t)]^{p_0}dt\int_0^{+\infty}[\hat{c}_\theta(s)s^{-\alpha}a(s)]^{p_\theta}ds, \\ K_1 &= \int_0^{+\infty}[\hat{c}_{1\#}(t)t^{1-\theta}\Phi(t)]^{p_1}dt\int_0^{+\infty}[\hat{c}_\theta(s)s^{-\alpha}a(s)]^{p_\theta}ds, \end{aligned}$$

and therefore

$$L^{p_\theta,\alpha}_{\hat{c}_\theta} \subset (L^{p_0,\alpha_0}_{\hat{c}_0}, L^{p_1,\alpha_1}_{\hat{c}_1})_{\theta,p_0,p_1,c_0,c_1} \quad (6.21)$$

with continuous injective mapping. The lemma is proved.

Now we need a result which is (partly) an extension to weighted spaces of the reiteration theorem of Lions and Peetre [21].

We introduce the space

$$\mathbf{X}_i = (\mathbf{A}_0,\mathbf{A}_1)^*_{\theta_i,r_i,\pi_i},\quad \theta_i\in(0,1),\ 1<r_i<+\infty, \pi_i\in\mathcal{H}(r_i),\ i=0,1.$$

defined by one of the methods of Proposition 6.2. We may claim

Proposition 6.7 *Assume* $0<\theta_0<\theta<\theta_1<1$. *Then the following equality holds with equivalent norms*

$$(\mathbf{A}_0,\mathbf{A}_1)_{\theta,p_0,p_1,c_0,c_1} = (\mathbf{X}_0,\mathbf{X}_1)_{\lambda,r_0,r_1,\pi_0,\pi_1}, \quad (6.22)$$

where

$$\theta=(1-\lambda)\theta_0+\lambda\theta_1,\quad \frac{1}{r_i}=\frac{1-\theta_i}{p_0}+\frac{\theta_i}{p_1},\quad \pi_i=c_0^{1-\theta_i}c_1^{\theta_i},\quad i=0,1. \tag{6.23}$$

Remark 6.8 Actually for the relation between θ and λ in (6.23), one has $-\lambda=\frac{\theta_0-\theta}{\theta_1-\theta_0}$, $1-\lambda=\frac{\theta_1-\theta}{\theta_1-\theta_0}$, then it is more convenient, thanks to Lemma 4.6 (homogeneity) to eventually works with the space $\Sigma(r_0,\eta_0,\pi_0,\mathbf{X}_0;r_1,\eta_1,\pi_1,\mathbf{X}_1)$ where $\eta_0=\theta_0-\theta=-(\theta_1-\theta_0)\lambda,\ \eta_1=\theta_1-\theta=(1-\lambda)(\theta_1-\theta_0)$.

Proof of the Proposition 6.7 (1) *Let* $a\in(\mathbf{A}_0,\mathbf{A}_1)_{\theta,r_0,r_1,c_0,c_1}$ then we may find, $u(t)\in\mathbf{X}$ *a.e. in* t, such that $a=\int_0^{+\infty}u(t)\frac{dt}{t}$ *in* $\mathbf{Y}$, with $t^{-\theta}u(t)\in L^{p_0}_{\hat{c}_{0\#}}$, $t^{1-\theta}u(t)\in L^{p_1}_{\hat{c}_{1\#}}$ and from Corollary 4.5 (4.22) gives

$$|u(t)|_{\mathbf{X}_i}\le\kappa_i\,|u(t)|_{\mathbf{A}_1}^{1-\theta_i}\,|u(t)|_{\mathbf{A}_1}^{\theta_i}$$

so that, using Remark 6.8 and because $\eta_i=-\theta_0(1-\theta_i)+(1-\theta)\theta_i$, with $\pi_i=c_0^{1-\theta_i}c_1^{\theta_i}$, one has

$$\left|t^{\eta_i}\hat{\pi}_{i\#(t)}u(t)\right|_{\mathbf{X}_i}^{r_i}\le\kappa_i^{r_i}(|t^{-\theta}\hat{c}_{0\#}(t)u(t)|_{A0}^{p_0})^{r_i(1-\theta_i)/p_0}(|t^{(1-\theta_i}\hat{c}_{1\#}(t)u(t)|_{A_1}^{p_1})^{r_i\theta_i/p_1}$$

and Hölder's inequality gives

$$t^{\eta_i}u(t)\in L^{r_i}_{\hat{\pi}_i}(\mathbf{X}_i),\quad i=0,1, \tag{6.24}$$

thus, thanks to Remark 6.8, one obtains

$$(\mathbf{A}_0,\mathbf{A}_1)_{\theta,p_0,p_1,c_0,c_1}\subset(\mathbf{X}_0,\mathbf{X}_1)_{\lambda,r_0,r_1,\pi_0,\pi_1} \tag{6.25}$$

(2) To prove the converse of (6.25) and to avoid any measurability problem,[8] a discrete representation may be considered, based on space that involves a sum instead of an integral[9]: $\forall n\in\mathbf{Z}$, since on a interval $e^n\le s\le e^{n+1}$, one has $K(e^n;a)\le K(s;a)\le eK(e^n;a)$, and since the measure of (e^n,e^{n+1}) for Haar's measure $\frac{ds}{s}$ is 1, on setting $c_n=c(e^{-n})$, we conclude

$$a\in(\mathbf{A}_0,\mathbf{A}_1)^*_{\theta,p,c}\ \textit{ is equivalent to } c_ne^{-n\theta}K(e^n;a)\in l^p(\mathbf{Z})$$

and

$$\left|c_ne^{-n\theta}K(e^n;a)\right|_{l^p(\mathbf{Z})}\ \textit{ is an equivalent norm on }\ (\mathbf{A}_0,\mathbf{A}_1)_{a,p,c}\,. \tag{6.26}$$

The following proof uses a procedure similar to the discretisation method developed in [21].

Assume $a\in(\mathbf{X}_0,\mathbf{X}_1)_{\lambda,r_0,r_1,\pi_0,\pi_1}$. Remark 6.8 permits v_0, v_1 to be found such that

$$a=v_0(s)+v_1(s),\quad s^{\eta_i}v_i\in L^{r_i}_{\hat{\pi}_{i\#}}(\mathbf{X}_i),\ i=0,1.$$

and in general

$$a=v_{0n}+v_{1n},|\,\pi_{in}e^{\eta_in}v_{in}|_{X_i}\in l^{r_i}(\mathbf{Z}). \tag{6.27}$$

But $\mathbf{X}_i\subset(\mathbf{A}_0,\mathbf{A}_1)_{\theta,\infty,\pi_i}$, and from (6.13), $\forall t_{in}>0$, v_{i0n},v_{i1n} can be found with $v_{in}=v_{i0n}+v_{i1n}$ such that

$$|v_{i0n}|_{\mathbf{A}_0}\le\kappa_it_{in}^{\theta_i}\,|v_{in}|_{\mathbf{A}_0},\quad |v_{i1n}|_{\mathbf{A}_1}\le\kappa_it_{in}^{-(1-\theta_i)}\,|v_{in}|_{\mathbf{A}_1} \tag{6.28}$$

8 As noticed by J. Peetre.

9 See also Tartar [31].

and consequently

$$\left|c_{0n}e^{-\theta n}v_{i0n}\right|^{p_0}_{\mathbf{A}_0} \le \kappa_i^{p_0}(\tau_{in})^{\theta_i p_0}\,|\tilde{v}_{in}|^{p_0}_{\mathbf{X}_i}$$
$$\left|c_{1n}e^{(1-\theta)n}v_{i1n}\right|^{p_1}_{\mathbf{A}_1} \le \kappa_i^{p_1}(\tau_{in})^{(\theta_i-1)p_1}\,|\tilde{v}_{in}|^{p_1}_{\mathbf{X}_i}$$

where

$$\tau_{in} = t_{in}e^{-n}\frac{c_{0n}}{c_{1n}}, \quad \tilde{v}_{in} = e^{\eta_i n}\pi_{in}v_{in}.$$

Then, one can choose τ_{in} such that $(\tau_{in})^{\theta_i p_0} = |\tilde{v}_{in}|^{r_i-p_0}_{\mathbf{X}_i}$ and we can check that $(\tau_{in})^{(\theta_i-1)p_1} = |\tilde{v}_{in}|^{r_i-p_1}_{\mathbf{X}_i}$, so that, one has

$$\left|c_{0n}e^{-\theta n}v_{0in}\right|^{p_0}_{\mathbf{A}_0} \le \kappa_i^{p_0}\left|e^{\eta_1 n}\pi_{in}v_{in})\right|^{r_i}_{\mathbf{X}_i}$$
$$\left|c_{1n}e^{(1-\theta)n}v_{1in}\right|^{p_1}_{\mathbf{A}_1} \le \kappa_i^{p_1}\,|\tilde{v}_i(s)|^{r_i}_{\mathbf{X}_i}$$

thus, setting

$$w_{0n} = v_{00n} + v_{10n}, \quad w_{1n} = v_{10n} + v_{11n}$$

we have

$$\left|c_{0n}e^{-\theta n}w_{0n}\right|_{\mathbf{A}_0} \in l^{p_0}(Z), \quad \left|c_{1n}e^{(1-\theta)n}w_{1n}\right|_{\mathbf{A}_1} \in l^{p_1}(Z) \tag{6.29}$$

and because $w_{0n} + w_{1n} = v_{0n} + v_{1n} = a$, we conclude that

$$a \in (\mathbf{A}_0, \mathbf{A}_1)_{\theta,p_0,p_1,c_0,c_1}. \tag{6.30}$$

The result follows from Remark 6.8 and the definition for λ.

6.3 The main theorem

Finally we can claim

Theorem 6.9 *We assume*: $0 < \theta < 1, 1 < p_i < \infty,\ c_i \in \mathcal{H}(p_i)$, *for* $i = 0, 1$, *satisfy* (6.14). *Then*

$$(\mathbf{A}_0, \mathbf{A}_1)_{\theta,p_0,p_1,c_0,c_1} = (\mathbf{A}_0, \mathbf{A}_1)_{\theta,p_\theta,p_\theta,c_\theta,c_\theta} = (\mathbf{A}_0, \mathbf{A}_1)^*_{\theta,p_\theta,c_\theta} \tag{6.31}$$
$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad c_\theta = c_0^{1-\theta}c_1^{\theta}. \tag{6.32}$$

Proof of Theorem 6.9 (1) Assume $a \in (\mathbf{A}_0, \mathbf{A}_1)_{\theta,p_0,p_1,c_0,c_1}$. Then, thanks to Propositions 6.2 and 6.7, there is a function $u(t)$ with values $a.e.$ in $\mathbf{X}$ such that

$$a = \int_0^{+\infty} u(t)\frac{dt}{t}, \quad t^{-\lambda}\left|s^{-\theta_i}K(s; u(t)\right|_{L^{r_i}_{\hat{\pi}_{i\#}}} \in L^{r_i}_{\hat{\pi}_{i\#}} \quad i = 0, 1.$$

Since one has

$$K(s; a) \le \int_0^{+\infty} K(s; u(t))\frac{dt}{t}$$

one deduces that

$$K(s; a) \in (L^{r_0,\theta_0}_{\hat{\pi}_{0\#}}, L^{r_1,\theta_1}_{p_{1\#}})_{\lambda,r_0,r_1,p_0,p_1}$$

which from Lemma 6.6 implies

$$s^{-\theta}K(s;a)\in L^{p_\lambda}_{\hat{\pi}_{\lambda\#}}\equiv L^{p_\theta}_{\hat{c}_{\theta\#}},$$

because from the definition (6.23) of λ:

$$\frac{1}{p_\lambda}=\frac{1-\lambda}{r_0}+\frac{\lambda}{r_1}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}=\frac{1}{p_\theta},\quad \pi_\lambda=\pi_0^{1-\lambda}\pi_1^{\lambda}=c_0^{1-\theta}c_1^{\theta}=c_\theta.$$

Thus $a\in(\mathbf{A}_0,\mathbf{A}_1)^K_{\theta,p_\theta,c_\theta}$ and its follows from Proposition 6.2 that

$$(\mathbf{A}_0,\mathbf{A}_1)_{\theta,p_0,p_1,c_0,c_1}\subset(\mathbf{A}_0,\mathbf{A}_1)^*_{\theta,p_\theta,c_\theta}\,. \tag{6.33}$$

(2) Assume that $a\in(\mathbf{A}_0,\mathbf{A}_1)^*$, and consider now the *J.W-method.* Then with the help of proposition (6.2), there exists a function u taking values in $\mathbf{X}$ satisfying

$$a=\int_0^{+\infty}u()\frac{dt}{t},\quad in\ \mathbf{Y},\quad s^{-\theta}J(s;u(s)\in L^{p_\theta}_{\hat{c}_{\theta\#}}.$$

On using Lemma 6.6, we can find two positive functions $j_o(s,t)$, $j_1\ (s,t)$ with $j_0(s,t)+j_1(s,t)=J(s;u(s))$, $t^{-\lambda}\left|s^{-\theta_0}j_0(s,t)\right|_{L^{r_0}_{\hat{\pi}_{0\#}}}\in L^{r_0}_{\hat{\pi}_{0\#}}$, $t^{1-\lambda}\left|s^{-\theta_1}j_1(s,t)\right|_{L^{r_1}_{\hat{\pi}_{1\#}}}\in L^{r_1}_{\hat{\pi}_{1\#}}$. so that the functions v_i, $i=0,1$, defined by

$$v_i(t)=\int_0^{+\infty}\frac{j_i(s,t)}{J(s;u(s))}u(s)\frac{ds}{s}=\int_0^{+\infty}g_i(s,t)\frac{ds}{s},$$

satisfy

$$v_0(t)+v_1(t)=\int_0^{+\infty}u(s)\frac{ds}{s}=a. \tag{6.34}$$

Consider now the spaces $X_i=(\mathbf{A}_0,\mathbf{A}_1)^J_{\theta_i,r_i,\pi_i}$ $i=0,1$. From the definition and the hypothesis we can easily check that

$$J(s;g_i(s,t))=j_i(s,t)$$

and therefore

$$t^{-\lambda}|v_0(t)|_{X_0}\in L^{r_0}_{\hat{\pi}_{0\#}},\quad t^{1-\lambda}\,|v_1(t)|_{X_1}\in L^{r_1}_{\hat{\pi}_{1\#}}.$$

Then Proposition 6.7 gives

$$(\mathbf{A}_0,\mathbf{A}_1)^*_{\theta,p_\theta,c_\theta}\subset(\mathbf{A}_0,\mathbf{A}_1)_{\theta,p_0,p_1,c_0,c_1}\,, \tag{6.35}$$

and Theorem 6.9 is proved.

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