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A new contractive mapping principle in fuzzy metric spaces

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Abstract In this paper we introduce a new fuzzy contraction mapping and prove that such mappings have fixed point in complete fuzzy metric spaces. We give an illustrative example. The result generalizes some existing results.

Keywords Fuzzy metric spaces · $\alpha - \psi$-Contraction · α-Admissible · Continuous t-norm · Fixed point

Mathematics Subject Classification 47H25 · 54H10

1 Introduction

In this paper we make a contribution to the fuzzy fixed point theory by providing a fixed point theorem for a new type of contraction mapping in fuzzy metric spaces. The new contraction is defined with the help of two functions. We call it $\alpha - \psi$-fuzzy contraction. The motivation for such a definition is derived from a recent work of Samet et al. [14] in the context of metric spaces and also from other works following it like those noted in [6,7]. The fuzzy metric we consider here is that which is as defined in [2]. Fixed point theory in such spaces has developed quite extensively through works like [1,4,9–11] amongst other works. Particularly, fuzzy extensions of the Banach's contraction have appeared in works like [1,5,12,15]. The reason behind this development is some salient features of this space, one of which is that the

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topology is Hausdorff topology, a feature which is considered very useful for a successful development of the metric fixed point theory.

2 Mathematical preliminaries

George and Veeramani in their paper [2] introduced the following definition of fuzzy metric space. We are concerned only with this definition of fuzzy metric space.

Definition 2.1 [2] The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:

(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and
(v) $M(x, y, .) \colon (0, \infty) \longrightarrow (0, 1]$ is continuous,

where $*$ is a continuous t-norm, that is, a continuous function $* \colon [0, 1]^2 \longrightarrow [0, 1]$ such that (i)$*$ is associative and commutative, (ii) $a * 1 = a$ for all $a \in [0, 1]$, (iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$ and r with $0 < r < 1$, the open ball $B(x, t, r)$ with center $x \in X$ is defined by

$$B(x, t, r) = \{y \in X \colon M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and r with $0 < r < 1$ such that $B(x, t, r) \subset A$. Let τ denote the family of all open subsets of X. Then τ is a topology and is called the topology on X induced by the fuzzy metric M. The topology τ is a Hausdorff topology [2]. In fact the Definition 2.1 is a modification of the definition given in [8] for ensuring Hausdorff topology of the space.

Definition 2.2 [2] Let $(X, M, *)$ be a fuzzy metric space.

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.
(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each ε with $0 < \varepsilon < 1$ and $t > 0$, there exists a positive integer n_0 such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.
(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

The following lemma was proved by Grabiec [3] for fuzzy metric spaces defined by Kramosil and Michalek [8]. The proof is also applicable to the fuzzy metric space given in Definition 2.1.

Lemma 2.3 [3] *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, .)$ is non-decreasing for all $x, y \in X$.*

Lemma 2.4 [13] *M is a continuous function on $X^2 \times (0, \infty)$.*

We use the following class of functions in our theorem.

Definition 2.5 (ψ*-function*) A function $\psi : [0, \infty) \to [0, \infty)$ is said to be a ψ- function if

(i) ψ is nondecreasing and continuous,
(ii) $\Sigma_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where $\psi^{n+1}(t) = \psi(\psi^n(t)), n \geq 1$.

We denote the family of such functions by Ψ. It is clear that if $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.

The following function is an example of a ψ-function:

$$\psi(t) = \begin{cases} t - \frac{t^2}{2}, & \text{if } t \in [0, 1], \\ \frac{1}{2}, & t > 1. \end{cases}$$

Definition 2.6 Let $(X, M, *)$ be a fuzzy metric space. Let $f : X \to X$ and $\alpha : X \times X \times (0, \infty) \to (0, \infty)$ be two mappings. The mapping f is α-admissible if

$$\alpha(x, y, t) \geq 1 \Rightarrow \alpha(fx, fy, t) \geq 1 \quad \text{for all } t > 0 \text{ and } x, y \in X.$$

Definition 2.7 Let $(X, M, *)$ be a fuzzy metric space and $f : X \to X$ be a mapping. The mapping f is an $\alpha - \psi$-contractive mapping if there exist two functions $\alpha : X \times X \times (0, \infty) \to (0, \infty)$, and $\psi \in \Psi$ such that for all $t > 0$ and $x, y \in X$ we have

$$\alpha(x, y, t) \left(\frac{1}{M(fx, fy, t)} - 1 \right) \leq \psi \left(\frac{1}{M(x, y, t)} - 1 \right). \tag{2.1}$$

Remark The above definition is a generalization of the contraction introduced by Gregori and Sapena [5]. If we take $\alpha(x, y, t) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t > 0$ and $k \in (0, 1)$, then we get the following contraction

$$\left(\frac{1}{M(fx, fy, t)} - 1 \right) \leq k \left(\frac{1}{M(x, y, t)} - 1 \right), \quad \text{for all } x, y \in X \text{ and } t > 0, \tag{2.2}$$

which has been studied in [5].

In the following we prove two lemmas which we use in the proof of our main theorem in the next section.

Lemma 2.8 *If $*$ is a continuous t-norm, and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences such that $\alpha_n \to \alpha, \gamma_n \to \gamma$ as $n \to \infty$, then $\overline{\lim}_{k\to\infty}(\alpha_k * \beta_k * \gamma_n) = \alpha * \overline{\lim}_{k\to\infty} \beta_k * \gamma$ and*

$$\underline{\lim}_{k\to\infty} (\alpha_k * \beta_k * \gamma_n) = \alpha * \underline{\lim}_{k\to\infty} \beta_k * \gamma.$$

Proof There exists $\{\beta_{n(p)}\} \subset \{\beta_n\}$ such that

$$\begin{aligned} \lim_{p\to\infty} \beta_{n(p)} &= \overline{\lim}_{k\to\infty} \beta_k = \beta (\text{say}). \text{ Then} \\ \alpha * \overline{\lim}_{k\to\infty} \beta_k * \gamma &= \lim_{p\to\infty} \alpha_{n(p)} * \lim_{p\to\infty} \beta_{n(p)} * \lim_{p\to\infty} \gamma_{n(p)} \\ &= \lim_{p\to\infty} (\alpha_{n(p)} * \beta_{n(p)} * \gamma_{n(p)}) \quad (\text{by the continuity property of } *) \\ &\leq \overline{\lim}_{k\to\infty} (\alpha_k * \beta_k * \gamma_k). \end{aligned} \tag{2.3}$$

We now show that the equality in (2.3) must hold. If not, then there exists a sequence of natural $\{n(q)\}$ such that

$$\begin{aligned}\alpha * \overline{\lim_{k\to\infty}} \beta_k * \gamma < \overline{\lim_{k\to\infty}} (\alpha_k * \beta_k * \gamma_k) &< \lim_{q\to\infty} (\alpha_{n(q)} * \beta_{n(q)} * \gamma_{n(q)}),\\ &= \lim_{q\to\infty} \alpha_{n(q)} * \lim_{q\to\infty} \beta_{n(q)} * \lim_{q\to\infty} \gamma_{n(q)},\\ &= \alpha * \lim_{q\to\infty} \beta_{n(q)} * \gamma.\end{aligned}$$

By the monotone property of $*$ we have that

$$\overline{\lim_{k\to\infty}} \beta_k < \overline{\lim_{q\to\infty}} \beta_{n(q)}, \quad \text{which is contradiction.}$$

Therefore, we conclude that $\alpha * \overline{\lim}_{k\to\infty} \beta_k * \gamma = \overline{\lim}_{k\to\infty}(\alpha_k * \beta_k * \gamma_k)$.

The other part of the lemma is similarly proved.

Lemma 2.9 *Let* $\{f(k,.)\colon [0,\infty) \to [0,1], k = 0,1,2,\ldots\}$ *be a sequence of functions such that* $f(k,.)$ *is continuous and monotone increasing for each* $k \geq 0$*. Then* $\overline{\lim}_{k\to\infty} f(k,t)$ *is a left continuous function in t and* $\underline{\lim}_{k\to\infty} f(k,t)$ *is a right continuous function in t.*

Proof Let $g(n,t) = \sup_{p\geq n} f(p,t)$. Then

$$\lim_{n\to\infty} g(n,t) = \overline{\lim_{k\to\infty}} f(k,t).$$

By the conditions of the lemma the above limit exists finitely. Let $\eta > 0$ be arbitrary. We can find $p \geq n$ such that

$$f(p,t) > \sup_{p\geq n} f(p,t) - \eta = g(n,t) - \eta, \quad \text{that is, } g(n,t) < \eta + f(p,t)$$

Since each $f(k,.)$ is monotone increasing for each k, $g(n,.)$ is also monotone increasing for each n. Then

$$\begin{aligned}g(n,t) - g(n,t-\eta) &\leq \eta + f(p,t) - \sup_{p\geq n} f(p,t-\eta)\\ &\leq \eta + f(p,t) - f(p,t-\eta)\\ &\leq \eta + \eta. \sup_{s\in[t-\eta,t]} f(p,s)\\ &\leq \eta + \eta.1 \quad \text{(since the range of f is within [0, 1])}\\ &= 2\eta.\end{aligned}$$

Taking $n \to \infty$ in the above inequality,

$$\begin{aligned}\lim_{n\to\infty} g(n,t) - \lim_{n\to\infty} g(n,t-\eta) &= \overline{\lim_{k\to\infty}} f(k,t) - \overline{\lim_{k\to\infty}} f(k,t-\eta)\\ &\leq 2\eta \to 0 \quad \text{as } \eta \to \infty.\end{aligned}$$

This establishes that $\overline{\lim}_{k\to\infty} f(k,t)$ is left continuous in t.

The other part of the lemma, that is, $\overline{\lim}_{k\to\infty} f(k,t)$ a right continuous function is similarly established.

We denote $O(x) = \{x, fx, f^2x, \ldots\}$

3 Main results

Theorem 3.1 *Let $(X, M, *)$ be a complete fuzzy metric space and let $f : X \to X$ be a $\alpha - \psi$-contractive mapping which satisfies the following conditions*:

(i) *f is α- admissible,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x, y, t) \geq 1$ for all $t > 0$ whenever $x, y \in O(x_0)$,*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \geq 1$ and for all $t > 0$, and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, t) \geq 1$ for all $n \geq 1$ and for all $t > 0$.*

Then the mapping f has a fixed point.

Proof By an assumption of the theorem there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ for all $t > 0$. We now construct a sequence $\{x_n\}$ in X as follows:
$x_1 = fx_0, x_2 = fx_1, x_3 = fx_2, \ldots,$ and, in general, for all $n \geq 1$,

$$x_n = fx_{n-1}. \tag{3.1}$$

Since f is α- admissible, for all $t > 0$, we have

$$\alpha(x_0, fx_0, t) = \alpha(x_0, x_1, t) \geq 1 \Rightarrow \alpha(fx_0, fx_1, t) = \alpha(x_1, x_2, t) \geq 1.$$

Again, for all $t > 0$, we have

$$\alpha(x_1, fx_1, t) = \alpha(x_1, x_2, t) \geq 1 \Rightarrow \alpha(fx_1, fx_2, t) = \alpha(x_2, x_3, t) \geq 1.$$

By continuing this above process, for all $t > 0$, we have

$$\alpha(x_n, x_{n+1}, t) \geq 1 \quad \text{for all } n \geq 1.$$

Now, for all $t > 0$, we have

$$\begin{aligned}
\left(\frac{1}{M(x_1, x_2, t)} - 1\right) &= \left(\frac{1}{M(fx_0, fx_1, t)} - 1\right) \\
&\leq \alpha(x_0, x_1, t)\left(\frac{1}{M(fx_0, fx_1, t)} - 1\right), \quad \text{since } [\alpha(x_0, x_1, t) \geq 1] \\
&\leq \psi\left(\frac{1}{M(x_0, x_1, t)} - 1\right) \text{[by (2.1)]}.
\end{aligned}$$

Again, for all $t > 0$, we obtain

$$\begin{aligned}
\left(\frac{1}{M(x_2, x_3, t)} - 1\right) &= \left(\frac{1}{M(fx_1, fx_2, t)} - 1\right) \\
&\leq \alpha(x_1, x_2, t)\left(\frac{1}{M(fx_1, fx_2, t)} - 1\right), \quad \text{since } [\alpha(x_1, x_2, t) \geq 1] \\
&\leq \psi\left(\frac{1}{M(x_1, x_2, t)} - 1\right) \text{[by (2.1)]} \\
&\leq \psi^2\left(\frac{1}{M(x_0, x_1, t)} - 1\right) \quad \text{(since } \psi \text{ in non-decreasing)}.
\end{aligned}$$

Repeating the above procedure, for all $t > 0$, we have

$$\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) \leq \psi^n\left(\frac{1}{M(x_0, x_1, t)} - 1\right).$$

Taking $n \to \infty$ in the above inequality, for all $t > 0$, we obtain

$$\lim_{n\to\infty}\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) \leq \lim_{n\to\infty}\psi^n\left(\frac{1}{M(x_0, x_1, t)} - 1\right) \to 0$$
$$\text{as } n \to \infty \text{ (by a property of } \psi).$$

That is,

$$\lim_{n\to\infty}\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) = 0.$$

Then, for all $t > 0$, we obtain

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1. \tag{3.2}$$

Next we show that $\{x_n\}$ is a Cauchy sequence in X. We suppose, if possible, that $\{x_n\}$ is not a Cauchy sequence in X. Then there exists some $\epsilon > 0$ and some λ with $0 < \lambda < 1$, for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with

$$n(k) > m(k) > k \tag{3.3}$$

such that

$$M(x_{m(k)}, x_{n(k)}, \epsilon) \leq (1 - \lambda), \tag{3.4}$$

for all positive integer k.

We may choose the $n(k)$ as the smallest integer exceeding $m(k)$ for which (3.4) holds. Then, for all positive integer k,

$$M(x_{m(k)}, x_{n(k)-1}, \epsilon) > (1 - \lambda) \tag{3.5}$$

Then, for all $k \geq 1, 0 < s < \frac{\epsilon}{2}$, we obtain,

$$\begin{aligned}(1 - \lambda) &\geq M(x_{m(k)}, x_{n(k)}, \epsilon)\\ &\geq M(x_{m(k)}, x_{m(k)-1}, s) * M(x_{m(k)-1}, x_{n(k)-1}, \epsilon - 2s)\\ &\quad *M(x_{n(k)-1}, x_{n(k)}, s).\end{aligned} \tag{3.6}$$

Let,

$$h_1(t) = \overline{\lim_{k\to\infty}} M(x_{m(k)-1}, x_{n(k)-1}, t), \quad t > 0. \tag{3.7}$$

Taking limit supremum on both sides of (3.6), using (3.2), and the properties of M and $*$, by Lemma 2.8, we obtain

$$(1 - \lambda) \geq 1 * \overline{\lim_{k\to\infty}} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon - 2s) * 1 = h_1(\epsilon - 2s). \tag{3.8}$$

Since M is bounded with range in [0, 1], continuous and, by Lemma 2.3, monotone increasing in the third variable t, it follows by an application of Lemma 2.9 that h_1, as given in (3.7) is continuous from the left.

Letting $s \to 0$ in (3.8), we obtain

$$\overline{\lim_{k\to\infty}} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) \leq (1 - \lambda). \tag{3.9}$$

Let,

$$h_2(t) = \underline{\lim_{k\to\infty}} M(x_{m(k)-1}, x_{n(k)-1}, t), \quad t > 0. \tag{3.10}$$

Again, for all $k \geq 1, s > 0$,

$$\begin{aligned} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon + s) &\geq M(x_{m(k)-1}, x_{m(k)}, s) * M(x_{m(k)}, x_{n(k)-1}, \epsilon) \\ &\geq M(x_{m(k)-1}, x_{m(k)}, s) * (1 - \lambda), \quad \text{[by (3.5)]}. \end{aligned} \tag{3.11}$$

Taking limit infimum as $k \to \infty$ in (3.11), by virtue of (3.2), we obtain

$$\begin{aligned} h_2(\epsilon + s) &= \underline{\lim}_{k \to \infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon + s) \\ &\geq \underline{\lim}_{k \to \infty} M(x_{m(k)-1}, x_{m(k)}, s) * (1 - \lambda) \\ &= 1 * (1 - \lambda) = (1 - \lambda). \end{aligned} \tag{3.12}$$

Since M is bounded with range in [0,1], continuous and, by Lemma 2.3, monotone increasing in the third variable t, it follows by an application of Lemma 2.9 that h_2, as given in (3.10) is continuous from the right.

Taking $s \to 0$ in the above inequality (3.12), we obtain

$$\underline{\lim}_{k \to \infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) \geq (1 - \lambda). \tag{3.13}$$

The inequalities (3.9) and (3.13) jointly imply that

$$\lim_{k \to \infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) = (1 - \lambda). \tag{3.14}$$

Again by (3.4),

$$\overline{\lim}_{k \to \infty} M(x_{m(k)}, x_{n(k)}, \epsilon) \leq (1 - \lambda). \tag{3.15}$$

Also for all $k \geq 1, s > 0$, we obtain

$$\begin{aligned} M(x_{m(k)}, x_{n(k)}, \epsilon + 2s) &\geq M(x_{m(k)}, x_{m(k)-1}, s) * M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) \\ &\quad * M(x_{n(k)-1}, x_{n(k)}, s) \end{aligned}$$

Taking limit infimum as $k \to \infty$ in the above inequality, using (3.2), (3.14) and the properties of M and $*$, by Lemma 2.8, we obtain

$$\underline{\lim}_{k \to \infty} M(x_{m(k)}, x_{n(k)}, \epsilon + 2s) \geq 1 * \underline{\lim}_{k \to \infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) * 1 = 1 - \lambda.$$

Since M is bounded with range in [0, 1], continuous and, by Lemma 2.3, monotone increasing in the third variable t, it follows by an application of Lemma 2.9 that $\underline{\lim}_{k \to \infty} M(x_{m(k)}, x_{n(k)}, t)$ is continuous function of t from the right.

Taking $s \to 0$ in the above inequality, and using Lemma 2.9, we obtain

$$\underline{\lim}_{k \to \infty} M(x_{m(k)}, x_{n(k)}, \epsilon) \geq (1 - \lambda), \tag{3.16}$$

Combining (3.15) and (3.16), we obtain

$$\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}, \epsilon) = (1-\lambda) \tag{3.17}$$

$$\begin{aligned}\left(\frac{1}{M(x_{m(k)}, x_{n(k)}, \epsilon)} - 1\right) &= \left(\frac{1}{M(fx_{m(k)-1}, fx_{n(k)-1}, \epsilon)} - 1\right)\\ &\leq \alpha(x_{m(k)-1}, x_{n(k)-1}, \epsilon)\left(\frac{1}{M(fx_{m(k)-1}, fx_{n(k)-1}, \epsilon)} - 1\right)\\ &\quad \text{(by condition (ii) of Theorem 3.1)}\\ &\leq \psi\left(\frac{1}{M(fx_{m(k)-1}, fx_{n(k)-1}, \epsilon)} - 1\right), \quad \text{by (2.1).}\end{aligned}$$

Taking $k \to \infty$ in the above inequality, we have

$$\left(\frac{1}{\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}, \epsilon)} - 1\right) \leq \psi\left(\frac{1}{\lim_{k\to\infty} M(fx_{m(k)-1}, fx_{n(k)-1}, \epsilon)} - 1\right).$$
$$\text{(since } \psi \text{ is continuous)}$$

Using (3.14) and (3.17), we have

$$\left(\frac{1}{1-\lambda} - 1\right) \leq \psi\left(\frac{1}{1-\lambda} - 1\right) < \left(\frac{1}{1-\lambda} - 1\right),$$

which is a contradiction.

Thus it is established that $\{x_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is complete, there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x. \tag{3.18}$$

Next, we show that x is a fixed point of f. Now, for all $t > 0$

$$\begin{aligned}\left(\frac{1}{M(x_{n+1}, fx, t)} - 1\right) &= \left(\frac{1}{M(fx_n, fx, t)} - 1\right)\\ &\leq \alpha(x_n, x, t)\left(\frac{1}{M(fx_n, fx, t)} - 1\right)\\ &\quad \text{[by condition (iii) of Theorem 3.1]}\\ &\leq \psi\left(\frac{1}{M(x_n, x, t)} - 1\right) \quad \text{[by (2.1)].}\end{aligned}$$

Taking $n \to \infty$ in the above inequality, for all $t > 0$, we have

$$\begin{aligned}\left(\frac{1}{\lim_{n\to\infty} M(x_{n+1}, fx, t)} - 1\right) &\leq \psi\left(\frac{1}{\lim_{k\to\infty} M(x_n, x, t)} - 1\right)\\ &= \psi(1-1)\\ &= \psi(0)\\ &= 0 \quad \text{(by the properties of } \psi\text{),}\end{aligned}$$

which implies that $\lim_{n\to\infty} M(x_{n+1}, fx, t) = 1$. Since M is continuous, in view of (3.18), we have $M(x, fx, t) = 1$, which implies that $fx = x$, that is, f has fixed point.

Example 3.2 Let $X = [0, \infty)$, for all $t > 0$, $M(x, y, t) = e^{-\frac{|x-y|}{t}}$ where $x, y \in X$ and $a * b = \min\{a, b\}$. Then $(X, M, *)$ is a complete fuzzy metric space. Let the mapping $f : X \to X$ be defined as follows:

$$fx = \begin{cases} \frac{x}{4}, & \text{if } x \in [0, 1], \\ 4, & \text{otherwise}, \end{cases}$$

and the mapping $\alpha : X \times X \times (0, \infty) \to (0, \infty)$ by

$$\alpha(x, y, t) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise}, \end{cases}$$

for all $x, y \in X$ and $\psi(t) = \frac{3}{4}t$.

Now let $x, y \in X$ be such that $\alpha(x, y, t) \geq 1$ for all $t > 0$. It then follows that $x, y \in [0, 1]$. Then by using the definition of f and α, we have $fx = \frac{x}{4} \in [0, 1]$, $fy = \frac{y}{4} \in [0, 1]$ and then $\alpha(fx, fy, t) = 1$ for all $t > 0$, which implies that f is α- admissible.

With any $x_0 \in [0, 1]$ we see that condition (ii) of Theorem 3.1 is satisfied. Also it is obvious that condition (iii) is also satisfied.

Now let at least one of x and y is not in [0, 1], then $\alpha(fx, fy, t) = 0$ and holds trivially. If x and y both are in [0, 1], then $\alpha(fx, fy, t) = 1$ and the inequality (2.1) holds. Then, by an application of Theorem 3.1, f has at least one fixed point. Here f has two fixed points 0 and 4.

Note It may be noted that the contraction of Gregori and Sapena (2.2) is not satisfied for given $0 < k < 1$. To see this we take $x = 1$ and $y = 1 + \frac{1}{n}$. Then for all $t > 0$, we have

$$M(x, y, t) = e^{-\frac{1}{nt}}, \quad \text{that is,} \quad \frac{1}{M(x, y, t)} - 1 = e^{\frac{1}{nt}} - 1$$

and

$$M(fx, fy, t) = e^{-\frac{\frac{1}{4}-4}{t}}, \quad \text{that is,} \quad \frac{1}{M(fx, fy, t) - 1} = e^{\frac{3.75}{t}} - 1.$$

In order that (2.2) is satisfied for fixed $0 < k < 1$, we must have, for $t > 0$,

$$\frac{e^{\frac{3.75}{t}} - 1}{e^{\frac{1}{nt}} - 1} \leq k.$$

But taking n sufficiently large, we see that the above inequality is violated. This shows that the contraction in the Theorem 3.1 is more generalized than the contraction of Gregori and Sapena [5]. Our theorem thus in an actual improvement over the result in [5].

Conclusion and open problem The inequality (2.1) can lead to a new metric inequality if we consider the fuzzy metric space as induced by a metric in the usual way. In that case we can have new fixed point results if we proceed similarly as in our theorems. The idea of the contraction introduced here can be extended to the case of more than one mappings. Also coupled contractions can be introduced following the same line.

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