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Convergence results related to the modified Kawahara equation

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Abstract We consider the modified Kawahara equation, which contains nonlinear dispersive effects. We prove that as the diffusion parameter tends to zero, the solutions of the dispersive equation converge to the discontinuous weak solutions of the scalar conservation law. The proof relies on deriving suitable a priori estimates together with an application of the compensated compactness method in the L^p setting.

Keywords Singular limit · Compensated compactness · Modified Kawahara equation · Entropy condition

Mathematics Subject Classification 35G25 · 35L65 · 35L05

1 Introduction

The evolution equation

$$\partial_t u + au^2 \partial_x u + c\partial_{xxxxx}^5 u = 0, \quad a, \ c \in \mathbb{R}, \tag{1.1}$$

is known as the modified Kawahara equation. It appears in the study of water waves with surface tension, in which the Bona number takes on the critical value, where the Bona number

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represents a dismension-less magnitude of surface tension in the shallow water region (see [3,14,16]).

To obtain the exact solutions, a number of methods have been proposed in the literature. Some of them are the solitary wave ansatz method, the inverse scattering, Hirotas bilinear method, homogeneous balance method, Lie group analysis, etc. Among the above, the Lie group analysis method, which is also called the symmetry method, is one of the most effective to determine solutions of nonlinear partial differential equations [15].

For example, in [2], the authors use Lie group analysis to obtain some exact solutions for (1.1), the Kawahara equation

$$\partial_t u + au\partial_x u + c\partial^5_{xxxxx} u = 0, \tag{1.2}$$

the Kawahara–Korteweg–de Vries equation

$$\partial_t u + au\partial_x u + b\partial^3_{xxx} u + c\partial^5_{xxxxx} u = 0, \tag{1.3}$$

the modified Kawahara–Korteweg–de Vries equation

$$\partial_t u + au^2\partial_x u + b\partial^3_{xxx} u + c\partial^5_{xxxxx} u = 0, \tag{1.4}$$

and, the Rosenau–Kawahara equation

$$\partial_t u + au\partial_x u + b\partial^3_{xxx} u + c\partial^5_{xxxxx} u + d\partial^5_{txxxx} u = 0, \tag{1.5}$$

where $u := u(t, x)$ is a real function, and $a,\ b,\ c,\ d \in \mathbb{R}$ are constants.

These equations occur in the theory of magneto-acoustic waves in plasmas and propagation of nonlinear water-waves in the long-wave length region as in the case of Korteweg–de Vries equation. Moreover, Eq. (1.2) is a model for small-amplitude gravity capillary waves on water of finite depth when the Weber number is close to $\frac{1}{3}$ (see [21]).

In [14], the author deduced (1.2) and (1.3) as a model for one-dimensional propagation of small-amplitude long waves in various problems of fluid dynamics and plasma physics. Moreover, Eq. (1.3) is known as the fifth-order Korteweg–de Vries equation, or the generalized Benney–Lin equation (see [1]).

In [18], the exp-function method has been used to find some exact solution for (1.5).

In [19], the authors proved that the solution of (1.2) converges to the solution of the Korteweg–de Vries equation

$$\partial_t u + au\partial_x u + b\partial^3_{xxx} u = 0. \tag{1.6}$$

Let u be a solution of (1.2) or (1.3), following [12], we consider the function

$$u_\beta(t, x) = u(\sqrt{\beta}\, t, \sqrt{\beta}\, x)$$

and study the behavior of u_β as $\beta \to 0$. Since u solves (1.2) or (1.3), we obtain the following equations for u_β

$$\partial_t u + au\partial_x u + c\beta^2\partial^5_{xxxxx} u = 0, \tag{1.7}$$

$$\partial_t u + au\partial_x u + b\beta\partial^3_{xxx} u + c\beta^2\partial^5_{xxxxx} u = 0. \tag{1.8}$$

In [6,7], the authors proved that the solutions of (1.7), and (1.8) converge to the weak solutions of the Burgers equation

$$\partial_t u + \partial_x u^2 = 0. \tag{1.9}$$

We consider (1.5), assume $a = 2,\ b,\ c = 0,\ d = 1$, re-scale as in [12], and get

$$\partial_t u + 2u\partial_x u + \beta^2 \partial^5_{txxxx} u = 0, \tag{1.10}$$

which is known as the Rosenau equation (see [23,24]). The existence and the uniqueness of solutions for (1.10) has been proved in [22]. In [5] the authors proved the convergence of the solution of (1.10) to the unique entropy solution. The same convergence result has been proven for the Rosenau–Korteweg–de Vries equation (see [4])

$$\partial_t u + 2u\partial_x u - \beta \partial^3_{xxx} u + \beta^2 \partial^5_{txxxx} u = 0. \tag{1.11}$$

In this paper, we consider (1.1). In particular, we consider the following equations

$$\partial_t u + \partial_x u^3 + \partial^5_{xxxxx} u = 0, \tag{1.12}$$

$$\partial_t u - \partial_x u^3 + \partial^5_{xxxxx} u = 0. \tag{1.13}$$

Arguing as in [12], we re-scale the equations as follows

$$\partial_t u + \partial_x u^3 + \beta^2 \partial^5_{xxxxx} u = 0, \tag{1.14}$$

$$\partial_t u - \partial_x u^3 + \beta^2 \partial^5_{xxxxx} u = 0. \tag{1.15}$$

Moreover, we augment (1.14) and (1.15) with the initial condition

$$u(0, x) = u_0(x),$$

on which we assume that

$$u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}). \tag{1.16}$$

We are interested in the no high frequency limit, therefore we send $\beta \to 0$ in (1.14)–(1.15), and obtain the scalar conservation laws

$$\partial_t u + \partial_x u^3 = 0, \tag{1.17}$$
$$\partial_t u - \partial_x u^3 = 0, \tag{1.18}$$

respectively. In particular as $\beta \to 0$, we have the converge to the entropy weak solutions of (1.17) and (1.18).

The argument behind the convergence of (1.14)–(1.17) and of (1.15)–(1.18) cannot be same because the two fifth order equations (1.14) and (1.15) experience different conserved quantities. On one side we have that both of them preserve the L^2 norm of the solution, indeed multiplying both equations by u and integrating over $\mathbb{R}$ we gain

$$\frac{d}{dt}\int_{\mathbb{R}} u^2 dx = 0.$$

On the other side only (1.14) preserves the L^4 norm of the solutions, indeed multiplying (1.14) by $u^3 + \beta^2 \partial^4_{xxxx} u$ and integrating over $\mathbb{R}$ we gain

$$\frac{d}{dt}\int_{\mathbb{R}} \left(\frac{u^4}{4} + \beta^2 \frac{\partial^2_{xx} u^2}{2} \right) dx = 0.$$

As a consequence, inspired by [8], we consider the following approximation of (1.14)

$$\partial_t u + \partial_x u^3 + \beta^2 \partial^5_{xxxxx} u = \varepsilon \partial^2_{xx} u - \beta^{\frac{3}{2}} \varepsilon \partial^4_{xxxx} u$$

and prove that as $\beta = o(\varepsilon^4)$ the solutions converge to the unique entropy solutions of (1.17). On the other side we approximate (1.15) with (see [6])

$$\partial_t u - \partial_x u^3 + \beta^2 \partial^5_{xxxxx} u = \varepsilon \partial^2_{xx} u - \beta\varepsilon \partial^4_{xxxx} u$$

and prove that as $\beta = \mathcal{O}(\varepsilon^6)$ the solutions converge to the unique entropy solutions of (1.18).

The same arguments and results hold for the modified Kawahara–Korteweg–de Vries equation (1.3), see [7].

The manuscript is organized as follows. In Sect. 2, we study the convergence of (1.14)–(1.17) and in Sect. 3 the one of (1.15)–(1.18).

2 The case $f(u) = u^3$

In this section, we consider (1.14). We augment (1.14) with the initial condition

$$u(0, x) = u_0(x), \tag{2.1}$$

on which we assume that (1.16) holds. Observe that if $\beta \to 0$, we have (1.17).

We study the dispersion-diffusion limit for (1.14). Therefore, following [8], we fix two small numbers ε, β and consider the following fifth order approximation

$$\begin{cases} \partial_t u_{\varepsilon,\beta} + \partial_x u^3_{\varepsilon,\beta} + \beta^2 \partial^5_{xxxxx} u_{\varepsilon,\beta} = \varepsilon \partial^2_{xx} u_{\varepsilon,\beta} - \beta^{\frac{3}{2}} \varepsilon \partial^4_{xxxx} u_{\varepsilon,\beta}, & t > 0,\ x \in \mathbb{R}, \\ u_{\varepsilon,\beta}(0, x) = u_{\varepsilon,\beta,0}(x), & x \in \mathbb{R}, \end{cases} \tag{2.2}$$

where $u_{\varepsilon,\beta,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned} & u_{\varepsilon,\beta,0} \to u_0 \quad \text{in } L^p_{loc}(\mathbb{R}),\ 1 \le p < 4, \text{ as } \varepsilon,\ \beta \to 0, \\ & \|u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \|u_{\varepsilon,\beta,0}\|^4_{L^4(\mathbb{R})} \le C_0, \quad \varepsilon, \beta > 0, \\ & \beta \|\partial_x u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta^2 \|\partial^2_{xx} u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \le C_0, \quad \varepsilon, \beta > 0, \end{aligned} \tag{2.3}$$

and C_0 is a constant independent on ε and β.

The main result of this section is the following theorem.

Theorem 2.1 *Assume that* (1.16) *and* (2.3) *hold. Fix $T > 0$, if*

$$\beta = \mathcal{O}(\varepsilon^4), \tag{2.4}$$

then, there exist two sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, with $\varepsilon_n, \beta_n \to 0$, and a limit function

$$u \in L^\infty((0, T); L^2(\mathbb{R}) \cap L^4(\mathbb{R})),$$

such that

(i) *$u_{\varepsilon_n,\beta_n} \to u$ strongly in $L^p_{loc}(\mathbb{R}^+ \times \mathbb{R})$, for each $1 \le p < 4$,*
(ii) *u is a distributional solution of* (1.17).
Moreover, if

$$\beta = o(\varepsilon^4), \tag{2.5}$$

(iii) *u is the unique entropy solution of* (1.17).

Let us prove some a priori estimates on $u_{\varepsilon,\beta}$, denoting with C_0 the constants which depend only on the initial data.

Lemma 2.1 *For each $t > 0$,*

$$\|u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\int_0^t \|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^{\frac{3}{2}}\varepsilon\int_0^t \|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} ds \le C_0. \tag{2.6}$$

Proof Multiplying (2.2) by $2u_{\varepsilon,\beta}$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\|u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} &= 2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta} dx\\ &= -6\int_{\mathbb{R}} u^3_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta} dx - 2\beta^2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial^5_{xxxxx} u_{\varepsilon,\beta} dx\\ &\quad + 2\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial^2_{xx} u_{\varepsilon,\beta} - 2\beta^{\frac{3}{2}}\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial^4_{xxxx} u_{\varepsilon,\beta} dx\\ &= -2\varepsilon\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\beta^{\frac{3}{2}}\varepsilon\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},\end{aligned}$$

that is

$$\frac{d}{dt}\|u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^{\frac{3}{2}}\varepsilon\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} = 0. \tag{2.7}$$

Integrating (2.7) on $(0, t)$, from (2.3), we have (2.6). □

Lemma 2.2 *Fix $T > 0$. Assume (2.4) holds. There exists $C_0 > 0$, independent on ε, β such that*

$$\|u_{\varepsilon,\beta}\|_{L^\infty((0,T)\times\mathbb{R})} \le C_0\beta^{-\frac{1}{2}}. \tag{2.8}$$

Moreover,

$$\begin{aligned}&\beta^2\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\varepsilon\int_0^t \|\partial^2_{xx} u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})} ds\\ &\quad + 2\beta^{\frac{7}{2}}\varepsilon\int_0^t \|\partial^3_{xxx} u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \le C_0.\end{aligned} \tag{2.9}$$

Proof Let $0 < t < T$. Multiplying (2.2) by $-\beta\partial^2_{xx}u_{\varepsilon,\beta}$, we have

$$\begin{aligned}&-\beta\partial^2_{xx}u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta} - 3\beta u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta} - \beta^3\partial^2_{xx}u_{\varepsilon,\beta}\partial^5_{xxxxx}u_{\varepsilon,\beta}\\ &\quad = -\beta\varepsilon(\partial^2_{xx}u_{\varepsilon,\beta})^2 + \beta^{\frac{5}{2}}\varepsilon\partial^2_{xx}u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta}.\end{aligned} \tag{2.10}$$

Since

$$\begin{aligned}-\beta\int_{\mathbb{R}}\partial^2_{xx}u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx &= \frac{\beta}{2}\frac{d}{dt}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ -\beta^3\int_{\mathbb{R}}\partial^2_{xx}u_{\varepsilon,\beta}\partial^5_{xxxxx}u_{\varepsilon,\beta}dx &= \beta^3\int_{\mathbb{R}}\partial^3_{xxx}u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta}dx = 0,\\ \beta^{\frac{5}{2}}\varepsilon\int_{\mathbb{R}}\partial^2_{xx}u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta}dx &= -\beta^{\frac{5}{2}}\varepsilon\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},\end{aligned}$$

integrating (2.10) on $\mathbb{R}$, we get

$$\begin{aligned}&\frac{\beta}{2}\frac{d}{dt}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta\varepsilon\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\quad + \beta^{\frac{5}{2}}\varepsilon\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} = 3\beta\int_{\mathbb{R}}u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}dx.\end{aligned} \tag{2.11}$$

Due to (2.4) and the Young inequality,

$$
\begin{aligned}
3\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}^2|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx &= 3\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}^2\left|\frac{\partial_x u_{\varepsilon,\beta}}{\beta^{\frac{1}{4}}\varepsilon^{\frac{1}{2}}}\right|\left|\beta^{\frac{1}{4}}\varepsilon^{\frac{1}{2}}\partial_{xx}^2 u_{\varepsilon,\beta}\right|dx\\
&\le \frac{3\beta^{\frac{1}{2}}}{2\varepsilon}\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad+\frac{3\beta^{\frac{3}{2}}\varepsilon}{2}\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C_0\varepsilon\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad+\frac{3\beta^{\frac{3}{2}}\varepsilon}{2}\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}\tag{2.12}
$$

It follows from (2.11) and (2.12) that

$$
\begin{aligned}
&\frac{\beta}{2}\frac{d}{dt}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta\varepsilon\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad+\beta^{\frac{5}{2}}\varepsilon\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C_0\varepsilon\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad+\frac{3\beta^{\frac{3}{2}}\varepsilon}{2}\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Integrating on $(0,t)$, from (2.3) and (2.6), we have

$$
\begin{aligned}
&\frac{\beta}{2}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta\varepsilon\int_0^t\|\partial_{xx}^2 u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\quad+\beta^{\frac{5}{2}}\varepsilon\int_0^t\|\partial_{xxx}^3 u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\le C_0\varepsilon\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\|\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\quad+\frac{3\beta^{\frac{3}{2}}\varepsilon}{2}\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\|\partial_{xx}^2 u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\le C_0(1+\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}).
\end{aligned}\tag{2.13}
$$

We prove (2.8). Due to (2.6), (2.13) and the Hölder inequality,

$$
\begin{aligned}
u_{\varepsilon,\beta}^2(t,x)&=2\int_{-\infty}^x u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx\le 2\int_{\mathbb{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}|dx\\
&\le 2\|u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}\\
&\le\frac{C_0}{\beta^{\frac{1}{2}}}\sqrt{(1+\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})})}.
\end{aligned}
$$

Hence,

$$
\|u_{\varepsilon,\beta}\|^4_{L^\infty((0,T)\times\mathbb{R})}\le\frac{C_0}{\beta}(1+\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}).
$$

Arguing as in [10, Lemma 2.1], we get (2.8).

Finally, we prove (2.9). It follows from (2.8) and (2.13) that

$$\frac{\beta}{2}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta\varepsilon\int_0^t\|\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds$$
$$+\beta^{\frac{5}{2}}\varepsilon\int_0^t\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\le C_0\beta^{-1},$$

which gives (2.9). □

Lemma 2.3 *Fix $T>0$. Assume (2.4). Then,*

(i) *the family $\{u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ is bounded in $L^\infty((0,T);L^4(\mathbb{R}))$;*
(ii) *the family $\{\beta\partial^2_{xx}u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ is bounded in $L^\infty((0,T);L^2(\mathbb{R}))$;*
(iii) *the families $\{\beta\varepsilon^{\frac{1}{2}}\partial^3_{xxx}u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta^{\frac{7}{4}}\varepsilon^{\frac{1}{2}}\partial^4_{xxxx}u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\varepsilon^{\frac{1}{2}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta^{\frac{3}{4}}\varepsilon^{\frac{1}{2}}u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ are bounded in $L^2((0,T)\times\mathbb{R})$).*

Proof Let $0<t<T$. Multiplying (2.2) by

$$u^3_{\varepsilon,\beta}+\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta},$$

we have

$$\begin{aligned}&(u^3_{\varepsilon,\beta}+\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta})\partial_t u_{\varepsilon,\beta}+3(u^3_{\varepsilon,\beta}+\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta})u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\\&\quad+\beta^2(u^3_{\varepsilon,\beta}+\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta})\partial^5_{xxxxx}u_{\varepsilon,\beta}\\&=\varepsilon(u^3_{\varepsilon,\beta}+\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta}-\beta^{\frac{3}{2}}\varepsilon(u^3_{\varepsilon,\beta}\\&\quad+\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta})\partial^4_{xxxx}u_{\varepsilon,\beta}.\end{aligned}\tag{2.14}$$

Since

$$\int_{\mathbb{R}}(u^3_{\varepsilon,\beta}+\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta})\partial_t u_{\varepsilon,\beta}dx=\frac{d}{dt}\left(\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})}+\beta^2\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right),$$
$$3\int_{\mathbb{R}}(u^3_{\varepsilon,\beta}+\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta})u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx=3\beta^2\int_{\mathbb{R}}u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta}dx,$$
$$\beta^2\int_{\mathbb{R}}(u^3_{\varepsilon,\beta}+\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta})\partial^5_{xxxxx}u_{\varepsilon,\beta}dx=-3\beta^2\int_{\mathbb{R}}u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta}dx,$$
$$\begin{aligned}\varepsilon\int_{\mathbb{R}}(u^3_{\varepsilon,\beta}+\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta}dx&=-3\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\quad-\beta^2\varepsilon\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},\end{aligned}$$
$$\begin{aligned}-\beta^{\frac{3}{2}}\varepsilon\int_{\mathbb{R}}(u^3_{\varepsilon,\beta}+\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta})\partial^4_{xxxx}u_{\varepsilon,\beta}dx&=3\beta^{\frac{3}{2}}\varepsilon\int_{\mathbb{R}}u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}dx\\&\quad-\beta^{\frac{7}{2}}\varepsilon\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},\end{aligned}$$

from an integration of (2.14) on $\mathbb{R}$, we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})}+\beta^2\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\&\quad+3\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\varepsilon\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\quad+\beta^{\frac{7}{2}}\varepsilon\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}=3\beta^{\frac{3}{2}}\varepsilon\int_{\mathbb{R}}u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}dx.\end{aligned}\tag{2.15}$$

Observe that

$$
\begin{aligned}
3\beta^{\frac{3}{2}}\varepsilon\int_{\mathbb{R}}u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\partial_{xxx}^3 u_{\varepsilon,\beta}dx &= -6\beta^{\frac{3}{2}}\varepsilon\int_{\mathbb{R}}u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2\partial_{xx}^2 u_{\varepsilon,\beta}dx\\
&\quad - 3\beta^{\frac{3}{2}}\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&= 2\beta^{\frac{3}{2}}\varepsilon\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx - 3\beta^{\frac{3}{2}}\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, it follows from (2.15) that

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})}+\beta^2\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\quad + 3\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2\varepsilon}{2}\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad + \beta^{\frac{7}{2}}\varepsilon\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+3\beta^{\frac{3}{2}}\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le 2\beta^{\frac{3}{2}}\varepsilon\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx.
\end{aligned}
\tag{2.16}
$$

Coclite and Karlsen [13, Lemma 4.2] says that

$$
\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx \le c_1\int_{\mathbb{R}}u_{\varepsilon,\beta}^2dx\int_{\mathbb{R}}(\partial_{xx}^2 u_{\varepsilon,\beta})^2dx, \tag{2.17}
$$

for some constant $c_1 > 0$. Hence, from (2.6), and (2.17), we have

$$
2\beta^{\frac{3}{2}}\varepsilon\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx \le C_0\beta^{\frac{3}{2}}\varepsilon\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.
$$

Therefore, from (2.16), we gain

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})}+\beta^2\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\quad + 3\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2\varepsilon}{2}\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad + \beta^{\frac{7}{2}}\varepsilon\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+3\beta^{\frac{3}{2}}\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C_0\beta^{\frac{3}{2}}\varepsilon\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (2.3), (2.6), and an integration on $\mathbb{R}$ that

$$
\begin{aligned}
&\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})}+\beta^2\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad + 3\varepsilon\int_0^t\|u_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds+\beta^2\varepsilon\int_0^t\|\partial_{xxx}^3 u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\quad + \beta^{\frac{7}{2}}\varepsilon\int_0^t\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds+3\beta^{\frac{3}{2}}\varepsilon\int_0^t\|u_{\varepsilon,\beta}(s,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\le C_0\beta^{\frac{3}{2}}\varepsilon\int_0^t\|\partial_{xx}^2 u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \le C_0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|u_{\varepsilon,\beta}(t,\cdot)\|_{L^4(\mathbb{R})} &\le C_0,\\
\beta\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})} &\le C_0,\\
\varepsilon\int_0^t \|u_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds &\le C_0,\\
\beta^2\varepsilon\int_0^t \|\partial_{xxx}^3 u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds &\le C_0,\\
\beta^{\frac{7}{2}}\varepsilon\int_0^t \|\partial_{xxxx}^4 u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds &\le C_0,\\
\beta^{\frac{3}{2}}\varepsilon\int_0^t \|u_{\varepsilon,\beta}(s,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds &\le C_0,
\end{aligned}
$$

for every $0 < t < T$. □

To prove Theorem 2.1. The following technical lemma is needed [20].

Lemma 2.4 *Let Ω be a bounded open subset of $\mathbb{R}^2$. Suppose that the sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ lies in a bounded subset of $\mathcal{M}_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$.

Moreover, we consider the following definition.

Definition 2.1 A pair of functions (η, q) is called an entropy–entropy flux pair if $\eta : \mathbb{R} \to \mathbb{R}$ is a C^2 function and $q : \mathbb{R} \to \mathbb{R}$ is defined by

$$q(u) = 3\int_0^u \xi^2\eta'(\xi)d\xi.$$

An entropy–entropy flux pair (η, q) is called convex/compactly supported if, in addition, η is convex/compactly supported.

We begin by proving the following result

Lemma 2.5 *Assume that (1.16), (2.3) and (2.4) hold. Then for any compactly supported entropy–entropy flux pair (η, q), there exist two sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, with $\varepsilon_n, \beta_n \to 0$, and a limit function*

$$u \in L^\infty((0,T); L^2(\mathbb{R}) \cap L^4(\mathbb{R})),$$

such that

$$u_{\varepsilon_n,\beta_n} \to u \quad \textit{in } L^p_{loc}((0,T)\times\mathbb{R}), \quad \textit{for each } 1 \le p < 4, \tag{2.18}$$

u is a distributional solution of (1.17). (2.19)

Proof Let us consider a compactly supported entropy–entropy flux pair (η, q). Multiplying (2.2) by $\eta'(u_{\varepsilon,\beta})$, we have

$$
\begin{aligned}
\partial_t\eta(u_{\varepsilon,\beta}) + \partial_x q(u_{\varepsilon,\beta}) &= \varepsilon\eta'(u_{\varepsilon,\beta})\partial_{xx}^2 u_{\varepsilon,\beta} - \beta^{\frac{3}{2}}\varepsilon\eta'(u_{\varepsilon,\beta})\partial_{xxxx}^4 u_{\varepsilon,\beta} - \beta^2\eta'(u_{\varepsilon,\beta})\partial_{xxxxx}^5 u_{\varepsilon,\beta}\\
&= I_{1,\varepsilon,\beta} + I_{2,\varepsilon,\beta} + I_{3,\varepsilon,\beta} + I_{4,\varepsilon,\beta} + I_{5,\varepsilon,\beta} + I_{6,\varepsilon,\beta},
\end{aligned}
$$

where

$$\begin{aligned}
I_{1,\,\varepsilon,\,\beta} &= \partial_x(\varepsilon\eta'(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}),\\
I_{2,\,\varepsilon,\,\beta} &= -\varepsilon\eta''(u_{\varepsilon,\beta})(\partial_x u_{\varepsilon,\beta})^2,\\
I_{3,\,\varepsilon,\,\beta} &= -\partial_x(\beta^{\frac{3}{2}}\varepsilon\eta'(u_{\varepsilon,\beta})\partial_{xxx}^3 u_{\varepsilon,\beta}),\\
I_{4,\,\varepsilon,\,\beta} &= \beta^{\frac{3}{2}}\varepsilon\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{xxx}^3 u_{\varepsilon,\beta},\\
I_{5,\,\varepsilon,\,\beta} &= -\partial_x(\beta^2\eta'(u_{\varepsilon,\beta})\partial_{xxxx}^4 u_{\varepsilon,\beta}),\\
I_{6,\,\varepsilon,\,\beta} &= \beta^2\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{xxxx}^4 u_{\varepsilon,\beta}.
\end{aligned}\tag{2.20}$$

Fix $T > 0$. Arguing as in [11, Lemma 3.2], we have that $I_{1,\,\varepsilon,\,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, and $\{I_{2,\,\varepsilon,\,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$.

We claim that

$$I_{3,\,\varepsilon,\,\beta} \to 0 \quad \text{in } H^{-1}((0,T)\times\mathbb{R}),\ T>0, \quad \text{as } \beta,\ \varepsilon \to 0.$$

Due to (2.4) and Lemma 2.3,

$$\begin{aligned}
&\|\beta^{\frac{3}{2}}\varepsilon\eta'(u_{\varepsilon,\beta})\partial_{xxx}^3 u_{\varepsilon,\beta}\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\quad\le \beta^3\varepsilon^2\|\eta'\|_{L^\infty(\mathbb{R})}\|\partial_{xxx}^3 u_{\varepsilon,\beta}\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\quad\le C_0\|\eta'\|_{L^\infty(\mathbb{R})}\beta^2\varepsilon \to 0.
\end{aligned}$$

We have

$$I_{4,\,\varepsilon,\,\beta} \to 0 \quad \text{in } L^1((0,T)\times\mathbb{R}),\ T>0, \quad \text{as } \beta \to 0.$$

Thanks to Lemmas 2.1, 2.3, and the Hölder inequality,

$$\begin{aligned}
&\|\beta^{\frac{3}{2}}\varepsilon\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{xxx}^3 u_{\varepsilon,\beta}\|_{L^1((0,T)\times\mathbb{R})}\\
&\quad\le \beta^{\frac{3}{2}}\varepsilon\|\eta''\|_{L^\infty(\mathbb{R})}\int_0^T\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial_{xxx}^3 u_{\varepsilon,\beta}|dtdx\\
&\quad\le \beta^{\frac{3}{2}}\varepsilon\|\eta''\|_{L^\infty(\mathbb{R})}\|\partial_x u_{\varepsilon,\beta}\|_{L^2((0,T)\times\mathbb{R})}\|\partial_{xxx}^3 u_{\varepsilon,\beta}\|_{L^2((0,T)\times\mathbb{R})}\\
&\quad\le C_0\|\eta''\|_{L^\infty(\mathbb{R})}\beta^{\frac{1}{2}} \to 0.
\end{aligned}$$

We get

$$I_{5,\,\varepsilon,\,\beta} \to 0 \quad \text{in } H^{-1}((0,T)\times\mathbb{R}),\ T>0, \quad \text{as } \varepsilon \to 0.$$

Due to (2.4) and Lemma 2.3,

$$\begin{aligned}
&\|\beta^2\eta'(u_{\varepsilon,\beta})\partial_{xxxx}^4 u_{\varepsilon,\beta}\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\quad\le \beta^4\|\eta'\|_{L^\infty(\mathbb{R})}\|\partial_{xxxx}^4 u_{\varepsilon,\beta}\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\quad= \|\eta'\|_{L^\infty(\mathbb{R})}\frac{\beta^{\frac{1}{2}}\beta^{\frac{7}{2}}\varepsilon}{\varepsilon}\|\partial_{xxxx}^4 u_{\varepsilon,\beta}\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\quad\le C_0\|\eta'\|_{L^\infty(\mathbb{R})}\varepsilon \to 0.
\end{aligned}$$

We show that

$$\{I_{6,\,\varepsilon,\,\beta}\}_{\varepsilon,\,\beta>0} \quad \text{is bounded in } L^1((0,T)\times\mathbb{R}),\ T>0,\ \text{as } \varepsilon \to 0.$$

Thanks to (2.4), Lemmas 2.1, 2.3 and the Hölder inequality,

$$\begin{aligned}
&\|\beta^2\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{xxxx}^4 u_{\varepsilon,\beta}\|_{L^1((0,T)\times\mathbb{R})}\\
&\le \beta^2\|\eta''\|_{L^\infty(\mathbb{R})}\int_0^T\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial_{xxxx}^4 u_{\varepsilon,\beta}|dtdx\\
&= \|\eta''\|_{L^\infty(\mathbb{R})}\frac{\beta^{\frac{1}{4}}\beta^{\frac{7}{4}}\varepsilon}{\varepsilon}\|\partial_x u_{\varepsilon,\beta}\|_{L^2(\mathbb{R})}\|\partial_{xxxx}^4 u_{\varepsilon,\beta}\|_{L^2(\mathbb{R})}\\
&\le C_0\|\eta''\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

Therefore, Eq. (2.18) follows from Lemma 2.4 and the L^p compensated compactness of [25].

We prove that u is a distributional solution of (1.17). Let $\phi \in C^2(\mathbb{R}^2)$ be a test function with compact support. We have to prove that

$$\int_0^\infty\int_{\mathbb{R}}\left(u\partial_t\phi + u^3\partial_x\phi\right)dtdx + \int_{\mathbb{R}}u_0(x)\phi(0,x)dx = 0. \tag{2.21}$$

We define

$$u_{\varepsilon_n,\,\beta_n} := u_n. \tag{2.22}$$

We have

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}}\left(u_n\partial_t\phi + u_n^3\partial_x\phi\right)dtdx + \int_{\mathbb{R}}u_{0,n}\phi(0,x)dx\\
&= -\varepsilon_n\int_0^\infty\int_{\mathbb{R}}u_n\partial_{xx}^2\phi dtdx\\
&\quad + \beta_n^{\frac{3}{2}}\varepsilon_n\int_0^\infty\int_{\mathbb{R}}u_n\partial_{xxxx}^4\phi dtdx\\
&\quad - \beta_n^2\int_0^\infty\int_{\mathbb{R}}u_n\partial_{xxxxx}^5\phi dtdx.
\end{aligned}$$

Therefore, (2.21) follows from (2.3) and (2.18). □

Following [17], we prove the following result.

Lemma 2.6 *Assume that* (1.16), (2.3) *and* (2.5) *hold. Then for any compactly supported entropy–entropy flux pair* $(\eta,\ q)$*, there exist two sequences* $\{\varepsilon_n\}_{n\in\mathbb{N}},\ \{\beta_n\}_{n\in\mathbb{N}}$*, with* $\varepsilon_n,\ \beta_n \to 0$*, and a limit function*

$$u \in L^\infty((0,T);L^2(\mathbb{R})\cap L^4(\mathbb{R})), \tag{2.23}$$

such that (2.18) *holds and*

$$u \text{ is the unique entropy solution of (1.17).} \tag{2.24}$$

Proof Let us consider a compactly supported entropy–entropy flux pair $(\eta,\ q)$. Multiplying (2.2) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned}
\partial_t\eta(u_{\varepsilon,\beta})+\partial_x q(u_{\varepsilon,\beta}) &= \varepsilon\eta'(u_{\varepsilon,\beta})\partial_{xx}^2 u_{\varepsilon,\beta} - \beta^{\frac{3}{2}}\varepsilon\eta'(u_{\varepsilon,\beta})\partial_{xxxx}^4 u_{\varepsilon,\beta} - \beta^2\eta'(u_{\varepsilon,\beta})\partial_{xxxxx}^5 u_{\varepsilon,\beta}\\
&= I_{1,\,\varepsilon,\,\beta} + I_{2,\,\varepsilon,\,\beta} + I_{3,\,\varepsilon,\,\beta} + I_{4,\,\varepsilon,\,\beta} + I_{5,\,\varepsilon,\,\beta} + I_{6,\,\varepsilon,\,\beta},
\end{aligned}$$

where $I_{1,\,\varepsilon,\,\beta},\ I_{2,\,\varepsilon,\,\beta},\ I_{3,\,\varepsilon,\,\beta},\ I_{4,\,\varepsilon,\,\beta},\ I_{5,\,\varepsilon,\,\beta},\ I_{6,\,\varepsilon,\,\beta}$ are defined in (2.20).

Fix $T > 0$. Arguing as in Lemma 2.5, $I_{1,\varepsilon,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, $\{I_{2,\varepsilon,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$, $I_{3,\varepsilon,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, $I_{4,\varepsilon,\beta} \to 0$ in $L^1((0,T)\times\mathbb{R})$, and $I_{5,\varepsilon,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$.

We claim

$$I_{6,\varepsilon,\beta} \to 0 \quad \text{in } L^1((0,T)\times\mathbb{R}),\ T > 0, \quad \text{as } \varepsilon \to 0.$$

By (2.5), Lemmas 2.1, 2.3, and the Hölder inequality,

$$\begin{aligned}
&\|\beta^2\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{xxxx}^4 u_{\varepsilon,\beta}\|_{L^1((0,T)\times\mathbb{R})}\\
&\quad = \frac{\beta^{\frac{1}{4}}\beta^{\frac{7}{4}}\varepsilon}{\varepsilon}\|\eta''\|_{L^\infty(\mathbb{R})}\int_0^T\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial_{xxxx}^4 u_{\varepsilon,\beta}|dtdx\\
&\quad \le \frac{\beta^{\frac{1}{4}}\beta^{\frac{7}{4}}\varepsilon}{\varepsilon}\|\eta''\|_{L^\infty(\mathbb{R})}\|\partial_x u_{\varepsilon,\beta}\|_{L^2((0,T)\times\mathbb{R})}\|\partial_{xxxx}^4 u_{\varepsilon,\beta}\|_{L^2((0,T)\times\mathbb{R})}\\
&\quad \le C_0\|\eta''\|_{L^\infty(\mathbb{R})}\frac{\beta^{\frac{1}{4}}}{\varepsilon} \to 0.
\end{aligned}$$

Therefore, Eq. (2.18) follows from Lemma 2.4 and the L^p compensated compactness of [25].

We conclude by proving that that u is the unique entropy solution of (1.17). Let us consider a compactly supported entropy–entropy flux pair (η, q), and $\phi \in C_c^2((0,\infty)\times\mathbb{R})$ a non-negative function. Fix $T > 0$. We have to prove that

$$\int_0^\infty\int_{\mathbb{R}}(\eta(u)\partial_t\phi + q(u)\partial_x\phi)\,dtdx \ge 0. \tag{2.25}$$

Due to (2.22), we have

$$\begin{aligned}
&-\int_0^\infty\int_{\mathbb{R}}(\eta(u_n)\partial_t\phi + q(u_n)\partial_x\phi)\,dtdx\\
&= \varepsilon_n\int_0^\infty\int_{\mathbb{R}}\partial_x\left(\eta'(u_n)\partial_x u_n\right)\phi dx - \varepsilon_n\int_0^\infty\int_{\mathbb{R}}\eta''(u_n)\left(\partial_x u_n\right)^2\phi dtdx\\
&\quad - \beta_n^{\frac{3}{2}}\varepsilon_n\int_0^\infty\int_{\mathbb{R}}\partial_x\left(\eta'(u_n)\partial_{xxx}^3 u_n\right)\phi dtdx + \beta_n^{\frac{3}{2}}\varepsilon_n\int_0^\infty\int_{\mathbb{R}}\eta''(u_n)\partial_x u_n\partial_{xxxx}^4 u_n\phi dtdx\\
&\quad - \beta_n^2\int_0^\infty\int_{\mathbb{R}}\partial_x\left(\eta'(u_n)\partial_{xxxx}^4 u_n\right)\phi dtdx + \beta_n^2\int_0^\infty\int_{\mathbb{R}}\eta''(u_n)\partial_x u_n\partial_{xxxx}^4 u_n dtdx\\
&\le -\varepsilon_n\int_0^\infty\int_{\mathbb{R}}\eta'(u_n)\partial_x u_n\partial_x\phi dtdx + \beta_n^{\frac{3}{2}}\varepsilon_n\int_0^\infty\int_{\mathbb{R}}\eta'(u_n)\partial_{xxx}^3 u_n\partial_x\phi dtdx\\
&\quad + \beta_n^{\frac{3}{2}}\varepsilon_n\int_0^\infty\int_{\mathbb{R}}\eta''(u_n)\partial_x u_n\partial_{xxx}^3 u_n\phi dtdx + \beta_n^2\int_0^\infty\int_{\mathbb{R}}\eta'(u_n)\partial_{xxxx}^4 u_n\partial_x\phi dsdx\\
&\quad + \beta_n^2\int_0^\infty\int_{\mathbb{R}}\eta''(u_n)\partial_x u_n\partial_{xxxx}^4 u_n\phi dtdx\\
&\le \varepsilon_n\|\eta'\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\partial_x u_n\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\|\partial_x\phi\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\\
&\quad + \beta_n^{\frac{3}{2}}\varepsilon_n\|\eta'\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\partial_{xxx}^3 u_n\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\|\partial_x\phi\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\\
&\quad + \beta_n^{\frac{3}{2}}\varepsilon_n\|\eta''\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\partial_x u_n\partial_{xxx}^3 u_n\|_{L^1(\mathrm{supp}\,(\phi))}\\
&\quad + \beta_n^2\|\eta'\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\partial_{xxxx}^4 u_n\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\|\partial_x\phi\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\\
&\quad + \beta_n^2\|\eta''\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\partial_x u_n\partial_{xxxx}^4 u_n\|_{L^1(\mathrm{supp}\,(\phi))}
\end{aligned}$$

$$\begin{aligned}&\leq \varepsilon_n\|\eta'\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\partial_x u_n\|_{L^2((0,T)\times\mathbb{R})}\|\partial_x\phi\|_{L^2((0,T)\times\mathbb{R})}\\&\quad+\beta_n^{\frac{3}{2}}\varepsilon_n\|\eta'\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\partial_{xxx}^3 u_n\|_{L^2((0,T)\times\mathbb{R})}\|\partial_x\phi\|_{L^2((0,T)\times\mathbb{R})}\\&\quad+\beta_n^{\frac{3}{2}}\varepsilon_n\|\eta''\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\partial_x u_n\partial_{xxx}^3 u_n\|_{L^1((0,T)\times\mathbb{R})}\\&\quad+\beta_n^2\|\eta'\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\partial_{xxxx}^4 u_n\|_{L^2((0,T)\times\mathbb{R})}\|\partial_x\phi\|_{L^2((0,T)\times\mathbb{R})}\\&\quad+\beta_n^2\|\eta''\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\|\partial_x u_n\partial_{xxxx}^4 u_n\|_{L^1((0,T)\times\mathbb{R})}.\end{aligned}$$

Equation (2.25) follows from (2.5), (2.18), and Lemmas 2.1 and 2.3. □

Proof of Theorem 2.1 Theorem 2.1 follows from Lemmas 2.5 and 2.6. □

3 The case $f(u) = -u^3$

In section, we consider (1.15), and assume (1.16) on the initial datum. Observe that if $\beta \to 0$, we have (1.18).

We study the dispersion-diffusion limit for (1.15). Therefore, following [8], we fix two small numbers $\varepsilon,\ \beta$ and consider the following fifth order approximation

$$\begin{cases}\partial_t u_{\varepsilon,\beta}-\partial_x u_{\varepsilon,\beta}^3+\beta^2\partial_{xxxxx}^5 u_{\varepsilon,\beta}=\varepsilon\partial_{xx}^2 u_{\varepsilon,\beta}-\beta\varepsilon\partial_{xxxx}^4 u_{\varepsilon,\beta}, & t>0,\ x\in\mathbb{R},\\ u_{\varepsilon,\beta}(0,x)=u_{\varepsilon,\beta,0}(x), & x\in\mathbb{R},\end{cases}\tag{3.1}$$

where $u_{\varepsilon,\beta,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned}&u_{\varepsilon,\,\beta,\,0}\to u_0\quad\text{in } L^p_{loc}(\mathbb{R}),\ 1\leq p<4,\ \text{as } \varepsilon,\ \beta\to 0,\\&\|u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})}+\|u_{\varepsilon,\beta,0}\|^4_{L^4(\mathbb{R})}\leq C_0,\quad \varepsilon,\beta>0,\\&\beta^{\frac{1}{2}}\varepsilon\|\partial_x u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})}+\beta^2\|\partial_{xx}^2 u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})}\leq C_0,\quad \varepsilon,\beta>0,\end{aligned}\tag{3.2}$$

and C_0 is a constant independent on ε and β.

The main result of this section is the following theorem.

Theorem 3.1 *Assume that* (1.16) *and* (2.3) *hold. Fix* $T>0$, *if*

$$\beta=\mathcal{O}(\varepsilon^6),\tag{3.3}$$

holds, then, there exist two sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, *with* $\varepsilon_n,\ \beta_n\to 0$, *and a limit function*

$$u\in L^\infty((0,T);L^2(\mathbb{R})\cap\ L^4(\mathbb{R})),$$

such that

(i) *Equation* (2.18) *holds,*
(ii) *u is the unique entropy solution of* (1.18).

Let us prove some a priori estimates on $u_{\varepsilon,\beta}$, denoting with C_0 the constants which depend only on the initial data.

Lemma 3.1 *For each* $t>0$,

$$\|u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\varepsilon\int_0^t\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta\varepsilon\int_0^t\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}ds\leq C_0.\tag{3.4}$$

Proof We begin by observing that

$$-2\beta\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_{xxxx}^4 u_{\varepsilon,\beta}dx = 2\beta\varepsilon\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta}\partial_{xxx}^3 u_{\varepsilon,\beta}dx = -2\beta\varepsilon\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Therefore, arguing as in Lemma 2.1, we have (3.4). □

Lemma 3.2 *Fix $T > 0$. We have that*

$$\|u_{\varepsilon,\beta}\|_{L^\infty((0,T)\times\mathbb{R})} \leq C_0\beta^{-\frac{1}{4}}\varepsilon^{-\frac{1}{2}}. \tag{3.5}$$

In particular,

$$\begin{aligned}\beta\varepsilon^2\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ 2\beta^{\frac{3}{2}}\varepsilon^3\int_0^t\|\partial_{xx}^2 u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\ &+ 2\beta^2\varepsilon^3\int_0^t\|\partial_{xxx}^3 u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \leq C_0.\end{aligned} \tag{3.6}$$

Proof Let $0 < t < T$. Multiplying (3.1) by $-2\beta^{\frac{1}{2}}\varepsilon\partial_{xx}^2 u_{\varepsilon,\beta}$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\beta^{\frac{1}{2}}\varepsilon\frac{d}{dt}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} &= -2\beta^{\frac{1}{2}}\varepsilon\int_{\mathbb{R}}\partial_{xx}^2 u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx\\ &= -6\beta^{\frac{1}{2}}\varepsilon\int_{\mathbb{R}} u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx + 2\beta^{\frac{5}{2}}\varepsilon\int_{\mathbb{R}}\partial_{xx}^2 u_{\varepsilon,\beta}\partial_{xxxxx}^5 u_{\varepsilon,\beta}dx\\ &\quad - 2\beta^{\frac{1}{2}}\varepsilon^2\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^{\frac{3}{2}}\varepsilon^2\int_{\mathbb{R}}\partial_{xx}^2 u_{\varepsilon,\beta}\partial_{xxxx}^4 u_{\varepsilon,\beta}dx\\ &= -6\beta^{\frac{1}{2}}\varepsilon\int_{\mathbb{R}} u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx - 2\beta^{\frac{5}{2}}\varepsilon\int_{\mathbb{R}}\partial_{xxx}^3 u_{\varepsilon,\beta}\partial_{xxxx}^4 u_{\varepsilon,\beta}dx\\ &\quad - 2\beta^{\frac{1}{2}}\varepsilon^2\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\beta^{\frac{3}{2}}\varepsilon^2\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &= -6\beta^{\frac{1}{2}}\varepsilon\int_{\mathbb{R}} u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx - 2\beta^{\frac{1}{2}}\varepsilon^2\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\quad - 2\beta^{\frac{3}{2}}\varepsilon^2\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Hence,

$$\begin{aligned}&\beta^{\frac{1}{2}}\varepsilon\frac{d}{dt}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta\varepsilon^2\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^{\frac{3}{2}}\varepsilon^2\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\quad = -6\beta^{\frac{1}{2}}\varepsilon\int_{\mathbb{R}} u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx.\end{aligned} \tag{3.7}$$

Due to the Young inequality,

$$\begin{aligned}6\beta^{\frac{1}{2}}\varepsilon\int_{\mathbb{R}} u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx &= 6\varepsilon\int_{\mathbb{R}} u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\beta^{\frac{1}{2}}\partial_{xx}^2 u_{\varepsilon,\beta}|dx\\ &\leq 6\varepsilon\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\beta^{\frac{1}{2}}\partial_{xx}^2 u_{\varepsilon,\beta}|dx\\ &\leq 3\varepsilon\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\quad + 3\beta\varepsilon\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, from (3.7), we gain

$$\beta^{\frac{1}{2}}\varepsilon\frac{d}{dt}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta\varepsilon^2\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^{\frac{3}{2}}\varepsilon^2\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}$$
$$\leq 3\varepsilon\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+3\beta\varepsilon\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, from (3.2) and (3.4), we have

$$\begin{aligned}&\beta^{\frac{1}{2}}\varepsilon\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta\varepsilon^2\int_0^t\|\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds+2\beta^{\frac{3}{2}}\varepsilon^2\int_{\mathbb{R}}\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\quad\leq C_0+3\varepsilon\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\|\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\qquad+3\beta\varepsilon\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0\|\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\quad\leq C_0(1+\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}).\end{aligned}\tag{3.8}$$

We prove (3.5). Due to (3.4), (3.8), and the Hölder inequality,

$$\begin{aligned}u^2_{\varepsilon,\beta}(t,x)&=2\int_{-\infty}^x u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx\leq 2\int_{\mathbb{R}}|u_{\varepsilon,\beta}||\partial_x u_{\varepsilon,\beta}|dx\\
&\leq\|u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}\\
&\leq\frac{C_0}{\beta^{\frac{1}{4}}\varepsilon^{\frac{1}{2}}}\sqrt{(1+\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})})}.\end{aligned}$$

Hence,

$$\|u_{\varepsilon,\beta}\|^4_{L^\infty((0,T)\times\mathbb{R})}\leq\frac{C_0}{\beta^{\frac{1}{2}}\varepsilon}(1+\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}).\tag{3.9}$$

Introducing the notation

$$y:=\|u_{\varepsilon,\beta}\|_{L^\infty((0,T)\times\mathbb{R})},\quad\beta^{\frac{1}{2}}\varepsilon:=\delta.\tag{3.10}$$

Then, Eq. (3.9) reads

$$y^4\leq\frac{C_0}{\delta}\left(1+y^2\right).\tag{3.11}$$

Arguing as in [10, Lemma 2.3], we have

$$y\leq C_0\delta^{-\frac{1}{2}}.\tag{3.12}$$

Equation (3.5) follows (3.10), and (3.11).

Finally, we prove (3.6). It follows from (3.5), and (3.8) that

$$\begin{aligned}&\beta^{\frac{1}{2}}\varepsilon\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta\varepsilon^2\int_0^t\|\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\quad+2\beta^{\frac{3}{2}}\varepsilon^2\int_{\mathbb{R}}\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\leq C_0\beta^{-\frac{1}{2}}\varepsilon^{-1},\end{aligned}$$

which gives (3.6). □

Following [9, Lemma 2.2], we prove the following result.

Lemma 3.3 *Fix $T > 0$, and assume (3.3). Then,*

(i) *the family $\{u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ is bounded in $L^{\infty}((0,T);L^4(\mathbb{R}))$;*
(ii) *the family $\{\beta\partial_{xx}^2 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ is bounded in $L^{\infty}((0,T);L^2(\mathbb{R}))$;*
(iii) *the families $\{\beta\varepsilon^{\frac{1}{2}}\partial_{xxx}^3 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}\partial_{xxxx}^4 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\varepsilon^{\frac{1}{2}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ are bounded in $L^2((0,T)\times\mathbb{R})$.*

Proof Let $0 < t < T$. Let A be a positive constant that which will be specified later. Multiplying (3.1) by $u_{\varepsilon,\beta}^3 + A\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta}$, we have

$$\begin{aligned}
&(u_{\varepsilon,\beta}^3 + A\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta})\partial_t u_{\varepsilon,\beta}\\
&\quad - 3(u_{\varepsilon,\beta}^3 + A\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta})u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\\
&\quad + \beta^2(u_{\varepsilon,\beta}^3 + A\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta})\partial_{xxxxx}^5 u_{\varepsilon,\beta}\\
&= \varepsilon(u_{\varepsilon,\beta}^3 + A\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta})\partial_{xx}^2 u_{\varepsilon,\beta}\\
&\quad - \beta\varepsilon(u_{\varepsilon,\beta}^3 + A\beta^2\varepsilon\partial_{xxxx}^4 u_{\varepsilon,\beta})\partial_{xxxx}^4 u_{\varepsilon,\beta}.
\end{aligned} \tag{3.13}$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}}(u_{\varepsilon,\beta}^3 + A\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta})\partial_t u_{\varepsilon,\beta}dx\\
&\quad = \frac{d}{dt}\left(\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|_{L^4(\mathbb{R})}^4 + \frac{A\beta^2}{2}\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2\right),\\
&\quad\ - 3\int_{\mathbb{R}}(u_{\varepsilon,\beta}^3 + A\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta})u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}dx\\
&\quad = -3A\beta^2\int_{\mathbb{R}}u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\partial_{xxxx}^4 u_{\varepsilon,\beta}dx,\\
&\beta^2\int_{\mathbb{R}}(u_{\varepsilon,\beta}^3 + A\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta})\partial_{xxxxx}^5 u_{\varepsilon,\beta}dx\\
&\quad = -3\beta^2\int_{\mathbb{R}}u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\partial_{xxxx}^4 u_{\varepsilon,\beta}dx,\\
&\varepsilon\int_{\mathbb{R}}(u_{\varepsilon,\beta}^3 + A\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta})\partial_{xx}^2 u_{\varepsilon,\beta}dx\\
&\quad = -3\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 - A\beta^2\varepsilon\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2,\\
&\quad\ - \beta\varepsilon\int_{\mathbb{R}}(u_{\varepsilon,\beta}^3 + A\beta^2\varepsilon\partial_{xxxx}^4 u_{\varepsilon,\beta})\partial_{xxxx}^4 u_{\varepsilon,\beta}dx\\
&\quad = 3\beta\varepsilon\int_{\mathbb{R}}u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\partial_{xxx}^3 u_{\varepsilon,\beta}dx - A\beta^3\varepsilon\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

an integration of (3.13) on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|_{L^4(\mathbb{R})}^4 + \frac{A\beta^2}{2}\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2\right)\\
&\quad + 3\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 + A\beta^2\varepsilon\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2
\end{aligned}$$

$$
\begin{aligned}
&+ A\beta^3\varepsilon\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&= 3A\beta^2\int_{\mathbb{R}} u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta}dx + 3\beta^2\int_{\mathbb{R}} u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta}dx\\
&+ 3\beta\varepsilon\int_{\mathbb{R}} u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}dx. \qquad (3.14)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&3\beta\varepsilon\int_{\mathbb{R}} u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}dx\\
&= -6\beta\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2\partial^2_{xx}u_{\varepsilon,\beta}dx - 3\beta\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&= 2\beta\varepsilon\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx - 3\beta\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, from (3.14), we gain

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{A\beta^2}{2}\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&+ 3\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + A\beta^2\varepsilon\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&+ A\beta^3\varepsilon\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 3\beta\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&= 3A\beta^2\int_{\mathbb{R}} u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta}dx + 3\beta^2\int_{\mathbb{R}} u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta}dx\\
&+ 2\beta\varepsilon\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx. \qquad (3.15)
\end{aligned}
$$

From (3.3),

$$
\beta \le D^2\varepsilon^6, \qquad (3.16)
$$

where D is a positive constant which will specified later. Due to (3.5), (3.3), and the Young inequality,

$$
\begin{aligned}
&3A\beta^2\int_{\mathbb{R}} u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}|\partial^4_{xxxx}u_{\varepsilon,\beta}|dx = A\beta^2\int_{\mathbb{R}}\left|\frac{3\sqrt{2}u^2_{\varepsilon,\beta}\partial_x u^2_{\varepsilon,\beta}}{\beta^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}}\right|\left|\frac{\beta^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\partial^4_{xxxx}u_{\varepsilon,\beta}}{\sqrt{2}}\right|dx\\
&\le \frac{9A\beta}{\varepsilon}\int_{\mathbb{R}} u^4_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2dx + \frac{A\beta^3\varepsilon}{4}\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le \frac{9A\beta}{\varepsilon}\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{A\beta^3\varepsilon}{4}\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le \frac{C_0A\beta^{\frac{1}{2}}}{\varepsilon^2}\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{A\beta^3\varepsilon}{4}\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C_0DA\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{A\beta^3\varepsilon}{4}\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
&3\beta^2\int_{\mathbb{R}} u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial^4_{xxxx}u_{\varepsilon,\beta}|dx = \int_{\mathbb{R}}\left|\frac{3\sqrt{2}\beta^{\frac{1}{2}}u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}}{\sqrt{A}\varepsilon^{\frac{1}{2}}}\right|\left|\frac{\sqrt{A}\beta^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}\partial^4_{xxxx}u_{\varepsilon,\beta}}{\sqrt{2}}\right|dx\\
&\le \frac{9\beta}{A\varepsilon}\int_{\mathbb{R}} u^4_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2dx + \frac{A\beta^3\varepsilon}{4}\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}
\end{aligned}
$$

$$\leq \frac{9\beta}{A\varepsilon}\|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})}\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{A\beta^3\varepsilon}{4}\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}$$
$$\leq \frac{C_0\beta^{\frac{1}{2}}}{A\varepsilon^2}\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{A\beta^3\varepsilon}{4}\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$
$$\leq \frac{C_0D\varepsilon}{A}\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{A\beta^3\varepsilon}{4}\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Therefore, from (3.15), we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})}+\frac{A\beta^2}{2}\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\&\quad+\left(3-C_0DA-\frac{C_0D}{A}\right)\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\quad+A\beta^2\varepsilon\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{A\beta^3\varepsilon}{2}\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\quad+3\beta\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\leq 2\beta\varepsilon\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx.\end{aligned}\tag{3.17}$$

We search $A,\ D$ such that

$$3-C_0DA-\frac{C_0D}{A}>0,$$

that is

$$C_0DA^2-3A+C_0D<0.\tag{3.18}$$

A does exist if and only if

$$9-4C_0^2D^2>0.\tag{3.19}$$

Choosing,

$$D=\frac{1}{C_0},\tag{3.20}$$

it follows from (3.18), and (3.20) that, there exist $0<A_1<A_2$ such that for every

$$A_1<A<A_2,\tag{3.21}$$

Equation (3.18) holds. Hence, from (3.17), and (3.21), we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})}+\frac{A\beta^2}{2}\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\&\quad+K_1\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+A\beta^2\varepsilon\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\quad+\frac{A\beta^3\varepsilon}{2}\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+3\beta\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\leq 2\beta\varepsilon\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx,\end{aligned}\tag{3.22}$$

where K_1 is a positive constant. Thanks to (2.17), and (3.4),

$$2\beta\varepsilon\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx\leq C_0\beta\varepsilon\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Thus, from (3.22), we gain

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})}+\frac{A\beta^2}{2}\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\&\quad+K_1\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+A\beta^2\varepsilon\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\quad+\frac{A\beta^3\varepsilon}{2}\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+3\beta\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\le C_0\beta\varepsilon\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Equations (3.2), (3.4), and an integration on $(0,t)$ give

$$\begin{aligned}&\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})}+\frac{A\beta^2}{2}\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\quad+K_1\varepsilon\int_0^t\|u_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds+A\beta^2\varepsilon\int_0^t\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\&\quad+\frac{A\beta^3\varepsilon}{2}\int_0^t\|\partial^4_{xxxx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds+3\beta\varepsilon\int_0^t\|u_{\varepsilon,\beta}(s,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\&\le C_0\beta\varepsilon\int_0^t\|\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\le C_0.\end{aligned}$$

Hence,

$$\begin{aligned}\|u_{\varepsilon,\beta}(t,\cdot)\|_{L^4(\mathbb{R})}&\le C_0,\\ \beta\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}&\le C_0,\\ \varepsilon\int_0^t\|u_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds&\le C_0,\\ \beta^2\varepsilon\int_0^t\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds&\le C_0,\\ \beta^3\varepsilon\int_0^t\|\partial^4_{xxxx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds&\le C_0,\\ \beta\varepsilon\int_0^t\|u_{\varepsilon,\beta}(s,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds&\le C_0,\end{aligned}$$

for every $0<t<T$. □

Lemma 3.4 *We have that*

$$\begin{aligned}&\beta^{\frac{1}{2}}\varepsilon\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^{\frac{1}{2}}\varepsilon^2\int_0^t\|\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\&\quad+2\beta^{\frac{3}{2}}\varepsilon^2\int_0^t\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\le C_0,\end{aligned}\tag{3.23}$$

for every $0<t<T$.

Proof Let $0<t<T$. Multiplying (3.1) by $-2\beta^{\frac{1}{2}}\varepsilon\partial^2_{xx}u_{\varepsilon,\beta}$, we have

$$\begin{aligned}&-2\beta^{\frac{1}{2}}\varepsilon\partial^2_{xx}u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}+6\beta^{\frac{1}{2}}\varepsilon u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}-\beta^{\frac{5}{2}}\varepsilon\partial^2_{xx}u_{\varepsilon,\beta}\partial^5_{xxxxx}u_{\varepsilon,\beta}\\&\quad=-2\beta^{\frac{1}{2}}\varepsilon^2(\partial^2_{xx}u_{\varepsilon,\beta})^2+2\beta^{\frac{3}{2}}\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta}.\end{aligned}\tag{3.24}$$

Since

$$-2\beta^{\frac{1}{2}}\varepsilon\int_{\mathbb{R}}\partial_{xx}^2 u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx=\beta^{\frac{1}{2}}\varepsilon\frac{d}{dt}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$
$$-2\beta^{\frac{5}{2}}\varepsilon\int_{\mathbb{R}}\partial_{xx}^2 u_{\varepsilon,\beta}\partial_{xxxxx}^5 u_{\varepsilon,\beta}dx=0,$$
$$2\beta^{\frac{3}{2}}\varepsilon^2\int_{\mathbb{R}}\partial_{xx}^2 u_{\varepsilon,\beta}\partial_{xxxx}^4 u_{\varepsilon,\beta}dx=2\beta^{\frac{3}{2}}\varepsilon^2\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

an integration of (3.24) on $\mathbb{R}$ gives

$$\begin{aligned}&\beta^{\frac{1}{2}}\varepsilon\frac{d}{dt}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^{\frac{1}{2}}\varepsilon^2\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^{\frac{3}{2}}\varepsilon^2\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&=6\beta^{\frac{1}{2}}\varepsilon\int_{\mathbb{R}}u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx.\end{aligned}\tag{3.25}$$

Due to the Young inequality,

$$\begin{aligned}6\beta^{\frac{1}{2}}\varepsilon\int_{\mathbb{R}}u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx&=6\varepsilon\int_{\mathbb{R}}\left|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\right|\left|\beta^{\frac{1}{2}}u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}\right|dx\\&\le 3\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+3\beta\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Hence, from (3.25), we get

$$\begin{aligned}&\beta^{\frac{1}{2}}\varepsilon\frac{d}{dt}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^{\frac{1}{2}}\varepsilon^2\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^{\frac{3}{2}}\varepsilon\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\le 3\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}+3\beta\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Integrating on $(0,t)$, from (3.2), and Lemma 3.3, we get (3.23). □

To prove Theorem 3.1 and Lemma 2.4 is needed. Moreover, we use the following definition

Definition 3.1 A pair of functions (η,q) is called an entropy–entropy flux pair if $\eta\colon\mathbb{R}\to\mathbb{R}$ is a C^2 function and $q:\mathbb{R}\to\mathbb{R}$ is defined by

$$q(u)=-3\int_0^u\xi^2\eta'(\xi)d\xi.$$

An entropy–entropy flux pair $(\eta,\ q)$ is called convex/compactly supported if, in addition, η is convex/compactly supported.

Proof Theorem 3.1 Let us consider a compactly supported entropy–entropy flux pair (η,q). Multiplying (3.1) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned}\partial_t\eta(u_{\varepsilon,\beta})-\partial_x q(u_{\varepsilon,\beta})&=\varepsilon\eta'(u_{\varepsilon,\beta})\partial_{xx}^2 u_{\varepsilon,\beta}-\beta\varepsilon\eta'(u_{\varepsilon,\beta})\partial_{xxxx}^4 u_{\varepsilon,\beta}-\beta^2\eta'(u_{\varepsilon,\beta})\partial_{xxxxx}^5 u_{\varepsilon,\beta}\\&=I_{1,\,\varepsilon,\,\beta}+I_{2,\,\varepsilon,\,\beta}+I_{3,\,\varepsilon,\,\beta}+I_{4,\,\varepsilon,\,\beta}+I_{5,\,\varepsilon,\,\beta}+I_{6,\,\varepsilon,\,\beta},\end{aligned}$$

where

$$\begin{aligned}I_{1,\,\varepsilon,\,\beta}&=\partial_x(\varepsilon\eta'(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}),\\I_{2,\,\varepsilon,\,\beta}&=-\varepsilon\eta''(u_{\varepsilon,\beta})(\partial_x u_{\varepsilon,\beta})^2,\\I_{3,\,\varepsilon,\,\beta}&=-\partial_x\left(\beta\varepsilon\eta'(u_{\varepsilon,\beta})\partial_{xxx}^3 u_{\varepsilon,\beta}\right),\\I_{4,\,\varepsilon,\,\beta}&=\beta\varepsilon\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{xxx}^3 u_{\varepsilon,\beta},\\I_{5,\,\varepsilon,\,\beta}&=-\partial_x(\beta^2\eta'(u_{\varepsilon,\beta})\partial_{xxxx}^4 u_{\varepsilon,\beta}),\\I_{6,\,\varepsilon,\,\beta}&=\beta^2\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{xxxx}^4 u_{\varepsilon,\beta}.\end{aligned}\tag{3.26}$$

Fix $T > 0$. Arguing as in Lemma 2.5, we have that $I_{1,\varepsilon,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, and $\{I_{2,\varepsilon,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$. We claim

$$I_{3,\varepsilon,\beta} \to 0 \quad \text{in } H^{-1}((0,T)\times\mathbb{R}),\ T>0, \quad \text{as } \beta,\ \varepsilon \to 0.$$

Due to Lemma 3.3,

$$\begin{aligned}
&\|\beta\varepsilon\eta'(u_{\varepsilon,\beta})\partial^3_{xxx}u_{\varepsilon,\beta}\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\quad\le \beta^2\varepsilon^2\|\eta'\|_{L^\infty(\mathbb{R})}\|\partial^3_{xxx}u_{\varepsilon,\beta}\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\quad\le C_0\|\eta'\|_{L^\infty(\mathbb{R})}\varepsilon \to 0.
\end{aligned}$$

We have

$$I_{4,\varepsilon,\beta} \to 0 \quad \text{in } L^1((0,T)\times\mathbb{R}),\ T>0, \quad \text{as } \beta \to 0.$$

Due to (3.3), (3.4), (3.23), and the Hölder inequality,

$$\begin{aligned}
&\|\beta\varepsilon\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}\|_{L^1((0,T)\times\mathbb{R})}\\
&\quad= \beta^{\frac14}\beta^{\frac34}\varepsilon\|\eta''\|_{L^\infty(\mathbb{R})}\int_0^T\!\!\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial^3_{xxx}u_{\varepsilon,\beta}|dtdx\\
&\quad\le \beta^{\frac14}\|\eta''\|_{L^\infty(\mathbb{R})}C_0 \to 0.
\end{aligned}$$

We get

$$I_{5,\varepsilon,\beta} \to 0 \quad \text{in } H^{-1}((0,T)\times\mathbb{R}),\ T>0, \quad \text{as } \varepsilon \to 0.$$

Due to (3.3), and Lemma 3.3,

$$\begin{aligned}
&\|\beta^2\eta'(u_{\varepsilon,\beta})\partial^4_{xxxx}u_{\varepsilon,\beta}\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\quad= \|\eta'\|_{L^\infty(\mathbb{R})}\frac{\beta\beta^3\varepsilon}{\varepsilon}\|\partial^4_{xxxx}u_{\varepsilon,\beta}\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\quad\le C_0\|\eta'\|_{L^\infty(\mathbb{R})}\varepsilon^5 \to 0.
\end{aligned}$$

We show that

$$I_{6,\varepsilon,\beta} \to 0 \quad \text{in } L^1((0,T)\times\mathbb{R}),\ T>0, \quad \text{as } \varepsilon \to 0.$$

Thanks to (2.4), (2.6), Lemma 3.3, and the Hölder inequality,

$$\begin{aligned}
&\|\beta^2\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta}\|_{L^1((0,T)\times\mathbb{R})}\\
&\quad\le \beta^2\|\eta''\|_{L^\infty(\mathbb{R})}\int_0^T\!\!\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial^4_{xxxx}u_{\varepsilon,\beta}|dtdx\\
&\quad= \|\eta''\|_{L^\infty(\mathbb{R})}\frac{\beta^{\frac32}\beta^{\frac12}\varepsilon}{\varepsilon}\|\partial_x u_{\varepsilon,\beta}\|_{L^2(\mathbb{R})}\|\partial^4_{xxxx}u_{\varepsilon,\beta}\|_{L^2(\mathbb{R})}\\
&\quad\le C_0\|\eta''\|_{L^\infty(\mathbb{R})}\varepsilon^2 \to 0.
\end{aligned}$$

Therefore, Eq. (2.18) follows from Lemma 2.4 and the L^p compensated compactness of [25].

Arguing as in Lemmas 2.5, and 2.6, we obtain that u is a distributional solution of (1.18), and u is the entropy solution of (1.18). Therefore, the proof is concluded. □

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