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Bèzier variant of the generalized Baskakov Kantorovich operators

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Abstract In this paper, we introduce the Bèzier variant of the generalized Baskakov Kantorovich operators. We establish a direct approximation theorem with the aid of the Ditzian–Totik modulus of smoothness and also study the rate of convergence for the functions having a derivative of bounded variation for these operators.

Keywords Bèzier operator · Modulus of continuity · Rate of convergence · Bounded variation

Mathematics Subject Classification 26A15 · 40A35 · 41A25 · 41A36

1 Introduction

In 1998, Mihesan [1] introduced the generalized Baskakov operators $B^*_{n,a}$ defined as

$$B^*_{n,a}(f;x)=\sum_{k=0}^{\infty}W^a_{n,k}(x)f\left(\frac{k}{n+1}\right),$$

where $W^a_{n,k}(x)=e^{\frac{-ax}{1+x}}\frac{p_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}$, $p_k(n,a)=\sum_{i=0}^{k}\binom{k}{i}(n)_i a^{k-i}$, and $(n)_0=1$, $(n)_i=n(n+1)\cdots(n+i-1)$, for $i\geq 1$. In Agrawal and Goyal [2], considered the Kantorovich modification of these operators for the function f defined on $C_\gamma[0,\infty):=\{f\in C[0,\infty):|f(t)|\leq M(1+t^\gamma), t\geq 0 \text{ for some } \gamma>0\}$ as follows:

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$$K_n^a(f;x) = (n+1)\sum_{k=0}^{\infty} W_{n,k}^a(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt,\ \ a \geq 0, \tag{1.1}$$

and discussed some direct results in weighted approximation, simultaneous approximation and statistical convergence for these operators. They also obtained the rate of convergence for functions having a derivative equivalent with a function of bounded variation. As a special case, for $a = 0$, these operators include the well known Baskakov–Kantorovich operators (see e.g. [3]).

Bojanic and Cheng [4,5] estimated the rate of convergence with derivatives of bounded variation for Bernstein and Hermite–Fejer polynomials by using different methods. Guo [6] studied it for the Bernstein–Durrmeyer polynomials by using Berry Esseen theorem. Subsequently, due to the pivotal role of the Bèzier basis functions in computer aided design and related fields, the researchers started working on the approximation behaviour of the Bèzier variant of various sequences of linear positive operators. In Zeng and Chen [7], initiated the study of the rate of convergence for the Bèzier variant of Bernstein Durrmeyer operators. Zeng and Tao [8] also introduced the Bèzier type Baskakov–Durrmeyer operators and estimated the rate of convergence. They termed these operators as integral type Lupas–Bèzier operators. Abel and Gupta [9] introduced the Bèzier variant of the Baskakov operators and then Gupta [10] estimated the convergence of Bèzier type Baskakov–Kantorovich operators and studied the rate of convergence. Guo et al. [11] proved the direct, inverse and equivalence approximation theorems with unified Ditzian–Totik modulus $\omega_{\phi^\lambda}(f,t)(0 \leq \lambda \leq 1)$. Several other Bèzier variants of summation–integral type operators were studied in [12–15] etc.

So, it is worthwhile to study the Bèzier variant of other sequences of operators. Furthermore, the recent work on different Bèzier type operators inspired us to investigate further in this direction.

The purpose of this paper is to introduce the Bèzier variant of the operators (1.1) and investigate a direct approximation theorem with the aid of the Ditzian–Totik modulus of smoothness and the rate of convergence for functions with derivatives of bounded variation.

2 Construction of operators

For $\theta \geq 1$, we now define the Bèzier variant of the operators (1.1) on $[0,\infty)$ as:

$$K_{n,\theta}^a(f;x) = (n+1)\sum\nolimits_{k=0}^{\infty} F_{n,k,a}^{(\theta)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \tag{2.1}$$

where $F_{n,k,a}^{(\theta)}(x) = [J_{n,k}^a(x)]^\theta - [J_{n,k+1}^a(x)]^\theta$ and $J_{n,k}^a(x) = \sum_{j=k}^{\infty} W_{n,j}^a(x)$, when $k \leq n$ and 0 otherwise.

Some important properties of $J_{n,k}^a(x)$ are as follows:

- $J_{n,k}^a(x) - J_{n,k+1}^a(x) = W_{n,k}^a(x), k = 0,1,2,3...;$
- $J_{n,0}^a(x) > J_{n,1}^a(x) > J_{n,2}^a(x) > \cdots > J_{n,n}^a(x) > 0, x \in [0,\infty).$

For every natural number k, $J_{n,k}^a(x)$ increases strictly from 0 to 1 on $[0,\infty)$.

The operators $K_{n,\theta}^a(f;x)$ also admit the integral representation

$$K_{n,\theta}^a(f;x) = \int_0^\infty M_{n,\theta}^a(x,t)f(t)dt, \tag{2.2}$$

where $M^a_{n,\theta}(x,t) := (n+1)\sum_{k=0}^{\infty} F^{(\theta)}_{n,k,a}(x)\chi_{n,k}(t)$, where $\chi_{n,k}(t)$ is the characteristic function of the interval $\left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$ with respect to $[0,\infty)$.

It is easily verified that for $\theta = 1$, the operators (2.1) reduce to (1.1), i.e. $K^a_{n,1}(f;x) = K^a_n(f;x)$.

3 Auxiliary results

Let $C_B[0,\infty)$ denote the space of all bounded and continuous functions on $[0,\infty)$ endowed with the norm

$$\|f\| = \sup_{x\in[0,\infty)} |f(t)|.$$

Lemma 1 *[2] For the rth order ($r \in \mathbb{N}\cup\{0\}$) moment of the operators (1.1), defined as $T^a_{n,r}(x) := K^a_n(t^r;x)$, we have*

$$T^a_{n,r}(x) = \frac{1}{r+1}\sum_{j=0}^{r}\binom{r+1}{j}\frac{1}{(n+1)^{r-j}}\upsilon^a_{n,j}(x),$$

where $\upsilon^a_{n,j}(x)$ is the jth order moment of the operators $B^*_{n,a}$.

Consequently, $T^a_{n,0}(x) = 1,\ T^a_{n,1}(x) = \frac{1}{n+1}\left(nx + \frac{ax}{1+x} + \frac{1}{2}\right)$,

$T^a_{n,2}(x) = \frac{1}{(n+1)^2}\left(n^2x^2 + n\left(x^2+2x+\frac{2ax^2}{1+x}\right) + \frac{a^2x^2}{(1+x)^2} + \frac{2ax}{1+x} + \frac{1}{3}\right)$, *and for each $x \in (0,\infty)$ and $r \in \mathbb{N}$, $T^a_{n,r}(x) = x^r + n^{-1}(p_r(x,a)+o(1))$, where $p_r(x,a)$ is a rational function of x depending on the parameters a and r.*

Lemma 2 *[2] For the rth order central moment of K^a_n, defined as*

$$u^a_{n,r}(x) := K^a_n((t-x)^r;x),$$

we have

(i) $u^a_{n,0}(x) = 1, u^a_{n,1}(x) = \frac{1}{n+1}\left(-x + \frac{ax}{1+x} + \frac{1}{2}\right)$

and $u^a_{n,2}(x) = \frac{1}{(n+1)^2}\left\{nx(x+1) - x(1-x) + \frac{ax}{1+x}\left(\frac{ax}{1+x} + 2(1-x)\right) + \frac{1}{3}\right\}$;

(ii) $u^a_{n,r}(x)$ is a rational function of x depending on the parameters a and r;

(iii) for each $x \in (0,\infty)$, $u^a_{n,r}(x) = O\left(\frac{1}{n^{[\frac{r+1}{2}]}}\right)$, as $n\to\infty$.

Remark 1 [2] From Lemma 2, for $\lambda > 1$, $x \in (0,\infty)$ and n sufficiently large, we have

$$K^a_n((t-x)^2;x) = u^a_{n,2}(x) \le \frac{\lambda\phi^2(x)}{n+1},\ \text{ where } \phi(x) = \sqrt{x(1+x)}.$$

Lemma 3 *For $f \in C_B[0,\infty)$, $\| K^a_n(f) \| \le \| f \|$.*

Proof From (1.1) and Lemma 2, the proof of this lemma is immediate. Hence the details are omitted. □

Lemma 4 *Let* $f \in C_B[0,\infty)$. *Then,* $\parallel K^a_{n,\theta}(f) \parallel \leq \theta \parallel f \parallel$.

Proof Using the well known inequality $|a^\beta - b^\beta| \leq \beta|a-b|$ with $0 \leq a, b \leq 1, \beta \geq 1$ and the definition of $F^{(\theta)}_{n,k,a}(x)$, we have, for $\theta \geq 1$

$$0 < [J^a_{n,k}(x)]^\theta - [J^a_{n,k+1}(x)]^\theta \leq \theta[J^a_{n,k}(x) - J^a_{n,k+1}(x)] = \theta\, W^a_{n,k}(x). \tag{3.1}$$

Hence, from the definition of the operator $K^a_{n,\theta}(f;x)$ and Lemma 3, we get

$$\parallel K^a_{n,\theta}(f;x) \parallel \leq \theta \parallel K^a_n(f) \parallel \leq \theta \parallel f \parallel .$$

□

4 Direct approximation theorem

In this section, first we recall the definitions of the Ditizian–Totik modulus of smoothness $\omega_{\phi^\tau}(f,t)$ and Peetre's $\mathcal{K}$—functional [16]. Let $\phi(x) = \sqrt{x(1+x)}$ and $f \in C[0,\infty)$. Here, we use moduli $\omega_{\phi^\tau}(f,t)$ which unify the classical modulus $\omega(f,t)$, $\tau = 0$ and the Ditzian–Totik modulus $\omega_\phi(f,t)$.

For $0 \leq \tau \leq 1$, we define

$$\omega_{\phi^\tau}(f,t) = \sup_{0\leq h\leq t}\ \sup_{x\pm\frac{h\phi^\tau(x)}{2}\in[0,\infty)} \left| f\left(x + \frac{h\phi^\tau(x)}{2}\right) - f\left(x - \frac{h\phi^\tau(x)}{2}\right)\right|$$

and the $\mathcal{K}$—functional

$$\mathcal{K}_{\phi^\tau}(f,t) = \inf_{g\in W_\tau}\{\parallel f-g \parallel + t \parallel \phi^\tau g' \parallel\},$$

where $W_\tau = \{g : g \in AC_{loc}; \parallel \phi^\tau g' \parallel < \infty\}$ and $\parallel . \parallel$ is the uniform norm on $C[0,\infty)$. It is proved that [16], $\omega_{\phi^\tau}(f,t) \sim \mathcal{K}_{\phi^\tau}(f,t)$, i.e. there exists a constant $M > 0$ such that

$$M^{-1}\omega_{\phi^\tau}(f,t) \leq \mathcal{K}_{\phi^\tau}(f,t) \leq M\omega_{\phi^\tau}(f,t). \tag{4.1}$$

Lemma 5 *For* $f \in W_\tau, \phi(x) = \sqrt{x(1+x)}, 0 \leq \tau \leq 1$ *and* $t, x > 0$, *we have*

$$\left|\int_x^t f'(u)du\right| \leq 2^\tau \left(x^{-\tau/2}(1+t)^{-\tau/2} + \phi^{-\tau}(x)\right)|t-x| \left\|\phi^\tau f'\right\|.$$

Proof By applying Hölder's inequality, we get

$$\left|\int_x^t f'(u)du\right| \leq \|\phi^\tau f'\| \left|\int_x^t \frac{du}{\phi^\tau(u)}\right| \leq \left\|\phi^\tau f'\right\| |t-x|^{1-\tau} \left|\int_x^t \frac{du}{\phi(u)}\right|^\tau. \tag{4.2}$$

Now,

$$\left|\int_x^t \frac{du}{\phi(u)}\right| \leq \left|\int_x^t \frac{du}{\sqrt{u}}\right| \left(\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1+t}}\right)$$

and

$$\left|\int_x^t \frac{du}{\sqrt{u}}\right| \leq \frac{2|t-x|}{\sqrt{x}}.$$

On using above estimates in (4.2) and then the inequality $|a+b|^r \le |a|^r + |b|^r, 0 \le r \le 1$, we obtain

$$\left|\int_x^t f'(u)du\right| \le \left\|\phi^\tau f'\right\| |t-x| \frac{2^\tau}{x^{\tau/2}} \left|\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1+t}}\right|^\tau$$
$$\le \left\|\phi^\tau f'\right\| |t-x| \frac{2^\tau}{x^{\tau/2}} \left((1+t)^{-\tau/2} + (1+x)^{-\tau/2}\right).$$

□

Lemma 6 *For any $s \ge 0$ and each $x \in [0, \infty)$, there holds the inequality*

$$K_n^a((1+t)^{-s}; x) \le C(s)(1+x)^{-s}, \tag{4.3}$$

where $C(s)$ is a constant dependent on s.

Proof For $x = 0$, the result holds from (1.1). For $x \in (0, \infty)$, using (3.1) we have

$$K_n^a((1+t)^{-s}; x) = (n+1)\sum_{k=0}^{\infty} W_{n,k}^a(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \frac{1}{(1+t)^s} dt.$$

We first observe that

$$(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \frac{1}{(1+t)^s} dt \le \left(1 + \frac{k}{n+1}\right)^{-s}.$$

Thus, we get

$$K_n^a((1+t)^{-s}; x) \le \frac{1}{(1+x)^s} \sum_{k=0}^{\infty} \frac{e^{\frac{-ax}{1+x}} p_k(n,a) x^k}{k!(1+x)^{n+k-s}} \left(1 + \frac{k}{n+1}\right)^{-s}. \tag{4.4}$$

On using the ratio test, we note that for each $x > 0$, the series on the right hand side (4.4) is convergent. This proves the desired result. □

Let $L_B[0, \infty)$ denote the space of all bounded and Lebesgue integrable functions on $[0, \infty)$.

Theorem 1 *For $f \in L_B[0, \infty)$, we have*

$$\left|K_{n,\theta}^a(f; x) - f(x)\right| \le C\omega_{\phi^\tau}\left(f, \frac{\phi^{1-\tau}(x)}{\sqrt{n+1}}\right). \tag{4.5}$$

Proof By the definition of $\mathcal{K}_{\phi^\tau}(f, t)$, for a fixed n, x and τ, we can choose $g = g_{n,x,\tau}$ such that

$$||f - g|| + \frac{\phi^{1-\tau}(x)}{\sqrt{n+1}} ||\phi^\tau g'|| \le 2\mathcal{K}_{\phi^\tau}\left(f, \frac{\phi^{1-\tau}(x)}{\sqrt{n+1}}\right). \tag{4.6}$$

Applying Lemma 3, we may write

$$\left|K_{n,\theta}^a(f; x) - f(x)\right| \le 2||f - g|| + |K_{n,\theta}^a(g; x) - g(x)|. \tag{4.7}$$

Using the representation $g(t) = g(x) + \int_x^t g'(u)du$ and Lemma 5, we obtain

$$\left|K_{n,\theta}^a(g;x)-g(x)\right| = \left|K_{n,\theta}^a\left(\int_x^t g'(u)du;x\right)\right|$$
$$\leq 2^\tau ||\phi^\tau g'|| \left\{\phi^{-\tau}(x)K_{n,\theta}^a(|t-x|;x) + x^{-\tau/2}K_{n,\theta}^a\left(\frac{|t-x|}{(1+t)^{\tau/2}};x\right)\right\}. \tag{4.8}$$

By using Cauchy-Schwarz inequality, (3.1) and Remark 1, we have

$$K_{n,\theta}^a(|t-x|;x) \leq \left(K_{n,\theta}^a((t-x)^2;x)\right)^{1/2} \leq \frac{\sqrt{\theta\lambda}\phi(x)}{\sqrt{n+1}}. \tag{4.9}$$

Similarly, from Lemma 6, we get

$$\begin{aligned} K_{n,\theta}^a\left(\frac{|t-x|}{(1+t)^{\tau/2}};x\right) &\leq \theta K_n^a\left(\frac{|t-x|}{(1+t)^{\tau/2}};x\right) \\ &\leq \theta\left(K_n^a((t-x)^2;x)\right)^{1/2}\left(K_n^a((1+t)^{-\tau};x)\right)^{1/2} \\ &\leq C_1\theta\frac{\sqrt{\lambda}\phi(x)}{\sqrt{n+1}}(1+x)^{-\tau/2}. \end{aligned} \tag{4.10}$$

By combining (4.8)–(4.10), we get

$$\left|K_{n,\theta}^a(g;x)-g(x)\right| \leq C_2||\phi^\tau g'||\frac{\phi^{1-\tau}(x)}{\sqrt{n+1}}. \tag{4.11}$$

Using (4.1), (4.6)–(4.7) and (4.11), we obtain the required relation (4.5). □

5 Rate of convergence

Let $f \in DBV_\gamma(0,\infty)$, $\gamma \geq 0$, be the class of all functions defined on $(0,\infty)$, having a derivative of bounded variation on every finite subinterval of $(0,\infty)$ and $|f(t)| \leq Mt^\gamma$, $\forall\ t > 0$.

We notice that the functions $f \in DBV_\gamma(0,\infty)$ possess a representation

$$f(x) = \int_0^x g(t)dt + f(0),$$

where $g(t)$ is a function of bounded variation on each finite subinterval of $(0,\infty)$.

Lemma 7 *Let $x \in (0,\infty)$, then for $\theta \geq 1$, $\lambda > 2$ and sufficiently large n, we have*

(i) $\xi_{n,\theta}^a(x,y) = \int_0^y M_{n,\theta}^a(x,t)dt \leq \dfrac{\theta\lambda}{n+1}\dfrac{\phi^2(x)}{(x-y)^2}$, $0 \leq y < x$,

(ii) $1-\xi_{n,\theta}^a(x,z) = \int_z^\infty M_{n,\theta}^a(x,t)dt \leq \dfrac{\theta\lambda}{n+1}\dfrac{\phi^2(x)}{(z-x)^2}$, $x < z < \infty$.

Proof (i) From (3.1) and Remark 1, we get

$$\begin{aligned} \xi_{n,\theta}^a(x,y) &= \int_0^y M_{n,\theta}^a(x,t)dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 M_{n,\theta}^a(x,t)dt \\ &\leq K_{n,\theta}^a((t-x)^2;x)\ (x-y)^{-2} \leq \theta K_n^a((t-x)^2;x)(x-y)^{-2} \\ &\leq \theta\frac{\lambda}{n+1}\frac{\phi^2(x)}{(x-y)^2}. \end{aligned}$$

The proof of (ii) is similar hence it is omitted.

Theorem 2 *Let $f \in DBV_{\gamma}(0,\infty), \theta \geq 1$ and let $\bigvee_c^d(f_x')$ be the total variation of f_x' on $[c,d] \subset (0,\infty)$. Then, for every $x \in (0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}
|K_{n,\theta}^a(f;x) - f(x)| \leq & \frac{\theta^{1/2}}{\theta+1}\sqrt{\frac{\lambda x(1+x)}{n+1}}|f'(x+)+\theta f'(x-)| \\
& + \frac{\theta^{3/2}}{\theta+1}\sqrt{\frac{\lambda x(1+x)}{n+1}}|f'(x+)-f'(x-)| \\
& + \theta\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}(f_x') + \frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f_x') \\
& + \theta\frac{\lambda(1+x)}{n+1}\sum_{k=0}^{[\sqrt{n}]}\bigvee_{x}^{x+(x/k)}(f_x') + \frac{x}{\sqrt{n}}\bigvee_{x}^{x+(x/\sqrt{n})}(f_x'),
\end{aligned}$$

where $\lambda > 2$, and the auxiliary function f_x' is defined by

$$f_x'(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x \\ 0, & t = x \\ f'(t) - f'(x+), & x < t < \infty \end{cases}.$$

Proof From the definition of the function $f_x'(t)$, for any $f \in DBV_{\gamma}(0,\infty)$, we may write

$$\begin{aligned}
f'(t) = & \frac{1}{\theta+1}\Big(f'(x+)+\theta f'(x-)\Big) + f_x'(t) \\
& + \frac{1}{2}\Big(f'(x+)-f'(x-)\Big)\Big(sgn(t-x)+\frac{\theta-1}{\theta+1}\Big) \\
& + \delta_x(t)\Big(f'(x) - \frac{1}{2}\Big(f'(x+)+f'(x-)\Big)\Big),
\end{aligned} \tag{5.1}$$

where

$$\delta_x(t) = \begin{cases} 1 & x = t \\ 0 & x \neq t \end{cases}.$$

From (2.2) and the fact that $\int_0^\infty M_{n,\theta}^a(x,t)dt = K_{n,\theta}^a(e_0;x) = 1$, we get

$$K_{n,\theta}^a(f;x) - f(x) = \int_0^\infty\Big(\int_x^t f'(u)du\Big)M_{n,\theta}^a(x,t)dt. \tag{5.2}$$

It is clear that

$$\int_0^\infty M_{n,\theta}^a(x,t)\int_x^t\Big[f'(x) - \frac{1}{2}\Big(f'(x+)+f'(x-)\Big)\Big]\delta_x(u)dudt = 0.$$

Thus, from (5.1), (5.2) and the Schwarz inequality for sufficiently large n, we have

$$\begin{aligned}
& \left|\int_0^\infty\Big(\int_x^t \frac{1}{\theta+1}\Big(f'(x+)+\theta f'(x-)\Big)du\Big)M_{n,\theta}^a(x,t)dt\right| \\
& \leq \frac{\sqrt{\theta}}{\theta+1}\Big|f'(x+)+\theta f'(x-)\Big|\sqrt{\frac{\lambda}{n+1}}\phi(x)
\end{aligned} \tag{5.3}$$

and by applying Cauchy–Schwarz inequality, we obtain

$$\left|\int_0^\infty \left(\int_x^t \frac{1}{2}\Big(f'(x+)-f'(x-)\Big)\Big(sgn(u-x)+\frac{\theta-1}{\theta+1}\Big)du\right)M_{n,\theta}^a(x,t)dt\right|$$
$$\le \frac{\theta}{\theta+1}\Big|f'(x+)-f'(x-)\Big|K_{n,\theta}^a\Big(|t-x|\,;x\Big)$$
$$\le \frac{\theta^{3/2}}{\theta+1}\Big|f'(x+)-f'(x-)\Big|\sqrt{\frac{\lambda}{n+1}}\phi(x). \tag{5.4}$$

By using Lemma 2, Remark 1 and considering (5.2)–(5.4) we obtain the following estimate

$$\begin{aligned}\left|K_{n,\theta}^a(f;x)-f(x)\right| &\le \left|U_{n,\theta}^a(f_x',x)+V_{n,\theta}^a(f_x',x)\right|\\ &\quad+\frac{\sqrt{\theta}}{\theta+1}\Big|f'(x+)+\theta f'(x-)\Big|\sqrt{\frac{\lambda}{n+1}}\phi(x)\\ &\quad+\frac{\theta^{3/2}}{\theta+1}\left|f'(x+)-f'(x-)\right|\sqrt{\frac{\lambda}{n+1}}\,\phi(x),\end{aligned} \tag{5.5}$$

where

$$U_{n,\theta}^a(f_x',x)=\int_0^x\left(\int_x^t f_x'(u)du\right)M_{n,\theta}^a(x,t)dt,$$

and

$$V_{n,\theta}^a(f_x',x)=\int_x^\infty\left(\int_x^t f_x'(u)du\right)M_{n,\theta}^a(x,t)dt.$$

Now, let us estimate the terms $U_{n,\theta}^a(f_x',x)$ and $V_{n,\theta}^a(f_x',x)$. Since $\int_c^d d_t\xi_{n,\theta}^a(x,t)\le 1$, for all $[c,d]\subseteq(0,\infty)$, using integration by parts and applying Lemma 7 with $y=x-(x/\sqrt{n})$, we have

$$\begin{aligned}\left|U_{n,\theta}^a(f_x',x)\right| &= \left|\int_0^x\int_x^t\left(f_x'(u)du\right)d_t\xi_{n,\theta}^a(x,t)\right|\\ &= \left|\int_0^x \xi_{n,\theta}^a(x,t)f_x'(t)dt\right|\\ &\le \theta\frac{\lambda\phi^2(x)}{n+1}\int_0^y\bigvee_t^x(f_x')(x-t)^{-2}dt+\int_y^x\bigvee_t^x(f_x')dt\\ &\le \theta\frac{\lambda\phi^2(x)}{n+1}\int_0^y\bigvee_t^x(f_x')(x-t)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^x(f_x').\end{aligned}$$

By the substitution of $u=x/(x-t)$, we obtain

$$\begin{aligned}\theta\frac{\lambda\phi^2(x)}{n+1}\int_0^{x-(x/\sqrt{n})}(x-t)^{-2}\bigvee_t^x(f_x')dt &= \theta\frac{\lambda(1+x)}{n+1}\int_1^{\sqrt{n}}\bigvee_{x-(x/u)}^x(f_x')du\\ &\le \theta\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}\ ]}\int_k^{k+1}\bigvee_{x-(x/u)}^x(f_x')du\\ &\le \theta\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_{x-(x/k)}^x(f_x').\end{aligned}$$

Hence we reach the following result

$$\left|U^a_{n,\theta}(f'_x,x)\right| \leq \theta\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_{x-(x/k)}^{x}\left(f'_x\right)+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f'_x). \tag{5.6}$$

Again, using integration by parts and applying Lemma 7 with $z=x+(x/\sqrt{n})$, we have

$$\begin{aligned}
|V^a_{n,\theta}(f'_x,x)| &= \left|\int_x^{\infty}\left(\int_x^t f'_x(u)du\right)d_t(1-\xi^a_{n,\theta}(x,t))\right|\\
&= \left|\int_x^z f'_x(t)(1-\xi^a_{n,\theta}(x,t))dt+\int_z^{\infty} f'_x(t)(1-\xi^a_{n,\theta}(x,t))dt\right|\\
&< \theta\frac{\lambda\phi^2(x)}{n+1}\int_z^{\infty}\bigvee_x^t(f'_x)(t-x)^{-2}dt+\int_x^z\bigvee_x^t(f'_x)dt\\
&\leq \theta\frac{\lambda\phi^2(x)}{n+1}\int_{x+(x/\sqrt{n})}^{\infty}\bigvee_x^t(f'_x)(t-x)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_x^{x+(x/\sqrt{n})}(f'_x). \qquad (5.7)
\end{aligned}$$

By the substitution of $u=x/(t-x)$ as in the estimate of $U^a_{n,\theta}(f'_x,x)$, we get

$$\begin{aligned}
\theta\frac{\lambda\phi^2(x)}{n+1}\int_{x+(x/\sqrt{n})}^{\infty}\bigvee_x^t(f'_x)(t-x)^{-2}dt &= \theta\frac{\lambda\phi^2(x)}{x(n+1)}\int_0^{\sqrt{n}}\bigvee_x^{x+(x/u)}(f'_x)du\\
&\leq \theta\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_x^{x+(x/k)}(f'_x). \qquad (5.8)
\end{aligned}$$

Now, combining (5.7)–(5.8), we obtain

$$|V^a_{n,\theta}(f'_x,x)| \leq \theta\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_x^{x+(x/k)}(f'_x)+\frac{x}{\sqrt{n}}\bigvee_x^{x+(x/\sqrt{n})}(f'_x). \tag{5.9}$$

By collecting the estimates (5.5), (5.6) and (5.9), we get the required result. This completes the proof of theorem. □

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References

1. Mihesan, V.: Uniform approximation with positive linear operators generated by generalized Baskakov method. Autom. Comput. Appl. Math. **7**(1), 34–37 (1998)
2. Agrawal, P.N., Goyal, M.: Generalized Baskakov Kantorovich operators, filomat. arXiv:1509.02328v1 (accepted)
3. Zhang, C., Zhu, Z.: Preservation properties of the Baskakov–Kantorovich operators. Comput. Math. Appl. **57**(9), 1450–1455 (2009)
4. Bojanic, R., Cheng, F.: Rate of convergence of Bernstein polynomials for functions with derivatives of bounded variation. J. Math. Anal. Appl. **141**(1), 136–151 (1989)
5. Bojanic, R., Cheng, F.: Rate of convergence of Hermite-Fejer polynomials for functions with derivatives of bounded variation. Acta. Math. Hung. **59**(1–2), 91–102 (1992)

6. Guo, S.: On the rate of convergence of the Durrmeyer operators for functions of bounded variation. J. Approx. Theory **51**, 183–192 (1987)
7. Zeng, X.M., Chen, W.: On the rate of convergence of the generalized Durrmeyer type operators for functions of bounded variation. J. Approx. Theory **102**, 1–12 (2000)
8. Zeng, X.M., Tao, W.: Rate of convergence of the integral type Lupas–Bèzier operators. Kyungpook Math. J. **43**, 593–604 (2003)
9. Abel, U., Gupta, V.: An estimate of the rate of convergence of a Bèzier variant of the Baskakov–Kantorovich operators for bounded variation functions. Demonstr. Math. **36**(1), 123–136 (2003)
10. Gupta, V.: An estimate on the convergence of Baskakov–Bèzier operators. J. Math. Anal. Appl. **312**, 280–288 (2005)
11. Guo, S., Qi, Q., Liu, G.: The central approximation theorems for Baskakov Bèzier operators. J. Approx. Theory **147**, 112–124 (2007)
12. Gupta, V.: Simultaneous approximation for Bèzier variant of Szász–Mirakyan–Durrmeyer operators. J. Math. Anal. Appl. **328**(1), 101–105 (2007)
13. Guo, S., Jiang, H., Qi, Q.: Approximation by Bèzier type of Meyer–König and Zeller operators. Comput. Math. Appl. **54**, 1387–1394 (2007)
14. Ispir, N., Yüksel, I.: On the Bèzier variant of Srivastava–Gupta operators. Appl. Math. E-Notes **5**, 129–137 (2005)
15. Agrawal, P.N., Kajla, A.: Bèzier variant of the generalized Baskakov–Durrmeyer type operators. Math. Sci. (accepted)
16. Ditzian, Z., Totik, V.: Moduli of Smoothness. Springer, New York (1987)