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On (p, q)-generalization of Szász-Mirakyan Kantorovich operators

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Abstract In this paper, we introduce Szász-Mirakyan Kantorovich type of operators based on (p, q)-calculus. Using Krovokin's type theorem, we show that operator converges uniformly. In second section, we study rate of convergence of operator using modulus of continuity and Peetre's K-functional. In last section, we give Voronovskaya type results for operator.

Keywords q-Calculus · (p, q)-Calculus · (p, q)-Szász-Mirakyan operator · Modulus of continuity · Peetre K-functional

Mathematics Subject Classification 41A25 · 41A35

1 Introduction

In 1987, Lupaş [1] introduced the first q-analogue of the classical Bernstein polynomials. After that, with rapid development of q-calculus, new q-analogue of various positive linear operators are introduced by many researchers (for details see [2]). For detail study of q-calculus one can refer to [3,4].

We begin by recalling certain notations of (p, q)-calculus (for detail see [5–7]). Let $0 < q < p \leq 1$. The (p, q)-integer $[n]_{p,q}$ and (p, q)-factorial $[n]_{p,q}!$ are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} \quad n = 0, 1, 2 \ldots$$

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$$[n]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q}\dots[n]_{p,q}, & \text{n} \geq 1 \\ 1, & \text{n} = 0 \end{cases}.$$

For integers $0 \leq k \leq n$, (p, q)-binomial is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

The (p, q)-polynomials expansion is

$$(x+y)^n_{p,q} = (x+y)(px+qy)(p^2x+q^2y)\cdots(p^{n-1}x+q^{n-1}y).$$

Let $f : R \to R$, then the (p, q)-derivative of function f is:

$$(D_{p,q}f) = \frac{f(px)-f(qx)}{(p-q)x} \quad x \neq 0, \quad (D_{p,q}f)(0) = f(0)$$

provided that f is differentiable at 0.

Let $f : C[0, a] \to R$, then the (p, q)-integration of a function f is defined by,

$$\int_0^a f(t)d_{p,q}t = (q-p)a\sum_{k=0}^{\infty}\frac{p^k}{q^{k+1}}f\left(\frac{p^k}{q^{k+1}}a\right), \quad when \quad \left|\frac{p}{q}\right| < 1,$$

and

$$\int_0^a f(t)d_{p,q}t = (p-q)a\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}a\right), \quad when \quad \left|\frac{p}{q}\right| > 1. \tag{1.1}$$

Two (p, q)-analogues of the exponential function (see [8]) are as:

$$e_{p,q}(x) = \sum_{n=0}^{\infty}\frac{p^{\frac{n(n-1)}{2}}x^n}{[n]_{p,q}!},$$

$$E_{p,q}(x) = \sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}x^n}{[n]_{p,q}!},$$

which satisfy the equality $e_{p,q}(x)E_{p,q}(-x) = 1$. For $p = 1$, all the notations of (p, q)-calculus are reduced to q-calculus.

Recently, Tuncer Acar [9] introduced a (p, q)-analogue of Szász-Mirakyan operators as: For $0 < q < p \leq 1$, $n \in \mathbb{N}$ and $f : [0, \infty) \to R$:

$$S_{n,p,q}(f; x) = \sum_{k=0}^{\infty} s_n(p, q; x) f\left(\frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}}\right),$$

where

$$s_n(p, q; x) = \frac{1}{E_{p,q}([n]_{p,q}x)}q^{\frac{k(k-1)}{2}}\frac{[n]^k_{p,q}x^k}{[k]_{p,q}!}.$$

Lemma 1 [9, Lemma 2] *Let* $0 < q < p \leq 1$ *and* $n \in \mathbb{N}$. *We have*

$$S_{n,p,q}(1; x) = 1,$$
$$S_{n,p,q}(t; x) = qx,$$
$$S_{n,p,q}(t^2; x) = pqx^2 + \frac{q^2x}{[n]_{p,q}}.$$

2 Construction of (p, q)-Szász-Mirakyan Kantorovich operators

Motivated by Tuncer Acar, we set (p, q)-Szász-Mirakyan Kantorovich operators for $0 < q < p \leq 1$, $n \in \mathbb{N}$ and $f : [0, \infty) \to \mathbb{R}$ as:

$$K_n^{(p,q)}(f; x) = [n]_{p,q} \sum_{k=0}^{\infty} s_n(p, q; x) p^{-k} q^{k-2} \int_{\frac{q^{-k+3}[k]_{p,q}}{[n]_{p,q}}}^{\frac{q^{-k+2}[k+1]_{p,q}}{[n]_{p,q}}} f(t) d_{p,q}t. \tag{2.1}$$

Lemma 2 *Let* $0 < q < p \leq 1$ *and* $n \in \mathbb{N}$, *we have*

$$q^{k-2} \int_{\frac{q^{-k+3}[k]_{p,q}}{[n]_{p,q}}}^{\frac{q^{-k+2}[k+1]_{p,q}}{[n]_{p,q}}} d_{p,q}t = \frac{p^k}{[n]_{p,q}},$$

$$q^{k-2} \int_{\frac{q^{-k+3}[k]_{p,q}}{[n]_{p,q}}}^{\frac{q^{-k+2}[k+1]_{p,q}}{[n]_{p,q}}} t d_{p,q}t = \frac{p^k q^{-k+2}([k+1]_{p,q} + q[k]_{p,q})}{(p+q)[n]_{p,q}^2},$$

$$q^{k-2} \int_{\frac{q^{-k+3}[k]_{p,q}}{[n]_{p,q}}}^{\frac{q^{-k+2}[k+1]_{p,q}}{[n]_{p,q}}} t^2 d_{p,q}t = \frac{p^k q^{-2k+4}([k+1]_{p,q}^2 + q[k]_{p,q}[k+1]_{p,q} + q^2[k]_{p,q}^2)}{(p^2 + pq + q^2)[n]_{p,q}^3}.$$

Proof Using the identity $[k+1]_{p,q} = p^k + q[k]_{p,q}$ and Eq. (1.1), lemma can be proved. □

Lemma 3 *Let* $0 < q < p \leq 1$ *and* $n \in N$. *We have*

$$K_n^{(p,q)}(1; x) = 1, \tag{2.2}$$

$$K_n^{(p,q)}(t; x) = qx + \frac{q^2}{[n]_{p,q}(p+q)}, \tag{2.3}$$

$$K_n^{(p,q)}(t^2; x) = pqx^2 + \frac{(2q^4 + 3pq^3 + p^2q^2)x}{(p^2 + pq + q^2)[n]_{p,q}} + \frac{q^4}{(p^2 + pq + q^2)[n]_{p,q}^2}. \tag{2.4}$$

Proof Using definition of operator, Lemmas 1 and 2, moments can be obtained as follows:

$$\begin{aligned} K_n^{(p,q)}(1; x) &= [n]_{p,q} \sum_{k=0}^{\infty} s_n(p, q; x) p^{-k} q^{k-2} \int_{\frac{q^{-k+3}[k]_{p,q}}{[n]_{p,q}}}^{\frac{q^{-k+2}[k+1]_{p,q}}{[n]_{p,q}}} d_{p,q}t \\ &= [n]_{p,q} \sum_{k=0}^{\infty} s_n(p, q; x) p^{-k} \frac{p^k}{[n]_{p,q}} \\ &= \sum_{k=0}^{\infty} s_n(p, q; x) = S_{n,p,q}(1; x) = 1. \end{aligned}$$

And using identity $[k+1]_{p,q} = q^k + p[k]_{p,q}$, we get

$$\begin{aligned}
K_n^{(p,q)}(t;x) &= [n]_{p,q} \sum_{k=0}^{\infty} s_n(p,q;x) p^{-k} q^{k-2} \int_{\frac{q^{-k+3}[k]_{p,q}}{[n]_{p,q}}}^{\frac{q^{-k+2}[k+1]_{p,q}}{[n]_{p,q}}} t d_{p,q} t \\
&= [n]_{p,q} \sum_{k=0}^{\infty} s_n(p,q;x) p^{-k} \frac{p^k q^{-k+2}([k+1]_{p,q} + q[k]_{p,q})}{(p+q)[n]_{p,q}^2} \\
&= \frac{1}{[n]_{p,q}(p+q)} \sum_{k=0}^{\infty} s_n(p,q;x) q^{-k+2}(q^k + (p+q)[k]_{p,q}) \\
&= \frac{q^2}{[n]_{p,q}(p+q)} \sum_{k=0}^{\infty} s_n(p,q;x) + \sum_{k=0}^{\infty} s_n(p,q;x) \frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}} \\
&= \frac{q^2}{[n]_{p,q}(p+q)} S_{n,p,q}(1;x) + S_{n,p,q}(t;x) \\
&= \frac{q^2}{[n]_{p,q}(p+q)} + qx.
\end{aligned}$$

Finally,

$$\begin{aligned}
&K_n^{(p,q)}(t^2;x) \\
&= [n]_{p,q} \sum_{k=0}^{\infty} s_n(p,q;x) p^{-k} q^{k-2} \int_{\frac{q^{-k+3}[k]_{p,q}}{[n]_{p,q}}}^{\frac{q^{-k+2}[k+1]_{p,q}}{[n]_{p,q}}} t^2 d_{p,q} t \\
&= [n]_{p,q} \sum_{k=0}^{\infty} s_n(p,q;x) p^{-k} \frac{p^k q^{-2k+4} \left([k+1]_{p,q}^2 + q[k]_{p,q}[k+1]_{p,q} + q^2[k]_{p,q}^2\right)}{(p^2+pq+q^2)[n]_{p,q}^3} \\
&= [n]_{p,q} \sum_{k=0}^{\infty} s_n(p,q;x) \frac{q^{-2k+4} \left((p^2+pq+q^2)[k]_{p,q}^2 + q^k(2p+q)[k]_{p,q} + q^{2k}\right)}{(p^2+pq+q^2)[n]_{p,q}^3} \\
&= \sum_{k=0}^{\infty} s_n(p,q;x) \frac{[k]_{p,q}^2}{q^{2k-4}[n]_{p,q}^2} + \frac{(2p+q)q^2}{(p^2+pq+q^2)[n]_{p,q}} \sum_{k=0}^{\infty} s_n(p,q;x) \frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}} \\
&\quad + \frac{q^4}{(p^2+pq+q^2)[n]_{p,q}^2} \sum_{k=0}^{\infty} s_n(p,q;x) \\
&= S_{n,p,q}(t^2;x) + \frac{(2p+q)q^2}{(p^2+pq+q^2)[n]_{p,q}} S_{n,p,q}(t;x) + \frac{q^4}{(p^2+pq+q^2)[n]_{p,q}^2} S_{n,p,q}(1;x) \\
&= pqx^2 + \frac{q^2 x}{[n]_{p,q}} + \frac{(2p+q)q^3 x}{(p^2+pq+q^2)[n]_{p,q}} + \frac{q^4}{(p^2+pq+q^2)[n]_{p,q}^2} \\
&= pqx^2 + \frac{(2q^4+3pq^3+p^2q^2)x}{(p^2+pq+q^2)[n]_{p,q}} + \frac{q^4}{(p^2+pq+q^2)[n]_{p,q}^2}.
\end{aligned}$$

□

Corollary 1 *Central moments* $\Phi_m^{(p,q)}(x) = K_n^{(p,q)}((t-x)^m; x)$ *for* $m = 1, 2$ *are:*

$$\Phi_1^{(p,q)}(x) = (q-1)x + \frac{q^2}{(p+q)[n]_{p,q}},$$

$$\Phi_2^{(p,q)}(x) = (pq - 2q + 1)x^2 + \left(\frac{2q^4 + 3pq^3 + p^2q^2}{(p^2+pq+q^2)[n]_{p,q}} - \frac{2q^2}{(p+q)[n]_{p,q}}\right)x + \frac{q^4}{(p^2+pq+q^2)[n]_{p,q}^2}.$$

Proof Using Lemma 3, central moments can be obtained directly. □

Remark 1 For $q \in (0,1)$ and $p \in (q, 1]$, by simple computations $\lim_{n\to\infty}[n]_{p,q} = 1/(p-q)$. In order to obtain results for order of convergence of the operator, we take $q_n \in (0,1)$, $p_n \in (q_n, 1]$ such that $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} q_n = 1$, so that $\lim_{n\to\infty} \frac{1}{[n]_{p_n,q_n}} = 0$. Such a sequence can always be constructed for example, we can take $q_n = 1 - 1/n$ and $p_n = 1 - 1/2n$, clearly $\lim_{n\to\infty} p_n^n = e^{-1/2}$, $\lim_{n\to\infty} q_n^n = e^{-1}$ and $\lim_{n\to\infty} \frac{1}{[n]_{p_n,q_n}} = 0$.

Theorem 2 *Let* $(p_n)_n$ *and* $(q_n)_n$ *be the sequence as defined in Remark 1. Then for each* $f \in C[0,\infty)$, $K_n^{(p_n,q_n)}(f; x)$ *converges uniformly to* f.

Proof By Korovkin theorem, it is sufficient to show that $\lim_{n\to\infty} \|K_n^{(p_n,q_n)}(t^m; x) - x^m\|_{C[0,\infty)} = 0$ for $m = 0, 1, 2$.

Using Eq. 2.2, result for $m = 0$ is trivial. For $m = 1$, result can be obtained using Eq. (2.3), as follows:

$$\begin{aligned}
\lim_{n\to\infty} \|K_n^{(p_n,q_n)}(t; x) - x\|_{C[0,\infty)} &= \lim_{n\to\infty} \left|\frac{q_n^2}{(p_n+q_n)[n]_{p,q}} + q_n x - x\right| \\
&\le \lim_{n\to\infty} \left|\frac{q_n^2}{(p_n+q_n)[n]_{p,q}}\right| + \lim_{n\to\infty} \left|q_n - 1\right| x \\
&= 0.
\end{aligned}$$

Finally, using Eq. (2.4), we get

$$\begin{aligned}
\lim_{n\to\infty} \|K_n^{(p_n,q_n)}(t^2; x) - x^2\|_{C[0,\infty)} &= \lim_{n\to\infty} \left| \frac{q_n^4}{(p_n^2 + q_n^2 + p_n q_n)[n]_{p_n,q_n}^2} \right. \\
&\quad \left. + \frac{(2q_n^4 + 3p_n q_n^3 + p_n^2 q_n^2)x}{(p_n^2 + p_n q_n + q_n^2)[n]_{p_n,q_n}} + p_n q_n x^2 - x^2 \right| \\
&\le \lim_{n\to\infty} \left| \frac{q_n^4}{(p_n^2 + q_n^2 + p_n q_n)[n]_{p_n,q_n}^2} \right| \\
&\quad + \lim_{n\to\infty} |p_n q_n - 1| x^2 \\
&\quad + \lim_{n\to\infty} \left| \frac{(2q_n^4 + 3p_n q_n^3 + p_n^2 q_n^2)x}{(p_n^2 + p_n q_n + q_n^2)[n]_{p_n,q_n}} \right| \\
&= 0.
\end{aligned}$$

□

3 Direct results

In this section, we give some local result for the operator. Let $C_B[0,\infty)$ be the space of all real valued continuous bounded functions defined on $[0,\infty)$. The norm on the space $C_B[0,\infty)$ is the supremum norm $\|f\| = \sup_{x\in[0,\infty)} f(x)$. Further, Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf_{g\in W^2}\{\|f-g\| + \delta\|g''\|\},$$

here $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By [10, p. 177, Theorem 2.4], there exists a positive constant $C > 0$ such that $K_2(f,\delta) \leq C\omega_2(f,\delta^{\frac{1}{2}})$, $\delta > 0$, where

$$\omega_2(f,\delta^{\frac{1}{2}}) = \sup_{0<h<\delta^{\frac{1}{2}},x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of continuity of function $f \in C_B[0,\infty)$. Also, for $f \in C_B[0,\infty)$ the usual modulus of continuity is given by

$$\omega(f,\delta^{\frac{1}{2}}) = \sup_{0<h<\delta^{\frac{1}{2}},x\in[0,\infty)} |f(x+h) - f(x)|.$$

Theorem 3 *Let $(p_n)_n$ and $(q_n)_n$ be the sequence as defined in Remark 1. Let $f \in C_B[0,\infty)$. Then for all $n \in \mathbb{N}$, there exists an absolute constant $C > 0$ such that*

$$|K_n^{(p_n,q_n)}(f;x) - f(x)| \leq C\omega_2(f,\delta_n(x)) + \omega(f,\alpha_n(x)),$$

where

$$\delta_n(x) = \left\{\Phi_2^{(p_n,q_n)}(x) + (\Phi_1^{(p_n,q_n)}(x))^2\right\}^{\frac{1}{2}}$$

and

$$\alpha_n(x) = \left|\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)} + (q_n-1)x\right|.$$

Proof For $x \in [0,\infty)$, we consider the auxiliary operators $K_n^*(f;x)$ defined by

$$K_n^*(f;x) = K_n^{(p_n,q_n)}(f;x) + f(x) - f\left(\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)} + q_n x\right).$$

Using above operator and Eq. (2.3), we have

$$\begin{aligned} K_n^*(t-x;x) &= K_n^{(p_n,q_n)}(t-x;x) - \left(\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)} + q_n x - x\right) \\ &= K_n^{(p_n,q_n)}(t;x) - xK_n^{(p_n,q_n)}(1;x) - \left(\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)} + q_n x\right) + x \\ &= 0 \end{aligned}$$

Let $x \in [0,\infty)$ and $g \in W^2$. Using the Taylor's formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

Applying K_n^* to both sides of the above equation, we have

$$\begin{aligned}
K_n^*(g;x)-g(x) &= K_n^*((t-x)g'(x);x)+K_n^*\left(\int_x^t (t-u)g''(u)du;x\right)\\
&= g'(x)K_n^*((t-x);x)+K_n^{(p_n,q_n)}\left(\int_x^t (t-u)g''(u)du;x\right)\\
&\quad -\int_x^{\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)}+q_nx}\left(\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)}+q_nx-u\right)g''(u)du\\
&= K_n^{(p_n,q_n)}\left(\int_x^t (t-u)g''(u)du;x\right)\\
&\quad -\int_x^{\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)}+q_nx}\left(\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)}+q_nx-u\right)g''(u)du.
\end{aligned}$$

On the other hand,

$$\left|\int_x^t (t-u)g''(u)du\right| \le \int_x^t |t-u||g''(u)|du \le \|g''\|\int_x^t |t-u|du \le (t-x)^2\|g''\|,$$

and

$$\begin{aligned}
&\left|\int_x^{\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)}+q_nx}\left(\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)}+q_nx-u\right)g''(u)du\right|\\
&\quad\le \left(\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)}+q_nx-x\right)^2\|g''\|.
\end{aligned}$$

Therefore, we can conclude that

$$\begin{aligned}
|K_n^*(g;x)-g(x)| &= \left|K_n^{(p_n,q_n)}\left(\int_x^t (t-u)g''(u)du;x\right)\right|\\
&\quad +\left|\int_x^{\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)}+q_nx}\left(\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)}+q_nx-u\right)g''(u)du\right|\\
&\le \|g''\|K_n^{(p_n,q_n)}((t-x)^2;x)+\left(\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)}+q_nx-x\right)^2\|g''\|\\
&= \delta_n^2(x)\|g''\|.
\end{aligned}$$

Also, we have

$$|K_n^*(f;x)| \le |K_n^{(p_n,q_n)}(f;x)|+2\|f\| \le 3\|f\|.$$

Therefore,

$$\begin{aligned}
&|K_n^{(p_n,q_n)}(f;x)-f(x)|\\
&\leq |K_n^*(f-g;x)-(f-g)(x)|+\left|f\left(\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)}+q_nx\right)-f(x)\right|\\
&\quad+|K_n^*(g;x)-g(x)|\\
&\leq |K_n^*(f-g;x)|+|(f-g)(x)|+\left|f\left(\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)}+q_nx\right)-f(x)\right|\\
&\quad+|K_n^*(g;x)-g(x)|\\
&\leq 4\|f-g\|+\omega\left(f;\left|\frac{q_n^2}{[n]_{p_n,q_n}(p_n+q_n)}+(q_n-1)x\right|\right)+\delta_n^2(x)\|g''\|.
\end{aligned}$$

Hence, taking the infimum on the right-hand side over all $g\in W^2$, we get

$$|K_n^{(p_n,q_n)}(f;x)-f(x)|\leq 4K_2(f,\delta_n^2(x))+\omega(f,\alpha_n(x)).$$

By using property of K-functional, we get

$$|K_n^{(p_n,q_n)}(f;x)-f(x)|\leq C\omega_2(f,\delta_n(x))+\omega(f,\alpha_n(x)).$$

Hence the theorem. □

We consider following class of functions: $H_{x^2}[0,\infty)=\{f:[0,\infty)\to\mathbb{R}:|f(x)|\leq M_f(1+x^2)$ here M_f is constant depending on the function $f\}$,

$C_{x^2}[0,\infty)=\left\{f\in H_{x^2}[0,\infty):f \text{ is continuous}\right\}$,

$C^*_{x^2}[0,\infty)=\left\{f\in C_{x^2}[0,\infty):\lim_{|x|\to\infty}\frac{f(x)}{1+x^2}\text{ is finite}\right\}$. The norm on the space $C^*_{x^2}[0,\infty)$ is defined as $\|f\|_{x^2}=\sup_{x\in[0,\infty)}\frac{f(x)}{1+x^2}$. We denote the modulus of continuity of f on closed interval $[0,a]$, $a>0$ as:

$$\omega_a(f;\delta)=\sup_{|t-x|\leq\delta,\ x,t\in[0,a]}|f(t)-f(x)|.$$

Theorem 4 *Let $(p_n)_n$ and $(q_n)_n$ be the sequence as defined in Remark 1. Then for $f\in C_{x^2}[0,\infty)$, $\omega_{a+1}(f;\delta)$ be its modulus of continuity on the interval $[0,a+1]\subset[0,\infty)$, $a>0$ and for every $n>1$,*

$$\|K_n^{(p_n,q_n)}(f;x)-f\|_{C[0,a]}\leq 6M_f(1+a^2)\lambda_n+2\omega_{a+1}(f;\sqrt{\lambda_n}),$$

here,

$$\lambda_n=(1-p_nq_n)a^2+\frac{1}{[n]_{p_n,q_n}(p_n+q_n)(p_n^2+p_nq_n+q_n^2)}\left(\frac{6a}{p_n+q_n}+\frac{1}{[n]_{p_n,q_n}}\right).$$

Proof For $x\in[0,a]$ and $t\geq 0$, we have (see [11, Equation 3.3])

$$\begin{aligned}|f(t)-f(x)|&\leq 6M_f(1+a^2)(t-x)^2\\&\quad+\omega_{a+1}(f;\delta_n)\left(\frac{|t-x|}{\delta_n}+1\right).\end{aligned}$$

By using above inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|K_n^{(p_n,q_n)}(f(t);x)-f(x)\|_{C[0,a]} &\leq K_n^{(p_n,q_n)}(|f(t)-f(x)|;x)\\ &\leq 6M_f(1+a^2)K_n^{(p_n,q_n)}((t-x)^2;x)\\ &\quad +\omega_{a+1}(f;\delta_n)\left(1+\frac{1}{\delta_n^2}K_n^{(p_n,q_n)}((t-x)^2;x)\right)^{1/2}.\end{aligned}$$

For $x \in [0, a]$, using Corollary 1,

$$\begin{aligned}&K_n^{(p_n,q_n)}((t-x)^2;x)\\ &= (p_nq_n-2q_n+1)x^2+\left(\frac{2q_n^4+3p_nq_n^3+p_n^2q_n^2}{(p_n^2+p_nq_n+q^2)[n]_{p_n,q_n}}-\frac{2q_n^2}{(p_n+q_n)[n]_{p_n,q_n}}\right)x\\ &\quad+\frac{q_n^4}{(p_n^2+p_nq_n+q_n^2)[n]_{p_n,q_n}^2}\\ &\leq (q_n(p_n-1)-q_n+1)a^2+\frac{6a}{(p_n^2+p_nq_n+q^2)[n]_{p_n,q_n}}+\frac{1}{(p_n^2+p_nq_n+q_n^2)[n]_{p_n,q_n}^2}\\ &\leq (q_n(1-p_n)-q_n+1)a^2+\frac{1}{(p_n^2+p_nq_n+q^2)[n]_{p_n,q_n}}\left(\frac{6a}{p_n+q_n}+\frac{1}{[n]_{p_n,q_n}}\right)\\ &\leq (1-p_nq_n)a^2+\frac{1}{(p_n^2+p_nq_n+q^2)[n]_{p_n,q_n}}\left(\frac{6a}{p_n+q_n}+\frac{1}{[n]_{p_n,q_n}}\right)=\lambda_n.\end{aligned}$$

Taking $\delta_n = \sqrt{\lambda_n}$, we will get the theorem. □

4 Voronovskaya type theorem

Theorem 5 *Let $0 < q_n < p_n \leq 1$, such that $p_n \to 1$, $p_n \to 1$, $p_n^n \to a$ and $q_n^n \to b$ as $n \to \infty$. For any $f \in C^*_{x^2}[0,\infty)$, such that $f', f'' \in C^*_{x^2}[0,\infty)$, we have*

$$\lim_{n\to\infty}[n]_{p_n,q_n}|K_n^{(p_n,q_n)}(f;x)-f(x)| = (\alpha x+1/2)f'(x)+x(\gamma x+1)f''(x)/2$$

uniformly on $[0, A]$ for any $A > 0$. Here $\alpha = \lim_{n\to\infty}[n]_{p_n,q_n}(q_n-1)$ and $\gamma = [n]_{p_n,q_n}\lim_{n\to\infty}(p_nq_n-2q_n+1)$.

Proof By the Taylor's formula, we have

$$f(t) = f(x)+(t-x)f'(x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2,$$

here $r(t, x)$ is reminder term and $\lim_{t\to x} r(t, x) = 0$. Therefore,

$$\begin{aligned}[n]_{p_n,q_n}(K_n^{(p_n,q_n)}(f;x)-f(x)) &= [n]_{p_n,q_n}f'(x)K_n^{(p_n,q_n)}((t-x);x)\\ &\quad+[n]_{p_n,q_n}\frac{f''(x)}{2}K_n^{(p_n,q_n)}((t-x)^2;x)\\ &\quad+[n]_{p_n,q_n}K_n^{(p_n,q_n)}(r(t,x)(t-x)^2;x).\end{aligned}$$

By the Cauchy-Schwartz inequality, we have

$$K_n^{(p_n,q_n)}(r(t,x)(t-x)^2;x) \leq \sqrt{K_n^{(p_n,q_n)}(r^2(t,x);x)}\sqrt{K_n^{(p_n,q_n)}((t-x)^4;x)}.$$

As $r(t,x) \in C^*_{x^2}[0,\infty)$, therefore by Theorem 2 and fact that $\lim_{t\to x} r(t,x) = 0$, we get

$$\lim_{n\to\infty} K_n^{(p_n,q_n)}(r^2(t,x);x) = r^2(x,x) = 0,$$

uniformly for any $x \in [0, A]$. Hence, by using above equality and positivity of linear operator, we have

$$\lim_{n\to\infty} [n]_{p_n,q_n} K_n^{(p_n,q_n)}(r(t,x)(t-x)^2;x) = 0,$$

Therefore,

$$\lim_{n\to\infty} [n]_{p_n,q_n}(K_n^{(p_n,q_n)}(f;x) - f(x)) = \lim_{n\to\infty} [n]_{p_n,q_n} f'(x) K_n^{(p_n,q_n)}((t-x);x) + \lim_{n\to\infty} [n]_{p_n,q_n} \frac{f''(x)}{2} K_n^{(p_n,q_n)}((t-x)^2;x). \tag{4.1}$$

Consider,

$$\begin{aligned}\lim_{n\to\infty} [n]_{p_n,q_n} K_n^{(p_n,q_n)}((t-x);x) &= \lim_{n\to\infty} [n]_{p_n,q_n} \Phi_1^{(p_n,q_n)}(x)\\ &= \lim_{n\to\infty} [n]_{p_n,q_n}(q_n - 1)x + 1/2\\ &= \alpha x + 1/2,\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\lim_{n\to\infty} [n]_{p_n,q_n} K_n^{(p_n,q_n)}((t-x)^2;x) &= \lim_{n\to\infty} [n]_{p_n,q_n} \Phi_2^{(p_n,q_n)}(x)\\ &= \lim_{n\to\infty} [n]_{p_n,q_n}(p_n q_n - 2q_n + 1)x^2 + x\\ &= x(\gamma x + 1).\end{aligned} \tag{4.3}$$

Hence, by using Eqs. (4.1), (4.2) and (4.3), we get the theorem. □

References

1. Lupaş, A.: A $q-$analogue of the Bernstein operator. University of Cluj-Napoca, Seminar on Numerical and Statistical Calculus **9**, 85–92 (1987)
2. Aral, A., Gupta, V., Agarwal, R.P.: Applications of $q-$calculus in operator theory. Springer (2013)
3. Ernst, T.: The history of q-calculus and a new method. U.U.D.M. Report 16, Uppsala, Departament of Mathematics, Uppsala University (2000)
4. Kac, V., Cheung, P.: Quantum calculus. Universitext, Springer-Verlag, New York (2002)
5. Sadjang, P.N.: On the fundamental theorem of $(p,q)-$Taylor formulas. arXiv:1309.3934 [math.QA]
6. Chakrabarti, R., Jagannathan, R.: A (p,q)-oscillator realization of two parameter quantum algebras. J. Phys. A Math. Gen. **24**, 711–718 (1991)
7. Sahai, V., Yadav, S.: Representations of two parameter quantum algebras and (p,q)-special functions. J. Math. Anal. Appl. **335**, 268–279 (2007)
8. Jagannathan, R., Rao, K.S., Two-parameter quantum algebras, twin-basic numbers and associated generalized hypergeometric series. In: Proceedings of the international conference on number theory and mathematical physics, pp. 20–21 (2005)
9. Acar, T.: (p,q)-genralization of Sźasz-Mirakyan operators. arXiv:1505.06839 [math.CA]
10. DeVore, R.A., Lorentz, G.G.: Construtive approximation. Springer, Berlin (1993)
11. Gupta, V., Aral, A.: Convergence of the q-analogue of Szász-Beta operators. Appl. Math. Comput. **216**, 374–380 (2010)