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Wellposedness of bounded solutions of the non-homogeneous initial boundary for the short pulse equation

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Abstract The short pulse equation provides a model for the propagation of ultra-short light pulses in silica optical fibers. It is a nonlinear evolution equation. In this paper the wellposedness of bounded solutions for the inhomogeneous initial boundary value problem associated to this equation is studied.

Keywords Existence · Uniqueness · Stability · Entropy solutions · Conservation laws · Short pulse equation · Boundary value problems

Mathematics Subject Classification 35G15 · 35L65 · 35L05 · 35A05

1 Introduction

The short pulse equation has the form

$$2\partial_{x_1}\partial_\phi A_0 + \chi^{(3)}\partial^2_{\phi\phi}A_0^3 + \frac{1}{c_2^2}A_0 = 0, \tag{1.1}$$

where A_0 is the light wave amplitude, $\phi = \frac{t-x}{\varepsilon}$, $x_1 = \varepsilon x$, ε is a small scale parameter, and $\chi^{(3)}$ is the third order magnetic susceptibility (1.1) was introduced recently by Schäfer and Wayne [22] as a model equation describing the propagation of ultra-short light pulses in silica optical fibers. It provides also an approximation of nonlinear wave packets in dispersive

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media in the limit of few cycles on the ultra-short pulse scale. Numerical simulations [5] show that the short pulse equation approximation to Maxwell's equations in the case when the pulse spectrum is not narrowly localized around the carrier frequency is better than the one obtained from the nonlinear Schrödinger equation, which models the evolution of slowly varying wave trains. Such ultra-short plays a key role in the development of future technologies of ultra-fast optical transmission of informations.

In [4] the author studied a new hierarchy of equations containing the short pulse equation (1.1) and the elastic beam equation, which describes nonlinear transverse oscillations of elastic beams under tension. He showed that the hierarchy of equations is integrable. He obtained the two compatible Hamiltonian structures and constructed an infinite series of both local and nonlocal conserved charges. Moreover, he gave the Lax description for both systems. The integrability and the existence of solitary wave solutions have been studied in [20,21].

Well-posedness and wave breaking for the short pulse equation have been studied in [17,22], respectively.

Boyd [3] (Table 4.1.2, p 212) shows that, for some polymers, $\chi^{(3)}$ is a negative constant. Therefore, (1.1) reads

$$2\partial_{x_1}\partial_\phi A_0 - k^2\partial^2_{\phi\phi}A_0^3 + \frac{1}{c_2^2}A_0 = 0, \quad \chi^{(3)} = -k^2. \tag{1.2}$$

Following [1,12,13,15], we consider the admensional form of (1.2)

$$\partial_x\left(\partial_t u + 3u^2\partial_x u\right) = u. \tag{1.3}$$

Indeed, multiplying (1.2) by $-c_2^2$, we have

$$-2c_2^2\partial_{x_1}\partial_\phi A_0 + c_2^2k^2\partial^2_{\phi\phi}A_0^3 = A_0 \tag{1.4}$$

Consider the following Robelo transformation (see [1,13,15]):

$$x_1 = D_1 t, \quad \phi = D_2 x, \tag{1.5}$$

where D_1 and D_2 are two constants that will be specified later. Therefore,

$$\partial_{x_1} = D_1\partial_t, \quad \partial_\phi = D_2\partial_x. \tag{1.6}$$

Taking $A_0(x_1, \phi) = u(t, x)$, it follows from (1.1) and (1.6) that

$$-2c_2^2 D_1 D_2\partial_x(\partial_t u) + 3c_2^2k^2D_2^2\partial_x\left(u^2\partial_x u\right) = u. \tag{1.7}$$

We choose D_1, D_2 so that

$$2c_2^2D_1D_2 = -1, \qquad c_2^2k^2D_2^2 = 1,$$

that is

$$D_1 = -\frac{k}{2c_2}, \quad D_2 = \frac{1}{c_2 k}. \tag{1.8}$$

Therefore, (1.3) follows from (1.7) and (1.8).

It is interesting to remind that equation (1.3) was proposed earlier in [19] in the context of plasma physic. Moreover, similar equations describe the dynamics of radiating gases [16,23].

We are interested in the initial-boundary value problem for this equation, so we augment (1.3) with the boundary condition

$$u(t, 0) = g(t), \qquad t > 0, \tag{1.9}$$

and the initial datum

$$u(0,x)=u_0(x), \qquad x>0, \tag{1.10}$$

on which we assume that

$$u_0\in L^\infty(0,\infty)\cap L^1(0,\infty), \quad \int_0^\infty u_0(x)dx=0. \tag{1.11}$$

On the function

$$P_0(x)=\int_0^x u_0(y)dy, \tag{1.12}$$

we assume that

$$\|P_0\|^2_{L^2(0,\infty)}=\int_0^\infty\left(\int_0^x u_0(y)dy\right)^2dx<\infty. \tag{1.13}$$

On the boundary datum g, we assume that

$$g(t)\in L^\infty(0,\infty). \tag{1.14}$$

Integrating (1.3) in $(0,x)$ we gain the integro-differential formulation of (1.3) (see [20])

$$\begin{cases}\partial_t u+3u^2\partial_x u=\int_0^x u(t,y)dy, & t>0,\ x>0,\\ u(t,0)=g(t), & t>0,\\ u(0,x)=u_0(x), & x>0,\end{cases} \tag{1.15}$$

that is equivalent to

$$\begin{cases}\partial_t u+3u^2\partial_x u=P, & t>0,\ x>0,\\ \partial_x P=u, & t>0,\ x>0,\\ u(t,0)=g(t), & t>0,\\ P(t,0)=0, & t>0,\\ u(0,x)=u_0(x), & x>0.\end{cases} \tag{1.16}$$

One of the main issues in the analysis of (1.16) is that the equation is not preserving the L^1 norm, as a consequence the nonlocal source term P and the solution u are a priori only locally bounded. Indeed, from (1.15) and (1.16) is clear that we cannot have any L^∞ bound without an L^1 bound. Since we are interested in the bounded solutions of (1.3), some assumptions on the decay at infinity of the initial condition u_0 are needed. The unique useful conserved quantities are

$$t\longmapsto\int u(t,x)dx=0, \qquad t\longmapsto\int u^2(t,x)dx.$$

In the sense that if $u(t,\cdot)$ has zero mean at time $t=0$, then it will have zero mean at any time $t>0$. In addition, the L^2 norm of $u(t,\cdot)$ is constant with respect to t. Therefore, we require that initial condition u_0 belongs to $L^2\cap L^\infty$ and has zero mean.

Due to the regularizing effect of the P equation in (1.16) we have that

$$u\in L^\infty((0,T)\times(0,\infty))\Longrightarrow P\in L^\infty(0,T;W^{1,\infty}(0,\infty)), \quad T>0. \tag{1.17}$$

Therefore, if a map $u \in L^\infty((0,T)\times(0,\infty))$, $T>0$, satisfies, for every convex map $\eta \in C^2(\mathbb{R})$,

$$\partial_t \eta(u) + \partial_x q(u) - \eta'(u)P \le 0, \qquad q(u) = \int^u 3\xi^2 \eta'(\xi)\, d\xi, \tag{1.18}$$

in the sense of distributions, then [11, Theorem 1.1] provides the existence of strong trace u_0^τ on the boundary $x=0$.

We give the following definition of solution (see [2]):

Definition 1.1 We say that $u \in L^\infty((0,T)\times(0,\infty))$, $T>0$, is an entropy solution of the initial-boundary value problem (1.3), (1.9), and (1.10) if for every nonnegative test function $\phi \in C^2(\mathbb{R}^2)$ with compact support, and $c \in \mathbb{R}$

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \Big(|u-c|\partial_t\phi + \text{sign}\,(u-c)\left(u^3-c^3\right)\partial_x\phi \Big) dt dx \\
&\quad + \int_0^\infty\int_0^\infty \text{sign}\,(u-c)\, P\phi dt dx \\
&\quad + \int_0^\infty \text{sign}\,(g(t)-c)\left((u_0^\tau(t))^3 - c^3\right)\phi(t,0) dt \\
&\quad + \int_0^\infty |u_0(x)-c|\phi(0,x) dx \ge 0,
\end{aligned} \tag{1.19}$$

where $u_0^\tau(t)$ is the trace of u on the boundary $x=0$.

The main result of this paper is the following theorem.

Theorem 1.1 *Assume* (1.11)*,* (1.13)*,* (1.14)*. The initial-boundary value problem* (1.3)*,* (1.9) *and* (1.10) *possesses an unique entropy solution u in the sense of Definition 1.1. Moreover, if u and v are two entropy solutions of* (1.3)*,* (1.9)*,* (1.10) *in the sense of Definition 1.1 the following inequality holds*

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^1(0,R)} \le e^{C(T)t}\,\|u(0,\cdot)-v(0,\cdot)\|_{L^1(0,R+C(T)t)}, \tag{1.20}$$

for almost every $0<t<T$*,* $R>0$*, and some suitable constant* $C(T)>0$.

The paper is organized as follows. In Sect. 2 we prove several a priori estimates on a vanishing viscosity approximation of (1.16). Those play a key role in the proof of our main result, that is given in Sect. 3

2 Vanishing viscosity approximation

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.16).

Fix a small number $\varepsilon>0$, and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following mixed problem

$$\begin{cases}
\partial_t u_\varepsilon + 3u_\varepsilon^2 \partial_x u_\varepsilon = P_\varepsilon + \varepsilon\partial_{xx}^2 u_\varepsilon, & t>0,\ x>0,\\
\partial_x P_\varepsilon = u_\varepsilon, & t>0,\ x>0,\\
u_\varepsilon(t,0) = g_\varepsilon(t), & t>0,\\
P_\varepsilon(t,0) = 0, & t>0,\\
u_\varepsilon(0,x) = u_{0,\varepsilon}(x), & x>0,
\end{cases} \tag{2.1}$$

where $u_{\varepsilon,0}$ and g_ε are $C^\infty(0,\infty)$ approximations of u_0 and g such that

$$\begin{aligned}
&u_{0,\varepsilon} \to u_0, \quad \text{a.e. and in } L^p(0,\infty),\ 1 \le p < \infty, ,\\
&P_{0,\varepsilon} \to P_0, \quad \text{in } L^2(0,\infty),\\
&g_\varepsilon \to g, \quad \text{a.e. and in } L^p_{loc}(0,\infty),\ 1 \le p < \infty,\\
&\left\|u_{\varepsilon,0}\right\|_{L^\infty(0,\infty)} \le \|u_0\|_{L^\infty(0,\infty)}, \quad \left\|u_{\varepsilon,0}\right\|_{L^2(0,\infty)} \le \|u_0\|_{L^2(0,\infty)},\\
&\left\|u_{\varepsilon,0}\right\|_{L^4(0,\infty)} \le \|u_0\|_{L^4(0,\infty)}, \quad \int_0^\infty u_{\varepsilon,0}(x)dx = 0,\\
&\left\|P_{\varepsilon,0}\right\|_{L^2(0,\infty)} \le \|P_0\|_{L^2(0,\infty)}, \quad \|g_\varepsilon\|_{L^\infty(0,\infty)} \le C_0,
\end{aligned} \tag{2.2}$$

and C_0 is a constant independent on ε.

Clearly, (2.1) is equivalent to the integro-differential problem

$$\begin{cases} \partial_t u_\varepsilon + 3u_\varepsilon^2 \partial_x u_\varepsilon = \int_0^x u_\varepsilon(t,y)dy + \varepsilon \partial_{xx}^2 u_\varepsilon, & t>0,\ x>0,\\ u_\varepsilon(t,0) = g_\varepsilon(t), & t>0,\\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x>0. \end{cases} \tag{2.3}$$

Let us prove some a priori estimates on u_ε and P_ε, denoting with C_0 the constants which depend only on the initial data, and $C(T)$ the constants which depend also on T.

Arguing as [9, Lemma 1], or [12, Lemma 2.2.1], we have the following result.

Lemma 2.1 *The following statements are equivalent*

$$\int_0^\infty u_\varepsilon(t,x)dx = 0, \quad t \ge 0, \tag{2.4}$$

$$\frac{d}{dt}\int_0^\infty u_\varepsilon^2 dx + 2\varepsilon \int_0^\infty (\partial_x u_\varepsilon)^2 dx = \frac{3}{2} g_\varepsilon^4(t) + 2\varepsilon g_\varepsilon(t)\partial_x u_\varepsilon(t,0), \quad t>0. \tag{2.5}$$

Proof Let $t>0$. We begin by proving that (2.4) implies (2.5). Multiplying (2.3) by u_ε, an integration on $(0,\infty)$ gives

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty u_\varepsilon^2 dx &= 2\int_0^\infty u_\varepsilon \partial_t u_\varepsilon dx\\
&= 2\varepsilon \int_0^\infty u_\varepsilon \partial_{xx}^2 u_\varepsilon dx - 6\int_0^\infty u_\varepsilon^3 \partial_x u_\varepsilon dx + 2\int_0^\infty u_\varepsilon \left(\int_0^x u_\varepsilon dy\right) dx\\
&= 2\varepsilon \partial_x u_\varepsilon(t,0) g_\varepsilon(t) - 2\varepsilon \int_0^\infty (\partial_x u_\varepsilon)^2 dx + \frac{3}{2} g_\varepsilon^4(t)\\
&\quad +2\int_0^\infty u_\varepsilon \left(\int_0^x u_\varepsilon dy\right) dx.
\end{aligned}$$

By (2.1),

$$2\int_0^\infty u_\varepsilon \left(\int_0^x u_\varepsilon dy\right) dx = 2\int_0^\infty P_\varepsilon \partial_x P_\varepsilon dx = P_\varepsilon^2(t,\infty).$$

Then,

$$\frac{d}{dt}\int_0^\infty u_\varepsilon^2 dx + 2\varepsilon \int_0^\infty (\partial_x u_\varepsilon)^2 dx = P_\varepsilon^2(t,\infty) + 2\varepsilon \partial_x u_\varepsilon(t,0) g_\varepsilon(t) + \frac{3}{2} g_\varepsilon^4(t). \tag{2.6}$$

Thanks to (2.4),

$$\lim_{x\to\infty} P_\varepsilon^2(t,x) = \left(\int_0^\infty u_\varepsilon(t,x)dx\right)^2 = 0. \tag{2.7}$$

(2.6) and (2.7) give (2.5).

Let us show that (2.5) implies (2.4). We assume by contradiction that (2.4) does not hold, namely:

$$\int_0^\infty u_\varepsilon(t,x)dx \neq 0.$$

By (1.16),

$$P_\varepsilon^2(t,\infty) = \left(\int_0^\infty u_\varepsilon(t,x)dx\right)^2 \neq 0.$$

Therefore, (2.6) gives

$$\frac{d}{dt}\int_0^\infty u_\varepsilon^2 dx + 2\varepsilon\int_0^\infty (\partial_x u_\varepsilon)^2 dx \neq 2\varepsilon\partial_x u_\varepsilon(t,0)g_\varepsilon(t) + \frac{3}{2}g_\varepsilon^4(t),$$

which is in contradiction with (2.5).

Lemma 2.2 *For each $t \geq 0$, (2.4) holds true. In particular, we have that*

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + 2\varepsilon\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)} ds \leq C_0(t+1) + 2\varepsilon\int_0^t g_\varepsilon(t)\partial_x u_\varepsilon(t,0)ds. \tag{2.8}$$

Proof We begin by observing that $\partial_t u_\varepsilon(t,0) = g'_\varepsilon(t)$, being $u_\varepsilon(t,0) = g_\varepsilon(t)$. It follows from (2.3) that

$$\begin{aligned}\varepsilon\partial_{xx}^2 u_\varepsilon(t,0) &= \partial_t u_\varepsilon(t,0) + 3u_\varepsilon^2(t,0)\partial_x u_\varepsilon(t,0) - \int_0^0 u_\varepsilon(t,x)dx\\ &= g'_\varepsilon(t) + 3g_\varepsilon^2(t)\partial_x u_\varepsilon(t,0).\end{aligned} \tag{2.9}$$

Differentiating (2.3) with respect to x, we have

$$\partial_x(\partial_t u_\varepsilon + 3u_\varepsilon^2\partial_x u_\varepsilon - \varepsilon\partial_{xx}^2 u_\varepsilon) = u_\varepsilon.$$

From (2.9), and being u_ε a smooth solution of (2.3), an integration over $(0,\infty)$ gives (2.4). Lemma 2.1 says that also (2.5) holds true. Therefore, integrating (2.5) on $(0,t)$, for (2.2), we have

$$\begin{aligned}&\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + 2\varepsilon\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)} ds\\ &\quad\leq \|u_0\|^2_{L^2(0,\infty)} + \frac{3}{2}\int_0^t g_\varepsilon^4(s)ds + 2\varepsilon\int_0^t g_\varepsilon(s)\partial_x u_\varepsilon(s,0)ds\\ &\quad\leq \|u_0\|^2_{L^2(0,\infty)} + \frac{3}{2}\|g_\varepsilon\|^4_{L^\infty(0,\infty)} t + 2\varepsilon\int_0^t g_\varepsilon(s)\partial_x u_\varepsilon(s,0)ds\\ &\quad\leq \|u_0\|^2_{L^2(0,\infty)} + C_0 t + 2\varepsilon\int_0^t g_\varepsilon(s)\partial_x u_\varepsilon(s,0)ds,\end{aligned}$$

which gives (2.8).

Lemma 2.3 *We have that*

$$\lim_{x\to\infty} F_\varepsilon(t,x) = \int_0^\infty P_\varepsilon(t,x)dx = \varepsilon\partial_x u_\varepsilon(t,0) - g_\varepsilon^3(t), \tag{2.10}$$

where

$$F_\varepsilon(t,x) = \int_0^x P_\varepsilon(t,y)dy. \tag{2.11}$$

Proof We begin by observing that, integrating on $(0,x)$ the second equation of (2.1), we get

$$P_\varepsilon(t,x) = \int_0^x u_\varepsilon(t,y)dy. \tag{2.12}$$

Differentiating (2.12) with respect to t, we have

$$\partial_t P_\varepsilon(t,x) = \int_0^x \partial_t u_\varepsilon(t,y)dy = \frac{d}{dt}\int_0^x u_\varepsilon(t,y)dy. \tag{2.13}$$

It follows from (2.4) and (2.13) that

$$\lim_{x\to\infty} \partial_t P_\varepsilon(t,x) = \frac{d}{dt}\int_0^\infty u_\varepsilon(t,x)dx = 0. \tag{2.14}$$

Integrating on $(0,x)$ the first equation of (2.1), thanks to (2.13), we have

$$\partial_t P_\varepsilon(t,x) + u_\varepsilon^3(t,x) - g_\varepsilon^3(t) - \varepsilon\partial_x u_\varepsilon(t,x) + \varepsilon\partial_x u_\varepsilon(t,0) = \int_0^x P_\varepsilon(t,y)dy. \tag{2.15}$$

It follows from the regularity of u_ε that

$$\lim_{x\to\infty}\left(u_\varepsilon^3(t,x)) - \varepsilon\partial_x u_\varepsilon(t,x)\right) = 0. \tag{2.16}$$

(2.14) and (2.16) give (2.10).

Arguing as in [8, Lemma2.3], we prove the following lemma.

Lemma 2.4 *Let $T > 0$. There exists a constant $C(T) > 0$, independent on ε, such that*

$$\begin{aligned}
&\|u_\varepsilon(t,\cdot)\|^4_{L^4(0,\infty)} + 2\,\|P_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\quad + 12\varepsilon\int_0^t \|u_\varepsilon(s,\cdot)\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)}\,ds\\
&\quad + 4\varepsilon\int_0^t \|u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)}\,ds + \varepsilon^2\int_0^t (\partial_x u_\varepsilon(s,0))^2\,ds \le C(T),
\end{aligned} \tag{2.17}$$

for every $0 \le t \le T$.

Proof Let $0 \le t \le T$. We begin by observing that (2.11) and (2.15) imply

$$\partial_t P_\varepsilon(t,x) = F_\varepsilon(t,x) - u_\varepsilon^3(t,x) + g_\varepsilon^3(t) + \varepsilon u_\varepsilon(t,x) - \varepsilon\partial_x u_\varepsilon(t,0). \tag{2.18}$$

Multiplying (2.18) by P_ε, an integration on $(0,\infty)$ gives

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty P_\varepsilon^2 dx &= 2\int_0^\infty P_\varepsilon\partial_t P_\varepsilon dx\\
&= 2\int_0^\infty P_\varepsilon F_\varepsilon dx - 2\int_0^\infty u_\varepsilon^3 P_\varepsilon dx + 2g_\varepsilon^3(t)\int_0^\infty P_\varepsilon dx\\
&\quad + 2\varepsilon\int_0^\infty \partial_x u_\varepsilon P_\varepsilon dx - 2\varepsilon\partial_x u_\varepsilon(t,0)\int_0^\infty P_\varepsilon dx.
\end{aligned} \tag{2.19}$$

By (2.1),

$$2\int_0^\infty \partial_x u_\varepsilon P_\varepsilon dx = -2\varepsilon\int_0^\infty u_\varepsilon \partial_x P_\varepsilon dx = -2\varepsilon \left\| u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}, \tag{2.20}$$

while, in light of (2.11) and (2.10),

$$\begin{aligned}2\int_0^\infty P_\varepsilon F_\varepsilon dx &= 2\int_0^\infty F_\varepsilon \partial_x F_\varepsilon dx\\ &= F^2_\varepsilon(t,\infty) = \left(\varepsilon\partial_x u_\varepsilon(t,0) - g^3_\varepsilon(t)\right)^2\\ &= \varepsilon^2\left(\partial_x u_\varepsilon(t,0)\right)^2 - 2\varepsilon\partial_x u_\varepsilon(t,0) g^3_\varepsilon(t) + g^6_\varepsilon(t).\end{aligned} \tag{2.21}$$

Using again (2.10),

$$\begin{aligned}-2\varepsilon\partial_x u_\varepsilon(t,0)\int_0^\infty P_\varepsilon dx &= -2\varepsilon^2\left(\partial_x u_\varepsilon(t,0)\right)^2 + 2\varepsilon\partial_x u_\varepsilon(t,0) g^3_\varepsilon(t),\\ 2g^3_\varepsilon(t)\int_0^\infty P_\varepsilon dx &= 2\varepsilon\partial_x u_\varepsilon(t,0) g^3_\varepsilon(t) - 2g^6_\varepsilon(t).\end{aligned} \tag{2.22}$$

(2.19), (2.20), (2.21) and (2.22) give

$$\begin{aligned}\frac{d}{dt}\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} =& -2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} - 2\int_0^\infty u^3_\varepsilon P_\varepsilon dx\\ & -\varepsilon^2\left(\partial_x u_\varepsilon(t,0)\right)^2 - g^6_\varepsilon(t) + 2\varepsilon\partial_x u_\varepsilon(t,0) g^3_\varepsilon(t),\end{aligned}$$

that is,

$$\begin{aligned}\frac{d}{dt}\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} &+ 2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + \varepsilon^2\left(\partial_x u_\varepsilon(t,0)\right)^2\\ &= -2\int_0^\infty u^3_\varepsilon P_\varepsilon dx - g^6_\varepsilon(t) + 2\varepsilon\partial_x u_\varepsilon(t,0) g^3_\varepsilon(t).\end{aligned} \tag{2.23}$$

Multiplying (2.1) by $2u^3_\varepsilon$, an integration on $(0,\infty)$ gives

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_0^\infty u^4_\varepsilon dx\right) &= 2\int_0^\infty u^3_\varepsilon \partial_t u_\varepsilon dx\\ &= -6\int_0^\infty u^5_\varepsilon\partial_x u_\varepsilon dx + 2\int_0^\infty u^3_\varepsilon P_\varepsilon dx + 2\varepsilon\int_0^\infty u^3_\varepsilon \partial^2_{xx} u_\varepsilon dx\\ &= g^6_\varepsilon(t) + 2\int_0^\infty u^3_\varepsilon P_\varepsilon dx + 2\varepsilon\partial_x u_\varepsilon(t,0) g^3_\varepsilon(t) - 6\varepsilon\int_0^\infty u^2_\varepsilon(\partial_x u_\varepsilon)^2 dx,\end{aligned}$$

that is

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(0,\infty)}\right) &+ 6\varepsilon\left\|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &= g^6_\varepsilon(t) + 2\int_0^\infty u^3_\varepsilon P_\varepsilon dx + 2\varepsilon\partial_x u_\varepsilon(t,0) g^3_\varepsilon(t).\end{aligned} \tag{2.24}$$

Adding (2.23), (2.24), we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(0,\infty)} + \left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\right)\\ &+ 6\varepsilon\left\|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &+ \varepsilon^2\left(\partial_x u_\varepsilon(t,0)\right)^2 = 4\varepsilon\partial_x u_\varepsilon(t,0) g^3_\varepsilon(t).\end{aligned} \tag{2.25}$$

Due to the Young inequality,

$$4\varepsilon\partial_x u_\varepsilon(t,0)g_\varepsilon^3(t) \le |\varepsilon\partial_x u_\varepsilon(t,0)|\left|4g_\varepsilon^3(t)\right| \le \frac{\varepsilon^2}{2}(\partial_x u_\varepsilon(t,0))^2 + 8g_\varepsilon^6(t). \tag{2.26}$$

It follows from (2.25), (2.26) that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|u_\varepsilon(t,\cdot)\|^4_{L^4(0,\infty)} + \|P_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\right)\\ &+6\varepsilon\|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + 2\varepsilon\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)}\\ &+\frac{\varepsilon^2}{2}(\partial_x u_\varepsilon(t,0))^2 \le 8g_\varepsilon^6(t).\end{aligned} \tag{2.27}$$

Integrating (2.27) on (0,t), by (2.2), we have

$$\begin{aligned}&\frac{1}{2}\|u_\varepsilon(t,\cdot)\|^4_{L^4(0,\infty)} + \|P_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + 6\varepsilon\int_0^t\|u_\varepsilon(s,\cdot)\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)}ds\\ &+2\varepsilon\int_0^t\|u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)}ds + \frac{\varepsilon^2}{2}\int_0^t(\partial_x u_\varepsilon(s,0))^2 ds\\ &\le \|u_0\|^4_{L^4(0,\infty)} + \|P_0\|^2_{L^2(0,\infty)} + 8\int_0^t g_\varepsilon^6(s)ds\\ &\le C_0 + 8\|g_\varepsilon\|^6_{L^\infty(0,\infty)}t \le C_0(1+8t),\end{aligned}$$

which gives (2.17).

Lemma 2.5 *Let $T>0$. There exists a constant $C(T)>0$, independent on ε, such that*

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + 2\varepsilon\int_0^t\|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)}ds \le C(T), \tag{2.28}$$

for every $0\le t\le T$. In particular, we have

$$\|P_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))} \le C(T). \tag{2.29}$$

Proof We begin by observing that, using the Young inequality,

$$2\varepsilon g_\varepsilon(t)\partial_x u_\varepsilon(t,0) \le 2|g_\varepsilon(t)||\varepsilon\partial_x u_\varepsilon(t,0)| \le g_\varepsilon^2(t) + \varepsilon^2(\partial_x u_\varepsilon(t,0))^2.$$

Therefore, in light of (2.2) and (2.17),

$$\begin{aligned}2\varepsilon\int_0^t g_\varepsilon(s)\partial_x u_\varepsilon(s,0)ds &\le 2\int_0^t|g_\varepsilon(t)||\varepsilon\partial_x u_\varepsilon(t,0)|dx\\ &\le \int_0^t g_\varepsilon^2(s)ds + \varepsilon^2\int_0^t(\partial_x u_\varepsilon(s,0))^2 ds\\ &\le \|g_\varepsilon\|^2_{L^\infty(0,\infty)}t + \varepsilon^2\int_0^t(\partial_x u_\varepsilon(s,0))^2 ds\\ &\le C_0 t + \varepsilon^2\int_0^t(\partial_x u_\varepsilon(s,0))^2 ds \le C(T).\end{aligned} \tag{2.30}$$

(2.28) follows from (2.8) and (2.30).

Finally, we prove (2.29). Due to (2.1), (2.17), (2.28) and the Hölder inequality,

$$\begin{aligned} P_\varepsilon^2(t,x) = 2\int_0^x P_\varepsilon \partial_x P_\varepsilon dy \le 2\int_0^\infty |P_\varepsilon||\partial_x P_\varepsilon| dx \\ \le 2\,\|P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}\,\|\partial_x P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \\ = 2\,\|P_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}\,\|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)} \le C(T). \end{aligned}$$

Therefore,

$$|P_\varepsilon(t,x)| \le C(T),$$

which gives (2.29).

Lemma 2.6 *Let* $T > 0$. *We have*

$$\|u_\varepsilon\|_{L^\infty((0,T)\times(0,\infty))} \le \|u_0\|_{L^\infty(0,\infty)} + C(T). \tag{2.31}$$

Proof Due to (2.1) and (2.29),

$$\partial_t u_\varepsilon + 3u_\varepsilon^2 \partial_x u_\varepsilon - \varepsilon \partial_{xx}^2 u_\varepsilon \le C(T).$$

Since the map

$$\mathcal{F}(t) := \|u_0\|_{L^\infty(0,\infty)} + C(T)t,$$

solves the equation

$$\frac{d\mathcal{F}}{dt} = C(T)$$

and

$$\max\{u_\varepsilon(0,x), 0\} \le \mathcal{F}(t), \qquad (t,x) \in (0,T)\times(0,\infty),$$

the comparison principle for parabolic equations implies that

$$u_\varepsilon(t,x) \le \mathcal{F}(t), \qquad (t,x) \in (0,T)\times(0,\infty).$$

In a similar way we can prove that

$$u_\varepsilon(t,x) \ge -\mathcal{F}(t), \qquad (t,x) \in (0,T)\times(0,\infty).$$

Therefore,

$$|u_\varepsilon(t,x)| \le \|u_0\|_{L^\infty(0,\infty)} + C(T)t \le \|u_0\|_{L^\infty(0,\infty)} + C(T)T,$$

which gives (2.31).

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Let us begin by proving the existence of a distributional solution to (1.3), (1.9), (1.10) satisfying (1.19).

Lemma 3.1 *Let* $T > 0$. *There exists a function* $u \in L^\infty((0,T)\times(0,\infty))$ *that is a distributional solution of* (1.16) *and satisfies* (1.19).

We construct a solution by passing to the limit in a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of viscosity approximations (2.1). We use the compensated compactness method [24].

Lemma 3.2 *Let $T>0$. There exists a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and a limit function $u\in L^\infty((0,T)\times(0,\infty))$ such that*

$$u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{loc}((0,T)\times(0,\infty)),\ 1\le p<\infty. \tag{3.1}$$

Moreover, we have

$$P_{\varepsilon_k} \to P \text{ a.e. and in } L^p_{loc}(0,T;W^{1,p}_{loc}(0,\infty)),\ 1\le p<\infty, \tag{3.2}$$

where

$$P(t,x)=\int_0^x u(t,y)dy, \qquad t\ge 0, \quad x\ge 0, \tag{3.3}$$

and (1.19) holds true.

Proof Let $\eta:\mathbb{R}\to\mathbb{R}$ be any convex C^2 entropy function, and $q:\mathbb{R}\to\mathbb{R}$ be the corresponding entropy flux defined by $q'(u)=3u^2\eta'(u)$. By multiplying the first equation in (2.1) with $\eta'(u_\varepsilon)$ and using the chain rule, we get

$$\partial_t\eta(u_\varepsilon)+\partial_x q(u_\varepsilon)=\underbrace{\varepsilon\partial^2_{xx}\eta(u_\varepsilon)}_{=:\mathcal{L}_{1,\varepsilon}}\underbrace{-\varepsilon\eta''(u_\varepsilon)\left(\partial_x u_\varepsilon\right)^2}_{=:\mathcal{L}_{2,\varepsilon}}\underbrace{+\eta'(u_\varepsilon)P_\varepsilon}_{=:\mathcal{L}_{3,\varepsilon}},$$

where $\mathcal{L}_{1,\varepsilon}$, $\mathcal{L}_{2,\varepsilon}$, $\mathcal{L}_{3,\varepsilon}$ are distributions. Let us show that

$$\mathcal{L}_{1,\varepsilon}\to 0 \quad in\ H^{-1}((0,T)\times(0,\infty)),\ T>0.$$

Since

$$\varepsilon\partial^2_{xx}\eta(u_\varepsilon)=\partial_x(\varepsilon\eta'(u_\varepsilon)\partial_x u_\varepsilon),$$

from (2.28) and Lemma 2.6,

$$\begin{aligned}\left\|\varepsilon\eta'(u_\varepsilon)\partial_x u_\varepsilon\right\|^2_{L^2((0,T)\times(0,\infty))}&\le\varepsilon^2\left\|\eta'\right\|^2_{L^\infty(J_T)}\int_0^T\|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)}ds\\&\le\varepsilon\left\|\eta'\right\|^2_{L^\infty(J_T)}C(T)\to 0,\end{aligned}$$

where

$$J_T=\left(-\|u_0\|_{L^\infty(0,\infty)}-C(T),\|u_0\|_{L^\infty(0,\infty)}+C(T)\right).$$

We claim that

$$\{\mathcal{L}_{2,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1((0,T)\times(0,\infty)),\ T>0.$$

Again by (2.28) and Lemma 2.6,

$$\begin{aligned}\left\|\varepsilon\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2\right\|_{L^1((0,T)\times(0,\infty))}&\le\left\|\eta''\right\|_{L^\infty(J_T)}\varepsilon\int_0^T\|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)}ds\\&\le\left\|\eta''\right\|_{L^\infty(J_T)}C(T).\end{aligned}$$

We have that

$$\{\mathcal{L}_{3,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1_{loc}((0,T)\times(0,\infty)),\ T>0.$$

Let K be a compact subset of $(0, T) \times (0, \infty)$. Using (2.29) and Lemma 2.6,

$$\begin{aligned}\left\|\eta'(u_\varepsilon)P_\varepsilon\right\|_{L^1(K)} &= \int_K |\eta'(u_\varepsilon)||P_\varepsilon|dtdx \\ &\leq \left\|\eta'\right\|_{L^\infty(J_T)} \|P_\varepsilon\|_{L^\infty(I_T)} |K|.\end{aligned}$$

Therefore, Murat's lemma [18] implies that

$$\{\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon)\}_{\varepsilon>0} \text{ lies in a compact subset of } H^{-1}_{\text{loc}}((0, T) \times (0, \infty)). \tag{3.4}$$

The L^∞ bound stated in Lemma 2.6, (3.4), and the Tartar's compensated compactness method [24] give the existence of a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and a limit function $u \in L^\infty((0, T) \times (0, \infty))$, $T > 0$, such that (3.1) holds.

(3.2) follows from (3.1), the Hölder inequality and the identity

$$P_{\varepsilon_k} = \int_0^x u_{\varepsilon_k} dy, \quad \partial_x P_{\varepsilon_k} = u_{\varepsilon_k}.$$

Finally, we prove (1.19).

Let $k \in \mathbb{N}$, $c \in \mathbb{R}$ be a constant, and $\phi \in C^\infty(\mathbb{R}^2)$ be a nonnegative test function with compact support. Multiplying the first equation of(2.1) by sign $(u_\varepsilon - c)$, we have

$$\begin{aligned}\partial_t |u_{\varepsilon_k} - c| &+ \partial_x \left(\text{sign}\left(u_{\varepsilon_k} - c\right)\left(u^3_{\varepsilon_k} - c^3\right)\right) \\ &- \text{sign}\left(u_{\varepsilon_k} - c\right) P_{\varepsilon_k} - \varepsilon_k \partial^2_{xx} |u_{\varepsilon_k} - c| \leq 0.\end{aligned}$$

Multiplying by ϕ and integrating over $(0, \infty)^2$, we get

$$\begin{aligned}&\int_0^\infty\int_0^\infty \left(|u_{\varepsilon_k} - c|\partial_t\phi + \left(\text{sign}\left(u_{\varepsilon_k} - c\right)\left(u^3_{\varepsilon_k} - c^3\right)\right)\partial_x\phi\right) dtdx \\ &\quad + \int_0^\infty\int_0^\infty \text{sign}\left(u_{\varepsilon_k} - c\right) P_{\varepsilon_k} dtdx - \varepsilon_k \int_0^\infty\int_0^\infty \partial_x |u_{\varepsilon_k} - c|\partial_x\phi dtdx \\ &\quad + \int_0^\infty |u_0(x) - c|\phi(0, x)dx + \int_0^\infty \text{sign}\left(g_{\varepsilon_k}(t) - c\right)\left(g^3_{\varepsilon_k}(t) - c^3\right)\phi(t, 0)dt \\ &\quad - \varepsilon_k \int_0^\infty \partial_x |u_{\varepsilon_k}(t, 0) - c|\phi(t, 0)dt \geq 0.\end{aligned}$$

Thanks to (2.2) and Lemmas 2.5 and 2.6, when $k \to \infty$, we have

$$\begin{aligned}&\int_0^\infty\int_0^\infty \left(|u - c|\partial_t\phi + \left(\text{sign}\,(u - c)\left(u^3 - c^3\right)\right)\partial_x\phi\right) dtdx \\ &\quad + \int_0^\infty\int_0^\infty \text{sign}\,(u - c)\, P dtdx + \int_0^\infty |u_0(x) - c|\phi(0, x)dx \\ &\quad + \int_0^\infty \text{sign}\,(g(t) - c)\left(g^3(t) - c^3\right)\phi(t, 0)dt \\ &\quad - \lim_{\varepsilon_k} \varepsilon_k \int_0^\infty \partial_x |u_{\varepsilon_k}(t, 0) - c|\phi(t, 0)dt \geq 0.\end{aligned}$$

We have to prove that (see [2])

$$\begin{aligned}&\lim_{\varepsilon_k} \varepsilon_k \int_0^\infty \partial_x |u_{\varepsilon_k}(t, 0) - c|\phi(t, 0)dt \\ &\quad = \int_0^\infty \text{sign}\,(g(t) - c)\left(g^3(t) - (u^\tau_0(t))^3\right)\phi(t, 0)dt.\end{aligned} \tag{3.5}$$

Let $\{\rho_\nu\}_{\nu\in\mathbb{N}} \subset C^\infty(\mathbb{R})$ be such that

$$0 \le \rho_\nu \le 1, \quad \rho_\nu(0) = 1, \quad |\rho'_\nu| \le 1, \quad x \ge \frac{1}{\nu} \quad \Longrightarrow \quad \rho_\nu(x) = 0. \tag{3.6}$$

Using $(t, x) \mapsto \rho_\nu(x)\phi(t, x)$ as test function for the first equation of (2.1) we get

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \left(u_{\varepsilon_k}\partial_t\phi\rho_\nu + u^3_{\varepsilon_k}\partial_x\phi\rho_\nu + u^3_{\varepsilon_k}\phi\rho'_\nu\right) dtdx + \int_0^\infty\int_0^\infty P_{\varepsilon_k}\phi\rho_\nu dtdx\\
&\quad - \varepsilon_k \int_0^\infty\int_0^\infty \partial_x u_{\varepsilon_k}\left(\partial_x\phi\rho_\nu + \phi\rho'_\nu\right) dtdx + \int_0^\infty u_0(x)\phi(0, x)\rho_\nu(x)dx\\
&\quad + \int_0^\infty g^3_{\varepsilon_k}(t)\phi(t, 0)dt - \varepsilon_k \int_0^\infty \partial_x u_{\varepsilon_k}(t, 0)\phi(t, 0)dt = 0.
\end{aligned}$$

As $k \to \infty$, we obtain that

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \left(u\partial_t\phi\rho_\nu + u^3\partial_x\phi\rho_\nu + u^3\phi\rho'_\nu\right) dtdx + \int_0^\infty\int_0^\infty P\phi\rho_\nu dtdx\\
&\quad + \int_0^\infty u_0(x)\phi(0, x)\rho_\nu dx + \int_0^\infty g^3(t)\phi(t, 0)dt\\
&= \lim_{\varepsilon_k} \varepsilon_k \int_0^\infty \partial_x u_{\varepsilon_k}(t, 0)\phi(t, 0)dt.
\end{aligned}$$

Sending $\nu \to \infty$, we get

$$\lim_{\varepsilon_k} \varepsilon_k \int_0^\infty \partial_x u_{\varepsilon_k}(t, 0)\phi(t, 0)dt = \int_0^\infty \left(g^3(t) - (u^\tau_0(t))^3\right)\phi(t, 0)dt.$$

Therefore, due to the strong convergence of g_{ε_k} and the continuity of g we have

$$\begin{aligned}
&\lim_{\varepsilon_k} \varepsilon_k \int_0^\infty \partial_x |u_{\varepsilon_k}(t, 0) - c|\phi(t, 0)dt\\
&\quad = \lim_{\varepsilon_k} \int_0^\infty \partial_x u_{\varepsilon_k}(t, 0)\text{sign}\left(u_{\varepsilon_k}(t, 0) - c\right)\phi(t, 0)dt\\
&\quad = \lim_{\varepsilon_k} \int_0^\infty \partial_x u_{\varepsilon_k}(t, 0)\text{sign}\left(g_{\varepsilon_k}(t) - c\right)\phi(t, 0)dt\\
&\quad = \int_0^\infty \text{sign}\,(g(t) - c)\left(g^3(t) - (u^\tau_0(t))^3\right)\phi(t, 0)dt,
\end{aligned}$$

that is (3.5).

Proof of Theorem 1.1 Lemma (3.2) gives the existence of an entropy solution u for (1.15), or equivalently (1.16).

We observe that, fixed $T > 0$, the solutions of (1.15), or equivalently (1.16), are bounded in $(0, T) \times \mathbb{R}$. Therefore, using [6, Theorem 1.1], u is unique and (1.20) holds true. □

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