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Invariance of flows in doubly-connected domains with the same modulus

Giovanni Cimatti

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Abstract We consider systems of elliptic partial differential equations in divergence form with Dirichlet's boundary conditions in doubly-connected domain of the plane with modulus μ. We prove an invariance property of the corresponding global flows in the class of domains with the same modulus. Applications are given to the problem of electrical heating of a conductor whose thermal and electrical conductivities depend on the temperature and to the flow of a viscous fluid in a porous medium, taking into account the Soret and Dufour's effects.

Keywords Systems of PDE in divergence form · Doubly-connected plane domain · Thermistor problem · Porous media · Darcy law · Soret effect · Dufour effect

Mathematics Subject Classification 30E30 · 35J66 · 35J57

1 Introduction

The solution of the problem

$$\Delta\varphi = 0 \text{ in } \mathcal{O}, \varphi(R_1, \theta) = 0, \varphi(R_2, \theta) = V,$$

where $\mathcal{O} = \{(\rho, \theta); 0 < R_1 < \rho < R_2, 0 \leq \theta < 2\pi\}$, is given by

$$\varphi(\rho) = V\frac{ln(\rho/R_1)}{ln(R_2/R_1)}.$$

Moreover,

$$I = \int_{\{\rho=R_2\}} \frac{d\varphi}{dn} ds = \frac{2\pi V}{ln(R_2/R_1)}. \tag{1.1}$$

If V represents the difference of potential applied to a metallic specimen $\mathcal{O}$, Eq. (1.1) tells us that the total current crossing $\{\rho = R_2\}$ depends, in addition to V, only on the ratio R_2/R_1.

G. Cimatti (✉)
Department of Mathematics, University of Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
e-mail: cimatti@dm.unipi.it

In this paper we prove that this property of invariance remains true for much more complex domains and systems of PDE.

Let us consider the plane bounded doubly-connected domain Ω with boundary formed by the two simple closed curves Γ_1 and Γ_2. We assume that physical state of Ω is determined by n parameters $u_i(x, y)$, $i = 1 \dots n$ via the n fluxes densities

$$\mathbf{q}_i = -\sum_{j=1}^{n} a_{ij}(u_1 \dots u_n)\nabla u_j. \tag{1.2}$$

The functions $a_{ij}(u_1 \dots u_n)$ are given and regular. The conservation laws

$$\nabla \cdot \mathbf{q}_i = 0, \ i = 1 \dots n \tag{1.3}$$

are assumed to hold. Supposing Dirichlet's boundary conditions for u_i on Γ_1 and Γ_2 we arrive at the following nonlinear boundary value problem

$$\nabla \cdot \left(\sum_{j=1}^{n} a_{ij}(u_1 \dots u_n)\nabla u_j \right) = 0 \quad \text{in } \Omega \tag{1.4}$$

$$u_i = u_i^{(1)} \text{ on } \Gamma_1, u_i = u_i^{(2)} \text{ on } \Gamma_2, \quad i = 1 \dots n, \tag{1.5}$$

where $u_i^{(1)}$ and $u_i^{(2)}$ are $2n$ functions given respectively on Γ_1 and Γ_2. In many cases what is of interest in the solutions of (1.4) and (1.5) are the n global quantities

$$I_i = \int_{\Gamma_1} \mathbf{q}_i \cdot \mathbf{n}\, ds, \tag{1.6}$$

where $\mathbf{n}$ denotes the unit vector normal to the boundary of Ω pointing outward with respect to Ω. In view of (1.3) we have

$$\int_{\Gamma_1} \mathbf{q}_i \cdot \mathbf{n}\, ds = -\int_{\Gamma_2} \mathbf{q}_i \cdot \mathbf{n}\, ds.$$

Example 1 Consider a very long cylinder whose cross-section is the doubly-connected domain Ω. The cylinder is made of a material capable of conducting heat and electricity. Between the two lateral surfaces a difference of potential V is applied and they are kept at a constant temperature. The thermal conductivity κ and the electric conductivity σ are given functions of the temperature u. In this problem are relevant the current density $\mathbf{J}$ given by the Ohm's law

$$\mathbf{J} = \mathbf{q}_1 = -\sigma(u)\nabla\varphi$$

and the density of the flow of energy [12]

$$\mathbf{q}_2 = -\kappa(u)\nabla u - \varphi\sigma(u)\nabla\varphi.$$

By the conservation of charge and energy we obtain the so-called thermistor problem, which has been thoroughly studied by many authors, [3,4,10] and recently by [6,7] among others. In particular in [9] the invariance by conformal mappings of the problem is noted. The thermistor problem reads

$$\nabla \cdot (\sigma(u)\nabla\varphi) = 0 \quad \text{in } \Omega$$
$$\nabla \cdot (\kappa(u)\nabla u + \varphi\sigma(u)\nabla\varphi) = 0 \quad \text{in } \Omega$$
$$\varphi = 0 \text{ on } \Gamma_1, \varphi = V \text{ on } \Gamma_2, u = 0 \text{ on } \Gamma_1, u = 0 \quad \text{on } \Gamma_2.$$

In this case

$$I_1 = \int_{\Gamma_1} \mathbf{q}_1 \cdot \mathbf{n}\, ds$$

gives the total electric current crossing Γ_1 in the unit time.

Example 2 We consider, as a second example, the heat and mass transfer occurring in a Darcy's flow in a porous medium occupying the cylinder of Example 1. The first flow is the velocity given by the Darcy's law [1]

$$\mathbf{v} = -K\nabla p, \tag{1.7}$$

where $p(x, y)$ is the pressure. We take into account the Soret and Dufour's effects [1], which in certain cases are not negligible [5]. Thus we have for the densities of the mass and heat flow

$$\mathbf{q}_m = -\beta\nabla c - S\nabla u + c\mathbf{v}, \mathbf{q}_h = -\kappa\nabla u - D\nabla c + u\mathbf{v},$$

where $c(x, y)$ is the concentration, $u(x, y)$ the temperature, β the Fick's coefficient, S and D the Soret and Dufour's coefficients. They all are supposed, as K, to be given positive functions of $p,\ c$ and u. If we take into account (1.7) we have

$$\mathbf{q}_m = -\beta\nabla c - S\nabla u - Kc\nabla p,\ \mathbf{q}_h = -\kappa\nabla u - D\nabla c - Ku\nabla p. \tag{1.8}$$

By (1.3) we obtain from (1.7) and (1.8) the boundary value problem

$$\begin{aligned}
&\nabla \cdot (K(c, u, p)\nabla p) = 0 \quad \text{in } \Omega\\
&\nabla \cdot (\beta(c, u, p)\nabla c + S(c, u, p)\nabla u + K(c, u, p)c\nabla p) = 0 \quad \text{in } \Omega\\
&\nabla \cdot (\kappa(c, u, p)\nabla u + D(c, u, p)\nabla u + K(c, u, p)u\nabla p) = 0 \quad \text{in } \Omega\\
&p = p^{(1)}, c = c^{(1)}, u = u^{(1)} \text{ on } \Gamma_1, p = p^{(2)},\ c = c^{(2)}, u = u^{(2)} \quad \text{on } \Gamma_2.
\end{aligned}$$

In this problem we are interested in the global quantities

$$Q_m = \int_{\Gamma_1} \mathbf{q}_m \cdot \mathbf{n}\, ds,\ Q_h = \int_{\Gamma_1} \mathbf{q}_h \cdot \mathbf{n} ds.$$

In this paper we prove a property of invariance of the quantities I_i defined in (1.6) which, together with the notion of modulus of a doubly-connected domain (see[2,8,14]), permits to compute I_i reformulating the problem (1.4) and (1.5) in a simpler doubly-connected domain of the same modulus. In Sect. 2 this method is applied to the problem of Example 1 and in Sect. 3 to Example 2.

2 Invariance properties

Let $w = f(z)$ be the conformal mapping $f(z) = \Phi(x, y) + \Psi(x, y), z = x + iy, w = X + iY$ such that $|f'(z)| > 0$ in $\bar{\Omega}$. Let $\omega = f(\Omega)$, $\gamma_1 = f(\Gamma_1)$, $\gamma_2 = f(\Gamma_2)$. Assume $u_i(x, y), i = 1 \ldots n$ to be a solution of problem (1.4), (1.5). We set $\mathcal{U}_i(X, Y) = u_i(x, y)$ and $X = \Phi(x, y)$, $Y = \Psi(x, y)$. Using the Cauchy–Riemann equations and their consequences we find

$$\Delta u_i = |f'(z)|^2 \Delta\mathcal{U}_i \tag{2.1}$$

$$\nabla u_i \cdot \nabla u_j = |f'(z)|^2 \nabla\mathcal{U}_i \cdot \nabla\mathcal{U}_j. \tag{2.2}$$

Thus we have, using (2.1) and (2.2),

$$\nabla\cdot\left(\sum_{j=1}^{n}a_{ij}(u_1\ldots u_n)\nabla u_j\right)=\sum_{j=1}^{n}a_{ij}(u_1\ldots u_n)\Delta u_j+\sum_{j,k=1}^{n}\frac{\partial a_{ij}}{\partial u_k}\nabla u_k\cdot\nabla u_j$$
$$=|f'(z)|^2\left(\sum_{j=1}^{n}a_{ij}(\mathcal{U}_1\ldots\mathcal{U}_n)\Delta\mathcal{U}_j+\sum_{j,k=1}^{n}\frac{\partial a_{ij}}{\partial u_k}\nabla\mathcal{U}_k\cdot\nabla\mathcal{U}_j\right)$$
$$=|f'(z)|^2\nabla\cdot\left(\sum_{j=1}^{n}a_{ij}(\mathcal{U}_1\ldots\mathcal{U}_n)\nabla\mathcal{U}_j\right).\tag{2.3}$$

Therefore the Eq. (1.4) is invariant under conformal mappings. Let $z=\tilde{z}(s)$ be the parametric representation of the curve Γ_1 in term of the arc length s and $w=f(\tilde{z}(s))$ the corresponding parametric representation of the curve γ_1 on the w plane. If S denotes the arc length on γ_1 we have

$$\frac{dS}{ds}=|f'(\tilde{z}(s))|.$$

After a simple calculation, we have, if $\mathbf{N}$ denotes the unit vector normal to ω,

$$\frac{du_i}{dn}=|f'(\tilde{z}(s))|\frac{d\mathcal{U}_i}{dN}.$$

This gives

$$\int_{\gamma_1}a_{ij}(\mathcal{U}_1\ldots\mathcal{U}_n)\frac{d\mathcal{U}_j}{dN}\,dS=\int_{\Gamma_1}a_{ij}(u_1\ldots u_n)\frac{du_j}{dn}\,ds.$$

Therefore we have

$$I_i=\int_{\Gamma_1}\mathbf{q}_i\cdot\mathbf{n}ds=\int_{\Gamma_1}\sum_{j=1}^{n}a_{ij}(u_1\ldots u_n)\frac{du_j}{dn}\,ds=\int_{\gamma_1}\sum_{j=1}^{n}a_{ij}(\mathcal{U}_1\ldots\mathcal{U}_n)\frac{d\mathcal{U}_j}{dN}\,dS.$$

Summing up we have the following

Theorem 2.1 *The boundary value problem*

$$\nabla\cdot\left(\sum_{j=1}^{n}a_{ij}(u_1\ldots u_n)\nabla u_j\right)=0\ \ in\ \Omega,\ u_i=u_i^{(1)}\ \ on\ \Gamma_1,\ u_i=u_i^{(2)}\ \ on\ \Gamma_2,\ \ i=1\ldots n$$

and the quantities

$$I_i=\int_{\Gamma_1}\mathbf{q}_i\cdot\mathbf{n}\,ds,\ \ i=1\ldots n,$$

are invariant under conformal mappings.

To apply this result we recall the definition and properties of the modulus of a doubly-connected domain [8]. If Ω is a doubly-connected domain bounded by two non-degenerate curves it is always possible to map Ω conformally in a one-to-one manner on the annulus $1<|w|<\mu$. The number μ, the modulus of Ω, is a characteristic constant of Ω, i.e. to every Ω there corresponds one and only one number $\mu>1$. This determines a partition into equivalence classes of all doubly-connected domains, in particular all the annuli of radii

$R_2 > R_1 > 0$, such that $\frac{R_2}{R_1} = \mu$ belongs to the same class of modulus μ. Hence, by Theorem 2.1 the global fluxes I_i related to problem (1.4), (1.5) can be computed solving the same problem in an annulus of radii 1 and μ.

3 Total flows in the problem of electric heating of a conductor

In this section we consider the problem of Example 1 i.e.

$$\nabla \cdot (\sigma(u)\nabla\varphi) = 0 \quad \text{in } \Omega \tag{3.1}$$

$$\nabla \cdot (\kappa(u)\nabla u + \varphi\sigma(u)\nabla\varphi) = 0 \quad \text{in } \Omega \tag{3.2}$$

$$\varphi = 0 \text{ on } \Gamma_1, \varphi = V \quad \text{on } \Gamma_2 \tag{3.3}$$

$$u = 0 \text{ on } \Gamma_1, u = 0 \quad \text{on } \Gamma_2. \tag{3.4}$$

If $\sigma(u) \in C^1(\mathbf{R}^1)$, $\kappa(u) \in C^1(\mathbf{R}^1)$, $\sigma(u) > 0$, $\kappa(u) > 0$ and

$$\int_0^\infty \frac{\kappa(t)}{\sigma(t)} dt = \infty \tag{3.5}$$

problem (3.1–3.4) has one and only one solution [3]. We wish to compute the total current

$$I = \int_{\Gamma_1} \mathbf{J} \cdot \mathbf{n}\, ds$$

crossing the device for a generic doubly-connected domain of modulus μ. Define

$$F(u) = \int_0^u \frac{\kappa(t)}{\sigma(t)} dt, \theta = \frac{\varphi^2}{2} + F(u).$$

By (3.5) F maps one-to-one $[0, \infty)$ onto $[0, \infty)$. In terms of θ, φ and u we can restate problem (3.1–3.4) as follows

$$\begin{aligned} \nabla \cdot (\sigma(u)\nabla\varphi) = 0 &\quad \text{in } \Omega \\ \nabla \cdot (\sigma(u)\nabla\theta) = 0 &\quad \text{in } \Omega \\ \varphi = 0 \text{ on } \Gamma_1, \varphi = V &\quad \text{on } \Gamma_2 \\ \theta = 0 \text{ on } \Gamma_1, \theta = \tfrac{V^2}{2} &\quad \text{on } \Gamma_2. \end{aligned} \tag{3.6}$$

The simple functional relation

$$\theta = \frac{V}{2}\varphi \tag{3.7}$$

exists between θ and φ (see [3]). Hence, we have

$$u = F^{-1}\left(\frac{V}{2}\varphi - \frac{\varphi^2}{2}\right) \tag{3.8}$$

thus (3.6) can be rewritten

$$\nabla \cdot \left(\sigma\left(F^{-1}\left(\frac{V}{2}\varphi - \frac{\varphi^2}{2}\right)\right)\nabla\varphi\right) = 0 \quad \text{in } \Omega \tag{3.9}$$

with the boundary conditions

$$\varphi = 0 \text{ on } \Gamma_1, \varphi = V \quad \text{on } \Gamma_2. \tag{3.10}$$

To this problem we can apply the Kirchhoff's reduction. More precisely, define

$$\psi = L(\varphi), \text{ where } L(\varphi) = \int_0^{\varphi} \sigma\left(F^{-1}\left(\frac{V}{2}\xi - \frac{\xi^2}{2}\right)\right) d\xi.$$

We have

$$\nabla\psi = \sigma\left(F^{-1}\left(\frac{V}{2}\varphi - \frac{\varphi^2}{2}\right)\right)\nabla\varphi$$

and, in view of (3.9) and (3.10),

$$\Delta\psi = 0 \text{ in } \Omega,\ \psi = 0 \text{ on } \Gamma_1,\ \psi = L(V) \text{ on } \Gamma_2.$$

Moreover, recalling (3.8) we obtain

$$\mathbf{J} = -\sigma(u)\nabla\varphi = -\nabla\psi. \tag{3.11}$$

If v is the solution of the problem

$$\Delta v = 0 \text{ in } \Omega,\ v = 0 \text{ on } \Gamma_1,\ v = 1 \text{ on } \Gamma_2$$

we have

$$\psi(x, y) = L(V)v(x, y).$$

Hence, by (3.11)

$$\mathbf{J} = -L(V)\nabla v$$

and

$$I = \int_{\Gamma_1} \mathbf{J}\cdot\mathbf{n}\, ds = -L(V)\int_{\Gamma_1} \frac{dv}{dn} ds. \tag{3.12}$$

On the other hand, I is invariant in the class of the doubly-connected domains of modulus μ. It is therefore enough to compute $\int_{\Gamma_1} \frac{dv}{dn} ds$ in the annulus of radii 1 and μ. We easily find

$$\int_{\Gamma_1} \frac{dv}{dn} ds = -\frac{2\pi}{\ln\mu}.$$

Hence, by (3.12)

$$I = \frac{2\pi L(V)}{\ln\mu}.$$

This gives the total current crossing any doubly-connected domain of modulus μ if all the others data in problem (3.1–3.4) remain unchanged.

Remark 3.1 In problem (3.1–3.4) it is interesting to consider also the density of the heat flow as given by the Fourier's law

$$\mathbf{q}_h = -\kappa(u)\nabla u$$

and the corresponding global quantities, a priori not necessarily equal,

$$Q_{\Gamma_1} = \int_{\Gamma_1} \mathbf{q}_h\cdot\mathbf{n}\, ds,\ Q_{\Gamma_2} = \int_{\Gamma_2} \mathbf{q}_h\cdot\mathbf{n}\, ds.$$

We have

$$\mathbf{q}_2 = -k(u)\nabla u - \varphi\sigma(u)\nabla\varphi = -\sigma(u)\nabla\theta$$

and, by (3.7),

$$\mathbf{q}_2 = -\sigma(u)\frac{V}{2}\nabla\varphi = \frac{V}{2}\mathbf{J}.$$

On the other hand,

$$\mathbf{q}_h = \mathbf{q}_2 - \varphi\mathbf{J}.$$

Hence, in view of the condition $\varphi = 0$ on Γ_1 and of (3.12)

$$Q_{\Gamma_1} = \int_{\Gamma_1} \mathbf{q}_h \cdot \mathbf{n}\, ds = \frac{V}{2}\int_{\Gamma_1} \mathbf{J}\cdot\mathbf{n}\, ds = \frac{\pi V L(V)}{\ln\mu}.$$

Moreover, since $\varphi = V$ on Γ_2 and $\int_{\Gamma_1} \mathbf{J}\cdot\mathbf{n}\, ds = -\int_{\Gamma_2} \mathbf{J}\cdot\mathbf{n}\, ds$ we have

$$Q_{\Gamma_2} = \int_{\Gamma_2} \mathbf{q}_h \cdot \mathbf{n}\, ds = -\frac{V}{2}\int_{\Gamma_2} \mathbf{J}\cdot\mathbf{n}\, ds = \frac{\pi V L(V)}{\ln\mu}.$$

4 Invariance properties in the Soret-Dufour's problem

Theorem 2.1 can be applied to the problem of Example 2 i.e.

$$\nabla\cdot(K(c,u,p)\nabla p) = 0 \quad \text{in } \Omega \tag{4.1}$$

$$\nabla\cdot(\beta(c,u,p)\nabla c + S(c,u,p)\nabla u + K(c,u,p)c\nabla p) = 0 \quad \text{in } \Omega \tag{4.2}$$

$$\nabla\cdot(\kappa(c,u,p)\nabla u + D(c,u,p)\nabla c + K(c,u,p)\nabla p) = 0 \quad \text{in } \Omega \tag{4.3}$$

$$p = p^{(1)},\ c = c^{(1)},\ u = u^{(1)} \ \text{ on } \Gamma_1,\ \ p = p^{(2)},\ c = c^{(2)},\ u = u^{(2)} \quad \text{on } \Gamma_2. \tag{4.4}$$

Therefore the total flows of mass and heat depend on Ω only via its modulus. In this section we consider a special case of problem (4.1–4.4). We suppose β, κ, D, S and K to be positive constants. We have

Theorem 4.1 *Let*

$$p^{(1)} \text{ and } p^{(2)} \in H^2(\Omega),\ c^{(1)},\ c^{(2)},\ u^{(1)} \text{ and } u^{(2)} \in H^1(\Omega). \tag{4.5}$$

Suppose

$$\left(\frac{S+D}{2}\right)^2 < \beta\kappa. \tag{4.6}$$

Then the problem

$$\nabla\cdot(K\nabla p) = 0 \quad \text{in } \Omega \tag{4.7}$$

$$\nabla\cdot(\beta\nabla c + S\nabla u + Kc\nabla p) = 0 \quad \text{in } \Omega \tag{4.8}$$

$$\nabla\cdot(\nabla u + D\nabla c + K\nabla p) = 0 \quad \text{in } \Omega \tag{4.9}$$

$$p = p^{(1)},\ c = c^{(1)}, u = u^{(1)} \text{ on } \Gamma_1, p = p^{(2)},\ c = c^{(2)}, u = u^{(2)} \quad \text{on } \Gamma_2 \tag{4.10}$$

has one and only one solution. Moreover, if all the data are of class C^∞ then also the corresponding solution is of class C^∞.

Proof We compute $p(x,y)$ from the problem

$$\Delta p = 0 \ \text{ in } \Omega,\ \ p = p^{(1)} \ \text{ on } \Gamma_1,\ \ p = p^{(2)} \ \text{ on } \Gamma_2. \tag{4.11}$$

By (4.5) we have $p \in H^2(\Omega)$. Denote c_b and u_b the solutions of the problems

$$\Delta c_b = 0 \quad \text{in } \Omega,\ c_b = c^{(1)} \quad \text{on } \Gamma_1,\ c_b = c^{(2)} \quad \text{on } \Gamma_2 \tag{4.12}$$

$$\Delta u_b = 0 \quad \text{in } \Omega,\ u_b = u^{(1)} \quad \text{on } \Gamma_1,\ u_b = u^{(2)} \quad \text{on } \Gamma_2. \tag{4.13}$$

Setting $h = c - c_b$ and $z = u - u_b$ we restate problem (4.7–4.10) with homogeneous boundary conditions

$$h \in H_0^1(\Omega),\ \nabla \cdot (\beta \nabla h + S\nabla z + Kh\nabla p) = -\nabla \cdot (Kc_b \nabla p) \tag{4.14}$$

$$z \in H_0^1(\Omega),\ \nabla \cdot (\kappa \nabla z + D\nabla h + Kz\nabla p) = -\nabla \cdot (Ku_b \nabla p). \tag{4.15}$$

Define the bilinear form

$$a((h,z),(v,w)) = \int_\Omega (\beta \nabla h \cdot \nabla v + S\nabla z \cdot \nabla v + Kh\nabla p \cdot \nabla v + \kappa \nabla z \cdot \nabla w$$
$$+ D\nabla h \cdot \nabla w + Kz\nabla p \cdot \nabla w)dxdy,$$

where $(h,z) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $(v,w) \in H_0^1(\Omega) \times H_0^1(\Omega)$. $a((h,z),(v,w))$ is coercive and bounded. In fact, we have by (4.11)

$$\int_\Omega h\nabla p \cdot \nabla h\, dxdy = -\int_\Omega \nabla \cdot (h\nabla p)h\, dxdy = -\int_\Omega h\nabla h \cdot \nabla p\, dxdy.$$

Thus

$$\int_\Omega h\nabla p \cdot \nabla h\, dxdy = 0.$$

Similarly

$$\int_\Omega z\nabla p \cdot \nabla z\, dxdy = 0.$$

Hence

$$a((h,z),(h,z)) = \int_\Omega \left(\beta|\nabla h|^2 + (S+D)\nabla h \cdot \nabla z + \kappa|\nabla z|^2\right)dxdy.$$

On the other hand, the matrix

$$\begin{pmatrix} \beta & 0 & \frac{S+D}{2} & 0 \\ 0 & \beta & 0 & \frac{S+D}{2} \\ \frac{S+D}{2} & 0 & \kappa & 0 \\ 0 & \frac{S+D}{2} & 0 & \kappa \end{pmatrix}$$

of the quadratic form

$$\beta h_x^2 + \beta h_y^2 + \kappa z_x^2 + \kappa z_y^2 + (S+D)h_x z_x + (S+D)h_y z_y$$

is definite positive since the determinants of the principal minors i.e.

$$\beta^2,\ \beta\left[\beta\kappa - \left(\frac{S+D}{2}\right)^2\right],\ \left[\beta\kappa - \left(\frac{S+D}{2}\right)^2\right]^2$$

are all positive by (4.6). Hence there exists a positive constant L such that

$$a((h,z),(h,z)) \geq L\int_\Omega \left(|\nabla h|^2 + |\nabla z|^2\right)dxdy. \tag{4.16}$$

It is also easy to prove that $a((h, z), (v, w))$ is bounded. Let us define the linear functional of $(H_0^1(\Omega) \times H_0^1(\Omega))'$

$$\langle f, (v, w)\rangle = -K \int_\Omega \Big(c_b \nabla p \cdot \nabla v + u_b \nabla p \cdot \nabla w\Big) dxdy.$$

We can rewrite (4.14), (4.15) as follows

$$(h, z) \in H_0^1(\Omega) \times H_0^1(\Omega),\ a((h, z), (v, w)) = \langle f, (v, w)\rangle \forall (v, w) \in H_0^1(\Omega) \times H_0^1(\Omega).$$

The Lax-Milgram lemma [13] can be applied and we conclude that (4.7–4.10) has one and only one solution. Using standard regularity results for elliptic system [11] this weak solution can be regularized.

Remark 4.1 If the condition $\left(\frac{S+D}{2}\right)^2 < \beta\kappa$ is not satisfied, problem (4.7–4.10) may not have solutions at all. This fact is already apparent in the one-dimensional counterpart of problem (4.7–4.10) i.e.

$$(\beta c' + Su' + ac)' = 0, c(0) = c^{(1)}, c(1) = c^{(2)}$$
$$(\kappa u' + Dc' + au)' = 0, u(0) = u^{(1)}, u(1) = u^{(2)},$$

where a is a given constant. In fact, suppose $\beta = \kappa = D = S = a = 1$. We have by difference $(c - u)' = 0$ which is not always compatible with the boundary conditions $c(0) = c^{(1)}, c(1) = c^{(2)}, u(0) = u^{(1)}, u(1) = u^{(2)}$.

Let (4.6) be satisfied and let us take in problem (4.7–4.10) $u^{(1)}, u^{(2)}, c^{(1)}, c^{(2)}$ as constants and $p^{(1)} = 0$, $p^{(2)} = \bar{P}$. We compute the total flows of mass and heat in a generic doubly-connected domain of modulus μ. For this goal we solve the problem in the annulus of radii 1 and μ. Denote ρ and θ the polar coordinates. Since the solution is unique by Theorem 4.1 a solution which depends only on ρ is the only possible one. Therefore, we have

$$p(\rho) = \frac{\bar{P}}{\ln \mu} \ln \rho. \tag{4.17}$$

In polar coordinates the Eqs. (4.8) and (4.9) become

$$\frac{d}{d\rho}\left(\beta\rho \frac{dc}{d\rho} + S\rho \frac{du}{d\rho} + mc\right) = 0 \tag{4.18}$$

$$\frac{d}{d\rho}\left(\kappa\rho \frac{du}{d\rho} + D\rho \frac{dc}{d\rho} + mu\right) = 0, \tag{4.19}$$

where

$$m = \frac{\bar{P}}{\ln \mu}.$$

From (4.6) and the inequality $\sqrt{DS} < \frac{S+D}{2}$ we obtain $DS - \beta\kappa \neq 0$. This permits to solve the system of ordinary differential Eqs. (4.18), (4.19). We obtain

$$u(\rho) = A_1 + A_2\rho^M + A_3\rho^L$$
$$c(\rho) = (2D)^{-1}\big[2\beta A_1 + 2DA_4 + A_3\left(\beta - \kappa - \sqrt{N}\right)\rho^L + A_2\left(\beta - \kappa + \sqrt{N}\right)\rho^M\big],$$

where

$$N = (\beta - \kappa)^2 + 4DS, L = \frac{m\left(\beta + \kappa + \sqrt{N}\right)}{2(DS - \beta\kappa)},\ M = \frac{m\left(\beta + \kappa - \sqrt{N}\right)}{2(DS - \beta\kappa)}.$$

The four constants of integration A_1, A_2, A_3 and A_4 are in a one-to-one correspondence with the boundary values $u^{(1)},\ u^{(2)},\ c^{(1)}, c^{(2)}$. They can easily be computed explicitly. Thus we obtain for the total flows of mass Q_m and heat Q_h in an arbitrary domain of modulus μ

$$Q_h = -2\pi\left(A_2 M + A_3 L\right)$$
$$Q_m = -2\pi(2D)^{-1}\left[2\beta A_1 + 2DA_4 + A_3 L\left(\beta - \kappa - \sqrt{N}\right) + A_2 M\left(\beta - \kappa + \sqrt{N}\right)\right].$$

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