Computational Particle Mechanics (2019) 6:297–309
https://doi.org/10.1007/s40571-018-0214-7

Unique solvability and stability analysis for incompressible smoothed particle hydrodynamics method

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Received: 10 July 2018 / Revised: 2 November 2018 / Accepted: 8 November 2018 / Published online: 17 November 2018

Abstract
The incompressible smoothed particle hydrodynamics (ISPH) method is a numerical method widely used for accurately and efficiently solving flow problems with free surface effects. However, to date there has been little mathematical investigation of properties such as stability or convergence for this method. In this paper, unique solvability and stability are mathematically analyzed for implicit and semi-implicit schemes in the ISPH method. Three key conditions for unique solvability and stability are introduced: a connectivity condition with respect to particle distribution and smoothing length, a regularity condition for particle distribution, and a time step condition. The unique solvability of both the implicit and semi-implicit schemes in two- and three-dimensional spaces is established with the connectivity condition. The stability of the implicit scheme in two-dimensional space is established with the connectivity and regularity conditions. Moreover, with the addition of the time step condition, the stability of the semi-implicit scheme in two-dimensional space is established. As an application of these results, modified schemes are developed by redefining discrete parameters to automatically satisfy parts of these conditions.

Keywords Incompressible smoothed particle hydrodynamics method · Incompressible Navier–Stokes equations · Unique solvability · Stability

1 Introduction

The smoothed particle hydrodynamics (SPH) method [6,11] is a kind of numerical method for solving partial differential equations and discretizing them in space using a weighted average of interactions between particles within a neighborhood defined by a smoothing length. For the incompressible Navier–Stokes equations, the incompressible smoothed particle hydrodynamics (ISPH) method, by which the equations are discretized by the SPH method in space and a semi-implicit projection method [4,7] in time, was developed by Cummins and Rudman [5]. The ISPH method has been widely used as a numerical method as it is able to accurately and efficiently solve flow problems with free surface effects [1,10,15]. Moreover, in order to simulate problems with high viscosity, an ISPH method that uses an implicit projection method has been developed [9].

However, there is almost no mathematical background on properties such as stability or convergence for the ISPH method. Although there are a few mathematical analyses for the SPH method or related particle methods, e.g., error estimates for the SPH method with particle volumes related to the vortex method [2,3,13] and error estimates for Poisson and heat equations of a generalized particle method [8], their results do not directly apply to the ISPH method. Hence, the identification of discrete parameter conditions necessary for obtaining stable results has had to rely on experimental studies [14].

This paper establishes the mathematical properties of unique solvability and stability for implicit and semi-implicit schemes in the ISPH method. We introduce three key conditions for unique solvability and stability: a connectivity condition with respect to the particle distribution and smoothing length, a regularity condition for the particle distribution, and a time step condition corresponding to viscous diffusion. Then, we show the unique solvability of both the implicit and semi-implicit schemes in two- and three-dimensional spaces

This work was supported by JSPS KAKENHI Grant Number 17K17585, JSPS A3 Foresight Program, and "Joint Usage/Research Center for Interdisciplinary Large-scale Information Infrastructures" in Japan (Project ID: jh180060-NAH).

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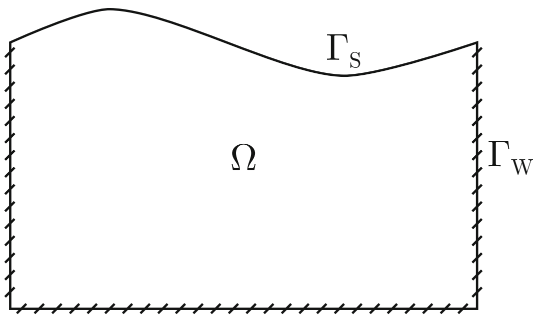

Fig. 1 Domain Ω and boundaries $\Gamma_{\rm W}$ and $\Gamma_{\rm S}$

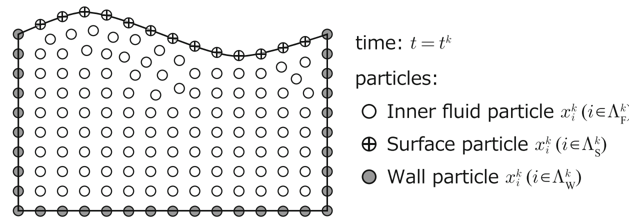

Fig. 2 Particle distribution

with the connectivity condition. We go on to prove the stability of velocity for the implicit scheme in two-dimensional space with the connectivity and regularity conditions. Further, we show the stability of velocity for the semi-implicit scheme in two-dimensional space with the addition of the time step condition. The main advantage of these results in the engineering sense is to clarify conditions required for stable computing in the ISPH method. As an application of these results, we introduce modified schemes with discrete parameters redefined to automatically satisfy the semi-regularity and time step conditions.

2 Incompressible smoothed particle hydrodynamics method

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with smooth boundary Γ. The boundary Γ is divided into two parts: a wall boundary $\Gamma_{\rm W} \subset \Gamma$ and a free surface boundary $\Gamma_{\rm S} := \Gamma \backslash \Gamma_{\rm W}$; see Fig. 1.

We consider the incompressible Navier–Stokes equations: Find $u : \Omega \times (0, T) \to \mathbb{R}^d$ and $p : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\mathrm{D}u}{\mathrm{D}t} = -\dfrac{1}{\rho}\nabla p + \nu \Delta u + f, & (x,t) \in \Omega \times (0,T), \\ \nabla \cdot u = 0, & (x,t) \in \Omega \times (0,T), \\ u = a, & x \in \Omega,\ t = 0, \\ u = 0, & (x,t) \in \Gamma_{\rm W} \times (0,T), \end{cases} \tag{1}$$

where $u : \Omega \times (0, T) \to \mathbb{R}^d$, $p : \Omega \times (0, T) \to \mathbb{R}$, $\rho > 0$, $\nu > 0$, $f : \Omega \times (0, T) \to \mathbb{R}^d$, and $a : \Omega \to \mathbb{R}^d$ denote velocity, pressure, density, kinematic viscosity, body force, and initial velocity of the fluid, respectively. Furthermore, $\mathrm{D}/\mathrm{D}t$ denotes the material derivative defined as $\mathrm{D}/\mathrm{D}t := \partial/\partial t + u \cdot \nabla$. The unknowns are the velocity u and pressure p. We assume the uniqueness and existence of a smooth solution for the incompressible Navier–Stokes equations (1).

We introduce the ISPH method. Let $\tau > 0$ be the time step. Let K be $K := \lfloor T/\tau \rfloor$, where $\lfloor T/\tau \rfloor$ denotes the greatest integer that is less than or equal to T/τ. For $k = 0, 1, \ldots, K$, the kth time t^k is defined as $t^k := k\,\tau$. For $N \in \mathbb{N}$, we define a particle distribution $\mathcal{X}_N$ as

$$\mathcal{X}_N := \left\{ x_i \in \Omega \cup \Gamma \;\middle|\; i = 1, 2, \ldots, N,\ x_i \neq x_j\ (i \neq j) \right\}. \tag{2}$$

We refer to $x_i \in \mathcal{X}_N$ as a particle. Let $\mathcal{X}_N^k$ and x_i^k be a particle distribution and an ith particle at t^k, respectively. Let $\Lambda_N := \{1, 2, \ldots, N\}$. Let $\Lambda_{\rm S}^k \subset \Lambda_N$ be the index set of particles judged to be on the free surface, $\Lambda_{\rm W}^k \subset \Lambda_N$ the index set of particles on the wall boundary, and $\Lambda_{\rm F}^k \subset \Lambda_N$ the index set of the other particles. We refer to x_i^k $(i \in \Lambda_{\rm F}^k)$, x_i^k $(i \in \Lambda_{\rm S}^k)$, and x_i^k $(i \in \Lambda_{\rm W}^k)$ as an inner fluid particle, a surface particle, and a wall particle, respectively; see Fig. 2.

For $N \in \mathbb{N}$, we define a particle volume set $\mathcal{V}_N$ as

$$\mathcal{V}_N := \left\{ \omega_i > 0 \;\middle|\; i = 1, 2, \ldots, N,\ \sum_{i=1}^{N} \omega_i = |\Omega| \right\}. \tag{3}$$

Here, $|\Omega|$ denotes the volume of Ω. We refer to $\omega_i \in \mathcal{V}_N$ as a particle volume. In the SPH method, by introducing a particle density ρ_i and particle mass m_i, the particle volume ω_i is generally given as $\omega_i = \rho_i / m_i$.

For $w : [0, \infty) \to \mathbb{R}$, we consider the following conditions:

$$w(r) \begin{cases} > 0, & 0 < r < r_0, \\ = 0, & r \geq r_0; \end{cases} \tag{4}$$

$$\dot{w}(r) \begin{cases} < 0, & 0 < r < r_0, \\ = 0, & r = 0 \text{ or } r \geq r_0; \end{cases} \tag{5}$$

$$\int_{\mathbb{R}^d} w(|x|)\mathrm{d}x = 1; \tag{6}$$

$$w \in C^2([0, \infty)). \tag{7}$$

Here, r_0 and $\dot{w}$ are a positive constant and the first derivative of w, respectively. We define a function set $\mathcal{W}$ as

$$\mathcal{W} := \{ w : [0, \infty) \to \mathbb{R} \mid w \text{ satisfies (4)–(7)} \}. \tag{8}$$

We refer to $w \in \mathcal{W}$ as a reference weight function. We define a smoothing length h as a positive number that satisfies $r_0 h > \min\{|x_i^k - x_j^k| \mid i \neq j\}$. For the reference weight function

$w \in \mathcal{W}$ and the smoothing length h, a weight function $w_h : [0,\infty) \to \mathbb{R}$ is defined as

$$w_h(r) := \frac{1}{h^d} w\left(\frac{r}{h}\right). \tag{9}$$

For example, in the SPH method, the following reference weight functions are often used: the cubic B-spline ($r_0 = 2$)

$$w(r) := \beta_d^{\text{cubic}} \begin{cases} 1 - \frac{3}{2}r^2 + \frac{4}{3}r^3, & 0 \le r < 1, \\ \frac{1}{4}(2-r)^3, & 1 \le r < 2, \\ 0, & 2 \le r, \end{cases} \tag{10}$$

and the quintic B-spline ($r_0 = 3$)

$$w(r) := \beta_d^{\text{quintic}} \begin{cases} (3-r)^5 - 6(2-r)^5 + 15(1-r)^5, & 0 \le r < 1, \\ (3-r)^5 - 6(2-r)^5, & 1 \le r < 2, \\ (3-r)^5, & 2 \le r < 3, \\ 0, & 3 \le r. \end{cases} \tag{11}$$

Here, β_d^{cubic} and β_d^{quintic} are constants dependent on d to satisfy condition (6) and are calculated as

$$\beta_d^{\text{cubic}} = \begin{cases} \frac{10}{7\pi}, & d = 2, \\ \frac{1}{\pi}, & d = 3; \end{cases} \qquad \beta_d^{\text{quintic}} = \begin{cases} \frac{7}{478\pi}, & d = 2, \\ \frac{1}{120\pi}, & d = 3. \end{cases} \tag{12}$$

For an index set $\Lambda \subset \Lambda_N$ and $k = 0, 1, \ldots, N$, a function space $V_N^k(\Lambda)$ is defined as

$$V_N^k(\Lambda) := \left\{ v : \{x_i^k \in \mathcal{X}_N^k\}_{i \in \Lambda} \to \mathbb{R} \right\}. \tag{13}$$

For $k = 0, 1, \ldots, K$, a function $f^k \in V_N^k(\Lambda_N)$ is defined as

$$f^k(x_i^k) := \begin{cases} f(x_i^k, t^k), & i \in \Lambda_{\mathrm{F}}^k \cup \Lambda_{\mathrm{S}}^k, \\ 0, & i \in \Lambda_{\mathrm{W}}^k. \end{cases} \tag{14}$$

Hereinafter, for a function v^k ($k = 0, 1, \ldots, K$) defined in $\mathcal{X}_N^k$, we denote $v^k(x_i^k)$ as v_i^k. Now, we consider the following two schemes in the ISPH method.

Implicit scheme: find $u^k \in V_N^k(\Lambda_N)^d$ ($k = 0, 1, \ldots, K$) and $p^k \in V_N^k(\Lambda_{\mathrm{F}}^k \cup \Lambda_{\mathrm{S}}^k)$ ($k = 1, 2, \ldots, K$) such that

$$u_i^0 = a(x_i^0), \qquad i = 1, 2, \ldots, N, \tag{I-a}$$

and for $k = 0, 1, \ldots, K-1$,

$$\begin{cases} \dfrac{\widetilde{u}_i^{k+1} - u_i^k}{\tau} = \nu \langle \Delta \widetilde{u} \rangle_i^{k+1} + f_i^k, & i \in \Lambda_{\mathrm{F}}^k \cup \Lambda_{\mathrm{S}}^k, \\ \widetilde{u}_i^{k+1} = 0, & i \in \Lambda_{\mathrm{W}}^k; \end{cases} \tag{I-b}$$

$$\begin{cases} \langle \Delta p \rangle_i^{k+1} = \dfrac{\rho}{\tau} \langle \nabla \cdot \widetilde{u} \rangle_i^{k+1}, & i \in \Lambda_{\mathrm{F}}^k, \\ p_i^{k+1} = 0, & i \in \Lambda_{\mathrm{S}}^k; \end{cases} \tag{I-c}$$

$$\begin{cases} \dfrac{u_i^{k+1} - \widetilde{u}_i^{k+1}}{\tau} = -\dfrac{1}{\rho} \langle \nabla p \rangle_i^{k+1}, & i \in \Lambda_{\mathrm{F}}^k \cup \Lambda_{\mathrm{S}}^k, \\ u_i^{k+1} = 0, & i \in \Lambda_{\mathrm{W}}^k. \end{cases} \tag{I-d}$$

Semi-implicit scheme: find $u^k \in V_N^k(\Lambda_N)^d$ ($k = 0, 1, \ldots, K$) and $p^k \in V_N^k(\Lambda_{\mathrm{F}}^k \cup \Lambda_{\mathrm{S}}^k)$ ($k = 1, 2, \ldots, K$) such that

$$u_i^0 = a(x_i^0), \qquad i = 1, 2, \ldots, N, \tag{SI-a}$$

and for $k = 0, 1, \ldots, K-1$,

$$\begin{cases} \dfrac{\widetilde{u}_i^{k+1} - u_i^k}{\tau} = \nu \langle \Delta u \rangle_i^{k} + f_i^k, & i \in \Lambda_{\mathrm{F}}^k \cup \Lambda_{\mathrm{S}}^k, \\ \widetilde{u}_i^{k+1} = 0, & i \in \Lambda_{\mathrm{W}}^k; \end{cases} \tag{SI-b}$$

$$\begin{cases} \langle \Delta p \rangle_i^{k+1} = \dfrac{\rho}{\tau} \langle \nabla \cdot \widetilde{u} \rangle_i^{k+1}, & i \in \Lambda_{\mathrm{F}}^k, \\ p_i^{k+1} = 0, & i \in \Lambda_{\mathrm{S}}^k; \end{cases} \tag{SI-c}$$

$$\begin{cases} \dfrac{u_i^{k+1} - \widetilde{u}_i^{k+1}}{\tau} = -\dfrac{1}{\rho} \langle \nabla p \rangle_i^{k+1}, & i \in \Lambda_{\mathrm{F}}^k \cup \Lambda_{\mathrm{S}}^k, \\ u_i^{k+1} = 0, & i \in \Lambda_{\mathrm{W}}^k. \end{cases} \tag{SI-d}$$

Here, the approximations of the derivatives are defined as

$$\langle \Delta u \rangle_i^k := 2 \sum_{j \neq i} \omega_j \frac{u_i^k - u_j^k}{|x_i^k - x_j^k|} \frac{x_i^k - x_j^k}{|x_i^k - x_j^k|} \cdot \nabla w_h(|x_i^k - x_j^k|), \tag{15}$$

$$\langle \Delta \widetilde{u} \rangle_i^{k+1} := 2 \sum_{j \neq i} \omega_j \frac{\widetilde{u}_i^{k+1} - \widetilde{u}_j^{k+1}}{|x_i^k - x_j^k|} \frac{x_i^k - x_j^k}{|x_i^k - x_j^k|} \cdot \nabla w_h(|x_i^k - x_j^k|), \tag{16}$$

$$\langle \nabla \cdot \widetilde{u} \rangle_i^{k+1} := \sum_{j=1}^{N} \omega_j (\widetilde{u}_j^{k+1} + \widetilde{u}_i^{k+1}) \cdot \nabla w_h(|x_i^k - x_j^k|), \tag{17}$$

$$\langle \nabla p \rangle_i^{k+1} := \sum_{j \in \Lambda_{\mathrm{F}}^k \cup \Lambda_{\mathrm{S}}^k} \omega_j (p_j^{k+1} - p_i^{k+1}) \nabla w_h(|x_i^k - x_j^k|), \tag{18}$$

$$\langle \Delta p \rangle_i^{k+1} := 2 \sum_{j \in \Lambda_{\mathrm{F}}^k \cup \Lambda_{\mathrm{S}}^k \setminus \{i\}} \omega_j \frac{p_i^{k+1} - p_j^{k+1}}{|x_i^k - x_j^k|} \frac{x_i^k - x_j^k}{|x_i^k - x_j^k|} \cdot \nabla w_h(|x_i^k - x_j^k|). \tag{19}$$

In both schemes, the particles are updated by

$$x_i^{k+1} = x_i^k + \tau u_i^{k+1}, \qquad i = 1, 2, \dots, N,\ k = 0, 1, \dots, K-1. \tag{20}$$

Equations (I-b) and (SI-b) are referred to as the prediction step, Eqs. (I-c) and (SI-c) as the pressure Poisson equation, and Eqs. (I-d) and (SI-d) as the correction step. The difference between these schemes is only the viscous term in the prediction steps (I-b) and (SI-b). Because (I-b) and (I-c) are implicit, we refer to the first scheme as the implicit scheme. In contrast, because only the equation for pressure (SI-c) is implicit, we refer to the second scheme as the semi-implicit scheme.

Remark 1 Although many approximation operators have been proposed for the ISPH method, the approximation operators (15)–(19) are chosen because they satisfy mathematical properties in Sect. 4.2. Analyses of the ISPH method using other approximation operators are left as future problems.

3 Key conditions for discrete parameters

In order to analyze the unique solvability and the stability for the implicit and semi-implicit schemes, we introduce three important conditions for discrete parameters: the connectivity, semi-regularity, and time step conditions.

Definition 1 (*h-connectivity condition*) For the smoothing length h, the particle distribution $\mathcal{X}_N^k$ satisfies the h-connectivity condition if for all $i \in \Lambda_{\mathrm{F}}^k$, there exist sequences $\{i_l\}_{l=1}^{\zeta}$ and $\{i_l^*\}_{l=1}^{\zeta^*} \subset \Lambda_N$ such that

$$\begin{aligned} &i_1 = i, \quad 0 < |x_{i_l}^k - x_{i_{l+1}}^k| < r_0 h\ (1 \le l < \zeta), \\ &\quad i_l \in \Lambda_{\mathrm{F}}^k\ (1 \le l < \zeta), \quad i_\zeta \in \Lambda_{\mathrm{S}}^k, \end{aligned} \tag{21}$$

$$\begin{aligned} &i_1^* = i, \quad 0 < |x_{i_l^*}^k - x_{i_{l+1}^*}^k| < r_0 h\ (1 \le l < \zeta^*), \\ &\quad i_l^* \in \Lambda_{\mathrm{F}}^k\ (1 \le l < \zeta^*), \quad i_{\zeta^*}^* \in \Lambda_{\mathrm{W}}^k. \end{aligned} \tag{22}$$

Definition 2 (*semi-regularity condition*) A family $\{((\{\mathcal{X}_N^k\}_{k=0}^K, \mathcal{V}_N, h, \tau)\}$ satisfies the semi-regularity condition if there exists a positive constant c_0 such that

$$\begin{aligned} &\max_{i=1,2,\dots,N} \left\{ \sum_{j=1}^N \omega_j |x_i^k - x_j^k| |\dot{w}_h(|x_i^k - x_j^k|)| \right\} \\ &\quad \le d + c_0 \tau, \quad k = 1, 2, \dots, K. \end{aligned} \tag{23}$$

Definition 3 (*time step condition*) A family $\{((\{\mathcal{X}_N^k\}_{k=0}^K, \mathcal{V}_N, h, \tau)\}$ satisfies the time step condition if there exists a constant δ $(0 < \delta < 1)$ such that

$$\begin{aligned} &\tau \le \frac{\delta}{2\nu} \left[\max_{i=1,2,\dots,N} \left(\sum_{j \ne i} \omega_j \frac{|\dot{w}_h(|x_i^k - x_j^k|)|}{|x_i^k - x_j^k|} \right) \right]^{-1}, \\ &\quad k = 1, 2, \dots, K. \end{aligned} \tag{24}$$

Remark 2 Consider a graph G whose vertex set is particle distribution $\mathcal{X}_N^k$ and whose edges are a pair (x_i^k, x_j^k) that satisfies $0 < |x_i^k - x_j^k| < r_0 h$, as shown, for example, in Fig. 3. By Definition 1, that the particle distribution $\mathcal{X}_N^k$ satisfies the h-connectivity condition is equivalent to all fluid particles having a path on G to a surface particle and a wall particle.

Remark 3 Because the approximation

$$\begin{aligned} &\max_{i=1,2,\dots,N} \left\{ \sum_{j=1}^N \omega_j |x_i^k - x_j^k| |\dot{w}_h(|x_i^k - x_j^k|)| \right\} \\ &\quad \approx \int_{\mathbb{R}^d} |y|\, |\dot{w}_h(|y|)|\, \mathrm{d}y = d \end{aligned} \tag{25}$$

holds, the family $\{((\{\mathcal{X}_N^k\}_{k=0}^K, \mathcal{V}_N, h, \tau)\}$ can satisfy the semi-regularity condition under appropriate settings of discrete parameters. In particular, as the left side of (23) becomes larger when a cohesion of particles occurs, the semi-regularity condition partially denotes a regularity of particle distributions.

Remark 4 The time step condition (24) corresponds to a constraint of the time step due to viscous diffusion. Experimentally, the constraint is given by

$$\tau \le \alpha \frac{h^2}{\nu}, \tag{26}$$

where the coefficient α is usually given as a value on the order of 0.1 [12,14]. In contrast, the time step condition (24) becomes

$$\begin{aligned} &\tau \le \frac{\delta}{2\nu} \left[\max_{i=1,2,\dots,N} \left(\sum_{j \ne i} \omega_j \frac{|\dot{w}_h(|x_i^k - x_j^k|)|}{|x_i^k - x_j^k|} \right) \right]^{-1} \\ &\quad < \widehat{\alpha} \frac{h^2}{\nu}, \end{aligned} \tag{27}$$

where $\widehat{\alpha}$ is defined by

$$\widehat{\alpha} := \frac{1}{2h^2} \left[\max_{i=1,2,\dots,N} \left(\sum_{j \ne i} \omega_j \frac{|\dot{w}_h(|x_i^k - x_j^k|)|}{|x_i^k - x_j^k|} \right) \right]^{-1}. \tag{28}$$

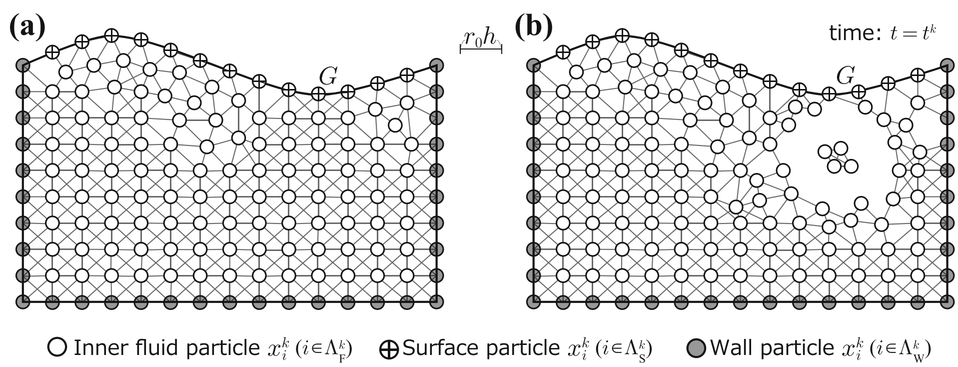

Fig. 3 Particle distributions $\mathcal{X}_N^k$ and their graphs G: the left and right sides show particle distributions **a** with and **b** without satisfying the h-connectivity condition, respectively

Therefore, $\widehat{\alpha}$ corresponds to α. Moreover, because the approximation

$$\begin{aligned}\widehat{\alpha} &= \frac{1}{2h^2}\left[\max_{i=1,2,\ldots,N}\left(\sum_{j\neq i}\omega_j \frac{|\dot{w}_h(|x_i^k - x_j^k|)|}{|x_i^k - x_j^k|}\right)\right]^{-1} \\ &\approx \frac{1}{2h^2}\left(\int_{\mathbb{R}^d}\frac{|\dot{w}_h(|y|)|}{|y|}\mathrm{d}y\right)^{-1} \\ &= \frac{1}{2}\left(\int_{\mathbb{R}^d}\frac{|\dot{w}(|y|)|}{|y|}\mathrm{d}y\right)^{-1} \end{aligned} \tag{29}$$

holds, we can estimate the approximate value of $\widehat{\alpha}$ for reference weight functions in advance. When w is the cubic B-spline (10), the approximate value of $\widehat{\alpha}$ is calculated as

$$\widehat{\alpha} \approx \frac{1}{2}\left(\int_{\mathbb{R}^d}\frac{|\dot{w}(|y|)|}{|y|}\mathrm{d}y\right)^{-1} = \begin{cases}\dfrac{7}{40} = 0.175, & d=2,\\ \dfrac{1}{6} \approx 0.167, & d=3,\end{cases} \tag{30}$$

and when w is the quintic B-spline (11) as

$$\widehat{\alpha} \approx \frac{1}{2}\left(\int_{\mathbb{R}^d}\frac{|\dot{w}(|y|)|}{|y|}\mathrm{d}y\right)^{-1} = \begin{cases}\dfrac{239}{924} \approx 0.259, & d=2,\\ \dfrac{1}{4} = 0.250, & d=3.\end{cases} \tag{31}$$

Because α is experimentally given as on the order of 0.1, the above approximate values of $\widehat{\alpha}$ agree well with the experimental values.

4 Unique solvability and stability

4.1 Unique solvability

First, we show the unique solvability for the implicit and semi-implicit schemes.

Theorem 1 *If particle distribution $\mathcal{X}_N^k$ satisfies the h-connectivity condition for all $k = 0, 1, \ldots, K-1$, then both the implicit and semi-implicit schemes have a unique solution.*

Proof Because (I-d), (SI-b), and (SI-d) are explicit, these are clearly solvable. Therefore, we prove the unique solvability of (I-b), (I-c), and (SI-c).

First, we show the unique solvability of the prediction step (I-b). We fix $k = 0, 1, \ldots, K$. Let N_F^k and N_S^k be the number of inner fluid particles and the number of surface particles, respectively, at time step t^k. We renumber the index of particles so that $i = 1, 2, \ldots, N_\mathrm{F}^k \in \Lambda_\mathrm{F}^k$ and $i = N_\mathrm{F}^k + 1, N_\mathrm{F}^k + 2, \ldots, N_\mathrm{F}^k + N_\mathrm{S}^k \in \Lambda_\mathrm{S}^k$. Let a_{ij} $(i, j = 1, 2, \ldots, N)$ be

$$a_{ij} := \begin{cases}0, & i=j,\\ -2\dfrac{x_i^k - x_j^k}{|x_i^k - x_j^k|^2}\cdot\nabla w_h(|x_i^k - x_j^k|), & i\neq j.\end{cases} \tag{32}$$

We define a matrix $A \in \mathbb{R}^{(N_\mathrm{F}^k+N_\mathrm{S}^k)\times(N_\mathrm{F}^k+N_\mathrm{S}^k)}$ and a vector $b \in \mathbb{R}^{N_\mathrm{F}^k+N_\mathrm{S}^k}$ respectively as

$$A_{ij} := \begin{cases}1 + \tau\nu\displaystyle\sum_{l=1}^{N}\omega_l a_{il}, & i=j,\\ -\omega_j a_{ij}, & i\neq j;\end{cases} \tag{33}$$

$$b_i := u_i^k + \tau f_i^k, \qquad i = 1, 2, \ldots, N_\mathrm{F}^k + N_\mathrm{S}^k. \tag{34}$$

Then, (I-b) is equivalent to the linear equations

$$A\mathrm{x} = b, \tag{35}$$

where $\mathrm{x}_i := \widetilde{u}_i^{k+1}$ $(i = 1, 2, \ldots, N_\mathrm{F}^k + N_\mathrm{S}^k)$. Therefore, it is sufficient to show that A is non-singular. From (5), we have

$$-2\frac{x_i^k - x_j^k}{|x_i^k - x_j^k|^2}\cdot\nabla w_h(|x_i^k - x_j^k|) = -2\frac{\dot{w}_h(|x_i^k - x_j^k|)}{|x_i^k - x_j^k|} \geq 0. \tag{36}$$

Therefore, as a_{ij} is nonnegative and $a_{ii}=0$, we have, for $i=1,2,\dots,N_{\mathrm{F}}^k+N_{\mathrm{S}}^k$,

$$|A_{ii}|-\sum_{j\neq i}|A_{ij}|=1+\tau\nu\sum_{l=1}^{N}\omega_l a_{il}-\sum_{j\neq i}\omega_j a_{ij}$$
$$=1+\tau\nu\sum_{l=N_{\mathrm{F}}^k+N_{\mathrm{S}}^k+1}^{N}\omega_l a_{il}>0. \tag{37}$$

Hence, A is a strictly diagonally dominant matrix. Thus, A is non-singular.

Next, we show the unique solvability of the Poisson equations (I-c) and (SI-c). We define matrices $\widehat{A}, D\in\mathbb{R}^{N_{\mathrm{F}}^k\times N_{\mathrm{F}}^k}$ and a vector $\widehat{b}\in\mathbb{R}^{N_{\mathrm{F}}^k}$ respectively as

$$\widehat{A}_{ij}:=\begin{cases}\displaystyle\sum_{l=1}^{N_{\mathrm{S}}^k}\frac{\omega_l}{\omega_i}a_{il}, & i=j,\\ -a_{ij}, & i\neq j;\end{cases}$$
$$D:=\mathrm{diag}(\omega_i);$$
$$\widehat{b}_i:=\frac{\rho}{\tau}\langle\nabla\cdot\widetilde{u}\rangle_i^{k+1},\qquad i=1,2,\dots,N_{\mathrm{F}}^k. \tag{38}$$

Then, (I-c) and (SI-c) are equivalent to

$$\widehat{A}D\widehat{\mathrm{x}}=\widehat{b}, \tag{39}$$

where $\widehat{\mathrm{x}}_i:=p_i^{k+1}\,(i=1,2,\dots,N_{\mathrm{F}}^k)$. As $\omega_i>0\,(i=1,2,\dots,N_{\mathrm{F}}^k)$, the diagonal matrix D is non-singular. Therefore, it is sufficient to prove that $\widehat{A}$ is non-singular. As $\widehat{A}$ is symmetric, we will prove that $\widehat{A}$ is a positive definite matrix. For all $\alpha\in\mathbb{R}^{N_{\mathrm{F}}^k}\backslash\{0\}$, we have

$$\sum_{i,j=1}^{N_{\mathrm{F}}^k}\alpha_i\alpha_j\widehat{A}_{ij}=2\sum_{1\le i<j\le N_{\mathrm{F}}^k}\alpha_i\alpha_j\widehat{A}_{ij}+\sum_{i=1}^{N_{\mathrm{F}}^k}\alpha_i^2\widehat{A}_{ii}$$
$$=-2\sum_{1\le i<j\le N_{\mathrm{F}}^k}\alpha_i\alpha_j a_{ij}+\sum_{i=1}^{N_{\mathrm{F}}^k}\alpha_i^2\sum_{k=1}^{N_{\mathrm{S}}^k}\frac{\omega_k}{\omega_i}a_{ik}$$
$$=\sum_{1\le i<j\le N_{\mathrm{F}}^k}\frac{(\omega_j\alpha_i-\omega_i\alpha_j)^2}{\omega_i\omega_j}a_{ij}$$
$$+\sum_{i=1}^{N_{\mathrm{F}}^k}\alpha_i^2\sum_{k=N_{\mathrm{F}}^k+1}^{N_{\mathrm{S}}^k}\frac{\omega_k}{\omega_i}a_{ik}. \tag{40}$$

As a_{ij} is nonnegative and ω_i is positive, (40) is nonnegative. For $a\in\mathbb{R}^{N_{\mathrm{F}}^k}\backslash\{0\}$, we set i such that $\alpha_i\neq 0$. Because of the particle distribution $\mathcal{X}_N^k$ with h-connectivity, we can take a sequence $\{i_k\}_{k=1}^{\zeta}$ such that the conditions given in (21) hold. As all terms of the last equation in (40) are nonnegative, we have

$$\sum_{i,j=1}^{N_{\mathrm{F}}^k}\alpha_i\alpha_j\widehat{A}_{ij}\ge\sum_{k=1}^{\zeta-1}\frac{(\omega_{i_{k+1}}\alpha_{i_k}-\omega_{i_k}\alpha_{i_{k+1}})^2}{\omega_{i_k}\omega_{i_{k+1}}}a_{i_ki_{k+1}}$$
$$+\frac{\omega_{i_\zeta}}{\omega_{i_{\zeta-1}}}\alpha_{i_\zeta}^2a_{i_{\zeta-1}i_\zeta}. \tag{41}$$

As $|x_{i_k}^k-x_{i_{k+1}}^k|<r_0h$, the value of $a_{i_ki_{k+1}}$ $(k=1,2,\dots,\zeta-1)$ is positive. Thus, if the right side of (41) equals zero, then $\alpha_{i_k}=0\,(k=1,2,\dots,\zeta)$. As this is inconsistent with $\alpha_i=\alpha_{i_1}\neq 0$, the right side of (41) is positive. Therefore, the matrix $\widehat{A}$ is a positive definite matrix. Consequently, the matrix $\widehat{A}$ is non-singular. $\square$

4.2 Discrete Sobolev norms and their mathematical properties

Next, we introduce some notation and show certain lemmas. For $\Lambda\subset\Lambda_N$, $m=1,2,\dots,d$, and $k=0,1,\dots,N$, we define a discrete inner product $(\cdot,\cdot)_\Lambda:V_N^k(\Lambda)^m\times V_N^k(\Lambda)^m\to\mathbb{R}$ and discrete L^2 norm $\|\cdot\|_{0,\Lambda}:V_N^k(\Lambda)^m\to\mathbb{R}$ as

$$(\phi,\varphi)_\Lambda:=\sum_{i\in\Lambda}\omega_i\,\phi_i\cdot\varphi_i, \tag{42}$$

$$\|\phi\|_{0,\Lambda}:=(\phi,\phi)_\Lambda^{1/2}=\left(\sum_{i\in\Lambda}\omega_i\,\phi_i^2\right)^{1/2}, \tag{43}$$

respectively. Moreover, we define a discrete H_0^1 semi-norm $|\cdot|_{1,\Lambda,k}:V_N^k(\Lambda)^m\to\mathbb{R}$ and discrete H_0^{-1} semi-norm $|\cdot|_{-1,\Lambda,k}:V_N^k(\Lambda)^m\to\mathbb{R}$ as

$$|\phi|_{1,\Lambda,k}:=\left(\sum_{i\in\Lambda}\omega_i\sum_{j\in\Lambda\backslash\{i\}}\omega_j\frac{|\phi_i-\phi_j|^2}{|x_i^k-x_j^k|}|\dot{w}_h(|x_i^k-x_j^k|)|\right)^{1/2}, \tag{44}$$

$$|\phi|_{-1,\Lambda,k}:=\sup_{\varphi\in V_N^k(\Lambda)^m\backslash\{0\}}\frac{(\phi,\varphi)_\Lambda}{|\varphi|_{1,\Lambda,k}}, \tag{45}$$

respectively. Then, we obtain the following lemmas:

Lemma 1 *For $\phi,\varphi\in V_N^k(\Lambda)^d$ $(\Lambda\subset\Lambda_N,\ k=0,1,\dots,K)$, we have*

$$(\phi,\varphi)_\Lambda\le\|\phi\|_{0,\Lambda}\|\varphi\|_{0,\Lambda}. \tag{46}$$

Proof The Cauchy–Schwarz inequality (see Appendix A) immediately yields inequality (46). $\square$

Lemma 2 *Assume the particle distribution $\mathcal{X}_N^k$ $(k=0,1,\dots,K)$ satisfies the h-connectivity condition and $\varphi\in$*

$V_N^k(\Lambda_N)^d$ $(k=0,1,\ldots,K)$ satisfies $\varphi_i=0$ for $i\in\Lambda_{\mathrm{W}}^k$. Then, we have, for $\phi\in V_N^k(\Lambda_N)^d$ $(k=0,1,\ldots,K)$,

$$(\phi,\varphi)_{\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}=(\phi,\varphi)_{\Lambda_N}\le|\phi|_{-1,\Lambda_N,k}|\varphi|_{1,\Lambda_N,k}. \tag{47}$$

Proof We first show the norm property: $\varphi=0\ \Leftrightarrow\ |\varphi|_{1,\Lambda_N,k}=0$. From the definition of discrete H^1 semi-norm (44), it is obvious that $\varphi=0\ \Rightarrow\ |\varphi|_{1,\Lambda_N,k}=0$. Assume $|\varphi|_{1,\Lambda_N,k}=0$. As the particle distribution $\mathcal{X}_N^k$ satisfies the h-connectivity condition, for any $i\in\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k$, we can take a sequence $\{i_k^*\}_{k=1}^{\zeta^*}$ such that

$$\begin{aligned}&i_1^*=i,\qquad 0<|x_{i_l^*}^k-x_{i_{l+1}^*}^k|<r_0h\ (1\le l<\zeta^*),\\&i_{\zeta^*}^*\in\Lambda_{\mathrm{W}}^k.\end{aligned} \tag{48}$$

Therefore, we have, for $i\in\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k$,

$$\begin{aligned}&|\varphi|_{1,\Lambda_N,k}^2\\&\quad=\sum_{i\in\Lambda_N}\omega_i\sum_{j\in\Lambda_N\setminus\{i\}}\omega_j\frac{|\varphi_i-\varphi_j|^2}{|x_i^k-x_j^k|}|\dot{w}_h(|x_i^k-x_j^k|)|\\&\quad\ge\sum_{l=1}^{\zeta^*-1}\omega_{i_l^*}\omega_{i_{l+1}^*}\frac{|\varphi_{i_l^*}-\varphi_{i_{l+1}^*}|^2}{|x_{i_l^*}^k-x_{i_{l+1}^*}^k|}|\dot{w}_h(|x_{i_l^*}^k-x_{i_{l+1}^*}^k|)|\ge 0.\end{aligned} \tag{49}$$

Then, because $|\varphi|_{1,\Lambda_N,k}=0$ and $\omega_{i_l^*}\omega_{i_{l+1}^*}|x_{i_l^*}^k-x_{i_{l+1}^*}^k|^{-1}$ $|\dot{w}_h(|x_{i_l^*}^k-x_{i_{l+1}^*}^k|)|>0$, we have $|\varphi_{i_l^*}-\varphi_{i_{l+1}^*}|=0$ for $l=1,2,\ldots,\zeta^*-1$. Moreover, because $\varphi_{i_{\zeta^*}}=0$, we obtain $\varphi_i=\varphi_{i_1}=0$. Because $i\in\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k$ is arbitrary, we obtain $\varphi=0$.

Next, we show (47). As $\varphi_i=0$ for $i\in\Lambda_{\mathrm{W}}^k$, we have $(\phi,\varphi)_{\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}=(\phi,\varphi)_{\Lambda_N}$. When $|\varphi|_{1,\Lambda_N,k}=0$, the norm property, $\varphi=0\ \Leftrightarrow\ |\varphi|_{1,\Lambda_N,k}=0$, yields

$$(\phi,\varphi)_{\Lambda_N}=|\phi|_{-1,\Lambda_N,k}|\varphi|_{1,\Lambda_N,k}=0. \tag{50}$$

When $|\varphi|_{1,\Lambda_N,k}\neq 0$, from the definition of discrete H^{-1} semi-norm (45), we obtain

$$|\phi|_{-1,\Lambda,k}=\sup_{\widehat{\varphi}\in V_N^k(\Lambda)^m\setminus\{0\}}\frac{(\phi,\widehat{\varphi})_\Lambda}{|\widehat{\varphi}|_{1,\Lambda,k}}\ge\frac{(\phi,\varphi)_\Lambda}{|\varphi|_{1,\Lambda,k}}. \tag{51}$$

Therefore, we conclude (47). □

Lemma 3 *For $\phi\in V_N^k(\Lambda)^d$ and $\psi\in V_N^k(\Lambda)$ $(\Lambda\subset\Lambda_N,\ k=0,1,\ldots,K)$, we have*

$$(\psi,\langle\nabla\cdot\phi\rangle^k)_\Lambda=-(\langle\nabla\psi\rangle^k,\phi)_\Lambda, \tag{52}$$
$$-(\psi,\langle\Delta\psi\rangle^k)_\Lambda=|\psi|_{1,\Lambda,k}^2. \tag{53}$$

Here, these approximations of derivatives are defined as

$$\langle\nabla\cdot\phi\rangle^k:=\sum_{j\in\Lambda}\omega_j\left(\phi_j+\phi_i\right)\cdot\nabla w_h(|x_i^k-x_j^k|), \tag{54}$$
$$\langle\nabla\psi\rangle^k:=\sum_{j\in\Lambda}\omega_j\left(\psi_j-\psi_i\right)\nabla w_h(|x_i^k-x_j^k|), \tag{55}$$
$$\begin{aligned}\langle\Delta\psi\rangle^k&:=2\sum_{j\in\Lambda\setminus\{i\}}\omega_j\frac{\psi_i-\psi_j}{|x_i^k-x_j^k|}\frac{x_i^k-x_j^k}{|x_i^k-x_j^k|}\\&\quad\cdot\nabla w_h(|x_i^k-x_j^k|).\end{aligned} \tag{56}$$

Proof First, we prove (52). As $\nabla w_h(|x_i^k-x_j^k|)=-\nabla w_h(|x_j^k-x_i^k|)$, we have

$$\begin{aligned}&(\psi,\langle\nabla\cdot\phi\rangle^k)_\Lambda\\&\quad=\sum_{i\in\Lambda}\omega_i\psi_i\sum_{j\in\Lambda}\omega_j\left(\phi_j+\phi_i\right)\cdot\nabla w_h(|x_i^k-x_j^k|)\\&\quad=\sum_{i\in\Lambda}\sum_{j\in\Lambda}\omega_i\omega_j\psi_i\left(\phi_j+\phi_i\right)\cdot\nabla w_h(|x_i^k-x_j^k|)\\&\quad=\frac{1}{2}\sum_{i\in\Lambda}\sum_{j\in\Lambda}\omega_i\omega_j(\psi_i-\psi_j)\left(\phi_j+\phi_i\right)\cdot\nabla w_h(|x_i^k-x_j^k|)\\&\quad=\sum_{i\in\Lambda}\omega_i\phi_i\cdot\sum_{j\in\Lambda}\omega_j(\psi_i-\psi_j)\nabla w_h(|x_i^k-x_j^k|)\\&\quad=-(\langle\nabla\psi\rangle^k,\phi)_\Lambda.\end{aligned} \tag{57}$$

Next, we prove (53). Let

$$J_{ij}:=\begin{cases}0, & i=j,\\ \dfrac{x_i^k-x_j^k}{|x_i^k-x_j^k|^2}\cdot\nabla w_h(|x_i^k-x_j^k|), & i\neq j.\end{cases} \tag{58}$$

Because $J_{ij}=-J_{ji}$, we obtain

$$\begin{aligned}-(\psi,\langle\Delta\psi\rangle^k)_\Lambda&=2\sum_{i\in\Lambda}\omega_i\psi_i\sum_{j\in\Lambda}\omega_j\left(\psi_i-\psi_j\right)J_{ij}\\&=2\sum_{i\in\Lambda}\sum_{j\in\Lambda}\omega_i\omega_j\psi_i\left(\psi_i-\psi_j\right)J_{ij}\\&=\sum_{i\in\Lambda}\sum_{j\in\Lambda}\omega_i\omega_j\left(\psi_i-\psi_j\right)^2J_{ij}\\&=|\psi|_{1,\Lambda,k}^2.\end{aligned} \tag{59}$$

□

Lemma 4 *Assume that a family $\{(\{\mathcal{X}_N^k\}_{k=1}^K,\mathcal{V}_N,h,\tau)\}$ satisfies the semi-regularity condition with c_0. Then, for $\psi\in V_N^k(\Lambda)$ $(\Lambda\subset\Lambda_N,\ k=0,1,\ldots,K)$, we have*

$$\|\langle\nabla\psi\rangle^k\|_{0,\Lambda}^2\le(d+c_0\tau)|\psi|_{1,\Lambda,k}^2. \tag{60}$$

Here, $\langle\nabla\psi\rangle^k$ is defined as in (55).

Proof From

$$\nabla w_h(|x_i^k - x_j^k|) = \frac{x_i^k - x_j^k}{|x_i^k - x_j^k|}\dot{w}_h(|x_i^k - x_j^k|) \tag{61}$$

and the Cauchy–Schwarz inequality (see Appendix A), we have

$$\begin{aligned}
&\|\langle\nabla\psi\rangle^k\|_{0,\Lambda}^2\\
&\quad= \sum_{i\in\Lambda}\omega_i\left(\sum_{j\in\Lambda}\omega_j\,(\psi_j-\psi_i)\,\nabla w_h(|x_i^k - x_j^k|)\right)^2\\
&\quad= \sum_{i\in\Lambda}\omega_i\left(-\sum_{j\in\Lambda}\omega_j\,(\psi_j-\psi_i)\,\frac{x_i^k - x_j^k}{|x_i^k - x_j^k|}|\dot{w}_h(|x_i^k - x_j^k|)|\right)^2\\
&\quad\le \sum_{i\in\Lambda}\omega_i\left(\sum_{j\in\Lambda}\omega_j|\psi_j-\psi_i||\dot{w}_h(|x_i^k - x_j^k|)|\right)^2\\
&\quad\le \sum_{i\in\Lambda}\omega_i\left(\sum_{j\in\Lambda}\omega_j\frac{|\psi_j-\psi_i|^2}{|x_i^k - x_j^k|}|\dot{w}_h(|x_i^k - x_j^k|)|\right)\\
&\qquad\times\left(\sum_{j=1}^{N}\omega_j|x_i^k - x_j^k||\dot{w}_h(|x_i^k - x_j^k|)|\right).
\end{aligned}\tag{62}$$

As the family $\{(\{\mathcal{X}_N^k\}_{k=1}^K, \mathcal{V}_N, h, \tau)\}$ satisfies the semi-regularity condition (23), we obtain (60). □

Lemma 5 *Assume that a family $\{(\{\mathcal{X}_N^k\}_{k=1}^K, \mathcal{V}_N, h, \tau)\}$ satisfies the time step condition with δ. Then, for $\phi \in V_N^k(\Lambda)^d$ $(\Lambda \subset \Lambda_N,\ k = 0, 1, \ldots, K)$, we have*

$$\|\langle\Delta\phi\rangle^k\|_{0,\Lambda}^2 \le \frac{2\delta}{\tau\nu}|\phi|_{1,\Lambda,k}^2, \qquad k = 0, 1, \ldots, K. \tag{63}$$

Here, the definition of $\langle\Delta\phi\rangle^k$ is analogous to that given for $\langle\Delta\psi\rangle^k$ in (56), with the vector function ψ replaced by the scalar function ϕ.

Proof From the Cauchy–Schwarz inequality (see Appendix A), we have

$$\begin{aligned}
&\|\langle\Delta\phi\rangle^k\|_{0,\Lambda}^2\\
&\quad= \sum_{i\in\Lambda}\omega_i\left(2\sum_{j\in\Lambda\setminus\{i\}}\omega_j\frac{\phi_i-\phi_j}{|x_i^k - x_j^k|}\frac{x_i^k - x_j^k}{|x_i^k - x_j^k|}\right.\\
&\qquad\left.\cdot\nabla w_h(|x_i^k - x_j^k|)\right)^2\\
&\quad= 4\sum_{i\in\Lambda}\omega_i\left(\sum_{j\in\Lambda\setminus\{i\}}\omega_j\frac{\phi_i-\phi_j}{|x_i^k - x_j^k|}|\dot{w}_h(|x_i^k - x_j^k|)|\right)^2\\
&\quad\le 4\sum_{i\in\Lambda}\omega_i\left(\sum_{j\in\Lambda\setminus\{i\}}\omega_j\frac{|\phi_i-\phi_j|^2}{|x_i^k - x_j^k|}|\dot{w}_h(|x_i^k - x_j^k|)|\right)\\
&\qquad\times\left(\sum_{j\in\Lambda\setminus\{i\}}\omega_j\frac{|\dot{w}_h(|x_i^k - x_j^k|)|}{|x_i^k - x_j^k|}\right)\\
&\quad\le 4\max_{i=1,2,\ldots,N}\left(\sum_{j\in\Lambda\setminus\{i\}}\omega_j\frac{|\dot{w}_h(|x_i^k - x_j^k|)|}{|x_i^k - x_j^k|}\right)|\phi|_{1,\Lambda,k}^2.
\end{aligned}\tag{64}$$

As the family $\{(\{\mathcal{X}_N^k\}_{k=1}^K, \mathcal{V}_N, h, \tau)\}$ satisfies the time step condition (24), we obtain (63). □

4.3 Stability for the implicit scheme

From the lemmas in the previous section, we obtain the following stability of velocity in two-dimensional space for the implicit scheme in the ISPH method.

Theorem 2 *Let $d = 2$. Let (u^{k+1}, p^{k+1}) be the solution of the implicit scheme in the ISPH method. Assume a family $\{(\{\mathcal{X}_N^k\}_{k=1}^K, \mathcal{V}_N, h, \tau)\}$ that satisfies the semi-regularity condition with c_0 and whose particle distribution $\mathcal{X}_N^k$ satisfies the h-connectivity condition. Then, there exists a positive constant c dependent only on T, ν, and c_0 such that*

$$\begin{aligned}
&\|u^{k+1}\|_{0,\Lambda_N}^2 \le c\left(\|a\|_{0,\Lambda_N}^2 + \sum_{l=0}^{k}\tau|f^l|_{-1,\Lambda_N,l}^2\right),\\
&\quad k = 0, 1, \ldots, K-1.
\end{aligned}\tag{65}$$

Proof From (I-d) and Lemma 4, we have

$$\begin{aligned}
&\|u^{k+1}\|_{0,\Lambda_N}^2\\
&\quad= \|u^{k+1}\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 = (u^{k+1}, u^{k+1})_{\Lambda_F^k\cup\Lambda_S^k}\\
&\quad= \left(\widetilde{u}^{k+1} - \frac{\tau}{\rho}\langle\nabla p\rangle^{k+1}, \widetilde{u}^{k+1} - \frac{\tau}{\rho}\langle\nabla p\rangle^{k+1}\right)_{\Lambda_F^k\cup\Lambda_S^k}\\
&\quad= \|\widetilde{u}^{k+1}\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 + \frac{\tau^2}{\rho^2}\|\langle\nabla p\rangle^{k+1}\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2\\
&\qquad- 2\frac{\tau}{\rho}(\langle\nabla p\rangle^{k+1}, \widetilde{u}^{k+1})_{\Lambda_F^k\cup\Lambda_S^k}.
\end{aligned}\tag{66}$$

From (I-c) and Lemmas 3–4, we have

$$\begin{aligned}
&\|\langle\nabla p\rangle^{k+1}\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2\\
&\quad\le (2 + c_0\tau)|p^{k+1}|_{1,\Lambda_F^k\cup\Lambda_S^k,k}^2\\
&\quad= -(2 + c_0\tau)(p^{k+1}, \langle\Delta p\rangle^{k+1})_{\Lambda_F^k\cup\Lambda_S^k}\\
&\quad= -(2 + c_0\tau)\frac{\rho}{\tau}(p^{k+1}, \langle\nabla\cdot\widetilde{u}\rangle^{k+1})_{\Lambda_F^k\cup\Lambda_S^k}
\end{aligned}$$

$$= (2 + c_0\tau)\frac{\rho}{\tau}(\langle\nabla p\rangle^{k+1}, \widetilde{u}^{k+1})_{\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}. \tag{67}$$

From (66)–(67) and Lemma 1, we have

$$\begin{aligned}
&\|u^{k+1}\|^2_{0,\Lambda_N}\\
&\le \|\widetilde{u}^{k+1}\|^2_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k} + c_0\tau\frac{\tau}{\rho}(\langle\nabla p\rangle^{k+1}, \widetilde{u}^{k+1})_{\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\\
&\le \|\widetilde{u}^{k+1}\|^2_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\\
&\quad + c_0\tau\frac{\tau}{\rho}\|\langle\nabla p\rangle^{k+1}\|_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\|\widetilde{u}^{k+1}\|_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\\
&= \|\widetilde{u}^{k+1}\|^2_{0,\Lambda_N} + c_0\tau\frac{\tau}{\rho}\|\langle\nabla p\rangle^{k+1}\|_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\|\widetilde{u}^{k+1}\|_{0,\Lambda_N}.
\end{aligned} \tag{68}$$

Moreover, from (67) and Lemma 1, we have

$$\begin{aligned}
&\|\langle\nabla p\rangle^{k+1}\|^2_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\\
&\le (2 + c_0\tau)\frac{\rho}{\tau}\|\langle\nabla p\rangle^{k+1}\|_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\|\widetilde{u}^{k+1}\|_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}.
\end{aligned} \tag{69}$$

Therefore, we have

$$\begin{aligned}
\|\langle\nabla p\rangle^{k+1}\|_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k} &\le (2 + c_0\tau)\frac{\rho}{\tau}\|\widetilde{u}^{k+1}\|_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\\
&= (2 + c_0\tau)\frac{\rho}{\tau}\|\widetilde{u}^{k+1}\|_{0,\Lambda_N}.
\end{aligned} \tag{70}$$

By combining (68) and (70), we obtain

$$\|u^{k+1}\|^2_{0,\Lambda_N} \le \{1 + c_0(2 + c_0\tau)\tau\}\|\widetilde{u}^{k+1}\|^2_{0,\Lambda_N}. \tag{71}$$

From (I-b) and Lemmas 1–3, we have

$$\begin{aligned}
&\|\widetilde{u}^{k+1}\|^2_{0,\Lambda_N}\\
&= \|\widetilde{u}^{k+1}\|^2_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k} = (\widetilde{u}^{k+1}, \widetilde{u}^{k+1})_{\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\\
&= (u^k + \tau\nu\langle\Delta\widetilde{u}\rangle^{k+1} + \tau f^k, \widetilde{u}^{k+1})_{\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\\
&= (u^k, \widetilde{u}^{k+1})_{\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k} + \tau(f^k, \widetilde{u}^{k+1})_{\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\\
&\quad + \tau\nu(\langle\Delta\widetilde{u}\rangle^{k+1}, \widetilde{u}^{k+1})_{\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\\
&\le \|u^k\|_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\|\widetilde{u}^{k+1}\|_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\\
&\quad + \tau|f^k|_{-1,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k,k}|\widetilde{u}^{k+1}|_{1,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k,k}\\
&\quad - \tau\nu|\widetilde{u}^{k+1}|^2_{1,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k,k}.
\end{aligned} \tag{72}$$

For α, $\beta \in \mathbb{R}$ and $s > 0$, the following inequality holds:

$$\alpha\beta \le \frac{s}{2}\alpha^2 + \frac{1}{2s}\beta^2. \tag{73}$$

Hence, by utilizing

$$\begin{aligned}
&\|u^k\|_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\|\widetilde{u}^{k+1}\|_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\\
&\quad\le \frac{1}{2}\|u^k\|^2_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k} + \frac{1}{2}\|\widetilde{u}^{k+1}\|^2_{0,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k}\\
&\quad= \frac{1}{2}\|u^k\|^2_{0,\Lambda_N} + \frac{1}{2}\|\widetilde{u}^{k+1}\|^2_{0,\Lambda_N},
\end{aligned} \tag{74}$$

$$\begin{aligned}
&|f^k|_{-1,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k,k}|\widetilde{u}^{k+1}|_{1,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k,k}\\
&\quad\le \frac{1}{4\nu}|f^k|^2_{-1,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k,k} + \nu|\widetilde{u}^{k+1}|^2_{1,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k,k},
\end{aligned} \tag{75}$$

we have

$$\begin{aligned}
\|\widetilde{u}^{k+1}\|^2_{0,\Lambda_N} \le{}& \frac{1}{2}\|u^k\|^2_{0,\Lambda_N} + \frac{1}{2}\|\widetilde{u}^{k+1}\|^2_{0,\Lambda_N}\\
&+ \frac{\tau}{4\nu}|f^k|^2_{-1,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k,k}.
\end{aligned} \tag{76}$$

Thus, we have

$$\begin{aligned}
\|\widetilde{u}^{k+1}\|^2_{0,\Lambda_N} &\le \|u^k\|^2_{0,\Lambda_N} + \frac{\tau}{2\nu}|f^k|^2_{-1,\Lambda_{\mathrm{F}}^k\cup\Lambda_{\mathrm{S}}^k,k}\\
&\le \|u^k\|^2_{0,\Lambda_N} + \frac{\tau}{2\nu}|f^k|^2_{-1,\Lambda_N,k}.
\end{aligned} \tag{77}$$

From (71) and (77), we have

$$\begin{aligned}
\|u^{k+1}\|^2_{0,\Lambda_N} \le{}& \|u^k\|^2_{0,\Lambda_N} + c_0(2 + c_0\tau)\tau\|u^k\|_{0,\Lambda_N}\\
&+ (1 + c_0(2 + c_0\tau)\tau)\frac{\tau}{2\nu}|f^k|^2_{-1,\Lambda_N,k}.
\end{aligned} \tag{78}$$

By replacing the index k with l in (78) and summing it over $l = 0$ to k, we have

$$\begin{aligned}
\|u^{k+1}\|^2_{0,\Lambda_N} \le{}& \|a\|^2_{0,\Lambda_N} + c_0(2 + c_0\tau)\sum_{l=0}^{k}\tau\|u^l\|_{0,\Lambda_N}\\
&+ \frac{1 + c_0(2 + c_0\tau)\tau}{2\nu}\sum_{l=0}^{k}\tau|f^l|^2_{-1,\Lambda_N,l}.
\end{aligned} \tag{79}$$

Because $\tau < T$, we have

$$\begin{aligned}
\|u^{k+1}\|^2_{0,\Lambda_N} \le{}& \|a\|^2_{0,\Lambda_N} + c_0(2 + c_0T)\sum_{l=0}^{k}\tau\|u^l\|_{0,\Lambda_N}\\
&+ \frac{1 + c_0(2 + c_0T)T}{2\nu}\sum_{l=0}^{k}\tau|f^l|^2_{-1,\Lambda_N,l}.
\end{aligned} \tag{80}$$

Consequently, Grönwall's inequality (see Appendix A) yields

$$\|u^{k+1}\|_{0,\Lambda_N}^2 \le \exp\left(c_0(2+c_0T)T\right) \times \left(\|a\|_{0,\Lambda_N}^2 + \frac{1+c_0(2+c_0T)T}{2\nu}\sum_{l=1}^{k}\tau|f^l|_{-1,\Lambda_N,l}^2\right). \tag{81}$$

Taking c as

$$c = \exp\left(c_0(2+c_0T)T\right)\max\left\{1, \frac{1+c_0(2+c_0T)T}{2\nu}\right\}, \tag{82}$$

we conclude (65). □

4.4 Stability for the semi-implicit scheme

In addition to the stability for the implicit scheme, we obtain the following stability of velocity in two-dimensional space for the semi-implicit scheme in the ISPH method.

Theorem 3 *Let $d = 2$. Let (u^{k+1}, p^{k+1}) be the solution of the semi-implicit scheme in the ISPH method. Assume a family $\{(\{\mathcal{X}_N^k\}_{k=1}^K, \mathcal{V}_N, h, \tau)\}$ that satisfies the semi-regularity condition with c_0 and time step condition with δ and whose particle distribution $\mathcal{X}_N^k$ $(k = 1, 2, \ldots, K)$ satisfies the h-connectivity condition. Then, there exists a positive constant c dependent only on T, ν, c_0, and δ such that*

$$\|u^{k+1}\|_{0,\Lambda_N}^2 \le c\left(\|a\|_{0,\Lambda_N}^2 + \tau\sum_{l=0}^{k}\tau\|f^l\|_{0,\Lambda_N}^2 + \sum_{l=0}^{k}\tau|f^l|_{-1,\Lambda_N,l}^2\right), \quad k = 0, 1, \ldots, K-1. \tag{83}$$

Proof By the same procedure as for the estimation of u^{k+1} in the proof of Theorem 2, we obtain

$$\|u^{k+1}\|_{0,\Lambda_N}^2 \le \{1 + c_0(2 + c_0\tau)\tau\}\|\widetilde{u}^{k+1}\|_{0,\Lambda_N}^2. \tag{84}$$

From (I-b) and Lemmas 1–2, we have

$$\begin{aligned}
\|\widetilde{u}^{k+1}\|_{0,\Lambda_N}^2
&= \|\widetilde{u}^{k+1}\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 = (\widetilde{u}^{k+1}, \widetilde{u}^{k+1})_{\Lambda_F^k\cup\Lambda_S^k} \\
&= (u^k + \tau\nu\langle\Delta u\rangle^k + \tau f^k, u^k + \tau\nu\langle\Delta u\rangle^k + \tau f^k)_{\Lambda_F^k\cup\Lambda_S^k} \\
&= \|u^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 + \tau^2\nu^2\|\langle\Delta u\rangle^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 \\
&\quad + \tau^2\|f^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 \\
&\quad + 2\tau\nu(u^k, \langle\Delta u\rangle^k)_{\Lambda_F^k\cup\Lambda_S^k} + 2\tau^2\nu(\langle\Delta u\rangle^k, f^k)_{\Lambda_F^k\cup\Lambda_S^k} \\
&\quad + 2\tau(f^k, u^k)_{\Lambda_F^k\cup\Lambda_S^k} \\
&\le \|u^k\|_{0,\Lambda_N}^2 + \tau^2\nu^2\|\langle\Delta u\rangle^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 + \tau^2\|f^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 \\
&\quad - 2\tau\nu|u^k|_{1,\Lambda_F^k\cup\Lambda_S^k,k}^2 \\
&\quad + 2\tau^2\nu\|\langle\Delta u\rangle^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}\|f^k\|_{0,\Lambda_F^k\cup\Lambda_S^k} \\
&\quad + 2\tau|f^k|_{-1,\Lambda_F^k\cup\Lambda_S^k,k}|u^k|_{1,\Lambda_F^k\cup\Lambda_S^k,k}.
\end{aligned} \tag{85}$$

From (73), for δ $(0 < \delta < 1)$, we have

$$\begin{aligned}
\|\langle\Delta u\rangle^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}\|f^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}
&\le \frac{\nu(1-\delta)}{4\delta}\|\langle\Delta u\rangle^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 \\
&\quad + \frac{\delta}{\nu(1-\delta)}\|f^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2,
\end{aligned} \tag{86}$$

$$\begin{aligned}
|f^k|_{-1,\Lambda_F^k\cup\Lambda_S^k,k}|u^k|_{1,\Lambda_F^k\cup\Lambda_S^k,k}
&\le \frac{1}{2\nu(1-\delta)}|f^k|_{-1,\Lambda_F^k\cup\Lambda_S^k,k}^2 \\
&\quad + \frac{\nu(1-\delta)}{2}|u^k|_{1,\Lambda_F^k\cup\Lambda_S^k,k}^2.
\end{aligned} \tag{87}$$

Hence, we have

$$\begin{aligned}
\|\widetilde{u}^{k+1}\|_{0,\Lambda_N}^2
&\le \|u^k\|_{0,\Lambda_N}^2 + \tau^2\frac{1+\delta}{1-\delta}\|f^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 \\
&\quad + \frac{\tau}{\nu(1-\delta)}|f^k|_{-1,\Lambda_F^k\cup\Lambda_S^k,k}^2 \\
&\quad + \tau\nu(1+\delta)\left(\frac{\tau\nu}{2\delta}\|\langle\Delta u\rangle^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 - |u^k|_{1,\Lambda_F^k\cup\Lambda_S^k,k}^2\right).
\end{aligned} \tag{88}$$

By Lemma 5, we have

$$\frac{\tau\nu}{2\delta}\|\langle\Delta u\rangle^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 - |u^k|_{1,\Lambda_F^k\cup\Lambda_S^k,k}^2 \le 0. \tag{89}$$

Hence, we obtain

$$\begin{aligned}
\|\widetilde{u}^{k+1}\|_{0,\Lambda_N}^2 &\le \|u^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 + \tau^2\frac{1+\delta}{1-\delta}\|f^k\|_{0,\Lambda_F^k\cup\Lambda_S^k}^2 \\
&\quad + \frac{\tau}{\nu(1-\delta)}|f^k|_{-1,\Lambda_F^k\cup\Lambda_S^k,k}^2 \\
&\le \|u^k\|_{0,\Lambda_N}^2 + \tau^2\frac{1+\delta}{1-\delta}\|f^k\|_{0,\Lambda_N}^2 \\
&\quad + \frac{\tau}{\nu(1-\delta)}|f^k|_{-1,\Lambda_N,k}^2.
\end{aligned} \tag{90}$$

From (84) and (90), we have

$$\begin{aligned}&\|u^{k+1}\|_{0,\Lambda_N}^2\\&\quad\le \|u^k\|_{0,\Lambda_N}^2 + c_0(2+c_0\tau)\tau\|u^k\|_{0,\Lambda_N}\\&\qquad+\{1+c_0(2+c_0\tau)\tau\}\\&\qquad\times\left\{\tau^2\frac{1+\delta}{1-\delta}\|f^k\|_{0,\Lambda_N}^2+\frac{\tau}{\nu(1-\delta)}|f^k|_{-1,\Lambda_N,k}^2\right\}.\end{aligned}\tag{91}$$

By replacing the index k with l in (91) and summing it over $l = 0$ to k, we have

$$\begin{aligned}&\|u^{k+1}\|_{0,\Lambda_N}^2\\&\quad\le \|a\|_{0,\Lambda_N}^2 + c_0(2+c_0\tau)\sum_{l=0}^{k}\tau\|u^l\|_{0,\Lambda_N}\\&\qquad+\frac{1+c_0(2+c_0T)T}{1-\delta}\\&\qquad\times\left\{\tau(1+\delta)\sum_{l=0}^{k}\tau\|f^l\|_{0,\Lambda_N}^2+\frac{1}{\nu}\sum_{l=0}^{k}\tau|f^l|_{-1,\Lambda_N,l}^2\right\}.\end{aligned}\tag{92}$$

From $\tau < T$, we have

$$\begin{aligned}&\|u^{k+1}\|_{0,\Lambda_N}^2\\&\quad\le \|a\|_{0,\Lambda_N}^2 + c_0(2+c_0T)\sum_{l=0}^{k}\tau\|u^l\|_{0,\Lambda_N}\\&\qquad+\frac{1+c_0(2+c_0T)T}{1-\delta}\\&\qquad\times\left\{\tau(1+\delta)\sum_{l=0}^{k}\tau\|f^l\|_{0,\Lambda_N}^2+\frac{1}{\nu}\sum_{l=0}^{k}\tau|f^l|_{-1,\Lambda_N,l}^2\right\}.\end{aligned}\tag{93}$$

Consequently, Grönwall's inequality (see Appendix A) yields

$$\begin{aligned}&\|u^{k+1}\|_{0,\Lambda_N}^2\\&\quad\le \exp\left(c_0(2+c_0T)T\right)\left\{\|a\|_{0,\Lambda_N}^2\right.\\&\qquad+\frac{1+c_0(2+c_0T)T}{1-\delta}\\&\qquad\left.\times\left(\tau(1+\delta)\sum_{l=0}^{k}\tau\|f^l\|_{0,\Lambda_N}^2+\frac{1}{\nu}\sum_{l=0}^{k}\tau|f^l|_{-1,\Lambda_N,l}^2\right)\right\}.\end{aligned}\tag{94}$$

Taking c as

$$\begin{aligned}c &= \exp\left(c_0(2+c_0T)T\right)\\&\quad\times\max\left\{1,\,(1+c_0(2+c_0T)T)\frac{1+\delta}{1-\delta},\right.\\&\qquad\left.\frac{1+c_0(2+c_0T)T}{\nu(1-\delta)}\right\},\end{aligned}\tag{95}$$

we conclude (83). □

4.5 Extension for modified schemes

We consider improving the implicit and semi-implicit schemes by utilizing our results. As the time step τ and particle volume set $\mathcal{V}_N = \{\omega_i\}$ are fixed in the previous sections, we consider the introduction of modified schemes with variable time step τ^k and particle volume set $\mathcal{V}_N^k = \{\omega_i^k\}$ defined so as to satisfy some key conditions.

For $k = 0, 1, \ldots, K-1$, let $\tau^k > 0$ be a variable time step satisfying

$$\sum_{k=0}^{K-1}\tau^k = T.\tag{96}$$

Then, the kth time t^k is defined as $t^k = 0\,(k = 0)$, and $t^{k+1} := t^k + \tau^k$ $(k = 0, 1, \ldots, K-1)$. We set the particle volume set $\mathcal{V}_N^k = \{\omega_i^k\}$ by a solution of the linear equation

$$A^k\omega^k = b^k,\tag{97}$$

where $A^k \in \mathbb{R}^{N\times N}$, $\omega^k \in \mathbb{R}^N$, and $b^k \in \mathbb{R}^N$ are

$$A_{ij}^k := |x_i^k - x_j^k||\dot{w}_h(|x_i^k - x_j^k|)|,\tag{98}$$

$$\omega^k := (\omega_1^k, \omega_2^k, \ldots, \omega_N^k)^{\mathrm{T}},\tag{99}$$

$$b^k := (d, d, \ldots, d)^{\mathrm{T}},\tag{100}$$

respectively. Then, because the condition

$$\sum_{j=1}^{N}\omega_j^k|x_i^k - x_j^k||\dot{w}_h(|x_i^k - x_j^k|)| = d,\qquad i = 1, 2, \ldots, N,\tag{101}$$

is satisfied, the semi-regularity condition (23) is automatically satisfied at t^k. Therefore, we obtain the following corollary:

Corollary 4 *Let $d = 2$. Let (u^{k+1}, p^{k+1}) be the solution of the modified implicit scheme, which is the implicit scheme whose time step τ and particle volume set $\mathcal{V}_N$ are replaced with variable time step τ^k and particle volume set $\mathcal{V}_N^k$. Assume for a family $\{(\{\mathcal{X}_N^k, \mathcal{V}_N^k\}_{k=1}^K, h, \tau^k)\}$ that its particle distribution*

$\mathcal{X}_N^k$ satisfies the h-connectivity condition and particle volume set $\mathcal{V}_N^k = \{\omega_i^k > 0\}$ exists for $k = 0, 1, \ldots, K$. Then, there exists a positive constant c dependent only on T, ν, and c_0 such that

$$\|u^{k+1}\|_{0,\Lambda_N}^2 \le c\left(\|a\|_{0,\Lambda_N}^2 + \sum_{l=0}^{k} \tau^l |f^l|_{-1,\Lambda_N,l}^2\right), \quad k = 0, 1, \ldots, K-1. \tag{102}$$

Moreover, for fixed constant δ $(0 < \delta < 1)$, we give τ as

$$\tau^k = \frac{\delta}{2\nu}\left[\max_{i=1,2,\ldots,N}\left(\sum_{j\neq i} \omega_j^k \frac{|\dot{w}_h(|x_i^k - x_j^k|)|}{|x_i^k - x_j^k|}\right)\right]^{-1}, \quad k = 1, 2, \ldots, K. \tag{103}$$

Then, the time step condition (24) is automatically satisfied at each time step. Therefore, we obtain the following corollary:

Corollary 5 *Let $d = 2$. Let (u^{k+1}, p^{k+1}) be the solution of the modified semi-implicit scheme, which is the semi-implicit scheme whose time step τ and particle volume set $\mathcal{V}_N$ are replaced with variable time step τ^k and particle volume set $\mathcal{V}_N^k$. Assume for a family $\{(\{\mathcal{X}_N^k, \mathcal{V}_N^k\}_{k=1}^K, h, \tau^k)\}$ that its particle distribution $\mathcal{X}_N^k$ satisfies the h-connectivity condition and particle volume set $\mathcal{V}_N^k = \{\omega_i^k > 0\}$ exists for $k = 0, 1, \ldots, K$. Then, there exists a positive constant c dependent only on T, ν, c_0, and δ such that*

$$\|u^{k+1}\|_{0,\Lambda_N}^2 \le c\left(\|a\|_{0,\Lambda_N}^2 + \sum_{l=0}^{k}(\tau^l)^2\|f^l\|_{0,\Lambda_N}^2 + \sum_{l=0}^{k} \tau^l |f^l|_{-1,\Lambda_N,l}^2\right), \quad k = 0, 1, \ldots, K-1. \tag{104}$$

5 Concluding remarks

We have analyzed the unique solvability and stability of the implicit and semi-implicit schemes in the incompressible smoothed particle hydrodynamics (ISPH) method. Three key conditions were introduced for our analysis, the three conditions on discrete parameters, which are the h-connectivity, semi-regularity, and time step conditions. With h-connectivity, the unique solvability of the implicit and semi-implicit schemes was obtained in two- and three-dimensional space. With the h-connectivity and semi-regularity conditions, the stability of velocity for the implicit scheme was established in two-dimensional space. Moreover, with the addition of the time step condition, the stability of velocity for the semi-implicit scheme was established in two-dimensional space. Thanks to these results, the conditions on discrete parameters sufficient for obtaining stable computing with the ISPH method are clarified.

As an application of these results, we introduced modified implicit and semi-implicit schemes by redefining discrete parameters. By introducing the modified particle volume set, which imposes an additional constraint condition at each step, the modified implicit scheme becomes stable without the semi-regularity condition. Moreover, by introducing the variable time step, which is updated according to the particle distribution and particle volume set, the modified semi-implicit scheme becomes stable without the semi-regularity and time step conditions.

As future work, we will extend the stability to that in three-dimensional space and with boundary conditions such as Neumann boundary conditions in the pressure Poisson equation. Moreover, we will investigate convergence for the ISPH method mathematically.

Compliance with ethical standards

Conflict of interest The author declares no conflicts of interest.

Appendix A Mathematical tools

Cauchy–Schwarz inequality

Let $M \in \mathbb{N}$. For all $a_i, b_i \in \mathbb{R}$ $(i = 1, 2, \ldots, M)$, the following, called the Cauchy–Schwarz inequality, holds:

$$\sum_{i=1}^{M} a_i b_i \le \left(\sum_{i=1}^{M} a_i^2\right)^{1/2}\left(\sum_{i=1}^{M} b_i^2\right)^{1/2}. \tag{105}$$

Grönwall's inequality

Let $M \in \mathbb{N}$. Assume that $a_i, b_i > 0$ $(i = 0, 1, \ldots, M)$, $c > 0$ satisfy the inequality

$$a_k \le a_0 + c + \sum_{j=0}^{k-1} a_j b_j, \qquad k = 1, 2, \ldots, M. \tag{106}$$

Then, the following, called Grönwall's inequality, holds:

$$a_k \le (a_0 + c)\prod_{j=0}^{k-1}(1 + b_j) \le (a_0 + c)\exp\left(\sum_{j=0}^{k-1} b_j\right), \quad k = 1, 2, \ldots, M. \tag{107}$$

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