



Measurable multiresolution systems, endomorphisms, and representations of Cuntz relations

Sergey Bezuglyi · Palle E. T. Jorgensen

Received: 19 June 2023 / Accepted: 28 August 2023 / Published online: 8 February 2024
© The Author(s) under exclusive license to Chapman University 2024

Abstract The purpose of this paper is to present new classes of function systems as part of multiresolution analyses. Our approach is representation theoretic, and it makes use of generalized multiresolution function systems (MRSs). It further entails new ideas from measurable endomorphisms dynamics. Our results yield applications that are not amenable to more traditional techniques used on metric spaces. As the main tool in our approach, we make precise new classes of generalized MRSs which arise directly from a dynamical theory approach to the study of surjective endomorphisms on measure spaces. In particular, we give the necessary and sufficient conditions for a family of functions to define generators of Cuntz relations. We find an explicit description of the set of generalized wavelet filters. Our results are motivated in part by analyses of sub-band filters in signal/image processing. But our paper goes further, and it applies to such wider contexts as measurable dynamical systems and complex dynamics. A unifying theme in our results is a new analysis of endomorphisms in general measure space, and its connection to multi-resolutions, to representation theory, and generalized wavelet systems.

Keywords Measure spaces · Endomorphisms · Multiresolution · Wavelet filters · Cuntz relations · Radon–Nikodym derivative · Markovian functions

Mathematics Subject Classification 46G12 · 37D45 · 20K30 · 22E66

Contents

1	Introduction	88
2	Basics on endomorphisms	90
2.1	Endomorphisms of a measure space	90
2.2	Measurable partition	91
2.3	Radon–Nikodym derivatives and Markovian functions	92
2.4	Properties of endomorphisms	93
2.5	Markovian functions	96

S. Bezuglyi · P. E. T. Jorgensen (✉)
University of Iowa, Iowa City, IA, USA
e-mail: palle-jorgensen@uiowa.edu

S. Bezuglyi
e-mail: sergii-bezuglyi@uiowa.edu

3	Operators generated by endomorphisms	97
3.1	Transfer operators and endomorphisms	97
3.2	Composition operators and Markovian functions	99
3.3	Radon–Nikodym derivatives and conditional expectations	103
4	Cuntz relations for invariant and quasi-invariant measures	104
4.1	Quasi-invariant measure	105
4.2	Invariant measure	108
5	The set of wavelet filters	110
5.1	Actions of loop groups on wavelet filters	110
5.2	Endomorphisms, wavelet filters and cyclic representations	113
	References	114

1 Introduction

The present paper is focused on relationships between the following objects: surjective endomorphisms of a measure space (X, \mathcal{B}, μ) , transfer operators in $L^2(X, \mathcal{B}, \mu)$, generalized wavelet filters, Markovian functions, and representations of the Cuntz relations.

Our analysis of transformations of a measure space, associated transfer operators, and representations of the Cuntz algebras derive from a new analysis of endomorphisms in general measure space, presented here. While various special cases of endomorphisms have been studied in earlier work, we call attention here to three new elements: (i) our context is that of the most general transformations of a measure spaces; (ii) we introduce a new harmonic analysis into the problem via representations of Cuntz algebras; and (iii) our representations of Cuntz algebras for the purpose arise directly from the endomorphism at hand, σ , and a choice of a quasi-invariant measure μ . From this, we then identify a new construction of an infinite-dimensional manifold \mathfrak{M} of generalized wavelet filters, and a transitive action of a canonical group \mathcal{G} , acting on \mathfrak{M} , and depending only on the given pair σ (endomorphisms), and μ (measure). The representations of the particular Cuntz algebra (depending on σ) in turn define endomorphisms of $B(L^2(\mu))$, the C^* -algebra of all bounded operators in $L^2(\mu)$, and with each non-commutative endomorphism extending the initial endomorphism σ .

We present a new class of function systems \mathfrak{M} as part of multiresolution analyses (MRAs), motivated in part by [1, 19, 50]. Our approach is based on methods of the representation theory, and it makes use of generalized iterated function systems related to a surjective endomorphism of a measure space. For background on the representation theory, we refer the readers to [29, 42, 44].

There are recent papers dealing with the general theme of representation and multilevel filters. We call attention especially to [4]. Compared to [4], the main new development is our present wider context here of the measurable category, and its focus on new applications to Borel and measurable dynamics. Our results for these generalized MRSs further entail new ideas from measurable dynamics, see, e.g., [2, 3, 5, 11]. While our general focus is on branching systems in the measurable category, our applications are not amenable to more traditional metric techniques such as [33, 49, 55]. We turn to new classes of generalized iterated function systems which arise directly from a more general dynamical theory approach via a systematic study of endomorphisms in measure spaces. We are motivated in part by analyses of sub-band filters in signal/image processing, see, e.g., [3, 6, 7, 25]. But our paper goes further, and it encompasses such wider contexts as measurable dynamical systems and complex dynamics. The corresponding literature on wavelet filters, representations of Cuntz algebras, iterated function systems, transfer operators, and other adjacent areas are very extensive; we mention here the following sources where the reader can find more details and alternative approaches: [13–16, 24–27, 27, 28, 37–40, 43, 45, 46].

As noted in the papers cited above, the traditional approach to iterated function systems, or more generally to semi-branching function systems, the starting point is typically a fixed system of maps that can be shown to admit limits in the form of attractors and measures invariant with respect to iterated function systems. These constructions are typically based on metric considerations, and they play a big role in such diverse applications as (fractal) harmonic analysis, graph Laplacian, boundaries, and analysis of geometries which are given by classes of self-

similarity. Cantor and Sierpinski constructions are cases in point. The corresponding IFS can be shown in turn admit realizations in shift dynamical systems.

Our present approach is the opposite: we begin with a consideration of endomorphisms in measure spaces (see the definition in Sect. 2.1), and of associated measurable partitions (Sect. 2.2). With this as starting point, we then introduce several bounded operators which will define representations of a non-abelian C^* -algebras given by generators and relations, called Cuntz algebras, Sects. 3 and 4. There are two advantages to this approach, (i) it allows for a much wider family of (generalized) MRSs, and (ii) it also offers new and direct tools for attacking the corresponding harmonic analysis questions. Finally, we give an explicit description of the set of generalized wavelet filters, see Sect. 5.

Measurable transformations of a standard measure space (X, \mathcal{B}, μ) is the central concept of the ergodic theory. Invertible transformations (automorphisms) and their properties have been extensively studied from various points of view. The study of non-invertible transformations (endomorphisms) has been less popular than that of automorphisms although their role in dynamics and adjoint areas is extremely important. In particular, they are used in the construction of iterated function systems (IFSs), transfer operators, and wavelet filters. We refer here to several recent books dealing with endomorphisms and their applications in the operator theory [10,18,30,35,41,51,58].

Using an analysis of endomorphisms of a standard measure space, we associate with every endomorphism several bounded operators acting in $L^2(X, \mathcal{B}, \mu)$. Non-singular endomorphisms σ of a measure space (X, \mathcal{B}, μ) are naturally divided into two classes: if $\mu = \mu \circ \sigma^{-1}$, then σ is called *measure-preserving*; if $\mu \sim \mu \circ \sigma^{-1}$, then the measure μ is called *quasi-invariant* with respect to σ . If additionally, $\mu \circ \sigma \sim \mu$ on $\sigma^{-1}(\mathcal{B})$, then σ is called *forward quasi-invariant*. In this case, there are Borel functions φ , called *Markovian* functions, such that

$$\int_X (f \circ \sigma)\varphi \, d\mu = \int_X f \, d\mu. \tag{1.1}$$

This relation is the basis for defining isometric operators: the *composition operator* $S_\sigma : f \mapsto f \circ \sigma$ in the case of an invariant measure μ and the *weighted composition operator* $S_\varphi : f \mapsto \sqrt{\varphi}(f \circ \sigma)$ for a quasi-invariant measure μ (here φ is Markovian). These operators, together with transfer operators, will play key roles in our constructions.

We formulate now our *main results* and outline the paper’s organization. In Sect. 2, we define the main objects of this paper. They are: a standard measure space, measurable partitions, canonical systems of measures, subjective endomorphisms, invariant and quasi-invariant measures, and Markovian functions. All these notions are used in the next sections. Section 2 should be viewed as a brief survey on endomorphisms and related notions. Section 3 is focusing on the study of linear operators generated by an endomorphism of a measure space. We define abstract transfer operators acting on bounded Borel functions and consider their properties. If the operator S_σ is considered in $L^2(X, \mathcal{B}, \mu)$ where μ is σ -invariant, then S_σ^* is a transfer operator which has interesting properties, see Theorems 3.9, 3.14, 3.15, and 3.16. In particular, the operator S_σ^* coincides with a transfer operator R_σ defined by the measurable partition into preimages of σ . We prove also similar results for weighted composition operators. Section 4 contains the principal theorems connecting wavelet filters with representations of the Cuntz relations. We recall that a family of isometries $\{T_i : i \in \Lambda\}$ defines the Cuntz relations if $\sum_{i \in \Lambda} T_i T_i^* = \mathbb{1}$ and the projections $T_i T_i^*$ are mutually orthogonal for different indexes where Λ is finite or countable. Let φ be a Markovian function and m a complex-valued function such that $S_\varphi^*(\sqrt{\varphi}|m|^2) = \mathbb{1}$. Define $T_m(f) = m S_\varphi(f)$. Then, T_m an isometry in $L^2(\mu)$. We prove the following results (see Theorem 4.11).

Theorem 1.1 *Let σ be an onto endomorphism of (X, \mathcal{B}, μ) where μ is quasi-invariant with respect to σ . Let $\{m_i : i \in \Lambda\}$ be a family of complex-valued functions. The operators $\{T_{m_i}, i \in \Lambda\}$ generate a representation of the Cuntz algebra $\mathcal{O}_{|\Lambda|}$ if and only if*

- (i) $S_\varphi^*(\sqrt{\varphi} \overline{m}_j m_i) = \delta_{ij} \mathbb{1}$,
- (ii) $\sum_{i \in \Lambda} m_i \mathbb{E}_\varphi(\overline{m}_i f) = f, \quad f \in L^2(\mu)$.

Here, $\mathbb{E}_\varphi = S_\varphi S_\varphi^*$ is the orthogonal projection from $L^2(\mu)$ onto a subspace \mathcal{H}_φ .

As a corollary, we have the following decomposition of $L^2(\mu)$ in the case of σ -invariant measure μ :

$$L^2(\mu) = \bigoplus_{i \in \Lambda} m_i L^2(X, \sigma^{-1}(\mathcal{B}), \mu).$$

In Sect. 5, we focus on finding a description of the families of functions $\underline{m} = (m_i) \in \mathfrak{M}_\varphi$ satisfying the above theorem. Let \mathcal{G} be the group of Borel functions with values in the unitary operators on $\ell^2(\Lambda)$. Then, we prove the following:

Theorem 1.2 (Theorem 5.5)

- (1) The set \mathfrak{M}_φ is isomorphic (as a set) to the loop group \mathcal{G} .
- (2) For every element $G = (g_{ij})$ of the loop group, there exists a wavelet filter \underline{m} such that $g_{ij} = S_\varphi^*(\sqrt{\varphi} m_i \bar{m}_j)$.

Let functions $(m_i : i \in \Lambda)$ satisfy the property.

$$S_\sigma^*(|m_i|^2) = \mathbb{1} \tag{1.2}$$

and $S_i(f) = m_i(f \circ \sigma)$ is an isometry on $L^2(\mu)$.

Theorem 1.3 (Theorem 5.8) *Let $(m_i : i \in \Lambda)$ be a set of cyclic vectors for a representation of $L^\infty(X, \sigma^{-1}(\mathcal{B}), \mu)$ on $L^2(\mu)$ that satisfies (1.2). Then, $\underline{m} = (m_i)$ is a wavelet filter if and only if $\sum_{i \in \Lambda} S_i^* S_i = \mathbb{I}$. In other words, $\underline{m} \in \mathfrak{M}$ if and only if the operators S_i are the generators of a representation of the Cuntz algebra $\mathcal{O}_{|\Lambda|}$.*

2 Basics on endomorphisms

In this section, we give basic definitions and facts from the theory of endomorphisms of a standard measure space (X, \mathcal{B}, μ) . The notion of an endomorphism is one of the central concepts of ergodic theory; the foundations and more advanced results on endomorphisms can be found in some pioneering papers in ergodic theory and in more recent papers and books, see, e.g., [8, 17, 20, 34–36, 51, 54] and the papers cited therein. The study of endomorphisms is mostly based on the notion of a measurable partition of a measure space and associated subalgebras of Borel sets. A systematic study of measurable dynamical systems based on applications of measurable partitions was initiated by Rokhlin in [52–54].

2.1 Endomorphisms of a measure space

We begin with the definitions of the main objects considered in the paper.

Let (X, \mathcal{B}) be a *standard Borel space*, i.e., (X, \mathcal{B}) is Borel isomorphic to a Polish space with the sigma-algebra of Borel sets. If μ is a non-atomic Borel positive measure on (X, \mathcal{B}) , then (X, \mathcal{B}, μ) is called a *standard Borel space*.

By an *endomorphism* σ of (X, \mathcal{B}, μ) (or (X, \mathcal{B})) we mean a measurable (or Borel) map of X onto itself (σ is surjective). We will discuss various properties of endomorphisms below. In particular, σ defines a partition of X into subsets $\{\sigma^{-1}(x) : x \in X\}$. Depending on the cardinality of the sets $\sigma^{-1}(x)$, we call σ either finite-to-one, or countable-to-one, or continuum-to-one. Without loss of generality, we can assume that the cardinality $|\sigma^{-1}(x)|$ is constant. The collection of sets $\sigma^{-1}(A)$, $A \in \mathcal{B}$, forms a σ -subalgebra of Borel sets which is denoted $\sigma^{-1}(\mathcal{B})$ (below it will be also denoted by \mathcal{A} to shorten formulas). In general, the set $\sigma(A)$ is not Borel for every $A \in \mathcal{B}$, but if σ is at most countable-to-one then $\sigma(A)$ is automatically Borel.

Let $End(X, \mathcal{B})$ denote the set (semigroup) of all *surjective endomorphisms* of a standard Borel space (X, \mathcal{B}) . By $M_1(X)$, we denote the set of all *Borel probability non-atomic measures*. An element of $M_1(X)$ will be simply called a *measure* in the paper. For $\sigma \in End(X, \mathcal{B})$ and $\mu \in M_1(X)$, we define the measure $\mu \circ \sigma^{-1}$ where $\mu \circ \sigma^{-1}(A) := \mu(\sigma^{-1}(A))$. Then, the map $\mu \mapsto \mu \circ \sigma^{-1}$ defines an action of $End(X, \mathcal{B})$ on $M_1(X)$. We will be

interested in the following cases: (i) the measure $\mu \circ \sigma^{-1}$ is equivalent to μ , and (ii) μ is σ -invariant. In case (i), we say that an endomorphism σ is *non-singular*, i.e.,

$$\mu(A) = 0 \iff \mu(\sigma^{-1}(A)) = 0, \quad \forall A \in \mathcal{B}. \tag{2.1}$$

In other words, the measure μ is called (backward) σ -*quasi-invariant*, in symbols, $\mu \circ \sigma^{-1} \sim \mu$. We use the notation $End(X, \mathcal{B}, \mu)$ to denote the semigroup of all surjective endomorphisms σ such that μ is quasi-invariant with respect to σ . For a fixed μ , the set $End(X, \mathcal{B}, \mu)$ contains the sub-semigroup End_μ which consists of the endomorphisms preserving μ , i.e., $\mu(A) = \mu \circ \sigma^{-1}(A)$. The set End_μ can be viewed as the stabilizer of the action of $End(X, \mathcal{B})$ on $M_1(X)$ at μ .

We will need also the notion of a (forward) *quasi-invariant measure* μ . This means that for every μ -measurable set A , the set $\sigma(A)$ is measurable and $\mu(A) = 0 \iff \mu(\sigma(A)) = 0$.

Lemma 2.1 *Let σ be a surjective endomorphism of a standard Borel space (X, \mathcal{B}) . Then, $M_1(X)$ always contains a σ -quasi-invariant measure μ .*

Proof Every endomorphism σ generates a countable Borel equivalence relation $E(\sigma)$ whose classes are the orbits of σ . By definition, $(x, y) \in E(\sigma)$ if there exist $m, n \in \mathbb{N}_0$ such that $\sigma^n(x) = \sigma^m(y)$. Quasi-invariant measures for σ coincide with quasi-invariant measures for $E(\sigma)$. Then, we can use [23, Proposition 3.1] where the existence of $E(\sigma)$ -quasi-invariant measures was proved. □

Example 2.2 Let $(X, \mathcal{B}, \mu) = \prod_{i \in \mathbb{N}} (X_i, \mathcal{B}_i, \mu_i)$. Define the left shift σ on X : $\sigma(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$. Then, σ is an endomorphism. If $|X_i| = N$ for all i , then σ is N -to-one. Clearly, this construction can give other types of endomorphisms classified by the cardinality of $\sigma^{-1}(x)$. If all $(X_i, \mathcal{B}_i, \mu_i) = (Y, \mathcal{C}, \nu)$ are the same and μ is the product-measure $\otimes_i \nu$, then μ is σ -invariant. The measure μ can be quasi-invariant with respect to the left shift σ if we use different measures μ_i . More details are in [22,36], and other papers of these authors.

2.2 Measurable partition

We refer to [53] (or [20]) for the definition of a measurable partition.

Let $\xi = \{C_\alpha : \alpha \in I\}$ be a *partition* of a standard probability measure space (X, \mathcal{B}, μ) such that $C_\alpha \in \mathcal{B}$ (the index set I can be either countable or uncountable; we focus on the case of an uncountable set). A Borel set A of the form $\bigcup_{\alpha \in I'} C_\alpha$, $I' \subset I$, is called a ξ -*set*. Let $\mathcal{B}(\xi)$ be the sigma-algebra generated by all ξ -sets.

A partition ξ is called *measurable* if $\mathcal{B}(\xi)$ contains a countable subset $\{D_j\}$ of ξ -sets that separates any two elements C, C' of ξ . This means that there exists D_i such that either $C \subset D_i$ and $C' \subset X \setminus D_i$ or $C' \subset D_i$ and $C \subset X \setminus D_i$. Let π be the natural projection from X to X/ξ , i.e., $\pi(x) = C_x$ where C_x is the element of ξ containing x . Using the projection $\pi : X \rightarrow X/\xi$, one can define a measure space $(X/\xi, \mathcal{B}/\xi, \mu_\xi)$ where $E \in \mathcal{B}/\xi$ if and only if the ξ -set $\pi^{-1}(E)$ is in \mathcal{B} and $\mu_\xi = \mu \circ \pi^{-1}$.

The following result was proved in [53].

Lemma 2.3 *A partition ξ of a standard measure space (X, \mathcal{B}, μ) is measurable if and only if $(X/\xi, \mathcal{B}/\xi, \mu_\xi)$ is a standard measure space.*

It is said that a partition ζ *refines* ξ (in symbols, $\xi < \zeta$) if every element C of ξ is a ζ -set. It turns out that every partition ζ has a *measurable hull*, that is a measurable partition ξ such that $\xi < \zeta$ and ξ is a maximal measurable partition with this property. If ξ_α is a family of measurable partitions, then their product $\bigvee_\alpha \xi_\alpha$ is a measurable partition ξ which is uniquely determined by the conditions: (i) $\xi_\alpha < \xi$ for all α , and (ii) if η is a measurable partition such that $\xi_\alpha < \eta$, then $\xi < \eta$. Similarly, one defines the intersection $\bigwedge_\alpha \xi_\alpha$ of measurable partitions. There is a one-to-one correspondence between the set of measurable partitions of a standard measure space (X, \mathcal{B}, μ) and the set of complete sigma-subalgebras \mathcal{B}' of \mathcal{B} .

The role of measurable partitions becomes clear from Theorem 2.5 given below. This famous result uses the notion of measure disintegration.

Definition 2.4 For a standard probability measure space (X, \mathcal{B}, μ) and a measurable partition ξ of X , it is said that a collection of measures $(\mu_C)_{C \in X/\xi}$ is a *system of conditional measures* with respect to (X, \mathcal{B}, μ) and ξ if

- (i) for each $C \in X/\xi$, μ_C is a measure on the sigma-algebra $\mathcal{B}_C := \mathcal{B} \cap C$ such that $(C, \mathcal{B}_C, \mu_C)$ is a standard probability measure space;
- (ii) for any $B \in \mathcal{B}$, the function $C \mapsto \mu_C(B \cap C)$ is μ_ξ -measurable;
- (iii) for any $B \in \mathcal{B}$,

$$\mu(B) = \int_{X/\xi} \mu_C(B \cap C) \, d\mu_\xi(C). \tag{2.2}$$

Condition (2.2) can be rewritten in the equivalent form:

$$\int_X f(x) \, d\mu(x) = \int_{X/\xi} \left(\int_C f(y) \, d\mu_C(y) \right) \, d\mu_\xi(C). \tag{2.3}$$

Measurable partitions are characterized by the following result.

Theorem 2.5 ([53]) *For any measurable partition ξ of a standard probability measure space (X, \mathcal{B}, μ) , there exists a unique system of conditional measures (μ_C) . Conversely, if $(\mu_C)_{C \in X/\xi}$ is a system of conditional measures with respect to $((X, \mathcal{B}, \mu), \xi)$, then ξ is a measurable partition.*

We apply Theorem 2.5 to the case of an endomorphism $\sigma \in \text{End}(X, \mathcal{B}, \mu)$. Let ξ_σ be the measurable partition of (X, \mathcal{B}, μ) into preimages $\sigma^{-1}(x) = C_x$ of points $x \in X$. Let (μ_C) be the system of conditional measures defined by ξ_σ . In the case when $(X/\xi, \mu_\xi)$ is isomorphic to (X, μ) (for example, when σ is the left shift or $\sigma : z \mapsto z^N$, $z \in \mathbb{T}^1$), we see that relations (2.2) and (2.3) have the form

$$\int_X f(x) \, d\mu(x) = \int_X \left(\int_C f(y) \, d\mu_x(y) \right) \, d\mu(x). \tag{2.4}$$

This decomposition is the key fact in our representation of the transfer operator generated by σ .

In most important cases, the disintegration of a measure is applied to probability (finite) measures. The problem of measure disintegration is discussed in many books and articles. We refer here to [12, 20, 31, 32, 48]. The case of an infinite sigma-finite measure was considered by several authors, see, e.g., [57].

Theorem 2.6 ([57]) *Let (X, \mathcal{B}, μ) and (Y, \mathcal{A}, ν) be standard measure spaces with sigma-finite measures, and suppose that $\pi : X \rightarrow Y$ is a measurable map. Let (X, \mathcal{B}, μ) and (Y, \mathcal{A}, ν) be as above. Suppose that $\widehat{\mu} = \mu \circ \pi^{-1} \ll \nu$. Then, there exists a unique system of conditional measures $(\nu_y)_{y \in Y}$ for μ . For ν -a.e., ν_y is a sigma-finite measure.*

2.3 Radon–Nikodym derivatives and Markovian functions

Suppose $\sigma \in \text{End}(X, \mathcal{B}, \mu)$. Recall that $\mu \circ \sigma^{-1} \sim \mu$ in this case. Then, we can define the *Radon–Nikodym derivative* $\rho_\mu(x) := \frac{d\mu \circ \sigma^{-1}}{d\mu}(x)$, which is a Borel function such that

$$\int_X f(\sigma x) \, d\mu = \int_X f(x) \rho_\mu(x) \, d\mu, \quad f \in L^1(\mu). \tag{2.5}$$

Setting $\rho_n(x) = \frac{d\mu \circ \sigma^{-n}}{d\mu}(x)$, we obtain the Radon–Nikodym *cocycle* satisfying the equation $\rho_{n+m}(x) = \rho_m(\sigma^n(x)) \rho_n(x)$.

Remark 2.7 Suppose that $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ and ν is a measure equivalent to μ , i.e., there exists a Borel function h such that $d\nu(x) = h(x)d\mu(x)$. Then, σ is also non-singular with respect to ν , and $\rho_\nu(x) = h(\sigma x)\rho_\mu(x)h^{-1}(x)$.

Every $\sigma \in \text{End}(X, \mathcal{B})$ defines a linear operator S_σ called a composition operator on the space of bounded Borel functions:

$$S_\sigma(f) = f \circ \sigma, \quad f \in \mathcal{F}(X, \mathcal{B}).$$

This operator is also known by the name of a *Koopman operator* when it is considered in a L^2 space.

The map $A \mapsto \mu(\sigma(A))$, $A \in \mathcal{B}$, defines a measure on the subalgebra $\sigma^{-1}(\mathcal{B})$. If μ is a *forward quasi-invariant* measure, then there exists a unique $\sigma^{-1}(\mathcal{B})$ -measurable function $\omega_\mu(x) = \frac{d\mu \circ \sigma}{d\mu}(x)$ such that

$$\int_X f(\sigma x) \omega_\mu(x) \, d\mu = \int_X f(x) \, d\mu, \quad f \in L^1(\mu). \tag{2.6}$$

It can be deduced from the uniqueness of the Radon–Nikodym derivative that

$$\omega_\mu(x) = \rho_\mu(\sigma(x))^{-1}$$

when these functions are considered as functions measurable with respect to $\sigma^{-1}(\mathcal{B})$.

The following statement is well-known and we omit its proof.

Lemma 2.8 (1) *The composition operator $S_\sigma : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(X, \sigma^{-1}(\mathcal{B}), \mu)$ is an isometry if and only if $\mu \circ \sigma^{-1} = \mu$.*

(2) *The operator S_σ on $L^2(\mu)$ is bounded if and only if there exists a constant $k > 0$ such that*

$$\frac{\mu(\sigma^{-1}(A))}{\mu(A)} \leq k, \quad A \in \mathcal{B}.$$

(3) *If μ is a forward quasi-invariant measure, then*

$$T_\sigma : f \mapsto \sqrt{\omega_\mu}(f \circ \sigma)$$

is an isometry from $L^2(X, \mathcal{B}, \mu)$ onto $L^2(X, \sigma^{-1}(\mathcal{B}), \mu)$.

2.4 Properties of endomorphisms

In this subsection, we collected the properties of endomorphisms of a measure space for the reader’s convenience. Here and below, we implicitly use the mod 0-convention which means that a property (formula, relation, etc) holds almost everywhere with respect to a fixed measure.

Definition 2.9 Let σ be a surjective endomorphism of (X, \mathcal{B}, μ) with quasi-invariant measure μ .

- (i) The endomorphism σ is called *conservative* if for any set A of positive measure there exists $n > 0$ such that $\mu(\sigma^n(A) \cap A) > 0$.
- (ii) The endomorphism σ is called *ergodic* if whenever A is σ -invariant, i.e., $\sigma^{-1}(A) = A$, then either A or $X \setminus A$ is of measure zero. Equivalently, σ is ergodic if, for a bounded Borel function f , the condition $f \circ \sigma = f$ implies that f is a constant mod 0.
- (iii) Any endomorphism $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ generates the sequence of subalgebras:

$$\mathcal{B} \supset \sigma^{-1}(\mathcal{B}) \cdots \supset \sigma^{-i}(\mathcal{B}) \supset \cdots$$

Then, $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ is called *exact* if

$$\mathcal{B}_\infty := \bigcap_{k \in \mathbb{N}} \sigma^{-k}(\mathcal{B}) = \{\emptyset, X\} \quad \text{mod } 0.$$

- (iv) The surjective endomorphism σ of a probability measure space (X, \mathcal{B}, μ) is called *full* (or *limsup full*) if $\lim_{n \rightarrow \infty} \mu(\sigma^n(B)) = 1$ (or $\limsup_{n \rightarrow \infty} \mu(\sigma^n(B)) = 1$) for every $B \in \mathcal{B}$, $\mu(B) > 0$.

(v) A non-singular endomorphism σ of (X, \mathcal{B}, μ) is said to be μ -recurrent if for every non- negative Borel function f , the function

$$\Sigma(f) = \sum_{n \geq 0} f(\sigma^n(x))\omega_n(x)$$

takes only the values 0 and ∞ μ -a.e. where $\omega_n(x) = \frac{d\mu \circ \sigma^n}{d\mu}(x)$.

In the next remark, we include several results illustrating the properties of endomorphisms given in Definition 2.9. We use some results from [34, 36, 56].

Remark 2.10 (1) Every exact endomorphism is ergodic. There are examples of ergodic endomorphisms which are not exact. As it is customary in ergodic theory, we can always assume, without loss of generality, that an endomorphism is ergodic.

- (2) There are examples of one-sided shifts (n -to-one endomorphisms) which are not exact.
- (3) We note that there are ergodic endomorphisms that are not conservative.
- (4) An endomorphism σ is recurrent with respect to a finite measure μ if and only if $\sum_{n \geq 0} \omega_n(x) = \infty$.
- (5) A μ -recurrent endomorphism is conservative.

Since every surjective endomorphism defines an isometry S_σ (or T_σ), see Lemma 2.8, then we can apply Wold’s theorem to these objects.

Theorem 2.11 (Wold’s theorem) *Let S be an isometric operator in a Hilbert space \mathcal{H} . Define*

$$\mathcal{H}_\infty = \bigcap_n S^n \mathcal{H},$$

and

$$\mathcal{H}_{shift} = N_{S^*} \oplus SN_{S^*} \oplus \dots \oplus S^k N_{S^*} \oplus \dots .$$

Then, the following statements hold.

- (1) The space \mathcal{H} is decomposed into the orthogonal direct sum

$$\mathcal{H} = \mathcal{H}_\infty \oplus \mathcal{H}_{shift}.$$

- (2)

$$\mathcal{H}_\infty = \{x \in \mathcal{H} : \|(S^*)^n x\| = \|x\|, \forall n \in \mathbb{N}\}.$$

- (3) The operator S restricted on \mathcal{H}_∞ is a unitary operator, and S is a unilateral shift in the space \mathcal{H}_{shift} .

For $\sigma \in \text{End}(X, \mathcal{B}, \mu)$, define

$$\mathcal{B}_\infty = \bigcap_{n=0}^\infty \sigma^{-n}(\mathcal{B}),$$

and let $\mathcal{A}_\sigma = \{A \in \mathcal{B} : \sigma^{-1}(A) = A\}$ be the subalgebra of σ -invariant subsets of X .

Let ζ be a partition of (X, \mathcal{B}, μ) into orbits of σ (recall that $x \overset{\zeta}{\sim} y$ if there are n, m such that $\sigma^n(x) = \sigma^m(y)$). Let η be the partition of (X, \mathcal{B}, μ) such that $x \overset{\eta}{\sim} y$ if there is n such that $\sigma^n(x) = \sigma^n(y)$. By ζ' and η' we denote the measurable halls of ζ and η , respectively.

If ϵ denotes the partition of X into points, then we have the sequence of decreasing measurable partitions $\{\sigma^{-i}(\epsilon)\}_{i=0}^\infty$:

$$\epsilon \succeq \sigma^{-1}(\epsilon) \succeq \sigma^{-2}(\epsilon) \dots .$$

As shown in [54], the following results hold:

$$\zeta' \preceq \eta', \quad \eta' = \bigwedge_n \sigma^{-n}(\epsilon)$$

and

$$\mathcal{A}(\zeta') = \mathcal{A}_\sigma, \quad \mathcal{A}(\eta') = \mathcal{B}_\infty.$$

In particular, σ is ergodic if the partition ζ' is trivial, and σ is exact if the partition η' is trivial.

Since η' is a measurable partition, we can define the quotient measure space $(Y, \nu) = (X/\eta', \mathcal{B}/\eta', \mu_{\eta'})$ where $\mathcal{B}/\eta' = \mathcal{B}_\infty$.

The following result is deduced from Wold’s theorem, see details in [10].

Corollary 2.12 (1) *Let $\pi : X \rightarrow Y$ be the natural projection. Then, there exists a measure-preserving automorphism $\tilde{\sigma} : (Y, \nu) \rightarrow (Y, \nu)$ such that $\tilde{\sigma}$ is an automorphic factor of σ , i.e.,*

$$\tilde{\sigma} \circ \pi = \pi \circ \sigma.$$

(2) *Let $S_\sigma : f \rightarrow f \circ \sigma$ be the isometry on $\mathcal{H} = L^2(\mu)$. Then, in the Wold decomposition $\mathcal{H} = \mathcal{H}_\infty \oplus \mathcal{H}_\infty^\perp$ for S_σ , we have*

$$\mathcal{H}_\infty = L^2(Y, \nu),$$

and the restriction of S_σ to \mathcal{H}_∞ corresponds to the unitary operator U defined by $\tilde{\sigma}$, $U(f) = f \circ \tilde{\sigma}$.

It turns out that every non-singular endomorphism is a factor of an invertible dynamical system. It is said that an automorphism $T \in \text{Aut}(Y, \mathcal{C}, \nu)$ is a *natural extension* of an endomorphism $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ if there exists a map $\tau : (Y, \mathcal{C}, \nu) \rightarrow (X, \mathcal{B}, \mu)$ such that

- (i) $\mathcal{C} = \bigvee_{n=0}^\infty T^n(\tau^{-1}\mathcal{B}), \quad \text{mod } 0,$
- (ii) there exists a measure $\nu' \sim \nu$ such that $\omega_{\nu'} = \omega_\mu \circ \tau$.

We recall an important fact saying that every non-singular endomorphism σ of (X, \mathcal{B}, μ) admits a natural extension, see [52,56], [8], and Example 2.13.

Example 2.13 Let σ be an onto endomorphism of a standard Borel space (X, \mathcal{B}) . Define the set \widehat{X} as follows: \widehat{X} is a subset of $X \times X \times X \times \dots$ such that

$$\widehat{x} = (x_i)_{i \geq 0} \in \widehat{X} \iff \sigma(x_{i+1}) = x_i \quad \forall i \geq 0.$$

The set \widehat{X} is often called the *solenoid* constructed by (X, σ) and denoted $\text{Sol}_{(X, \sigma)}$. Note that \widehat{X} is a closed subset in the product space $X \times X \times \dots$. Since \widehat{X} is Borel, it inherits the Borel structure $\widehat{\mathcal{B}}$ from the product space.

Define the map $\widehat{\sigma} : \widehat{X} \rightarrow \widehat{X}$ by setting

$$\widehat{\sigma}(x_0, x_1, x_2, \dots) = (\sigma(x_0), x_0, x_1, \dots).$$

Then, one can easily verify that $\widehat{\sigma}$ is a one-to-one Borel map of \widehat{X} onto itself, and the shift

$$\tau : (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots)$$

is inverse to $\widehat{\sigma}$, $\tau = \widehat{\sigma}^{-1}$.

Let $\pi_n : \widehat{X} \rightarrow X$ be the projection from \widehat{X} onto the n -th coordinate: for $\widehat{x} = (x_n)$, set $\pi_n(\widehat{x}) = x_n, n \geq 0$. Then, π_n can be extended to a map $f \mapsto f \circ \pi_n$ from $\mathcal{F}(X, \mathcal{B})$ to $\mathcal{F}(\widehat{X}, \widehat{\mathcal{B}})$. It follows from the above definitions that $\pi_{n+1}\widehat{\sigma}(\widehat{x}) = \pi_n(\widehat{x})$, and π_0 is a factor map from $(\widehat{X}, \widehat{\sigma})$ to (X, σ) , i.e.,

$$\pi_0\widehat{\sigma} = \sigma\pi_0.$$

Proposition 2.14 *Let μ be a Borel continuous measure on a standard Borel space (X, \mathcal{B}) which is quasi-invariant with respect to an endomorphism σ . Let the measure \mathbb{P} on $(\widehat{X}, \widehat{\mathcal{B}})$ be defined by the relation $\mathbb{P} \circ \pi_0^{-1} = \mu$. Then, the map*

$$L^2(\mu) \ni f \xrightarrow{V_0} f \circ \pi_0 \in L^2(\mathbb{P})$$

is an isometry. Moreover, the operator $U : L^2(\mathbb{P}) \rightarrow L^2(\mathbb{P})$ defined by the formula

$$U = V_0 S_\sigma V_0^*$$

is an isometry.

2.5 Markovian functions

In this subsection, we consider a class of functions defined by an endomorphism $\sigma \in \text{End}(X, \mathcal{B}, \mu)$. The following definition is motivated by relation (2.6).

Definition 2.15 Let σ be an onto endomorphism of (X, \mathcal{B}, μ) . A function $\varphi \in \mathcal{F}(X, \mathcal{B})$ satisfying

$$\int_X (f \circ \sigma) \varphi \, d\mu = \int_X f \, d\mu \tag{2.7}$$

for all f in $L^1(\mu)$ is called a *Markovian function*. The set of all Markovian functions is denoted by $M(\sigma, \mu)$. We denote $M_2(\sigma, \mu) = M(\sigma, \mu) \cap L^2(\mu)$.

The Markovian functions were considered in a series of papers [22,34,36], and others.

Remark 2.16 (1) The set $M(\sigma, \mu)$ is convex.

(2) Suppose that μ is a forward quasi-invariant measure for $\sigma \in \text{End}(X, \mathcal{B}, \mu)$. Then, the set of Markovian functions $M(\sigma, \mu)$ is not empty because $\omega_\mu \in M(\sigma, \mu)$ due to relation (2.6).

(3) For two equivalent measures μ and ν , we discuss relation between the functions ω_μ and ω_ν in Theorem 3.15. As above, the function ω_μ generates a cocycle by setting $\omega_n(x) = \frac{d\mu \circ \sigma^n}{d\mu}(x)$ (where $\omega_1 = \omega_\mu$).

(4) Let g be a function from $L^1(\mu)$. It was shown in [34] that, for the measure $d\nu = g d\mu$, the following holds:

$$\int_X f(\sigma x) \frac{g \omega_\nu}{g \circ \sigma}(x) \, d\mu = \int_X f(x) \, d\mu, \quad f \in L^1(\mu).$$

This means that the function $\frac{g \omega_\nu}{g \circ \sigma}$ is Markovian with respect to (σ, μ) .

Based on the facts from Remark 2.16, we prove the following result.

Proposition 2.17 *Let $\sigma \in \text{End}(X, \mathcal{B}, \mu)$. Suppose a measure ν is equivalent to μ and $g = \frac{d\nu}{d\mu}$. Then,*

$$M(\sigma, \nu) = g^{-1} M(\sigma, \mu)(g \circ \sigma).$$

Proof For a function $\varphi \in M(\sigma, \mu)$, show that $h = g^{-1} \varphi(g \circ \sigma) \in M(\sigma, \nu)$. For any $f \in L^1(\nu)$, we have

$$\begin{aligned} \int_X (f \circ \sigma) g^{-1} \varphi(g \circ \sigma) \, d\nu &= \int_X (fg) \circ \sigma \varphi \, d\mu \\ &= \int_X fg \, d\mu \\ &= \int_X f \, d\nu. \end{aligned}$$

This shows that the function h is in $M(\sigma, \nu)$.

Conversely, let φ be a Markovian function from $M(\sigma, \nu)$. Since $d\nu = g d\mu$,

$$\begin{aligned} \int_X fg \, d\mu &= \int_X (f \circ \sigma)\varphi g \, d\mu \\ &= \int_X (f \circ \sigma)(g \circ \sigma)[(g \circ \sigma)^{-1}\varphi g] \, d\mu. \end{aligned}$$

The latter means that $(g \circ \sigma)^{-1}\varphi g \in M(\sigma, \mu)$. □

Lemma 2.18 *Let $\varphi_1, \dots, \varphi_k$ be Markovian functions from $M(\sigma, \mu)$. Then, the function $\psi = (\varphi_k \circ \sigma^{k-1}) \cdots (\varphi_2 \circ \sigma)\varphi_1$ belongs to $M(\sigma^k, \mu)$.*

Proof We compute

$$\begin{aligned} \int_X (f \circ \sigma^k)\psi \, d\mu &= \int_X [(f \circ \sigma^{k-1}) \circ \sigma(\varphi_k \circ \sigma^{k-2}) \circ \sigma \cdots (\varphi_2 \circ \sigma)]\varphi_1 \, d\mu \\ &= \int_X (f \circ \sigma^{k-1})(\varphi_k \circ \sigma^{k-2}) \cdots (\varphi_3 \circ \sigma)\varphi_2 \, d\mu \\ &\quad \dots\dots\dots \\ &= \int_X f \, d\mu. \end{aligned}$$

□

3 Operators generated by endomorphisms

This section considers several linear operators naturally defined by surjective endomorphisms of (X, \mathcal{B}, μ) . These operators act in $L^2(\mu)$ and other functional spaces.

3.1 Transfer operators and endomorphisms

We define a transfer operator in general settings using only the Borel structure of the space (X, \mathcal{B}) . Transfer operators are extensively studied for various dynamical systems applying the properties of phase spaces.

Definition 3.1 Let $\mathcal{F}(X, \mathcal{B})$ be the set of all bounded Borel functions¹ and let $R : \mathcal{F}(X, \mathcal{B}) \rightarrow \mathcal{F}(X, \mathcal{B})$ be a linear operator. Then, R is called a *transfer operator* if it satisfies the following properties:

- (i) $f \geq 0 \implies R(f) \geq 0$ (i.e., R is a positive operator);
- (ii) for any Borel functions $f, g \in \mathcal{F}(X, \mathcal{B})$, the *pull-out property* holds

$$R((f \circ \sigma)g) = fR(g). \tag{3.1}$$

To emphasize that a transfer operator R is defined by an onto endomorphism σ , we will also write R as (R, σ) .

Let $\mathbb{1}$ be a function on (X, \mathcal{B}) that takes the only value 1. If $R(\mathbb{1})(x) > 0$ for all $x \in X$, then we say that R is a *strict transfer operator*. If $R(\mathbb{1}) = \mathbb{1}$, then the transfer operator R is called *normalized*.

Every transfer operator R defines an action on the set of probability measures $M_1(X)$: given $\mu \in M_1(X)$, set

$$(\mu R)(f) = \int_X R(f) \, d\mu, \quad f \in \mathcal{F}(X, \mathcal{B}).$$

If $\mu = \mu R$, then μ is called R -invariant. A measure $\mu \in M_1(X)$ is called *strongly invariant* with respect to a transfer operator (R, σ) if $\mu \sigma$ -invariant and $\mu R = \mu$.

Restrictions of transfer operators on Banach or Hilbert spaces give more possibilities to study their properties.

¹ In this section, we will consider real-valued functions for definiteness; the case of complex-valued functions can be done similarly.

Lemma 3.2 *Let (X, \mathcal{B}, μ) be a probability standard measure space. Suppose that (R, σ) is a normalized transfer operator acting in $L^2(X, \mathcal{B}, \mu)$ where $\mu \in M_1(X)$. Then, μ is strongly invariant with respect to (R, σ) if and only if $R^*(\mathbb{1}) = \mathbb{1}$.*

Proof Here and below, we denote by $\langle \cdot, \cdot \rangle_\mu$ the inner product in $L^2(\mu)$. Since R is normalized, we can write

$$\langle f, \mathbb{1} \rangle_{\mu \circ \sigma^{-1}} = \langle f \circ \sigma, R^*(\mathbb{1}) \rangle_\mu = \langle R(f \circ \sigma), \mathbb{1} \rangle_\mu = \langle f R(\mathbb{1}), \mathbb{1} \rangle_\mu = \langle f, \mathbb{1} \rangle_\mu,$$

that is μ is σ -invariant if $R^*(\mathbb{1}) = \mathbb{1}$.

Similarly, we see that the condition $R^*(\mathbb{1}) = 1$ is equivalent to $\mu = \mu R$:

$$\langle f, \mathbb{1} \rangle_{\mu R} = \langle R(f), \mathbb{1} \rangle_\mu = \langle f, R^*(\mathbb{1}) \rangle_\mu = \langle f, \mathbb{1} \rangle_\mu.$$

□

Example 3.3 Let $\sigma \in \text{End}(X, \mathcal{B}, \mu)$, $\mu(X) = 1$, and let the partition ξ_σ of X be defined by preimages $\{\sigma^{-1}(x) : x \in X\}$ of σ . We give an example of a transfer operator R_σ which is determined by the system of conditional measures $\{\mu_C\}$ (see Sect. 2.2) over the partition ξ_σ of X . Define a linear operator R_σ acting on Borel bounded functions over the standard probability measure space (X, \mathcal{B}, μ) by setting

$$R_\sigma(f)(x) := \int_{C_x} f(y) \, d\mu_{C_x}(y) \tag{3.2}$$

where $C_x = \sigma^{-1}(x)$.

Lemma 3.4 *The operator $R_\sigma : \mathcal{F}(X, \mathcal{B}) \rightarrow \mathcal{F}(X, \mathcal{B})$ defined by (3.2) is a transfer operator.*

Proof Clearly, R_σ is a positive normalized operator. To see that (3.1) holds, we calculate

$$\begin{aligned} R_\sigma((f \circ \sigma)g)(x) &= \int_{C_x} (f \circ \sigma)(y)g(y) \, d\mu_{C_x} \\ &= f(x) \int_{C_x} g(y) \, d\mu_{C_x}(y) \\ &= f(x)(R_\sigma g)(x). \end{aligned}$$

Here, we used the fact that $f(\sigma(y)) = f(x)$ for $y \in C_x = \sigma^{-1}(x)$. □

For an onto endomorphism σ acting on the space (X, \mathcal{B}, μ) , we consider the subalgebra $\mathcal{A} = \{\sigma^{-1}(B) : B \in \mathcal{B}\}$ of \mathcal{B} . It is a well-known fact that there exists the conditional expectation $\mathbb{E}_\sigma : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(X, \sigma^{-1}(\mathcal{B}), \mu)$. To simplify the formulas, we will use also the following notations: $\mathcal{A} = \sigma^{-1}(\mathcal{B})$, $L^2(\mu) = L^2(X, \mathcal{B}, \mu)$, and $L^2(\mu_{\mathcal{A}}) = L^2(X, \sigma^{-1}(\mathcal{B}), \mu)$. Below, we will describe the operator \mathbb{E}_σ explicitly in terms of the composition operator S_σ .

It turns out that, for every transfer operator R , one can define another operator which is, in some sense, analogous to the conditional expectation. For this, let (R, σ) be a normalized transfer operator. We define

$$E : \mathcal{F}(X, \mathcal{B}) \rightarrow \mathcal{F}(X, \sigma^{-1}(\mathcal{B})) : f \mapsto R(f) \circ \sigma. \tag{3.3}$$

We discuss the properties of the operator E in Proposition 3.5. Some of them are proved in [10].

Proposition 3.5 *Let R be a normalized transfer operator and $E(f) = R(f) \circ \sigma$. Then, the following properties hold:*

- (1) *E is positive and $E^2 = E$,*

$$E(\mathcal{F}(X, \mathcal{B})) = \mathcal{F}(X, \sigma^{-1}(\mathcal{B})),$$

$$E|_{\mathcal{F}(X, \sigma^{-1}(\mathcal{B}))} = id, \\ R \circ E \circ R = R^2 \text{ and } R \circ E = R.$$

(2) For $S_\sigma(f) = f \circ \sigma$, we have

$$(RS_\sigma)(f) = f, \quad (S_\sigma R)(f) = E(f).$$

(3) A bounded Borel function f belongs to $\mathcal{F}(X, \sigma^{-1}(\mathcal{B}))$ if and only if there exists a function $g \in \mathcal{F}(X, \mathcal{B})$ such that $f = g \circ \sigma$.

Proof These properties are proved directly. We only check that $R \circ E \circ R = R^2$. Indeed,

$$R[E(R(f))] = R[R(R(f) \circ \sigma)] = R^2(f)R(\mathbb{1}) = R^2(f).$$

□

Corollary 3.6 *Let R be the transfer operator defined in (3.2). Then, the conditional expectation $E : \mathcal{F}(X, \mathcal{B}) \rightarrow \mathcal{F}(X, \sigma^{-1}(\mathcal{B})) : f \mapsto R(f) \circ \sigma$ acts by the formula*

$$E(f) = R(f) \circ \sigma = \int_{\sigma^{-1}(\sigma(x))} f(y) \, d\mu_{C_{\sigma(x)}}(y).$$

3.2 Composition operators and Markovian functions

We recall our notation: σ is a surjective endomorphism of a probability measure space (X, \mathcal{B}, μ) , $S_\sigma : f \mapsto f \circ \sigma$ is the composition operator, and $M(\sigma, \mu)$ is the set of Markovian functions.

Let $t = t(x)$ be a bounded Borel function, and μ a σ -invariant probability measure on (X, \mathcal{B}) . We define the operator P_t on $L^2(\mu)$ by setting

$$P_t(f) = t(f \circ \sigma), \quad f \in L^2(\mu).$$

We call the operator P_t a *weighted composition operator*. Clearly, P_t is a bounded operator in $L^2(\mu)$.

Theorem 3.7 (1) *For $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ as above, consider the composition operator S_σ in the Hilbert space $L^2(\mu)$. Then, the adjoint operator S_σ^* acts by the formula:*

$$S_\sigma^*(g) = \frac{(g d\mu) \circ \sigma^{-1}}{d\mu}, \quad g \in L^2(\mu_{\mathcal{A}}). \tag{3.4}$$

(2) *The adjoint operator S_σ^* is a transfer operator*

$$S_\sigma^*(g(f \circ \sigma)) = f S_\sigma^*(g).$$

The transfer operator S_σ^ is normalized if and only if μ is σ -invariant.*

(3) *For a function $t \in \mathcal{F}(X, \mathcal{B})$, the adjoint operator P_t^* is a non-normalized transfer operator.*

Proof (1) For functions $f, g \in L^2(\mu)$, we have

$$\begin{aligned} \langle S_\sigma f, g \rangle_\mu &= \int_X (f \circ \sigma)g \, d\mu \\ &= \int f (g d\mu) \circ \sigma^{-1} \\ &= \int f \frac{(g d\mu) \circ \sigma^{-1}}{d\mu} \, d\mu \\ &= \langle f, S_\sigma^* g \rangle_\mu \end{aligned}$$

which proves (3.4).

(2) To prove the pull-out property, we compute, for arbitrary functions $f, g, h \in L^2(\mu)$,

$$\begin{aligned} \int_X h S_\sigma^*((f \circ \sigma)g) \, d\mu &= \int_X S_\sigma(h)(f \circ \sigma)g \, d\mu \\ &= \int_X ((hf) \circ \sigma) g \, d\mu \\ &= \int_X hf S_\sigma^*(g) \, d\mu. \end{aligned}$$

Finally, we see that

$$\int_X f \, d\mu \circ \sigma^{-1} = \int_X (f \circ \sigma) \mathbb{1} \, d\mu = \int_X f S_\sigma^*(\mathbb{1}) \, d\mu,$$

and $S_\sigma^*(\mathbb{1}) = \mathbb{1} \iff \mu \circ \sigma^{-1} = \mu$.

(3) The same proof as in (1) gives the formula

$$P_t^*(g) = \frac{(tg \, d\mu) \circ \sigma^{-1}}{d\mu}, \quad g \in L^2(\mu_{\mathcal{A}}).$$

This shows that P_t^* is not normalized.

To prove that P_t^* satisfies the pull-out property, we write

$$\begin{aligned} \langle h, P_t^*(g(f \circ \sigma)) \rangle_\mu &= \langle P_t(h), g(f \circ \sigma) \rangle_\mu \\ &= \langle t(h \circ \sigma), g(f \circ \sigma) \rangle_\mu \\ &= \langle P_t(hf), g \rangle_\mu \\ &= \langle h, f P_t^*(g) \rangle_\mu. \end{aligned}$$

□

Corollary 3.8 (1) Let $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ be such that $\mu = \mu \circ \sigma^{-1}$. Then, the conditional expectation $\mathbb{E}_\sigma : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(X, \mathcal{A}, \mu_{\mathcal{A}})$ can be represented as $S_\sigma S_\sigma^*$.

(2) If μ is forward quasi-invariant, then the conditional expectation \mathbb{E}_σ coincides with $T_\sigma T_\sigma^*$ where the isometry $T_\sigma(f) = \sqrt{\omega_\mu}(f \circ \sigma)$ is defined in Lemma 2.8.

Proof The fact that $(S_\sigma S_\sigma^*)^2 = S_\sigma S_\sigma^*$ follows from the identity $S_\sigma^* S_\sigma = \mathbb{1}$.

Next, we verify that $\langle f, S_\sigma S_\sigma^*(h) \rangle_\mu = \langle f, h \rangle_\mu$ for $h \in L^2(\mu_{\mathcal{A}})$. Recall that if $f \in L^2(\mu_{\mathcal{A}})$, then there exists $g \in L^2(\mu)$ such that $f = g \circ \sigma$. Then, using the fact that S_σ^* is a normalized transfer operator (Theorem 3.7), we have

$$\begin{aligned} \langle g \circ \sigma, S_\sigma S_\sigma^*(h) \rangle_\mu &= \langle S_\sigma^*(g \circ \sigma), S_\sigma^*(h) \rangle_\mu \\ &= \langle g S_\sigma^*(\mathbb{1}), S_\sigma^*(h) \rangle_\mu \\ &= \langle g, S_\sigma^*(h) \rangle_\mu \\ &= \langle g \circ \sigma, h \rangle_\mu \\ &= \langle f, h \rangle_\mu \end{aligned}$$

The orthogonality of the projection $S_\sigma S_\sigma^*$ is obtained from the relation

$$\langle S_\sigma S_\sigma^* f, g \circ \sigma \rangle_\mu = \langle S_\sigma^* f, g \rangle_\mu = \langle f, g \circ \sigma \rangle_\mu.$$

It proves that $\mathbb{E}_\sigma = S_\sigma S_\sigma^*$.

(2) The case of a quasi-invariant measure μ is considered similarly. We note that ω_μ is a Borel function measurable with respect to $\sigma^{-1}(\mathcal{B})$. Hence, every function $f \in L^2(\mu_{\mathcal{A}})$ there exists a function $g \in L^2(\mu)$ such that $f = \sqrt{\omega_\mu}(g \circ \sigma)$. Then, we repeat the above calculations. We leave the details to the reader. □

We associated with every endomorphism $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ two transfer operators R and S_σ^* . It turns out that they coincide in $L^2(\mu)$.

Theorem 3.9 *Let $\sigma \in \text{End}(X, \mathcal{B}, \mu)$. Then, the transfer operators R_σ and S_σ^* coincide in $L^2(\mu)$, where R_σ is defined in (3.2) and S_σ^* satisfies (3.4).*

Proof We compute $S_\sigma^*(f)$ using the disintegration of μ with respect to the conditional measures μ_x on $C_x = \sigma^{-1}(x)$:

$$\begin{aligned} \langle S(g), f \rangle_\mu &= \int_X (g \circ \sigma) f \, d\mu \\ &= \int_X \left(\int_{C_x} g(\sigma(y)) f(y) \, d\mu_x(y) \right) d\mu(x) \\ &= \int_X g(x) \left(\int_{C_x} f(y) \, d\mu_x(y) \right) d\mu(x) \\ &= \langle g, S_\sigma^*(f) \rangle_\mu \end{aligned}$$

From the latter, we see that

$$S_\sigma^*(f) = \int_{C_x} f(y) \, d\mu_x(y) = R_\sigma(f).$$

□

Remark 3.10 (1) It follows from Lemma 3.2 and Theorem 3.9 that, for the transfer operator S_σ^* in $L^2(\mu)$, the measure μ is strong invariant if and only if it is σ -invariant.

(2) Theorem 3.9 implies that $E(f) = R(f) \circ \sigma$ coincides with \mathbb{E}_σ for $R = S_\sigma^*$.

We now consider weighted composition operators where the weight function is Markovian; for consistency, we will write P_φ for such a weighted composition operator. We will continue discussing the properties of weighted composition operators in the next section.

Let $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ be a surjective endomorphism, and μ is a forward quasi-invariant measure. Consider the operator

$$P_\varphi : f \rightarrow \varphi(f \circ \sigma) \tag{3.5}$$

which is formally defined in $\mathcal{F}(X, \mathcal{B})$. It can be also written as

$$P_\varphi(f) = M_\varphi S_\sigma(f), \quad f \in \mathcal{F}(X, \mathcal{B})$$

where M_φ is the multiplication operator.

Lemma 3.11 *Let σ, φ , and P_φ be as above. Then, φ is a Markovian function ($\varphi \in M(\sigma, \mu)$) if and only if μ is P_φ -invariant, i.e., $\mu P_\varphi = \mu$ where*

$$\mu P_\varphi(f) = \int_X P_\varphi(f) \, d\mu.$$

In particular, $\mu = \mu P_{\omega_\mu}$.

Proof Indeed, if φ is Markovian, then

$$\int_X P_\varphi(f) \, d\mu = \int_X (f \circ \sigma) \varphi \, d\mu = \int_X f \, d\mu$$

which means that μ is P_φ -invariant. The converse statement also follows from the relation above. □

Lemma 3.12 *Let $\varphi \in M(\sigma, \mu)$ and P_φ a weighted composition operator. Then, $P_\varphi : L^1(\mu) \rightarrow L^1(\mu_A)$ and $P_{\sqrt{\varphi}} : L^2(\mu) \rightarrow L^2(\mu_A)$ are isometric operators.*

Proof Straightforward. \square

Proposition 3.13 *Let $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ and φ is a function from $L^2(\mu)$. Then, φ is a Markovian function if and only if $S_\sigma^*(\varphi) = \mathbb{1}$.*

Proof Let $f \in L^2(\mu)$. Then, the result follows from the following relations:

$$\begin{aligned} \int_X f \, d\mu &= \int_X (f \circ \sigma) \varphi \, d\mu \\ &= \int S_\sigma(f) \varphi \, d\mu \\ &= \int_X f S_\sigma^*(\varphi) \, d\mu. \end{aligned}$$

\square

Theorem 3.14 *Let P_φ be defined in $L^2(\mu)$ according to (3.5) where $\varphi > 0$.*

(1) *The following statements are equivalent:*

- (i) P_φ is an isometry in $L^2(\mu)$;
- (ii) the composition operator S_σ is an isometry in $L^2(\varphi\mu)$;
- (iii) $S_\sigma^*(\varphi) = \mathbb{1}$;
- (iv)

$$\frac{(\varphi d\mu) \circ \sigma^{-1}}{d\mu} = 1 \text{ a.e.}$$

(2) *If φ is a positive Borel function from $L^2(\mu)$, then the adjoint operator P_φ^* is a transfer operator and P_φ^* is normalized if and only if φ is Markovian.*

Proof (1) The proof of the first statement uses the arguments given in the proofs of Theorems 3.13 and 3.7. We leave the details for the reader.

(2) It is clear that P_φ^* is positive because we have the following formula for P_φ^* :

$$P_\varphi^*(g) = \frac{(\varphi g d\mu) \circ \sigma^{-1}}{d\mu}.$$

Show that it satisfies the pull-out property. Since $P_\varphi = M_\varphi S_\sigma$ and S_σ^* is a transfer operator, we obtain

$$\begin{aligned} P_\varphi^*((f \circ \sigma)g) &= S_\sigma^* M_\varphi^*[(f \circ \sigma)g] \\ &= S_\sigma^*[\bar{\varphi}g(f \circ \sigma)] \\ &= f S_\sigma^*(\bar{\varphi}g) \\ &= f S_\sigma^* M_\varphi^*(g) \\ &= f P_\varphi^*(g). \end{aligned}$$

To finish the proof, we note that

$$\begin{aligned} \int_X (f \circ \sigma) \varphi \, d\mu &= \int_X P_\varphi(f) \mathbb{1} \, d\mu \\ &= \int_X f P_\varphi^*(\mathbb{1}) \, d\mu. \end{aligned}$$

Hence,

$$\int_X (f \circ \sigma) \varphi \, d\mu = \int_X f \, d\mu$$

if and only if $P_\varphi^*(\mathbb{1}) = \mathbb{1}$. \square

3.3 Radon–Nikodym derivatives and conditional expectations

As above, $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ is an onto endomorphism with quasi-invariant measure μ . Equation (2.6) defines a uniquely determined Radon–Nikodym derivative ω_μ which is $\sigma^{-1}(\mathcal{B})$ -measurable function.

Let \mathbb{E}_σ denote the conditional expectation from $L^2(\mu)$ onto $L^2(\mu_{\mathcal{A}})$, where $\mathcal{A} = \sigma^{-1}(\mathcal{B})$ and $\mu_{\mathcal{A}}$ is the projection of μ onto the sigma-algebra \mathcal{A} . We recall that $\mathbb{E}_\sigma = S_\sigma S_\sigma^*$ for σ -invariant measure μ and $\mathbb{E}_\sigma = T_\sigma T_\sigma^*$ for σ -quasi-invariant measure μ , see Corollary 3.8.

Suppose that ν is another measure on (X, \mathcal{B}) which is equivalent to the measure μ . In the next theorem, we show how Markovian functions with respect to the measures μ and ν are related (see also Remark 2.7).

Theorem 3.15 *Let $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ and $d\nu(x) = h(x)d\mu(x)$ where $h(x) > 0$ μ -a.e. Let $\psi \in M(\sigma, \nu)$ and $\varphi \in M(\sigma, \mu)$. Then,*

$$\mathbb{E}_\sigma(h)\psi = \varphi(h \circ \sigma). \tag{3.6}$$

Hence, relation (3.6) establishes a one-to-one correspondence between the sets of Markovian functions $M(\sigma, \mu)$ and $M(\sigma, \nu)$.

Proof Since the both sides of (3.6) are measurable with respect to $\sigma^{-1}(\mathcal{B})$, it suffices to prove that, for every function $g \in \mathcal{F}(X, \mathcal{B})$,

$$\int_X (g \circ \sigma)\mathbb{E}_\sigma(h) \psi \, d\mu = \int_X (g \circ \sigma)(h \circ \sigma) \varphi \, d\mu. \tag{3.7}$$

We compute the left-hand side and the right-hand side in (3.7) separately using the definition of Markovian functions and the properties of conditional expectations. For the RHS:

$$\begin{aligned} \int_X (g \circ \sigma)(h \circ \sigma) \varphi \, d\mu &= \int_X (gh) \circ \sigma \varphi \, d\mu \\ &= \int_X gh \, d\mu \\ &= \int_X g \, d\nu. \end{aligned}$$

For the LHS, we use the fact that \mathbb{E}_σ is the conditional expectation and the function $(g \circ \sigma)\psi$ is $\sigma^{-1}(\mathcal{B})$ -measurable:

$$\begin{aligned} \int_X (g \circ \sigma)\mathbb{E}_\sigma(h) \psi \, d\mu &= \int_X (g \circ \sigma)h \psi \, d\mu \\ &= \int_X (g \circ \sigma) \psi \, d\nu \\ &= \int_X g \, d\nu. \end{aligned}$$

□

Let $R = (R, \sigma)$ be a transfer operator where σ is an onto endomorphism of (X, \mathcal{B}, μ) . Recall that we have defined in (3.3) the operator $E = R(f) \circ \sigma : \mathcal{F}(X, \mathcal{B}) \rightarrow \mathcal{F}(X, \sigma^{-1}(\mathcal{B}))$ which is an analog of the conditional expectation \mathbb{E}_σ . In the following statement, we find out under what conditions on the measure μ and the transfer operator R the operator E coincides with the genuine conditional expectation \mathbb{E}_σ .

Theorem 3.16 *In the setting formulated above, the operator $E = R(f) \circ \sigma$ coincides with $\mathbb{E}_\sigma = S_\sigma S_\sigma^*$ in $L^2(\mu)$ if and only if*

$$R(f) \frac{d\mu \circ \sigma^{-1}}{d\mu} = \frac{(f d\mu) \circ \sigma^{-1}}{d\mu}, \quad f \in L^2(\mu). \tag{3.8}$$

Relation (3.8) can be also written as $\rho_\mu R(f) = S_\sigma^*(f)$.

Proof It was shown in Proposition 3.5 that $E^2 = E$. It remains to find out under what conditions the relation $E = E^*$ hold. Clearly, it is equivalent to the property

$$\int_X gE(f) \, d\mu = \int_X gf \, d\mu, \quad \forall g \in \mathcal{F}(X, \sigma^{-1}(\mathcal{B})). \quad (3.9)$$

Representing g as $h \circ \sigma$ ($h \in \mathcal{F}(X, \mathcal{B})$), we obtain that (3.9) is equivalent to

$$\int_X (h \circ \sigma)R(f) \circ \sigma \, d\mu = \int_X (h \circ \sigma)f \, d\mu$$

or

$$\int_X hR(f) \, d\mu \circ \sigma^{-1} = \int_X h(fd\mu) \circ \sigma^{-1}.$$

This means that

$$R(f) = \frac{(fd\mu) \circ \sigma^{-1}}{d\mu \circ \sigma^{-1}}$$

which is equivalent to (3.8). This proves that $E = \mathbb{E}_\sigma$ if and only if $S_\sigma^*(f) = \rho_\mu R(f)$.

We note that if μ is a σ -invariant measure, then

$$R(f) = \frac{(fd\mu) \circ \sigma^{-1}}{d\mu}$$

which coincides with $S_\sigma^*(f)$ by (3.4). It follows then that $\mathbb{E}_\sigma = E = R(f) \circ \sigma$ in the case of σ -invariant measure μ . Moreover, it is obvious that the condition $R(f) = S_\sigma^*(f)$ implies the invariance of μ with respect to σ . \square

Corollary 3.17 *In notation given above, the operator $E_{\rho_\mu} = (\rho_\mu R) \circ \sigma$ coincides with \mathbb{E}_σ .*

Proof We noted that $\rho_\mu R$ is a transfer operator coinciding with S_σ^* , and therefore we can define $E_{\rho_\mu} = S_\sigma R_{\rho_\mu}$. By Corollary 3.8, we obtain that $S_\sigma R_{\rho_\mu} = S_\sigma S_\sigma^* = \mathbb{E}_\sigma$ which proves the statement. \square

Remark 3.18 The results of Proposition 2.17 and Theorem 3.15 can be interpreted as follows.

Let G denote the group $\mathcal{F}_+(X, \mathcal{B})$ of Borel bounded strictly positive functions. Then, G acts on the set $\{M(\sigma, \nu) : \nu \sim \mu\}$. This action $\alpha = \{\alpha_f : f \in G\}$ is defined by the rule:

$$\alpha_f(\varphi) = \frac{f \circ \sigma}{f} \varphi, \quad \varphi \in M(\sigma, \mu).$$

Clearly,

$$\alpha_f(M(\sigma, \mu)) = M(\sigma, \nu),$$

where $d\nu = fd\mu$, and $\alpha_f \alpha_g = \alpha_{fg}$. The action α is free in the sense that it satisfies the property $M(\sigma, \mu) \cap M(\sigma, \nu) = \emptyset$ if $\nu \sim \mu$. Moreover, α is transitive.

4 Cuntz relations for invariant and quasi-invariant measures

Starting with the measurable category, and disintegration of the appropriate measures, we showed above that careful choice of Hilbert spaces allows for a powerful tool in the analysis of endomorphisms and branching systems (in the measurable setting). In more detail, the steps from transformations in measure space to L^2 spaces and operators are often called “passing to the Koopman operators”. In our context, the non-commutativity for the operators under consideration is captured well with the Cuntz relations, or rather their representations; see [9, 21, 47]. Recall that, following J. Cuntz, for every n , one introduces a C^* -algebra $\mathcal{O}_{|\Lambda|}$ defined by a system of $|\Lambda|$ generators T_i . These generators may be realized as operators in Hilbert space, say H as follows: The relations (Cuntz relations) state that the T_i system is represented by isometries with orthogonal ranges in H such that the sum of these ranges is H . (Think of the subspaces as sub-bands.) In other words, via the isometries, H arises as an orthogonal sum of copies of itself. As a C^* -algebra, $\mathcal{O}_{|\Lambda|}$ is simple. Its representations are important, and they play a crucial role in the study of self-similar dynamics and self-similar geometries.

4.1 Quasi-invariant measure

Let $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ where μ is a forward and backward quasi-invariant measure. We recall that, in this case, the corresponding Radon–Nikodym derivatives ω_μ and ρ_μ are well-defined functions satisfying (2.6) and (2.5). In the remaining sections of this paper, we will consider L^2 -spaces of complex-valued functions.

Let φ be a positive Markovian function, $\varphi \in M(\sigma, \mu)$. Then, we define a *weighted composition operator* S_φ acting on $L^2(\mu)$ by

$$S_\varphi(f) = \sqrt{\varphi}(f \circ \sigma) \tag{4.1}$$

Equivalently, $S_\varphi = M_{\sqrt{\varphi}}S_\sigma$ where $M_{\sqrt{\varphi}}$ denotes the operator of multiplication, and S_σ is the composition operator.

We mention two important particular cases of (4.1) when (a) $\varphi = \omega_\mu$ and (b) $\varphi = \mathbb{1}$. Case (b) occurs if and only if $\mu \circ \sigma^{-1} = d\mu$.

Lemma 4.1 (1) *Let φ be a Markovian function from $M_2(\sigma, \mu) = M(\sigma, \mu) \cap L^2(\mu)$, and S_σ the composition operator. Then, $S_\sigma^*(\varphi) = \mathbb{1}$.*

(2) *The function φ is Markovian with respect to σ and μ if and only if $(\varphi d\mu) \circ \sigma^{-1} = \mu$.*

Proof The first statement follows from the definition of a Markovian function:

$$\int_V f \, d\mu = \int_X (f \circ \sigma)\varphi \, d\mu = \int_X S_\sigma(f)\varphi \, d\mu = \int_V f S_\sigma^*(\varphi) \, d\mu.$$

The second statement is a reformulation of relation (2.7). □

Lemma 4.2 *The operator S_φ is an isometry in $L^2(\mu)$.*

Proof It follows from (2.7) that the function $\sqrt{\varphi}(f \circ \sigma) \in L^2(\mu)$ if $f \in L^2(\mu)$. Since φ is Markovian, we have

$$\begin{aligned} \langle S_\varphi(f), S_\varphi(g) \rangle_\mu &= \int_X \sqrt{\varphi}(f \circ \sigma)\sqrt{\varphi}(\bar{g} \circ \sigma) \, d\mu \\ &= \int_X (f\bar{g} \circ \sigma)\varphi \, d\mu \\ &= \int_X f\bar{g} \, d\mu, \quad f, g \in L^2(\mu). \end{aligned}$$

□

Theorem 4.3 (1) *For $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ as above and the operator S_φ , the adjoint operator S_σ^* acts by the formula:*

$$S_\varphi^*(g) = \frac{(g\sqrt{\varphi}d\mu) \circ \sigma^{-1}}{d\mu}, \quad g \in L^2(\mu). \tag{4.2}$$

(2) *The adjoint operator S_φ^* is a transfer operator satisfying the pull-out property:*

$$S_\varphi^*(g(f \circ \sigma)) = f S_\varphi^*(g). \tag{4.3}$$

The operator S_φ^ is normalized if and only if μ is σ -invariant.*

Proof (1) For functions $f, g \in L^2(\mu)$, we have

$$\int_X \sqrt{\varphi}(f \circ \sigma)g \, d\mu = \int f (\sqrt{\varphi}g d\mu) \circ \sigma^{-1} = \int f \frac{(g d\mu) \circ \sigma^{-1}}{d\mu} \, d\mu$$

which proves (4.2).

(2) The operator S_φ^* is obviously positive. To prove the pull-out property, we compute, for arbitrary functions $f, g, h \in L^2(\mu)$,

$$\begin{aligned} \int_X h S_\varphi^*((f \circ \sigma)g) \, d\mu &= \int_X S_\varphi(h)(f \circ \sigma)g \, d\mu \\ &= \int_X \sqrt{\varphi}(hf) \circ \sigma \, g \, d\mu \\ &= \int_X S_\varphi(hf)g \, d\mu \\ &= \int_X hf S_\varphi^*(g) \, d\mu. \end{aligned}$$

This proves that (4.3) holds.

We see that

$$S_\varphi^*(\mathbb{1}) = \frac{(\sqrt{\varphi}d\mu) \circ \sigma^{-1}}{d\mu}.$$

Hence, S_φ^* is normalized if and only if $\mu = \mu \circ \sigma^{-1}$ and $\varphi = \mathbb{1}$. This can be proved as follows:

$$\int_X f \, d\mu \circ \sigma^{-1} = \int_X (f \circ \sigma)\mathbb{1} \, d\mu = \int_X f S_\sigma^*(\mathbb{1}) \, d\mu,$$

$$\text{and } S_\sigma^*(\mathbb{1}) = \mathbb{1} \iff \mu \circ \sigma^{-1} = \mu. \quad \square$$

Remark 4.4 One can easily check that, for $\varphi = \omega_\mu$,

$$S_{\omega_\mu}^*\left(\frac{1}{\sqrt{\omega_\mu}}\right) = \rho_\mu.$$

We recall that \mathcal{A} denotes the subalgebra $\sigma^{-1}(\mathcal{B})$ and $\mu_{\mathcal{A}}$ denotes the restriction of μ onto \mathcal{A} . It is an important observation that a function f is \mathcal{A} -measurable if and only if there exists a \mathcal{B} -measurable function g such that $f = g \circ \sigma$.

For a fixed Markovian function φ , consider the subspace \mathcal{H}_φ of function spanned by $\sqrt{\varphi}$ and \mathcal{A} -measurable functions:

$$\mathcal{H}_\varphi = \{\sqrt{\varphi}(f \circ \sigma) : f \in L^2(\mu)\}$$

Proposition 4.5 *Let σ, φ , and S_φ be as above. Then,*

- (1) $\mathbb{E}_\varphi := S_\varphi S_\varphi^*$ is an orthogonal projection from $L^2(\mu)$ onto \mathcal{H}_φ ;
- (2) $\mathbb{E}_\varphi((f \circ \sigma)g) = (f \circ \sigma)\mathbb{E}_\varphi(g)$
- (3) $S_\varphi^*(f\mathbb{E}_\varphi(g)) = S_\varphi^*(g)S_\varphi^*(f\sqrt{\varphi})$.

Proof (1) Let $f, g \in L^2(\mu)$. Then, we write

$$\begin{aligned} \langle S_\varphi S_\varphi^* f, \sqrt{\varphi}(g \circ \sigma) \rangle_\mu &= \langle S_\varphi S_\varphi^* f, S_\varphi g \rangle_\mu \\ &= \langle S_\varphi^* f, g \rangle_\mu \\ &= \langle f, S_\varphi g \rangle_\mu \\ &= \langle f, \sqrt{\varphi}(g \circ \sigma) \rangle_\mu. \end{aligned}$$

Hence, $(f - S_\varphi S_\varphi^* f) \perp \mathcal{H}_\varphi$.

For (2), we first note that $S_\varphi(fg) = \sqrt{\varphi}(g \circ \sigma)(f \circ \sigma) = S_\varphi(g)S_\sigma(f)$, and then

$$\begin{aligned} \mathbb{E}_\varphi((f \circ \sigma)g) &= S_\varphi S_\varphi^*((f \circ \sigma)g) \\ &= S_\varphi(S_\varphi^*(g)f) \\ &= (S_\varphi S_\varphi^*)(g)S_\sigma(f) \\ &= (f \circ \sigma)\mathbb{E}_\varphi(g). \end{aligned}$$

For (3), we use the pull-out property of S_φ^* :

$$\begin{aligned} S_\varphi^*(f\mathbb{E}_\varphi(g)) &= S_\varphi^*(fS_\varphi S_\varphi^*(g)) \\ &= S_\varphi^*(f\sqrt{\varphi}(S_\varphi^*(g)\circ\sigma)) \\ &= S_\varphi^*(g)S_\varphi^*(f\sqrt{\varphi}). \end{aligned}$$

□

Let $\{m_i : i \in \Lambda\}$ be a collection of complex-valued functions from $L^2(\mu)$. We fix a Markovian function φ . For every $i \in \Lambda$, we define

$$T_{m_i}(f) = m_i\sqrt{\varphi}(f\circ\sigma) = M_{m_i}S_\varphi(f) \tag{4.4}$$

where M_{m_i} is the multiplication operator. Then, T_{m_i} is an operator acting from $L^2(\mu)$ onto $m_i\mathcal{H}_\varphi$.

Lemma 4.6 *The operator $T_m(f) = m\sqrt{\varphi}(f\circ\sigma)$ is bounded on $L^2(\mu)$ if and only if $|m|^2 \in L^\infty(\mu)$.*

Proof For $f \in L^2(\mu)$, we have

$$\begin{aligned} \|T_m(f)\|^2 &= \int_X (m\sqrt{\varphi}(f\circ\sigma))(\overline{m}\sqrt{\varphi}(\overline{f}\circ\sigma)) \, d\mu \\ &= \int_X |m|^2(|f|^2\circ\sigma)\varphi \, d\mu \\ &\leq \sup |m|^2 \int_X (|f|^2\circ\sigma)\varphi \, d\mu \\ &= \sup |m|^2\|f\|^2. \end{aligned}$$

□

Lemma 4.7 *The operator T_m is an isometry in $L^2(\mu)$ if and only if*

$$S_\varphi^*(\sqrt{\varphi}|m|^2) = \mathbb{1}.$$

Proof We note that $T_m^*(f) = S_\varphi^*M_{\overline{m}}(f)$. Using the pull-out property for S_φ^* , we can write

$$T_m^*T_m(f) = S_\varphi^*M_{\overline{m}}(m_i\sqrt{\varphi}(f\circ\sigma)) = fS_\varphi^*(|m|^2\sqrt{\varphi})$$

where $f \in L^2(\mu)$. This proves the lemma. □

Remark 4.8 It follows from Lemma 4.7 that T_m is an isometry if and only if

$$\mathbb{E}_\varphi(\sqrt{\varphi}|m|^2) = \sqrt{\varphi}.$$

Lemma 4.9 *The operators T_m, S_φ , and \mathbb{E}_φ satisfy the properties:*

$$T_{m_1}^*T_{m_2}(f) = fS_\varphi^*(\sqrt{\varphi}\overline{m_1}m_2), \tag{4.5}$$

$$T_{m_1}T_{m_2}^*(f) = m_1\mathbb{E}_\varphi(\overline{m_2}f). \tag{4.6}$$

Proof Indeed, we have

$$\begin{aligned} T_{m_1}^*T_{m_2}(f) &= T_{m_1}^*(\sqrt{\varphi}m_2(f\circ\sigma)) \\ &= S_{m_2}^*M_{\overline{m_1}}(\sqrt{\varphi}m_2(f\circ\sigma)) \\ &= fS_{m_2}^*(\sqrt{\varphi}\overline{m_1}m_2), \end{aligned}$$

and

$$T_{m_1}T_{m_2}^*(f) = M_{m_1}S_\varphi S_\varphi^*M_{\overline{m_2}}(f) = m_1\mathbb{E}_\varphi(\overline{m_2}f).$$

□

Remark 4.10 Let g be a bounded positive Borel function. Take a Markovian function $\varphi \in M(\sigma, \mu)$ and consider $\psi = (g \circ \sigma)\varphi g^{-1}$. By Proposition 2.17 we see that $\psi \in M(\sigma, \nu)$ where $d\nu = g d\mu$. Then,

$$S_\psi = M_{\sqrt{g}}^{-1} S_\varphi M_{\sqrt{g}}.$$

Then, a direct computation shows that S_ψ is an isometry in $L^2(\nu)$ where $d\nu = g d\mu$.

Denote by \tilde{T}_m the operator acting on $L^2(\nu)$: $f \mapsto m S_\psi(f)$. By Lemma 4.7, \tilde{T}_m is an isometry in $L^2(\nu)$ if and only if $S_\psi^*(\sqrt{\psi}|m|^2) = \mathbb{1}$. It follows from the definition of the operators T_m and \tilde{T}_m that

$$\tilde{T}_m = M_{\sqrt{g}}^{-1} T_m M_{\sqrt{g}}.$$

Theorem 4.11 *Let σ be an onto endomorphism of (X, \mathcal{B}, μ) where μ is quasi-invariant with respect to σ . Let $\{m_i : i \in \Lambda\}$ be a family of complex-valued functions. The operators $\{T_{m_i}, i \in \Lambda\}$ generate a representation of the Cuntz algebra $\mathcal{O}_{|\Lambda|}$ if and only if*

- (i) $S_\varphi^*(\sqrt{\varphi} \bar{m}_j m_i) = \delta_{ij} \mathbb{1}$,
- (ii) $\sum_{i \in \Lambda} m_i \mathbb{E}_\varphi(\bar{m}_i f) = f, \quad f \in L^2(\mu)$.

Proof We first note that condition (i) of the theorem implies that the operators T_{m_i} are isometries because of Lemma 4.7. The Cuntz relations for isometries $\{T_{m_i}\}_{i \in \Lambda}$ mean that

$$\sum_{i \in \Lambda} T_{m_i} T_{m_i}^* = \mathbb{1}$$

and the projections $T_{m_i} T_{m_i}^*$ are mutually orthogonal.

We will show that, for $f, g \in L^2(\mu)$ and $i \neq j$, the vectors $T_{m_i} T_{m_i}^*(f)$ and $T_{m_j} T_{m_j}^*(g)$ are orthogonal if and only if condition (i) of the theorem holds. In the following computation, we use (4.5).

$$\begin{aligned} \langle T_{m_i} T_{m_i}^*(f), T_{m_j} T_{m_j}^*(g) \rangle_\mu &= \langle (T_{m_j}^* T_{m_i}) T_{m_i}^*(f), T_{m_j}^*(g) \rangle_\mu \\ &= \langle S_\varphi^*(\sqrt{\varphi} \bar{m}_j m_i)(\xi), \eta \rangle_\mu \end{aligned}$$

where $\xi = T_{m_i}^*(f)$ and $\eta = T_{m_j}^*(g)$. Since f, g are arbitrary functions, the left-hand side is zero if and only if $S_\varphi^*(\sqrt{\varphi} \bar{m}_j m_i) = \delta_{ij} \mathbb{1}$.

Next, we use (4.6) to see when the identity operator is decomposed in the sum of orthogonal projections $T_{m_i} T_{m_i}^*$. It follows from Lemma 4.9 that, for any $f \in L^2(\mu)$,

$$\sum_{i \in \Lambda} T_{m_i} T_{m_i}^*(f) = \sum_{i \in \Lambda} m_i \mathbb{E}_\varphi(\bar{m}_i f)$$

Hence, the property $\sum_{i \in \Lambda} T_{m_i} T_{m_i}^* = \mathbb{1}$ is equivalent to condition (ii) of the theorem. □

Corollary 4.12 *It follows from Theorem 4.11 that*

$$L^2(\mu) = \bigoplus_{i \in \Lambda} m_i \mathcal{H}_\varphi.$$

In other words, $\{m_i : i \in \Lambda\}$ is an orthogonal module basis for $L^2(\mu)$ over $\mathcal{H}_\varphi = \sqrt{\varphi} L^2(\mu_{\mathcal{A}})$ where $\mathcal{A} = \sigma^{-1}(\mathcal{B})$.

4.2 Invariant measure

In what follows, we will consider the case of a σ -invariant measure μ .

Lemma 4.13 *Let $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ be an onto endomorphism. Then, if μ is σ -invariant probability measure, then the only $\sigma^{-1}(\mathcal{B})$ -measurable Markovian function is the constant function $\mathbb{1}$.*

Proof Indeed, let φ be a Markovian function. Then, we have

$$\int_X (f \circ \sigma)\varphi \, d\mu = \int_X f \, d\mu = \int_X f \, d(\mu \circ \sigma)^{-1} = \int_X (f \circ \sigma) \, d\mu.$$

Since f is arbitrary, we get $\varphi = \mathbb{1}$ on $\sigma^{-1}(\mathcal{B})$. □

We will apply the constructions of Sect. 4.1 to the case when $\varphi = \mathbb{1}$. Because the results of Sect. 4.1 are proved in more general settings, we will just formulate the corresponding facts without proof.

Let $\underline{m} = \{m_i : i \in \Lambda\}$ be a collection of complex-valued bounded functions, and let

$$S_{m_i}(f) = m_i(f \circ \sigma), \quad f \in L^2(\mu)$$

be the corresponding weighted composition operators.

In this lemma, we collect the properties of S_{m_i} .

Lemma 4.14 *Let $f, g \in L^2(\mu)$.*

- (1) S_m^* is an isometry on $L^2(\mu)$ if and only if $S_\sigma^*(|m|^2) = \mathbb{1}$.
- (2) $\mathbb{E}_\sigma := S_\sigma S_\sigma^*$ is the conditional expectation from $L^2(\mu)$ onto $L^2(\mu_{\mathcal{A}})$.
- (3) $E_\sigma((f \circ \sigma)g) = (f \circ \sigma)\mathbb{E}_\sigma(g)$.
- (4) $S_\sigma^*(f\mathbb{E}_\sigma(g)) = S_\sigma^*(f)S_\sigma^*(g)$.
- (5) $S_{m_1}^*S_{m_2}(f) = S_\sigma^*(\overline{m_1}m_2)f$.
- (6) $S_{m_1}S_{m_2}^*(f) = m_1\mathbb{E}(\overline{m_2}f)$.

Here is the modified version of Theorem 4.11.

Theorem 4.15 *Let σ be a surjective endomorphism of (X, \mathcal{B}, μ) with σ -invariant measure μ . Let $\{m_i : i \in \Lambda\}$ be a family of bounded complex-valued functions. The operators $\{S_{m_i}, i \in \Lambda\}$ generate a representation of the Cuntz algebra $\mathcal{O}_{|\Lambda|}$ if and only if*

- (i) $S_\sigma^*(\overline{m_j}m_i) = \delta_{ij}\mathbb{1}$,
- (ii) $\sum_{i \in \Lambda} m_i\mathbb{E}_\sigma(\overline{m_i}f) = f, \quad f \in L^2(\mu)$.

It follows from Theorem 4.15 that the Hilbert space $L^2(\mu)$ admits the decomposition

$$L^2(\mu) = \bigoplus_{i \in \Lambda} m_i L^2(\mu_{\mathcal{A}})$$

if and only if the operators S_{m_i} are the generators of a representation of the Cuntz algebra. In other words, $\{m_i : i \in \Lambda\}$ is an orthogonal module basis for $L^2(\mu)$ over $L^2(\mu_{\mathcal{A}})$ where $\mathcal{A} = \sigma^{-1}(\mathcal{B})$.

We finish this section with a discussion of the relations between the operators S_σ, S_φ and $\mathbb{E}_\sigma, \mathbb{E}_\varphi$ where φ is a Markovian function.

Proposition 4.16 *The following formulas hold: for $g \in L^2(\mu)$*

- (1) $S_\varphi^*(g) = S_\sigma^*(\sqrt{\varphi}g),$ (2) $\mathbb{E}_\varphi(g) = \sqrt{\varphi}\mathbb{E}_\sigma(\sqrt{\varphi}g).$

Proof For (1), take any functions $f, g \in L^2(\mu)$ and compute

$$\begin{aligned} \langle f, S_\varphi^*(g) \rangle_\mu &= \langle S_\varphi^*(f), g \rangle_\mu \\ &= \langle \sqrt{\varphi}(f \circ \sigma), g \rangle_\mu \\ &= \langle S_\sigma(f), \sqrt{\varphi}g \rangle_\mu \\ &= \langle f, S_\sigma^*(\sqrt{\varphi}g) \rangle_\mu. \end{aligned}$$

To see that (2) is true, we recall that $S_\varphi(f) = \sqrt{\varphi} S_\sigma(f)$, and then we can write $\mathbb{E}_\varphi(f) = S_\varphi S_\varphi^*(f) = S_\varphi S_\sigma^*(\sqrt{\varphi} f) = \sqrt{\varphi} S_\sigma S_\sigma^*(\sqrt{\varphi} f) = \sqrt{\varphi} \mathbb{E}_\sigma(\sqrt{\varphi} f)$.

□

The results of Proposition 4.16 will be used in the next section.

5 The set of wavelet filters

In this section, we answer the question about the structure of the set of *wavelet filters*, i.e., we describe the set of functions satisfying the conditions of Theorem 4.11. We consider two different approaches: (i) it will be shown that there the set \mathfrak{M} is isomorphic to the so called loop group \mathcal{G} ; (ii) using the decomposition into cyclic representations, we describe elements $\underline{m} \in \mathfrak{M}$ as the collection of cyclic vectors for the representation of $L^\infty(\mathcal{A})$ in $L^2(X, \mathcal{B}, \mu)$.

5.1 Actions of loop groups on wavelet filters

Consider bounded operators $B(l^2(\Lambda))$ in the Hilbert space $l^2(\Lambda)$ where Λ is a countably infinite set. If $\{e_i : i \in \Lambda\}$ is the canonical orthonormal basis in $l^2(\Lambda)$, then we define the infinite $|\Lambda| \times |\Lambda|$ matrix $\widehat{A} = (a_{ij})$ by setting $a_{ij} = \langle e_i, A e_j \rangle$. This observation will allow us to work with matrix notation in the computations below.

Let \mathcal{U} be the group of all unitary operators in $B(l^2(\Lambda))$. Denote by \mathcal{G} the group of Borel functions on (X, \mathcal{B}) with values in \mathcal{U} (we recall that \mathcal{U} is a Polish group). We use the notation $G = (g_{ij}(x))$ for elements of \mathcal{G} . Then, every entry $g_{ij}(x)$ is a Borel complex-valued matrix. The group \mathcal{G} is called the *loop group*. We remark that for $G \in \mathcal{G}$, $G^*G = \mathbb{I}$ where the matrix $G^* = \overline{G}^T$. We will use the relation

$$\sum_{l \in \Lambda} \overline{g_{li}} g_{lj} = \delta_{ij} \tag{5.1}$$

where $G = (g_{ij}) \in \mathcal{G}$.

Let φ be a Markovian function. We recall that, in this case, the operator S_φ is isometric. We denote the set of generalized wavelet filters by

$$\mathfrak{M}_\varphi := \{ \underline{m} = (m_i)_{i \in \Lambda} : S_\varphi^*(\sqrt{\varphi} m_i \overline{m}_j) = \delta_{ij} \mathbb{1}, \sum_{i \in \Lambda} m_i \mathbb{E}_\varphi(\overline{m}_j f) = f \}. \tag{5.2}$$

We consider simultaneously the case of σ -invariant measure μ and the corresponding operators S_σ and \mathbb{E}_σ , see Sect. 4 for properties of these operators. In this case, we use the set

$$\mathfrak{M}_\sigma := \{ \underline{m} = (m_i)_{i \in \Lambda} : S_\sigma^*(m_i \overline{m}_j) = \delta_{ij} \mathbb{1}, \sum_{i \in \Lambda} m_i \mathbb{E}_\sigma(\overline{m}_j f) = f \}. \tag{5.3}$$

It turns out that the group \mathcal{G} acts on the sets \mathfrak{M}_φ and \mathfrak{M}_σ . Indeed, for a fixed $\underline{m} \in \mathfrak{M}_\varphi$ (or $\underline{m} \in \mathfrak{M}_\sigma$) and $G \in \mathcal{G}$, we define $\underline{m}^G := (m_i^G(x) : i \in \Lambda)$ by the formula

$$m_i^G(x) = \sum_{j \in \Lambda} (\overline{g_{ji}} \circ \sigma)(x) m_j(x) \tag{5.4}$$

or $\underline{m}^G = (G^* \circ \sigma) \underline{m}$ in a short form.

In the next statements, we will study the properties of this action of \mathcal{G} on the sets of wavelet filters. For definiteness, we formulate these results for the set \mathfrak{M}_φ . The same proofs work for the action of \mathcal{G} on \mathfrak{M}_σ , we will omit them. We will show in the next lemmas that: (i) the set \mathfrak{M}_φ is invariant with respect to the action of group \mathcal{G} ; (ii) formula (5.4) defines a group action on \mathfrak{M}_φ ; (iii) this action of \mathcal{G} is free and transitive.

Lemma 5.1 *If $\underline{m} \in \mathfrak{M}_\varphi$, then $\underline{m}^G \in \mathfrak{M}_\varphi$.*

Proof We will verify that the family of functions $\underline{m}^G = (m_i^G(x) : i \in \Lambda)$ satisfies conditions (i) and (ii) of Theorem 4.11. For (i), we use (5.1) and the fact that S_φ^* is a transfer operator satisfying the pull-out property:

$$\begin{aligned} S_\varphi^*(\sqrt{\varphi} m_i^G \overline{m}_j^G) &= S_\varphi^* \left(\sqrt{\varphi} \sum_{k \in \Lambda} (\overline{g}_{ki} \circ \sigma) m_k \cdot \sum_{l \in \Lambda} \overline{(g_{jl} \circ \sigma) m_l} \right) \\ &= \sum_{k \in \Lambda} \sum_{l \in \Lambda} \overline{g}_{ki} g_{lj} S_\varphi^*(\sqrt{\varphi} \overline{m}_l m_k) \\ &= \sum_{k \in \Lambda} \sum_{l \in \Lambda} \overline{g}_{ki} g_{lj} \delta_{kl} \\ &= \sum_{k \in \Lambda} \overline{g}_{ki} g_{kj} \\ &= \delta_{ij}. \end{aligned}$$

For (ii), let $f \in L^2(\mu)$, then

$$\begin{aligned} \sum_{i \in \Lambda} m_i^G \mathbb{E}_\varphi(\overline{m}_i^G f) &= \sum_{i \in \Lambda} \sum_{k \in \Lambda} (\overline{g}_{ki} \circ \sigma) m_k \mathbb{E}_\varphi \left(\sum_{l \in \Lambda} (g_{li} \circ \sigma) \overline{m}_l f \right) \\ &= \sum_{i \in \Lambda} \sum_{k \in \Lambda} \sum_{l \in \Lambda} (\overline{g}_{ki} \circ \sigma) (g_{li} \circ \sigma) m_k \mathbb{E}_\varphi(\overline{m}_l f) \\ &= \sum_{i \in \Lambda} \sum_{k \in \Lambda} \sum_{l \in \Lambda} ((\overline{g}_{ki} g_{li}) \circ \sigma) m_k \mathbb{E}_\varphi(\overline{m}_l f) \\ &= \sum_{i \in \Lambda} \sum_{k \in \Lambda} \delta_{kl} m_k \mathbb{E}_\varphi(\overline{m}_l f) \\ &= \sum_{k \in \Lambda} m_k \mathbb{E}_\varphi(\overline{m}_k f) \\ &= f. \end{aligned}$$

We used here the equality $GG^* = \mathbb{I}$ or $\sum_i \overline{g}_{ki} g_{li} = \delta_{kl}$. □

In the next lemma, we show that (5.4) defines an action of the group \mathcal{G} on \mathfrak{M} .

Lemma 5.2 For every $\underline{m} \in \mathfrak{M}_\varphi$ and every $G, H \in \mathcal{G}$,

$$(\underline{m}^G)^H = \underline{m}^{GH}, \quad \underline{m}^{\mathbb{I}} = \underline{m}. \tag{5.5}$$

Proof Indeed, if \mathbb{I} is the identity matrix, then $\underline{m}^{\mathbb{I}} = \underline{m}$ for every $\underline{m} \in \mathfrak{M}_\varphi$.

Let $G, H \in \mathfrak{M}_\varphi$. Show that (5.5) holds:

$$\begin{aligned} (m_i^G)^H &= \sum_{j \in \Lambda} (\overline{h}_{ji} \circ \sigma) m_j^G \\ &= \sum_{j \in \Lambda} (\overline{h}_{ji} \circ \sigma) \sum_{k \in \Lambda} (\overline{g}_{kj} \circ \sigma) m_k \\ &= \sum_{k \in \Lambda} \sum_{j \in \Lambda} ((\overline{g}_{ij} \overline{h}_{jk}) \circ \sigma) m_k \\ &= \sum_{k \in \Lambda} (\overline{f}_{ik} \circ \sigma) m_k \\ &= m_i^{GH}, \end{aligned}$$

where the entries of GH are denoted by (f_{ik}) . □

Lemma 5.3 *Let φ be a positive Markovian function.*

- (1) *The map $\Phi : \underline{m} \mapsto \sqrt{\varphi} \underline{m}$ defines an isomorphism between the sets \mathfrak{M}_φ and \mathfrak{M}_σ .*
 (2) *The map Φ implements the conjugation of actions of \mathcal{G} on the sets \mathfrak{M}_σ and \mathfrak{M}_φ .*

Proof (1) We will show that if \underline{m} satisfies (5.2), then $\sqrt{\varphi} \underline{m}$ belongs to \mathfrak{M}_σ , i.e., (5.3) holds. For this, we check that

$$S_\sigma^*(\sqrt{\varphi} m_i \sqrt{\varphi} \bar{m}_j) = S_\varphi^*(\sqrt{\varphi} m_i \bar{m}_j) = \delta_{ij}, \quad i, j \in \Lambda,$$

and, using Proposition 4.16,

$$\begin{aligned} \sum_{i \in \Lambda} \sqrt{\varphi} m_i \mathbb{E}_\sigma(\sqrt{\varphi} \bar{m}_i f) &= \sum_{i \in \Lambda} \sqrt{\varphi} m_i S_\sigma^*(\sqrt{\varphi} \bar{m}_i f) \\ &= \sum_{i \in \Lambda} m_i S_\varphi S_\varphi^*(\bar{m}_i f) \\ &= f. \end{aligned}$$

- (2) It can be checked directly that, for every $G \in \mathcal{G}$ and $\underline{m} \in \mathfrak{M}_\varphi$,

$$\Phi \underline{m}^G = (\Phi \underline{m})^G.$$

□

Lemma 5.4 *Let φ be a positive Markovian function. The action of \mathcal{G} on \mathfrak{M}_φ defined in (5.4) is free and transitive.*

Proof It follows from Lemma 5.3 that it suffices to show that the action of \mathcal{G} on \mathfrak{M}_σ is free and transitive.

Let $\underline{m} = (m_i)$ and $\underline{n} = (n_j)$ be two elements of the set \mathfrak{M}_σ . Define an infinite matrix G by setting

$$g_{ij} = S_\varphi^*(m_i \bar{n}_j), \quad i, j \in \Lambda.$$

We show that $\underline{m}^G = \underline{n}$ to prove that the action is transitive. Indeed,

$$\begin{aligned} m_i^G &= \sum_{k \in \Lambda} (\bar{g}_{k,i} \circ \sigma) m_k \\ &= \sum_{k \in \Lambda} (S_\sigma^*(\bar{m}_k n_i) \circ \sigma) m_k \\ &= \sum_{k \in \Lambda} S_\sigma S_\sigma^*(\bar{m}_k n_i) m_k \\ &= \sum_{k \in \Lambda} \mathbb{E}_\sigma(\bar{m}_k n_i) m_k \\ &= n_i. \end{aligned}$$

We used here relation (5.3).

To see that this action is free, we assume that $\underline{m}^G = \underline{m}$ for some $\underline{m} \in \mathfrak{M}_\sigma$ and $G \in \mathcal{G}$. Then,

$$\sum_{k \in \Lambda} (\bar{g}_{ki} \circ \sigma) m_k = m_i, \quad i \in \Lambda.$$

Multiply both sides by \bar{m}_j and apply the operator S_σ^* :

$$\sum_{k \in \Lambda} S_\sigma^*((\bar{g}_{ki} \circ \sigma) m_k \bar{m}_j) = S_\sigma^*(m_i \bar{m}_j)$$

Use the pull-out property and (5.3):

$$\sum_{k \in \Lambda} \bar{g}_{ki} S_\sigma^*(m_k \bar{m}_j) = \delta_{ij}.$$

Hence,

$$\sum_{k \in \Lambda} \bar{g}_{ki} \delta_{kj} = \delta_{ij}$$

and $\bar{g}_{li} = \delta_{ij}$. This proves that $G = \mathbb{I}$ and the action is free. □

The following theorem gives a complete description of the set \mathfrak{M}_φ . This result immediately follows from the proven Lemmas 5.1–5.4.

Theorem 5.5 (1) *The set \mathfrak{M}_φ is isomorphic (as a set) to the loop group \mathcal{G} .*

(2) *For every element $G = (g_{ij})$ of the loop group, there exists a wavelet filter \underline{m} such that $g_{ij} = S_\varphi^*(\sqrt{\varphi} m_i \bar{m}_j)$.*

5.2 Endomorphisms, wavelet filters and cyclic representations

In this subsection, we will use another approach to describe the set \mathfrak{M} of wavelet filters associated with an endomorphism σ of (X, \mathcal{B}, μ) . We will assume here that σ is measure-preserving.

Let $\mathcal{A} = \sigma^{-1}(\mathcal{B})$ and $\mu_{\mathcal{A}}$ is the restriction of μ onto the sigma-subalgebra \mathcal{A} . Denote by \mathfrak{A} the set $L^\infty(\mu_{\mathcal{A}})$ of bounded $\sigma^{-1}(\mathcal{B})$ -measurable functions. The $*$ -algebra \mathfrak{A} acts on $L^2(\mu)$ by multiplication operators: for every $f \in L^2(\mu)$ and $\gamma \in \mathfrak{A}$

$$f \mapsto \gamma f = M_\gamma(f).$$

This formula defines a representation π of \mathfrak{A} in $L^2(\mu)$. Using the theorem about the decomposition of non-degenerate representations into orthogonal cyclic representations, we can write

$$L^2(\mu) = \bigoplus_{i \in \Lambda} \mathcal{H}_i, \quad \pi = \bigoplus_{i \in \Lambda} \pi_i \tag{5.6}$$

where π_i is a cyclic representation and \mathcal{H}_i is the closure of $\{\pi_i(\gamma)h_i : \gamma \in \mathfrak{A}\}$. The vectors $h_i, i \in \Lambda$, are called cyclic. The set of indexes Λ is countably infinite.

We will use the cyclic vectors (h_i) to construct a wavelet filter $\underline{m} = (m_i)$ satisfying the conditions of Theorem 4.15.

Lemma 5.6 *There are functions (g_i) such that the cyclic vectors $(m_i = h_i(g_i \circ \sigma))$ for representations of \mathfrak{A} have the property*

$$S_\sigma^*(|m_i|^2) = \mathbb{1}. \tag{5.7}$$

Proof We note that for every function $\gamma \in \mathfrak{A}$ there exists a function $g \in L^\infty(\mu)$ such that $\gamma = g \circ \sigma$. Hence, if h_i is a cyclic vector for a representation of \mathfrak{A} , then m_i is cyclic, too.

We will determine the function g_i such that $m_i = h_i(g_i \circ \sigma)$ satisfies the statement of the lemma. For this, we compute

$$S_\sigma^*(|m_i|^2) = S_\sigma^*(|h_i(g_i \circ \sigma)|^2) = |g_i|^2 S_\sigma^*(|h_i|^2),$$

and take g_i such that the above expression equals 1 a.e. □

For functions $f \in L^2(\mu)$, we define the operators

$$S_i(f) = m_i(f \circ \sigma) = M_{m_i} S_\sigma(f). \tag{5.8}$$

Lemma 5.7 *Let the functions (m_i) satisfy (5.7). Then, the operators $S_i, i \in \Lambda$, are isometries satisfying the property*

$$S_\sigma^*(m_i \bar{m}_j) = \delta_{ij}, \quad i, j \in \Lambda. \tag{5.9}$$

Proof We first note that $S_i^* = S_\sigma^* M_{\bar{m}_i}$, and

$$S_i^* S_i = \mathbb{I} \iff S_\sigma^* (|m_i|^2) = \mathbb{I}.$$

Next, we use the fact that cyclic representations, which are defined by the distinct vectors m_i and m_j , are mutually orthogonal. Applying the fact that the transfer operator S_σ^* has the pull-out property, we compute, for any $\gamma_1, \gamma_2 \in \mathfrak{A}$,

$$\begin{aligned} \langle \gamma_1 m_i, \gamma_2 m_j \rangle_\mu &= \langle (g_1 \circ \sigma) m_i, (g_2 \circ \sigma) m_j \rangle_\mu \\ &= \langle (g_1 \circ \sigma) m_i \bar{m}_j, S_\sigma(g_2) \rangle_\mu \\ &= \langle S_\sigma^*((g_1 \circ \sigma) m_i \bar{m}_j), g_2 \rangle_\mu \\ &= \langle g_1 S_\sigma^*(m_i \bar{m}_j), g_2 \rangle_\mu. \end{aligned}$$

Hence, the inner product is zero if and only if (5.9) holds. □

Theorem 5.8 *Let $(m_i : i \in \Lambda)$ be a set of cyclic vectors satisfying (5.7). Then, $\underline{m} = (m_i)$ is a wavelet filter if and only if $\sum_{i \in \Lambda} S_i^* S_i = \mathbb{I}$. In other words, $\underline{m} \in \mathfrak{M}$ if and only if the operators S_i are the generators of a representation of the Cuntz algebra $\mathcal{O}_{|\Lambda|}$.*

Proof To prove the theorem, we use the result of Theorem 4.15. For this, we should check that $\{S_i : i \in \Lambda\}$ are the operators satisfying conditions (i) and (ii) of Theorem 4.15. Condition (i) is proved in Lemma 5.7. To verify (ii), we observe that, for any $f \in L^2(\mu)$ and $i \in \Lambda$,

$$\mathbb{E}_\sigma(f \bar{m}_i) m_i = S_i S_i^*(f).$$

Indeed,

$$\begin{aligned} \mathbb{E}_\sigma(f \bar{m}_i) m_i &= S_i S_i^*(f) = S_\sigma S_\sigma^*(f \bar{m}_i) m_i = S_i S_i^*(f) \\ &= S_i (S_\sigma^* M_{\bar{m}_i}(f)) \\ &= S_i S_i^*(f). \end{aligned}$$

It follows that $\sum_{i \in \Lambda} S_i^* S_i = \mathbb{I}$ if and only if condition (ii) holds. This proves the theorem. □

Corollary 5.9 *Let $\sigma \in \text{End}(X, \mathcal{B}, \mu)$ where μ is a probability measure. Let $(m_i : i \in \Lambda)$ be a set of cyclic vectors satisfying (5.6) and (5.7). Then, for a.e. $x \in X$,*

$$\sum_{i \in \Lambda} |m_i(x)|^2 < \infty.$$

Proof The result follows from the fact that

$$\sum_{i \in \Lambda} \|m_i\|_\mu^2 = 1.$$

Indeed, this relation is obtained when the function $\mathbb{1}(x)$ is substituted into the cyclic decomposition (5.6). □

Data availability This paper does not contain Data.

References

1. Alpay, D., Colombo, F., Sabadini, I., Schneider, B.: Beurling–Lax type theorems and Cuntz relations. *Linear Algebra Appl.* **633**, 152–212 (2022)
2. Alpay, D., Jorgensen, P., Lewkowicz, I.: Characterizations of families of rectangular, finite impulse response, para-unitary systems. *J. Appl. Math. Comput.* **54**(1–2), 395–423 (2017)
3. Alpay, D., Jorgensen, P., Lewkowicz, I.: W -Markov measures, transfer operators, wavelets and multiresolutions. In: *Frames and harmonic analysis*, *Contemp. Math.*, vol. 706, pp. 293–343. Amer. Math. Soc., Providence (2018)

4. Alpay, D., Jorgensen, P., Lewkowicz, I.: Representation theory and multilevel filters. *J. Appl. Math. Comput.* **69**(2), 1599–1657 (2023)
5. Andrianov, P.A.: Multidimensional periodic discrete wavelets. *Int. J. Wavelets Multiresolut. Inf. Process.* **20**(2), 2150053 (2022)
6. Baggett, L.W., Larsen, N.S., Packer, J.A., Raeburn, I., Ramsay, A.: Direct limits, multiresolution analyses, and wavelets. *J. Funct. Anal.* **258**(8), 2714–2738 (2010)
7. Baggett, L.W., Merrill, K.D., Packer, J.A., Ramsay, A.B.: Probability measures on solenoids corresponding to fractal wavelets. *Trans. Am. Math. Soc.* **364**(5), 2723–2748 (2012)
8. Bénéteau, C.: A natural extension of a nonsingular endomorphism of a measure space. *Rocky Mt. J. Math.* **26**(4), 1261–1273 (1996)
9. Bezuglyi, S., Jorgensen, P.E.T.: Representations of Cuntz–Krieger relations, dynamics on Bratteli diagrams, and path-space measures. In: *Trends in Harmonic Analysis and Its Applications*, *Contemp. Math.*, vol. 650, pp. 57–88. Amer. Math. Soc., Providence (2015)
10. Bezuglyi, S., Jorgensen, P.E.T.: Transfer operators, endomorphisms, and measurable partitions, *Lecture Notes in Mathematics*, vol. 2217. Springer, Cham (2018)
11. Bhat, M.Y., Dar, A.H.: Fractional vector-valued nonuniform MRA and associated wavelet packets on $L^2(\mathbb{R}, \mathbb{C}^M)$. *Fract. Calc. Appl. Anal.* **25**(2), 687–719 (2022)
12. Bogachev, V.I.: *Measure Theory*, vols. I. II. Springer, Berlin (2007)
13. Bratteli, O., Jorgensen, P.E.T.: Endomorphisms of $\mathcal{B}(\mathcal{H})$. II. Finitely correlated states on \mathcal{O}_n . *J. Funct. Anal.* **145**(2), 323–373 (1997)
14. Bratteli, O., Jorgensen, P.E.T.: A connection between multiresolution wavelet theory of scale N and representations of the Cuntz algebra \mathcal{O}_N . In: *Operator Algebras and Quantum Field Theory (Rome, 1996)*, pp. 151–163. Int. Press, Cambridge (1997)
15. Bratteli, O., Jorgensen, P.E.T.: Isometries, shifts, Cuntz algebras and multiresolution wavelet analysis of scale N . *Integr. Equ. Oper. Theory* **28**(4), 382–443 (1997)
16. Bratteli, O., Jorgensen, P.E.T.: Iterated function systems and permutation representations of the Cuntz algebra. *Mem. Am. Math. Soc.* **139**(663), x+89 (1999)
17. Bruin, H., Hawkins, J.: Rigidity of smooth one-sided Bernoulli endomorphisms. *N. Y. J. Math.* **15**, 451–483 (2009)
18. Bruin, H.: *Topological and ergodic theory of symbolic dynamics*, Graduate Studies in Mathematics, vol. 228. American Mathematical Society, Providence (2022)
19. Christoffersen, N.J., Dutkay, D.E.: Representations of Cuntz algebras associated to random walks on graphs. *J. Oper. Theory* **88**(1), 139–170 (2022)
20. Cornfeld, I.P., Fomin, S.V., Sinaĭ, Y.G.: *Ergodic theory*, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 245. Springer, New York (1982). Translated from the Russian by A. B. Sosinskii
21. Cuntz, J.: Simple C^* -algebras generated by isometries. *Commun. Math. Phys.* **57**(2), 173–185 (1977)
22. Dajani, K.G., Hawkins, J.M.: Examples of natural extensions of nonsingular endomorphisms. *Proc. Am. Math. Soc.* **120**(4), 1211–1217 (1994)
23. Dougherty, R., Jackson, S., Kechris, A.S.: The structure of hyperfinite Borel equivalence relations. *Trans. Am. Math. Soc.* **341**(1), 193–225 (1994)
24. Dutkay, D.E., Jorgensen, P.E.T.: Hilbert spaces built on a similarity and on dynamical renormalization. *J. Math. Phys.* **47**(5), 053504 (2006)
25. Dutkay, D.E., Jorgensen, P.E.T.: Martingales, endomorphisms, and covariant systems of operators in Hilbert space. *J. Oper. Theory* **58**(2), 269–310 (2007)
26. Dutkay, D.E., Jorgensen, P.E.T.: Monic representations of the Cuntz algebra and Markov measures. *J. Funct. Anal.* **267**(4), 1011–1034 (2014)
27. Dutkay, D.E., Jorgensen, P.E.T.: The role of transfer operators and shifts in the study of fractals: encoding-models, analysis and geometry, commutative and non-commutative. In: *Geometry and Analysis of Fractals*, *Springer Proc. Math. Stat.*, vol. 88, pp. 65–95. Springer, Heidelberg (2014)
28. Dutkay, D.E., Jorgensen, P.E.T.: Representations of Cuntz algebras associated to quasi-stationary Markov measures. *Ergod. Theory Dyn. Syst.* **35**(7), 2080–2093 (2015)
29. Dutkay, D.E., Jorgensen, P.E.T., Silvestrov, S.: Decomposition of wavelet representations and Martin boundaries. *J. Funct. Anal.* **262**(3), 1043–1061 (2012)
30. Eisner, T., Farkas, B., Haase, M., Nagel, R.: *Operator theoretic aspects of ergodic theory*. Graduate Texts in Mathematics, vol. 272. Springer, Cham (2015)
31. Fabec, R.C.: Induced group actions, representations and fibered skew product extensions. *Trans. Am. Math. Soc.* **301**(2), 489–513 (1987)
32. Fabec, R.C.: *Fundamentals of infinite dimensional representation theory*. Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 114. Chapman & Hall/CRC, Boca Raton (2000)
33. Feng, D.-J., Simon, K.: Dimension estimates for C^1 iterated function systems and repellers. Part II. *Ergod. Theory Dyn. Syst.* **42**(11), 3357–3392 (2022)
34. Hawkins, J.M.: Amenable relations for endomorphisms. *Trans. Am. Math. Soc.* **343**(1), 169–191 (1994)
35. Hawkins, J.: *Ergodic dynamics—from basic theory to applications*, Graduate Texts in Mathematics, vol. 289. Springer, Cham (2021)
36. Hawkins, J.M., Silva, C.E.: Noninvertible transformations admitting no absolutely continuous σ -finite invariant measure. *Proc. Am. Math. Soc.* **111**(2), 455–463 (1991)

37. Jorgensen, P.E.T., Kornelson, K., Shuman, K.: Harmonic analysis of iterated function systems with overlap. *J. Math. Phys.* **48**(8), 083511 (2007)
38. Jorgensen, P.E.T., Kornelson, K.A., Shuman, K.L.: Iterated function systems, moments, and transformations of infinite matrices. *Mem. Am. Math. Soc.* **213**(1003), x+105 (2011)
39. Jorgensen, P.E.T.: A duality for endomorphisms of von Neumann algebras. *J. Math. Phys.* **37**(3), 1521–1538 (1996)
40. Jorgensen, P.E.T.: Ruelle operators: functions which are harmonic with respect to a transfer operator. *Mem. Am. Math. Soc.* **152**(720), viii+60 (2001)
41. Jorgensen, P.E.T.: Analysis and probability: wavelets, signals, fractals, Graduate Texts in Mathematics, vol. 234. Springer, New York (2006)
42. Jorgensen, P.E.T.: Harmonic analysis, CBMS Regional Conference Series in Mathematics, vol. 128. American Mathematical Society, Providence (2018). Smooth and non-smooth, Published for the Conference Board of the Mathematical Sciences
43. Jorgensen, P.E.T., Paolucci, A.M.: States on the Cuntz algebras and p -adic random walks. *J. Aust. Math. Soc.* **90**(2), 197–211 (2011)
44. Jorgensen, P.E.T., Song, M.-S.: Markov chains and generalized wavelet multiresolutions. *J. Anal.* **26**(2), 259–283 (2018)
45. Jorgensen, P., Tian, F.: Transfer operators, induced probability spaces, and random walk models. *Markov Process. Relat. Fields* **23**(2), 187–210 (2017)
46. Jorgensen, P., Tian, F.: Dynamical properties of endomorphisms, multiresolutions, similarity and orthogonality relations. *Discrete Contin. Dyn. Syst. Ser. S* **12**(8), 2307–2348 (2019)
47. Jorgensen, P., Tian, J.: Noncommutative boundaries arising in dynamics and representations of the Cuntz relations. *Numer. Funct. Anal. Optim.* **41**(5), 571–620 (2020)
48. Kechris, A.S.: Classical descriptive set theory. Graduate Texts in Mathematics, vol. 156. Springer, New York (1995)
49. Medhi, R., Viswanathan, P.: On the code space and Hutchinson measure for countable iterated function system consisting of cyclic ϕ -contractions. *Chaos Solitons Fractals* **167**, 113011 (2023)
50. Picklo, M.J., Ryan, J.K.: Enhanced multiresolution analysis for multidimensional data utilizing line filtering techniques. *SIAM J. Sci. Comput.* **44**(4), A2628–A2650 (2022)
51. Przytycki, F., Urbański, M.: Conformal fractals: ergodic theory methods. London Mathematical Society Lecture Note Series, vol. 371. Cambridge University Press, Cambridge (2010)
52. Rohlin, V.A.: Selected topics from the metric theory of dynamical systems. *Uspehi Matem. Nauk (N.S.)* **2**(30), 57–128 (1949)
53. Rohlin, V.A.: On the fundamental ideas of measure theory. *Mat. Sbornik N.S.* **25**(67), 107–150 (1949)
54. Rohlin, V.A.: Exact endomorphisms of a Lebesgue space. *Izv. Akad. Nauk SSSR Ser. Mat.* **25**, 499–530 (1961)
55. Roychowdhury, L., Roychowdhury, M.K.: Quantization for a probability distribution generated by an infinite iterated function system. *Commun. Korean Math. Soc.* **37**(3), 765–800 (2022)
56. Silva, C.E.: On μ -recurrent nonsingular endomorphisms. *Isr. J. Math.* **61**(1), 1–13 (1988)
57. Simmons, D.: Conditional measures and conditional expectation; Rohlin's disintegration theorem. *Discrete Contin. Dyn. Syst.* **32**(7), 2565–2582 (2012)
58. Urbański, M., Roy, M., Munday, S.: Non-invertible dynamical systems. Vol. 1. Ergodic theory—finite and infinite, thermodynamic formalism, symbolic dynamics and distance expanding maps, De Gruyter Expositions in Mathematics, vol. 69. De Gruyter, Berlin (2022)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.