Quantum Stud.: Math. Found. (2022) 9:71–92
https://doi.org/10.1007/s40509-021-00257-7

CHAPMAN UNIVERSITY | INSTITUTE FOR QUANTUM STUDIES

REGULAR PAPER

Check for updates

On the discrete Heisenberg group and commutative modular variables in quantum mechanics: II. Synchronization of unitary actions and homological Abelianization

Elias Zafiris · Albrecht von Müller

Received: 20 April 2021 / Accepted: 21 September 2021 / Published online: 29 September 2021
© The Author(s) under exclusive license to Chapman University 2021

Abstract Since, the quantum modular variables are encoded in terms of one-parameter unitary groups, we are led to a careful re-evaluation of the Heisenberg group, from where the commutation relations emerge from. In particular, the re-evaluation pertains to the discrete Heisenberg group from the perspective of its genuine descent from a more fundamental layer of structure targeting the origin of the non-commutativity of quantum observables. Due to the fact that the modular variables commute, they give rise to an integrality condition inherent to the structure of the discrete Heisenberg group. In this manner, this group should mediate in the structural transition from non-commutativity to its integral Abelian shadow, which is qualified symplectically via the non-squeezing theorem. In particular, the integrality of symplectic area pertains to the global topology of the torus, meaning that the phase space of the 2-d Abelian shadow is topologically toroidal and isomorphic to the modular lattice $\frac{\mathbb{R}^2}{\mathbb{Z}^2}$, such that it is universally covered by $\mathbb{R}^2$, where $\mathbb{Z}^2$ is the free Abelian group in two generators acting by integer translations. The main conclusion of this work is that the structural transition from the non-Abelian-free fundamental group of based loops Θ_2, expressing the synchronization condition of the joint action of modular variables, to the Abelian-free homology group of cycles $\mathbb{Z}^2$, where the entangled symplectic area pertains, factorizes though the discrete Heisenberg group. This elucidates the role and status of modular variables in quantum mechanics and constitutes a viable theoretical explanation of the nature and appearance of quantum interference phenomena underlying the significance of the notion of a geometric phase.

Keywords Topological link · Borromean rings · Symplectic area · Abelianization · Covering space · Chern class · Weyl commutation · 1-parameter unitary group · Free group · Geometric phase · Heisenberg group · Torus

1 Introduction

A global complex-valued geometric phase factor is thought of as the "memory" of a quantum system undergoing a "cyclic evolution" after coming back to its original physical state [1,6,24,30]. The state is identified with a ray in the complex projective Hilbert space of states, which equivalently, can be considered as a real phase space equipped with a symplectic form. In this way, the interpretation of the global phase factor refers to the symplectic area enclosed by the loop along which the transition takes place.

E. Zafiris (✉) · A. von Müller
Parmenides Foundation, Center for the Conceptual Foundations of Science, Kirchplatz 1, Pullach, 82049 Munich, Germany
e-mail: elias.zafiris@parmenides-foundation.org

We pointed out in [32] that the interpretation of the geometric phase factor should follow from the principle of calculation of the transition probability from an initial state to a final one when one or more potential intervening states are considered, for instance under the insertion of projection filters that can even be non-simultaneously realizable, like in the case of the double slit experiment. In particular, leveraging on the fact that the transition probability remains invariant under both, local phase transformations, and complex conjugation, it turns out that the transition probability is always associated with a process "evolving" around an imaginary loop in "time".

This invites caution in interpreting the real parameter t indexing the one-parameter unitary group of "evolution" as a classical temporal parameter. The reason is that for each description of a quantum system in terms of a state vector "evolving" along the positive direction of the real parameter t, there exists a physically indistinguishable description in terms of the conjugate state vector "evolving" along the negative direction of the real parameter t. Therefore, thinking in terms of the corresponding transition amplitudes, if one of them is associated with a process "evolving" forward in "time", its indistinguishable complex conjugate amplitude is associated with a process "evolving" backward in "time". This symmetry boils down to the fact that the transition probability derived by squaring is associated with a process "evolving" around an imaginary loop in "time". Taking this symmetry seriously, it leads to the correct evaluation of the transition probabilities in the case of the double slit experiment. This conceptual stance constitutes a viable alternative to the "two-state vector formalism", after Aharonov, Bergmann and Lebowitz [2,3]. It also makes justice to Schwinger's "closed time path formalism"[21], not in the sense of an efficient calculation tool, but in reflecting something of a deeper significance that is intertwined with the notion of a global geometric phase.

Most interestingly, the above symmetry seems to be inter-related with another powerful idea put forward by Aharonov and collaborators [4,5,25] in trying to decipher the quantum interference pattern of the double slit experiment. What they figured out is that the deciphering of the interference pattern actually pertains to modular operators of both position and momentum, instead of the standard corresponding observables. These modular operators are qualified as non-local, such that it becomes possible to detect a relative phase difference, something that it is not feasible using local self-adjoint operators. But the most remarkable property of these modular quantum variables is that they commute in contradistinction to their standard counterparts.

The quantum modular variables pertaining to conjugate observables can be properly encoded in terms of one-parameter unitary groups acting jointly on the phase space. This paves the way for considering the continuous group action of $\mathbb{R}^2$, to be though of as the product of the group $\mathbb{R}$ with its Fourier dual group $\mathbb{R}$, identified as such with $\mathbb{R}$. Due to the consideration of modular variables, we are led to consider two dual real-valued parameters reciprocally related to each other via Planck's constant. The constancy of the product of these reciprocally related real variables is expressed geometrically by means of a rectangular hyperbola. In this context, we showed in Part I of this work [33] that $\frac{h}{2}$ expresses the minimal indistinguishable invariant area—identified under the hyperbola—in the variances of these conjugate variables if they are thought of as symplectic ones.

Thus, there exists a fundamental symplectic area scale, such that, any variable positional length scale that is extended or contracted reciprocally with respect to the corresponding momentum scale preserves their product, which is qualified by means of the invariant indistinguishable symplectic area modulo $\mathbb{Z}$ due to modularity. The underlying reason is due to "Gromov's non-squeezing theorem" [11], leading to the conclusion that the 2-d symplectic Abelian shadow of the symplectic ball of radius $R = \sqrt{\hbar}$ in the $2n$-phase space of the conjugate position and momenta has symplectic area equal to $\frac{h}{2} = \pi\hbar$, corresponding to a linear area-preserving transformation, or bigger than this otherwise. In this manner, all $\mathbb{R}$-valued symplectic areas differing by an integer, induce the same S^1-valued geometric phase. Hence, the geometric phase pertains to the area $\pi\hbar$ of the 2-d symplectic Abelian shadow of the symplectic ball of radius $R = \sqrt{\hbar}$ modulo $\mathbb{Z}$.

The above conclusion leads to a revived interest on Weyl's view of the quantum kinematical space in terms of an Abelian group of unitary ray rotations [29], and in particular the role that the discrete Heisenberg group plays in this conundrum. Taking into account Weyl's group-theoretic commutation relations

$$W(x, y)W(x', y') = e^{(i/2\hbar)\omega}W(x + x', y + y')$$

characteristic of a projective unitary representation [15,26,29]—where

$$W : (x, y) \mapsto e^{(i/2\hbar)x \cdot y} V_x U_y$$

denotes the normalized continuous group homomorphism of $\mathbb{R}^2$ into the projective unitary group of the Hilbert space, ω is the symplectic form on $\mathbb{R}^2$, and V_x and U_y are one-parameter unitary groups infinitesimally generated by the non-commuting observables P and Q correspondingly—, and applied to the modular variables, we immediately realize that the commutativity condition requires that the complex exponential phase factor should be unity, meaning that the symplectic area should be integral in units of Planck's constant. The constancy of the product of the reciprocally co-related real symplectic variables, expressed geometrically by means of area preservation under the corresponding rectangular hyperbola, is a pre-requisite for the satisfaction of the integrality condition. This vindicates our standpoint that there is no fundamental length or momentum scale, but a fundamental symplectic area scale, which bears the semantics of the indivisible quantum of action and characterizes the deformation invariance structure of quantum mechanics under joint complementary actions.

Since, the integrality condition constitutes the dually co-related global one-parameter unitary group actions of position and momentum commutative in symplectic phase space, the corresponding modular observables of position and momentum share joint eigenstates, subject to the rectangular hyperbola constraint. Due to the fact that the modular variables are $\frac{\mathbb{R}}{\mathbb{Z}}$-valued, the cell of the modular lattice $\frac{\mathbb{R}^2}{\mathbb{Z}^2}$ cannot be experimentally distinguished. Thus, modular variables, although commutative, they are not deterministic due to complete uncertainty of the areal winding number. In particular, the integrality of symplectic area pertains to the global topology of the torus in this case, meaning that the global phase space should be topologically toroidal and universally covered by $\mathbb{R}^2$, where $\mathbb{Z}^2$ is the free Abelian fundamental group in two generators. Therefore, it is important to understand the emergence and instrumental role of this modular Abelian structure from the subsumed non-commutative algebraic structure of all observables in quantum theory.

We will show that the discrete Heisenberg group plays an instrumental role in this respect that explicates the function of the modular variables. Since, the quantum modular variables are encoded in terms of one-parameter unitary groups we are led to a careful re-evaluation of the Heisenberg group, from where the canonical commutation relations of the standard quantum variables emerge from. In particular, the re-evaluation pertains to the discrete Heisenberg group, this time not from the perspective of restriction of real variables to integer ones according to the usual presentations, but from the perspective of its genuine descent from a more fundamental layer of structure targeting the origin of the non-commutativity of quantum observables. The important issue is that, since the modular variables commute, they give rise to an integrality condition inherent to the structure of the Heisenberg group. In this sense, the discrete Heisenberg group should mediate in the structural transition from non-commutativity to its integral Abelian shadow.

In this frame of thinking, it proves indispensable, first of all, to adopt a topological viewpoint on qualifying non-commutativity. The idea is to think of an one-parameter unitary group infinitesimally generated by an observable as a based loop up to continuous deformation within the algebra of all observables commuting with it, and thus, sharing the same spectral resolution. Since an one-parameter unitary group preserves the degree of distinguishability induced by the spectral resolution of the observable it is generated from, the corresponding based loop may be considered in relation to a topological circle constituting a barrier, such that the loop cannot be contracted to its base point upon passing through this circle with a prescribed orientation. Therefore, the topological circle encodes the resolving capacity of the considered observable in terms of its simultaneously compatible projective resolution, such that the non-contractibility of a respective based loop upon passage through this circle enciphers the awareness and preservation of the degree of distinguishability afforded by the corresponding resolution.

At the next stage, although projection filters corresponding to conjugate observables are not simultaneously realizable, there might be a condition of joint synchronization with respect to an area bounding cycle in the projective space, though of as a symplectic phase space, which can be derived from the composition of based loops through the process of Abelianization. More concretely, if we consider conjugate observables, then the corresponding one-parameter unitary groups may be thought of in relation to non-directly linked disjoint topological circles from a homotopy-theoretic viewpoint, where each one of them preserves the degree of distinguishability afforded by the corresponding resolving observable spectrum. A composite based loop passing through both circles with some orientation is qualified as a synchronizing loop under the constraint that it gives rise to a symplectic area bounding cycle on the space of rays.

The synchronization condition in this way should descend from the possible linking property of based oriented loops in $\mathbb{R}^3$ excluding the contraction barriers B imposed by two disjoint topological circles. In particular, the synchronization condition should reflect the homological congruence of an area bounding square loop in $\mathbb{R}^2$ emerging through the reduction of the fundamental group of $\mathbb{R}^3 \backslash B$ to the first homology group of this space, according to the Hurewicz theorem [12]. The necessity of 3-D space comes from the fact that based oriented loops are required to pass through a topological circle with a prescribed orientation. From the perspective of homotopic deformation, the exclusion of these two topological circles from 3-D space gives rise to the free non-commutative group in two generators Θ_2 as its corresponding fundamental group, whose Abelianization is the free Abelian group in two generators $\mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}^2$. The crucial fact in this context is that all realizations of the free group Θ_2 are isomorphic, thus in principle, Abelianization can be transferred in the quantum setting under appropriate qualifying conditions.

We will show that these qualifying conditions are essentially based on the articulation of $\mathbb{R}^3$ in terms of the group-theoretic structure of the Heisenberg group, and in particular, on the instrumental role of the discrete Heisenberg group in unraveling the transition from the non-Abelian homotopy of based loops to the Abelian homology of cycles. Reciprocally, and in relation to this structural transition, the discrete Heisenberg group transcribes the synchronization condition in a discrete Abelian context, which nevertheless can be decoded only by homological means through the integrality qualification of the symplectic structure, when applied to the commutative modular variables of position and momentum in quantum mechanics.

The source of synchronization is derived via the universal realization of the free group in two non-commuting generators Θ_2 as the fundamental group of $\mathbb{R}^3 \backslash B$, where B refers to the impenetrable barrier for based loops posed by two disjoint topological circles under the intended semantics of our interpretation. We demonstrate that the source is a topological link identified as the "Borromean rings" [7,10,27], which is algebraically expressed through the group-theoretic commutator of the free group Θ_2. In this way, upon representation in the quantum-theoretic setting constrained by the Heisenberg group for the satisfaction of unitarity, the group commutator reduces to a central element in the Heisenberg group, which is physically interpreted as a geometric phase, or equivalently, as a symplectic area through the underlying symplectic structure constituting the Heisenberg group as a principal fiber bundle.

Therefore, if we think of the Borromean topological link in temporal terms, as providing the synchronization condition of based oriented loops, expressed through the commutator of the free non-Abelian group Θ_2, then upon the representation of Θ_2 in the Hilbert space in terms of one-parameter unitary groups generated infinitesimally by conjugate observables, and in relation to the unitary representation of the Heisenberg group, the synchronization condition is the root that gives rise to the entangled symplectic area characteristic of an interference pattern.

The main conclusion of this work is that the structural transition from the non-Abelian-free fundamental group of based loops (where the synchronization applies) to the Abelian free homology group of cycles (where entangled symplectic area pertains) factorizes though the discrete Heisenberg group. This elucidates the role and status of modular variables in quantum mechanics and constitutes a viable theoretical explanation of the nature and appearance of quantum interference phenomena underlying the significance of the notion of a geometric phase. Henceforth, the Borromean topological link becomes observable in terms of the geometric phase pertaining to modular variables, only after the qualification of $\mathbb{R}^3$ by means of the symplectic structure of the Heisenberg group, and the homotopy-to-homology process of Abelianization being subordinate to its factorization through the discrete Heisenberg group.

2 Principal fiber bundle structure of the Heisenberg group and the horizontal distribution

Due to the fact that the 2-D symplectic Abelian shadow of the symplectic ball of radius $R = \sqrt{\hbar}$ in the $2n$-phase space of the conjugate position and momenta has area A equal to $\frac{h}{2} = \pi\hbar$, corresponding to a linear area-preserving transformation, we are naturally led to consider the central extension of the Abelian group $\mathbb{R}^2$ by the multiplicative Abelian group of complex phases S^1, which gives rise to the Heisenberg group. It is enunciating to view the Heisenberg group $\mathbb{H}$ as a principal fiber bundle with base space $\mathbb{R}^2$ and structure group S^1. In this manner, we may define the notions of vertical and horizontal subspaces at a point $\zeta := (\upsilon, \tau)$ of $\mathbb{H}$.

The vertical subspace V_ζ is the set of vectors in $T_\zeta \mathbb{H}$ that are tangent to the fiber S^1 passing through ζ, and it is clearly one-dimensional. The horizontal subspace H_ζ is the orthogonal complement of V_ζ in $T_\zeta \mathbb{H}$, such that the distribution of two-dimensional horizontal subspaces is invariant under the action of group multiplication on the left, and transverse to the fibers of this principal bundle.

Equivalently, the above defines a connection on the total space of the Heisenberg group $\mathbb{H}$. The natural projection

$$\pi : \mathbb{H} \to \mathbb{R}^2$$

induces an isomorphism between H_ζ and $T_\upsilon \mathbb{R}^2$, that is:

$$H_\zeta \cong T_\upsilon \mathbb{R}^2$$

for any representative ζ in $\mathbb{H}$ of a vector υ in $T_\upsilon \mathbb{R}^2$. Furthermore, we may express a connection on the total space of $\mathbb{H}$ by means of a differential 1-form σ defined on this space, and taking values in the Heisenberg Lie algebra $\mathbf{h}$.

More precisely, for each point ζ in $\mathbb{H}$, the projection $T_\zeta \mathbb{H} \to V_\zeta \cong \mathbf{h}$ induced by the preceding direct sum decomposition of T_ζ defines such a connection 1-form σ with values in $\mathbf{h}$. Clearly, we obtain the identification of the horizontal subspace T_ζ, for each point ζ in $\mathbb{H}$, as the space where $\sigma(\xi) = 0$ for any vector ξ in T_ζ, i.e. for any tangent vector ξ at ζ.

Next, we define a symplectic form ω on the vector space $H_1 \subseteq \mathbf{h}$, where the center of $\mathbf{h}$ has been identified with $\mathbb{R}$, by setting:

$$\omega(z, z') := [z, z'] \in \mathbb{R}$$

for $z, z' \in H_1$. Clearly, this structure can be translated to other points ζ in $\mathbb{H}$ by group multiplication on the left. By the natural projection $\pi : \mathbb{H} \to \mathbb{R}^2$ the above symplectic structure can be restricted to $\mathbb{R}^2$, denoted by ω again.

Conversely, utilizing the distribution of two-dimensional horizontal subspaces in $\mathbb{H}$ we can lift paths from the base $\mathbb{R}^2$ to $\mathbb{H}$ as follows: Consider a path $\gamma : [0, 1] \to \mathbb{R}^2$ together with a point $x \in \pi^{-1}\gamma(0)$. Then, there exists a unique horizontal lift $\tilde{\gamma} : [0, 1] \to \mathbb{H}$, such that: $\tilde{\gamma}(0) = x$, and $\pi \circ \tilde{\gamma}(s) = \gamma(s)$ for all $s \in [0, 1]$.

In particular, if γ is a closed path, i.e. a loop in $\mathbb{R}^2$, then $\tilde{\gamma}$ is not a loop in $\mathbb{H}$, but it is precisely the symplectic structure on $\mathbb{R}^2$ that gives the information regarding the extent that the lifted path fails to be a loop. More precisely, the vertical distance between $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ belonging to the vertical subspace V_1 of the fiber $\pi^{-1}\gamma(1)$ equals the absolute value of the symplectic area enclosed by the loop $\gamma \in \mathbb{R}^2$, whereas, the sign of the symplectic area informs us which one between $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ lies above the other.

Clearly the same prescription holds for the simply-connected universal covering Heisenberg group $\bar{\mathbb{H}}$, which is diffeomorphic with the Heisenberg Lie algebra. The important conclusion, according to the preceding, is that the vertical distance between $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ provides a measure of distinguishability between them by means of the absolute value of the symplectic area enclosed by the loop γ in the base space $\mathbb{R}^2$. Notice that this vertical distance is encoded in terms of a real number measuring area located either at V_1 of the fiber $\pi^{-1}\gamma(1)$ of the Heisenberg group $\mathbb{H}$, or equivalently, at the fiber $\bar{\pi}^{-1}\gamma(1)$ of the simply connected $\bar{\mathbb{H}}$ which is isomorphic to $\mathbb{R}$.

3 Loop-bounded signed symplectic area and the geometric phase

We may extend the above frame of thinking as follows: we consider a path α in the universal covering Heisenberg group $\bar{\mathbb{H}}$, or equivalently, in the Heisenberg Lie algebra $\mathbf{h}$, characterized by the triplet of path coordinates $(\alpha_1, \alpha_2, \alpha_3)$. We say that a path α is horizontal, if all of its tangent vectors lie in the distribution of two-dimensional horizontal spaces. This condition is satisfied if and only if:

$$\acute{\alpha_3} = \frac{1}{2}(\alpha_1\acute{\alpha_2} - \alpha_2\acute{\alpha_1})$$

Then, for any path (α_1, α_2) from $(0, 0)$ to (x, y) in $\mathbb{R}^2$, there is a unique horizontal lift $(\alpha_1, \alpha_2, \alpha_3)$ in **h**. More precisely, the lifted path connects $(0, 0, 0)$ to (x, y, ε), where the third coordinate ε is given by the signed symplectic area of the region Σ in $\mathbb{R}^2$ bounded by the loop $\partial\Sigma$ formed from (α_1, α_2) and a straight line from $(0, 0)$ to (x, y). Clearly, the straight line contributes zero to the third coordinate, so all the contribution comes from the horizontal lift of the plane path (α_1, α_2) from $(0, 0)$ to (x, y).

More concretely, a simple application of Stokes theorem gives:

$$\varepsilon = \frac{1}{2}\oint_{\partial\Sigma}(\alpha_1\acute{\alpha_2} - \alpha_2\acute{\alpha_1}) = \int\int_{\Sigma} \mathrm{d}x \wedge \mathrm{d}y$$

which is the signed symplectic area of the region Σ in $\mathbb{R}^2$. Physically, this is equivalent to the action S of the connection 1-form

$$\sigma = \mathrm{d}w - \frac{1}{2}(x\mathrm{d}y - y\mathrm{d}x),$$

$$S = \varepsilon = \oint_{\partial\Sigma} \sigma,$$

whose kernel at any point ζ belongs to the horizontal space at this point, along the loop $\partial\Sigma$ bounding the plane region Σ. In this manner, the closed and exact 2-form:

$$\kappa = \mathrm{d}\sigma = \mathrm{d}x \wedge \mathrm{d}y$$

is identified as the curvature form of the above connection displayed in the standard real two-dimensional phase space Darboux form. Its existence captures the non-integrability of the defined horizontal distribution. In this sense the horizontality condition, or equivalently the connection, depicts an Abelian subgroup of the Heisenberg group, or equivalently, a commutative subalgebra of the Heisenberg algebra.

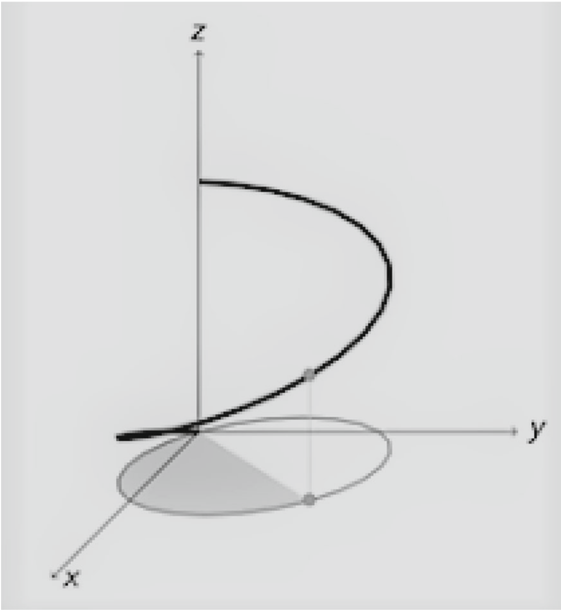
Note that any two points in **h** can be connected by a horizontal path, which is the unique horizontal lift of a plane path (α_1, α_2). In general, we may consider concatenations of piecewise linear paths on $\mathbb{R}^2$, for instance the concatenation of the piecewise linear path from $(0, 0)$ to (x, y) and the path from (x, y) to $(x + x', y + y')$. Then, similarly we consider the straight line connecting $(0, 0)$ to $(x + x', y + y')$ to obtain a loop bounding the formed plane region. The Heisenberg group element we obtain by the above prescription starting at $(0, 0, 0)$ is given by the triplet $(x + x', y + y', \varepsilon)$ where ε is the signed symplectic area bounded by the above loop. Notice that the area is zero if (x, y) and (x', y') are collinear. In this manner, the universal covering group composition rule encodes the sum of the piecewise linear increments in the base space $\mathbb{R}^2$, as well as the total generated signed symplectic area, which is encoded as the third coordinate.

The fundamental group of the Heisenberg group $\mathbb{H}$, as the total space of the principal fiber bundle with base space $\mathbb{R}^2$ and structure group S^1, is given by the additive group of the integers $\mathbb{Z}$, in accordance with the fact that the universal simply connected Hesisenberg group $\bar{\mathbb{H}}$, diffeomorphic with the Heisenberg Lie algebra **h**, is the total space of the corresponding principal fiber bundle with base space $\mathbb{R}^2$ and structure group $\mathbb{R}$. This is simply a consequence of the group isomorphism $S^1 \cong \mathbb{R}/\mathbb{Z}$.

In addition, due to the universal covering property of $\bar{\mathbb{H}}$, it can be thought of as a flat principal fiber bundle over $\mathbb{H}$ with discrete structure group $\mathbb{Z}$. This equivalently means that all points on an $\mathbb{R}$-fiber of $\bar{\mathbb{H}}$ differing by an integer project onto the same point of the corresponding S^1-fiber of $\mathbb{H}$. Physically speaking, the semantics of this fact is that all $\mathbb{R}$-valued signed symplectic areas of regions Σ in $\mathbb{R}^2$ differing by an integer, induce the same S^1-valued geometric phase, i.e.

$$\varepsilon + \mathbb{Z} = \oint_{\partial\Sigma} \sigma + \mathbb{Z} = \int\int_{\Sigma} dx \wedge dy + \mathbb{Z} = \int\int_{\Sigma} \kappa + \mathbb{Z} \cong \tau.$$

At this stage, it is crucial to reflect on the fact that a connection in the total space of the Heisenberg group $\mathbb{H}$, or its universal covering group $\bar{\mathbb{H}}$ depicts an Abelian subgroup, or equivalently, a commutative subalgebra of the Heisenberg algebra. It is precisely with respect to this commutative substructure and area bounding loops therein that the third coordinate is qualified as an S^1-valued geometric phase in the former case, and as an $\mathbb{R}$-valued signed symplectic area in the latter case, together with their previously described interrelation. In this manner, the non-commutativity encoded in the Heisenberg group is decoded via a connection to the information of a geometric phase or of a signed symplectic area with respect to the commutative substructure depicted by the employed connection.



Equivalently, the non-commutativity in the Heisenberg group appears as an artifact emanating from the existence of the symplectic area measure expressing exactly the global non-integrability of the horizontal distribution, i.e. the non-global extendibility of the commutative structure, defined by this connection. In turn, the non-commutativity in the nilpotent Heisenberg group may be considered as being not a fundamental and genuine form of non-commutativity, but merely as one that can be legitimately traded for a maximal extension of some local commutative substructure together with the associated geometric phase or signed symplectic area bounded by loops therein.

4 Source of non-commutativity and indirect topological linking

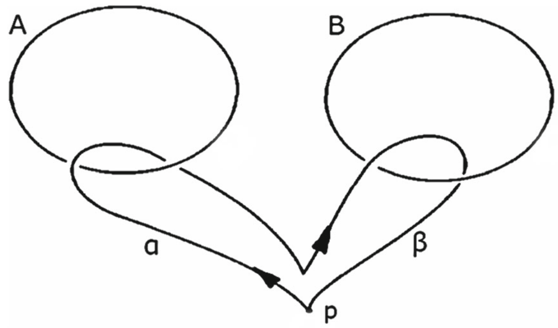
A natural question arising in the context of the above conclusion is what is precisely the source of genuine non-commutativity in the three-dimensional (3-D) space $\mathbb{R}^3$, which is the set-theoretic domain of definition of the universal covering Heisenberg group. The strategy here is to approach the problem of non-commutativity displayed in the group-structure of the Heisenberg group from an opposite conceptual pole in comparison to the former case. In particular, the aim is to describe the source of genuine non-commutativity in 3-D space excluding two disjoint topological circles according to the schema developed in Section 5, and then, understand the Heisenberg group structure as encoding a degenerate form of non-commutativity, induced essentially by the nilpotency condition together with the area-bounding loop symplectic semantics. This is particularly important in relation to the Hilbert space representation of the Heisenberg group pertaining to the description of a quantum system obeying the Weyl commutation relations.

For this purpose, we consider a loop in 3-D space $\mathbb{R}^3$ as an unknotted tame closed curve, that is, as a polygonal curve that can be continuously deformed in 3-D space $\mathbb{R}^3$ until it lies flat on a plane $\mathbb{R}^2$ without intersecting itself. The objective is to study the group structure formed by all possible loops, specified as above, with respect to the existence of two separate barriers, modelled in terms of topological circles in 3-D space. We immediately realize that a group structure can be legitimately defined only if these loops can be based at the same reference point.

In particular, the anticipated group structure should pertain to the number of times a based loop passes through a barrier together with the pertinent orientation of this passage.

According to the above, a based loop means simply that it starts and ends at a fixed reference point p of the 3-D space. The orientation of the loop can be thought of in terms of an observer, which is fixed at the point p, such that: If the loop passes through the barrier A one time with direction away from the observer it is denoted by α^1, whereas, if it passes one time with direction toward the observer it is denoted by α^{-1}. We note that any other loop with the same properties like α can be continuously deformed to the loop α. Thus, the algebraic symbol α actually denotes the equivalence class $[\alpha]$ of all loops of kind α, passing through the barrier A once with the prescribed orientation. Conceptually, passage of a loop through the barrier, which prevents the contraction of the loop at the base point, means awareness and preservation of the distinguishability between the inside and the outside of the considered topological circle.

Taking into account the algebraic encoding of based oriented loops in relation to topological circles in 3-D space, we can define the composition of two oriented loops under the proviso that they are based on the same base point p in 3-D space. Notice that the composition operation $\alpha \circ \beta$ of the p-based oriented loops α and β in relation to circles A and B correspondingly, is not a commutative operation, meaning that the order of composition is not allowed to be reversed. Clearly, the rule of composition produces a based oriented loop $\alpha \circ \beta$ in 3-D space in relation to the circles A and B in the prescribed order. We think of the composition rule $\alpha \circ \beta$ as the non-commutative multiplicative product of the oriented loops α and β based at the same point p in 3-D space, which we may simply denote as $\alpha\beta$. It is immediate to verify that the above defined multiplication is an associative operation.



Having established the closure of the elements of the generic form χ under non-commutative associative multiplication as previously, we look for the existence of an identity element, as well as for the existence of inverses with respect to this operation. There is an obvious candidate for each based oriented loop α, namely the loop α^{-1}, where the orientation has been reversed. If we consider the compositions $\alpha \circ \alpha^{-1}$, $\alpha^{-1} \circ \alpha$ we obtain in both cases as a multiplication product the based loop at the same point, which does not pass through any circle at all. Thus, we name the latter loop as the multiplicative identity 1 in our algebraic structure, such that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$. It is also easy to verify that $1\alpha = \alpha 1 = 1$. We conclude that if we consider two types of p-based oriented loops as generators in relation to the barriers A and B, denoted by the symbols α and β respectively with the prescribed orientation and obeying no further constraints, we form a non-commutative free group in two generators, denoted by Θ_2. The equality sign in the non-commutative group Θ is interpreted topologically as an equivalence relation of p-based oriented loops under continuous deformation. By making use of the multiplication operation in Θ_2 we may form any permissible string of symbols in this group, which can be reduced into an irreducible form by using only the group-theoretic relations $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$, $\alpha\alpha = \alpha^2$, and so on.

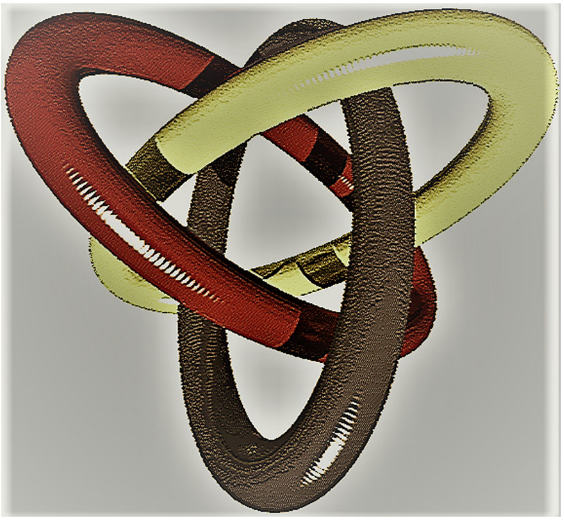
The claim is that the non-commutative free group in two generators Θ_2, represented in 3-D space according to the preceding, expresses a genuine and non-reducible type of non-commutativity. For this purpose, it is indispensable to examine the behavior of the group commutator and explicate its semantics. The group-theoretic commutator induced by the generators of Θ_2:

$$[\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta$$

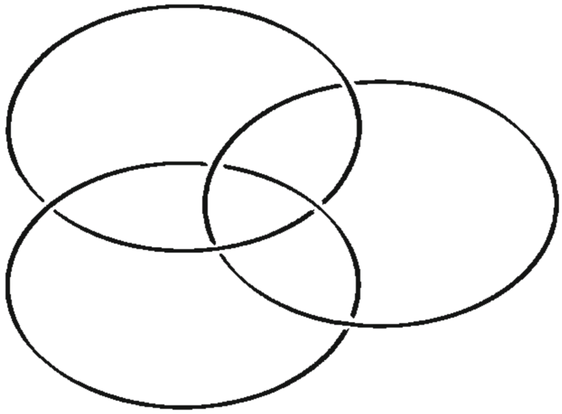
produces an irreducible non-commutative string of symbols in Θ_2. This string represents a new based loop γ as a product loop composed by the ordered composition of the based oriented loops $\alpha \circ \beta^{-1} \circ \alpha^{-1} \circ \beta$. The crucial observation is that deletion of both symbols α and α^{-1}, an operation that corresponds to removing or cutting the circle A, reduces the group commutator to the identity 1. Clearly, the same behavior is encountered symmetrically for both β and β^{-1}.

The process of cutting topological circles from a constellation is utilized in the theory of topological links to express a property called splittability. The notion of a topological link is based on the underlying idea of connectivity among a collection of loops. A topological N-link is a collection of N loops, defined in the same way as above, where N is a natural number. Regarding the connectivity of a collection of N loops, the crucial property is the property of splittability of the corresponding N-link. We say that a topological N-link is splittable if it can be deformed continuously, such that part of the link lies within B and the rest of the link lies within C, where B, C denote mutually exclusive solid spheres (balls). Intuitively, the property of splittability of an N-link means that the link can come at least partly apart without cutting. Complete splittability means that the link can come completely apart without cutting. On the other side, non-splittability means that not even one of the involved loops, or any pair of them, or any combination of them, can be separated from the rest without cutting.

From the viewpoint of the theory of topological links, the Borromean link constitutes an interlocking family of three loops, such that if any one of them is cut at a point and removed, then the remaining two loops become completely unlinked [9,10,12–14], see also [27,28] for related work. The Borromean link describes topologically the constellation known as the “Borromean rings”, consisting of three rings, i.e. topological circles, which are linked together in such a way that each of the rings lies completely over one of the other two, and completely under the other, as it is shown at the picture below:



The Borromean link is characterized topologically by the property of splittability as follows: The Borromean link is a non-splittable 3-link (because it consists of three loops), such that every 2-sublink of this 3-link is completely splittable. It is clear that it is a non-splittable 3-link because not even one of the three loops, or any pair of them, can be separated from the rest without cutting. A 2-sublink is simply any sub-collection of two loops obtained by erasing the loop that does not belong to this sub-collection. Since, the Borromean link is characterized by the property that if we erase any one of the three interlocking loops, then the remaining two loops become unlinked, it is clear that every 2-sublink of the non-splittable 3-link is completely splittable.

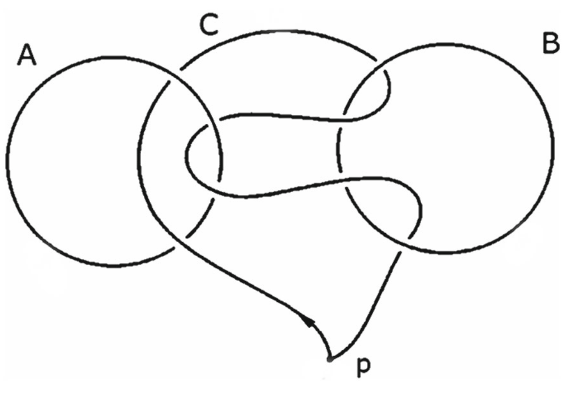


We will show that the topological splittability information incorporated in the specification of the Borromean link is precisely encoded in algebraic terms via in the non-commutative group-structure of the free group Θ_2. The property of irreducibility of a string of symbols in the group Θ_2 is the guiding idea for the algebraic encoding of the Borromean link.

The crucial observation is that algebraic irreducibility in Θ_2 can be used to model the topological property of non-splittability of a 3-link, where complete splittability of all 2-sublinks is encoded by the unique identity element of Θ_2. In particular, the group-theoretic commutator induced by the generators of Θ_2:

$$[\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta$$

produces an irreducible non-commutative string of symbols in Θ_2. This string represents a new based loop γ as a product loop composed by the ordered and non-reversible composition of the based oriented loops $\alpha \circ \beta^{-1} \circ \alpha^{-1} \circ \beta$. We call the product loop γ the Borromean loop and the formula or multiplicative string $\alpha\beta^{-1}\alpha^{-1}\beta$ in Θ_2 the Borromean loop formula.



The algebraic irreducibility of the group commutator $[\alpha, \beta^{-1}]$ in the free group Θ_2 encodes the topological non-splittability property of the Borromean 3-link. We noticed above that deletion of both α and α^{-1} (corresponding to removal of the circle A) reduces the formula to the identity 1 (and the same happens symmetrically for both β and β^{-1}). This fact models algebraically in the terms of Θ_2 that every 2-sublink of the Borromean 3-link is completely splittable.

We conclude that the topological splittability, that is, the connectivity information of the Borromean 3-link can be completely encoded in terms of the algebraic structure of the non-commutative multiplicative free group in two generators Θ_2. In particular, the group-theoretic commutator $[\alpha, \beta^{-1}]$ in Θ_2, encodes algebraically the modular gluing condition of the based oriented loops α and β^{-1} (with respect to the circles A and B, respectively, in the prescribed orientation), and therefore the non-splittability of the Borromean 3-link, together with the complete splittability of all 2-sublinks of this 3-link.

In view of the established representability of the free group in two non-commuting generators, Θ_2 in the quantum space of state vectors, in terms of one-parameter unitary groups infinitesimally generated by complementary observables, the group-theoretic commutator in this setting transfers the Borromean linking property in the quantum regime. Recall that an one-parameter unitary group infinitesimally generated by an observable should be thought of topologically as a loop based at a reference state vector, up to continuous deformation within the algebra of all observables commuting with it, and thus sharing the same spectral resolution. Since, an one-parameter unitary group preserves the degree of distinguishability induced by the spectral resolution of the observable it is generated from, the corresponding based loop is considered correspondingly in relation to a topological circle constituting a barrier, preventing the loop from contraction to its base point upon passing through this circle with a prescribed orientation. In this setting, the topological circle encodes the resolving capacity of the considered observable in terms of its simultaneously compatible projection filters, such that the non-contractibility of the based loop upon passage through this circle enciphers the awareness and preservation of the degree of distinguishability afforded by the corresponding resolution.

This is instrumental in thinking about the notion of quantum probability assigned to a transition by means of amplitudes. The crucial issue is that quantum transition probability is always bearing the symmetry of a process "evolving" around a loop in "time". This is the heuristic standpoint from which we aim to interpret the semantics of the Borromean linking expressed in terms of one-parameter unitary groups acting on a reference state vector. The idea is that under the standpoint of this symmetry, the pertinent topological linking property gives rise to a precise synchronization condition of one-parameter unitary groups generated by complementary observables with respect to a reference state vector on which they act on. Thus, although simultaneous applicability is not feasible, synchronization becomes possible. This is the anchor via which the semantics of the geometric phase can be unfolded.

5 Homological Abelianization through square boundary loops

We note that the Borromean topological link is characterized by threefold symmetry. In the algebraic terms of the group Θ_2 this is reflected on the fact that if we consider any two of the based loops α, β^{-1}, γ, then the third is expressed by the group commutator of the other two. The threefold symmetry of the Borromean link may be broken by reducing the free non-commutative group on two generators Θ_2 to the free nilpotent group on two generators of nilpotent class 2.

More concretely, we may choose the based loops α, β^{-1} such that $\gamma = [\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta$ and impose the relations $[\alpha, \gamma] = [\beta^{-1}, \gamma] = 1$. Since in the reduced group all threefold and higher commutators vanish, γ belongs to the center of this group. Moreover, the non-commutativity of the nilpotent group descends from the non-commutativity of the non-Abelian-free group in two generators.

It is worth examining the type of non-commutativity displayed by the nilpotent group in more detail. In the case of the free group Θ_2 we have interpreted the group commutator as the algebraic encoding of the Boromean topological link formed among the based loops α, β^{-1} and $\gamma = [\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta$, or equivalently, among the barriers A, B, and the commutator product loop γ. The key idea is that the loop γ depends on the order of the constituting actions α, β^{-1}, α^{-1}, β, and thus, cannot be deformed to the identity loop in Θ_2.

In other words γ is homotopically non-trivial, since it is not reducible or contractible to the identity. We identify this homotopy-theoretic fact with the source of genuine non-commutativity, and concomitantly, with the source of the topological linking characterizing the connectivity of the Borromean rings. Breaking the threefold symmetry, according to the above, amounts to two things: First, there takes place a distinction or marking between topological circles and product loops, or equivalently, between barriers and product actions preserving the distinguishability with respect to them; Second, the group commutator product action is always a central element in the reduced nilpotent group. Thus, we obtain a degenerate type of non-commutativity, which is based exclusively on central characters.

It may be still represented in the pictorial only form of the Borromean link under the proviso that the barriers are now marked and distinguished from the commutator product loop, which is equivalent to breaking the intrinsic three-fold symmetry. The question now is if the semantics, which has been assigned to the Heisenberg group before through the symplectic interpretation, can be recovered through the opposite pole of genuine non-commutativity.

For this purpose, it is possible to formulate a simple argument that hopefully sheds light on the above issue, and simultaneously paves the way for scrutinizing the apparent type of non-commutativity encountered in quantum theory via the corresponding representation of the nilpotent group in the Hilbert space of quantum states. We have deduced that the non-commutative ordered product

$$\gamma = [\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta$$

is not contractible to the identity due to the homotopic non-deformability of the commutator product loop to a trivial loop. Equivalently, γ belongs to the non-trivial homotopy class of the fundamental group defined on the complement of the disjoint topological circles A and B.

Conversely, we realize that γ is actually reducible to the identity if α commutes with β^{-1}. Hence, the vanishing of the commutator γ amounts to the Abelianization of the fundamental group, which in turn is identified with the Abelian first homology group. This means that the reduction of the fundamental group to the Abelian homology group depicts a commutative group structure identified as an Abelian subgroup of the nilpotent group. Concomitantly, this commutativity criterion is equivalent to the connection-induced horizontality condition in the principal bundle displaying the nilpotent group over the base space defined by the above commutative subgroup. We can see this immediately from the commutativity condition:

$$\gamma = [\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta = 1,$$

which in homology means that γ defines a boundary, that is, the extent that two homologous cycles differ by. This is exactly the form of the homological area boundary that is realized on the nilpotent Heisenberg group, or its universal covering group in terms of two-dimensional translations taking place in the horizontal distribution.

To gain a better insight, it is worth looking at the semantics of the commutativity condition from the perspective of topological links. The commutativity condition amounts to the degeneration of the Borromean link, and thus of its nilpotent avatar, into a Hopf link, meaning that the marked barriers appear in the homological Abelian shadow as directly linked. In this interesting way, the commutativity condition bears non-trivial topological aspects, since this type of Hopf link can be actually realized on the non-simply connected surface of a torus $S^1 \times S^1$ given that α, β^{-1}, define a homology basis for a torus if linked like this.

In turn, this indicates that from a commutative viewpoint, the universal covering Heisenberg group itself should be considered as a universal covering space of $S^1 \times S^1 \times S^1$, which by means of its corresponding principal bundle instantiation, invites attention to the significance of the discrete Heisenberg group $\mathbb{H}_{\mathbb{Z}}$ in this context.

6 Discrete Heisenberg group and integral signed symplectic area

The discrete Heisenberg group $\mathbb{H}_{\mathbb{Z}}$ as a set is simply:

$$\mathbb{H}_{\mathbb{Z}} := \mathbb{Z}^2 \times \mathbb{Z},$$

and the composition law is defined as follows:

$$(x, y, w)(x', y', w') := (x + x', y + y', w + w' + \frac{1}{2}\omega),$$

where the variables (x, y, w) admit only integer values. Note that the order two nilpotency condition is still characteristic of the discrete Heisenberg group $\mathbb{H}_{\mathbb{Z}}$, and thus, the Baker–Campbell–Hausdorff formula simplifies as follows:

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$$

We deduce that the following sequence of groups is actually a short exact sequence:

$$0 \to \mathbb{Z} \hookrightarrow \mathbb{H}_{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}^2 \to 0$$

where the image of $\mathbb{Z}$ is central in $\mathbb{H}_{\mathbb{Z}}$. The above means that the discrete Heisenberg group $\mathbb{H}_{\mathbb{Z}}$ is the uniquely defined central extension of $\mathbb{Z}^2$ by $\mathbb{Z}$.

For simplicity in the notation, we denote the non-Abelian-free group in two generators Θ_2 as $\Theta_2 := \oplus := \oplus_1$, whence its commutator normal subgroup $[\Theta_2, \Theta_2]$ is denoted as $[\Theta_2, \Theta_2] := \oplus_2$, and so on for higher commutators. We note that $\oplus_1 \supset \oplus_2 \supset \oplus_3 \supset \ldots$, and the Abelianization of $\oplus$ is given by the quotient $\oplus_1/\oplus_2$. Hence, the discrete Heisenberg group $\mathbb{H}_{\mathbb{Z}}$ is actually identified up to isomorphism with the quotient $\oplus_1/\oplus_3$. More precisely, since $\oplus_2 \supset \oplus_3$ the Abelianization homomorphism $\oplus \to \mathbb{Z}^2$, where $\mathbb{Z}^2 = \oplus_1/\oplus_2$, induces a surjective homomorphism:

$$\mathbb{H}_{\mathbb{Z}} = \oplus_1/\oplus_3 \twoheadrightarrow \mathbb{Z}^2 = \oplus_1/\oplus_2$$

whose kernel is the normal subgroup of $\mathbb{H}_{\mathbb{Z}}$ generated by the equivalence class of the commutators $\gamma = [\alpha, \beta^{-1}]$, denoted by the same symbol.

We also notice that any homomorphism $h : \oplus \to \mathbb{Z}$ factors through $\mathbb{H}_{\mathbb{Z}}$, due to the fact that $\oplus_2 \supset \oplus_3$. By a slight abuse of notation we denote these homomorphisms by the same symbol if their domain is restricted to $\mathbb{H}_{\mathbb{Z}}$. Then, for any element ϱ in $\mathbb{H}_{\mathbb{Z}}$, the image $h(\varrho)$ is valued in the integers. We consider two homomorphisms $\bar{x} : \oplus \to \mathbb{Z}$, and $\bar{y} : \oplus \to \mathbb{Z}$, such that:

$$\bar{x}\bar{y} : \oplus \to \mathbb{Z}^2$$

amounts to the Abelianization homomorphism whose kernel is generated by the equivalence class of the commutators $\gamma = [\alpha, \beta^{-1}]$, as previously. We denote by $\bar{x}(\varrho) := x$ and $\bar{y}(\varrho) := y$ the integer images of these homomorphisms for an element ϱ in $\mathbb{H}_{\mathbb{Z}}$. Then, the group $[\mathbb{H}_{\mathbb{Z}}]$ of upper triangular matrices with integer entries of the form:

$$\begin{bmatrix} 1 & x & w \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \Pi(x, y, w)$$

constitutes a representation of the group $\mathbb{H}_{\mathbb{Z}}$. To see how the above works, we may make a convenient gauge choice, such that α and β^{-1} in $\oplus$ are mapped to $\Pi(1, 0, 0)$ and $\Pi(0, 1, 0)$ correspondingly in $[\mathbb{H}_{\mathbb{Z}}]$. Then, the commutator $\gamma = [\alpha, \beta^{-1}]$ is mapped to $\Pi(0, 0, 1)$ being central in $[\mathbb{H}_{\mathbb{Z}}]$. Moreover, γ^k is mapped to $\Pi(0, 0, k)$, which for $k \neq 0$ is always non-trivial.

Next, we consider the lattice $\mathbb{Z}^2$ inside the plane $\mathbb{R}^2$. Let ϱ be the discrete Heisenberg group avatar of a free group $\oplus$ string $\chi_1\chi_2 \ldots \chi_\nu$ in in $\alpha, \beta, \alpha^{-1}, \beta^{-1}$. Each such string gives rise to a path in the plane $\mathbb{R}^2$ as follows: If we consider the origin $(0, 0)$, then we may take the path from $(0, 0)$ to one of the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(-1, -1)$ depending on χ_1 being one of $\alpha, \beta, \alpha^{-1}, \beta^{-1}$ correspondingly. We extend this path in the same manner by joining successively to it the paths corresponding to $\chi_2, \ldots, \chi_{\nu-1}$.

Therefore, we obtain a composite path from the origin $(0, 0)$ to a lattice point described by the integral coordinates (l, m) in the plane $\mathbb{R}^2$. At the final stage ν of this process, we extend the path arrived at the point (l, m) with a final one terminating at the integral coordinates $(l, m) + (1, 0)$, or $(l, m) + (0, 1)$, or $(l, m) + (-1, 0)$, or $(l, m) + (-1, -1)$, depending on χ_ν being one of $\alpha, \beta, \alpha^{-1}, \beta^{-1}$ correspondingly. We may select the gauge $\bar{x}(\varrho) := x = 0$, and $\bar{y}(\varrho) := y = 0$, depicting the integer images of the $\mathbb{Z}$-valued homomorphisms $\bar{x}$, and $\bar{y}$, for an element ϱ in $\mathbb{H}_{\mathbb{Z}}$, so that the obtained path is actually a loop o in the plane $\mathbb{R}^2$. The loop o bounds a region in $\mathbb{R}^2$, whose oriented area is expressed by the integral value w in this gauge. Notice that the integral value w is a measure of signed area, depending on the orientation of the loop o, which is positive for the counterclockwise orientation and negative for the clockwise one.

In more detail, we may consider the integral unit square $\boxdot$ in the plane $\mathbb{R}^2$ described by the vertices (l, m), $(l+1, m)$, $(l, m+1)$, $(l+1, m+1)$. Then, we may define the $\mathbb{Z}$-valued winding number $\vartheta(l, m)$ of a loop o around the undisclosed center of $\boxdot$. In this manner, the signed area of the region bounded by the loop o in the plane $\mathbb{R}^2$ is given by:

$$w = \sum_{(l,m)} \vartheta(l, m)$$

We emphasize at this point that the commutativity condition:

$$\gamma = [\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta = 1,$$

interpreted homologically means that γ defines a boundary, i.e. an area bounding loop of the above form o.

Moreover, this is tantamount to the homological area boundary that is realized on the nilpotent Heisenberg group or its universal covering group in terms of two-dimensional translations taking place in the horizontal distribution. From the above analysis, we have shown that this interpretation is actually rooted in the explicit consideration of the lattice $\mathbb{Z}^2$ inside $\mathbb{R}^2$. In this vein, $\mathbb{R}^2$ stands for the universal covering space of the torus $S^1 \times S^1$, where $\mathbb{Z}^2$ plays the role of the fundamental group of the latter. In turn, the torus $S^1 \times S^1$ provides the realization surface of the Hopf link formed by the homology basis α, β^{-1}, instantiated via the commutativity condition as the homological Abelian shadow of the Borromean link, or more precisely, of its nilpotent Heisenberg avatar.

The homological interpretation pertaining to the modular variables is based on the natural cellular structure of $\mathbb{R}^2$ if we consider the lattice $\mathbb{Z}^2$ inside it, according to the above discussion. Then, an integral unit square $\boxdot$ in the plane $\mathbb{R}^2$ is a face of this cellular complex, whose vertices are points on the lattice and edges are paths joining these vertices. Clearly, there exists a free action of the group $\mathbb{Z}^2$ on this cellular complex expressed through horizontal and vertical integer translations. This action is actually the free group action of the fundamental group $\mathbb{Z}^2$ of the torus $S^1 \times S^1$ on its universal covering space $\mathbb{R}^2$, where each $\mathbb{R}$-copy is to be thought of as a simply-connected spiral unfolding in discrete steps, counted by windings, the corresponding loop in the homology basis.

An interesting observation emerges if we consider the topological graph G obtained by the union of all edges and vertices of the above cellular complex. In particular, the quotient of this topological graph G by the action of $\mathbb{Z}^2$ restricted to it, is equivalent to a bouquet of two topological circles ∞, whose fundamental group is the non-Abelian-free group in two generators $\oplus := \oplus_1$. Then, the topological graph G instantiates the Abelian Galois cover of the bouquet of two circles ∞ that corresponds precisely to the commutator normal subgroup $[\oplus_1, \oplus_1] := \oplus_2$ of $\oplus$.

Consequently, a free group $\oplus$ string $\chi_1\chi_2 \dots \chi_\nu$ in α, β, α^{-1}, β^{-1}, whose discrete Heisenberg group avatar is ϱ, lifts to an 1-chain in the homological Abelian shadow, described by the Abelian Galois cover, and depicted in terms of a path starting at the origin of the topological graph G. Selecting again the gauge $\bar{x}(\varrho) := x = 0$, and $\bar{y}(\varrho) := y = 0$, the above 1-chain becomes an 1-cycle, which in turn, due to the contractibility of the plane, is actually a boundary, i.e. an area bounding loop in the plane. Thus, we obtain again the signed area w of the region bounded by the boundary loop o in the plane $\mathbb{R}^2$.

7 Qualifying the interference pattern through integrality

Recall that the Heisenberg group $\mathbb{H}$, defined as a set by the product $\mathbb{H} := \mathbb{R}^2 \times S^1$, and endowed with the group composition law:

$$(v, \tau)(v', \tau') := (v + v', (\tau + \tau') \cdot \mu(v : v')) = (v + v', (\tau + \tau') \cdot e^{(i/2\hbar)\omega}),$$

where $v := (x, y)$ in $\mathbb{R}^2$, such that the following sequence of groups is exact:

$$1 \to S^1 \hookrightarrow \mathbb{H} \twoheadrightarrow \mathbb{R}^2 \to 0,$$

admits a genuine unitary representation in the Hilbert space:

$$W_\mu : (v, \tau) \mapsto \tau \cdot W(v)$$

such that:

$$W_\mu(1, \tau) = \tau, \quad W_\mu(v, 1) = v.$$

Moreover, the correspondence $W \to W_\mu$ gives rise to a bijection between the set of unitary representations of the group $\mathbb{H}$ and the projective unitary representation of $\mathbb{R} \times \mathbb{R}$, with multiplier $\mu(v : v')$. Thus, since the Heisenberg group $\mathbb{H}$ is represented unitarily on the Hilbert space $\mathcal{H}$, it preserves both the complex-valued inner product and the unit sphere of normalized state vectors in this space $\mathbf{S}\mathcal{H} := \mathfrak{U} = \{|\psi\rangle \in \mathcal{H} | \langle\psi|\psi\rangle = 1\}$.

Taking into account the fact that $\mathbf{S}\mathcal{H}$ constitutes a principal fiber bundle over the space of rays $\mathbf{P}\mathcal{H}$ with structure group S^1,

$$pr : \mathbf{S}\mathcal{H} \to \mathbf{P}\mathcal{H},$$
$$|\psi\rangle \longmapsto pr(|\psi\rangle) = \Psi = |\psi\rangle\langle\psi|,$$

if the continuous Abelian group action of $\mathbb{R}^2$ on the space of rays $\mathbf{P}\mathcal{H}$ instantiates the symmetry group of this space, then the continuous non-Abelian group action of the Heisenberg group $\mathbb{H}$ on the unit sphere $\mathbf{S}\mathcal{H}$ shall instantiate the symmetry group of the latter space correspondingly. Since $\mathbb{H}$ is a central extension of $\mathbb{R}^2$ by S^1, if we project the symmetry action of $\mathbb{H}$ on the unit sphere $\mathbf{S}\mathcal{H}$ onto the symmetry action of $\mathbb{R}^2$ on the space of rays $\mathbf{P}\mathcal{H}$, then the structure group S^1 of the principal fiber bundle $\mathbf{S}\mathcal{H}$ over $\mathbf{P}\mathcal{H}$ is naturally identified with the central normal subgroup S^1 of the Heisenberg group central extension. In other words, since the action of the central subgroup S^1 of the Heisenberg group $\mathbb{H}$ leaves the fibers of the principal bundle $pr : \mathbf{S}\mathcal{H} \to \mathbf{P}\mathcal{H}$ invariant it can be naturally identified with the structure group of this principal bundle. This is important because it shows that the origin of the S^1-invariance of unit state vectors emanates from the subgroup of central characters of the Heisenberg group in its function as a symmetry group of the unit sphere $\mathbf{S}\mathcal{H}$.

In this state of affairs, we remind that an one-parameter unitary group infinitesimally generated by an observable is to be thought of as a based loop up to continuous deformation that can be oriented in two possible ways. In particular, if the observable is a single projector, this based loop may be identified with the orbit of its action on a reference state vector, i.e. the lift of a triangular loop on the space of rays with a distinguished reference vertex. It is instructive to recall that a unitary transformation is an automorphism of the Hilbert space of state vectors preserving the inner product structure.

The inner product between two state vectors, interpreted as the transition amplitude from one to the other, actually pertains to the space of rays. From the latter viewpoint, the inner product expresses the degree of overlap between the corresponding rays or projection operators, where a projection operator bears the function of a potential distinguishability filter, for instance it can distinguish a slit. In this interpretation, the overlap between two rays amounts to the degree of objective indistinguishability between the corresponding states. Thus, a unitary transformation is a transformation which is preserving the degree of objective indistinguishability between states of a quantum system given that a measurement is not taking place.

Concomitantly, the real-valued continuously varying parameter in an one-parameter group of unitary operators associated bijectively with an observable is a parameter indexing continuously the preservation of this degree of objective indistinguishability between quantum states. Therefore, what accounts for the existence of different oriented based loops after the specification of a reference vertex is the existence of non-simultaneously realizable potential filters instantiated by sharply incompatible Boolean frames of projection operators.

The change of perspective enunciated by thinking of one-parameter unitary groups in terms of based loops under the specification of a fixed reference vertex is the fact that different based loops may be composed together in a particular order and reveal connectivity properties that are not visible from the perspective of sharply incompatible, or more precisely, conjugate local Boolean frames. In this context, the meaning of topological connectivity acquires a temporal semantics under the standpoint that quantum transition probability is always bearing the symmetry of a process "evolving" around a loop in "time". In this manner, this pertinent topological linking property transcribes to a synchronization condition of one-parameter unitary groups generated by complementary observables with respect to a reference state vector on which they act on. This synchronization condition gives rise via the process of homological Abelianization, which factors through the symplectic structure of the discrete Heisenberg group, to signed symplectic areas formed by translations expressed in terms of modular conjugate observables, thus bounded by square loops. In turn, these integral symplectic areas give rise to the geometric phase characteristic of the interference pattern on the screen.

The underlying motivation for this consideration comes from the fundamental role that the Borromean link plays in deciphering the Weyl-Heisenberg commutation relations in quantum mechanics. The notion of a link of loops pertains to the domain of topological connectivity, which is qualified temporally in terms of synchronization of one-parameter unitary groups, delineating precisely the type of objective indistinguishability encountered in quantum theory. More concretely, the existence of the Borromean link as a non-splittable 3-link described by the non-vanishing commutator $\gamma = [\alpha, \beta^{-1}]$ in the free non-Abelian group of two generators $\oplus$, is indicative of the topological entanglement between the barriers A, B, and the commutator product loop γ. In turn, this type of entanglement is strictly subordinate to the composition order of the based loop actions $\alpha, \beta^{-1}, \alpha^{-1}, \beta$, which is the hindsight of genuine non-commutativity. The basic notion here is that the entanglement-inducing composite loop γ cannot be deformed to the identity loop in $\oplus$, i.e. it is homotopically non-contractible to the identity.

Due to the representation of the group $\oplus$ in the Hilbert space of states of a quantum system in terms of one-parameter unitary groups, the above setting can be literally transferred in the context of the double slit experiment under the proviso that: (i) There is a distinction or marking between slits and product loops, or equivalently, between slits as projection filters and product actions as composite one-parameter unitary groups with respect to these slits; and (ii) the group commutator product action plays the role of a central element in the symplectic Heisenberg group reflection of the broken threefold symmetry of the Borromean link due to the above imposed distinctions in (i). Note that, in the case of the Borromean link, the act of cutting any one of the three entangled loops leads to complete splittability of the remaining 2-link. Analogously, in the Heisenberg reflection, the act of measurement at any one of the two marked slits does not preserve the degree of indistinguishability between states, and the corresponding unitary group action breaks down, meaning that the interference pattern of the double slit experiment disappears.

It is essential at this point to examine in more detail what is actually involved in the specification of the interference pattern characteristic of the double slit experiment from the proposed topological perspective. The claim is that the interference pattern should be understood as a pattern of double periodicity pertaining to modular observables, i.e. the commuting one-parameter unitary groups of position and momentum, or equivalently, their based loops with respect to the choice of the initial reference state vector. This is based on the conception of the screen where the interference pattern is displayed as the homological Abelian shadow of the non-reducible commutator in the Heisenberg group. In other words, the insertion of a screen is tantamount to a process of Abelianization pertaining to both the unitary translation actions of the position and momentum observables. More precisely, the screen forms a homological area boundary that is realized on the representation of the nilpotent Heisenberg group, or its universal covering group, in terms of two-dimensional unitary translations taking place in the horizontal distribution.

The semantics of this homological Abelian shadow is clarified if we think in link-theoretic terms. In this manner, the commutativity condition amounts to the appearance of the nilpotent Heisenberg reflection of the Borromean link as a Hopf link, which in turn, is realizable on a torus, given that the pertinent based loops give rise to a homology basis for the torus, if linked like this. This is precisely what is accomplished by the discrete Heisenberg group. The aftermath is that on the homological Abelian shadow of modular position and momentum, the marked slits appear as directly linked or entangled, giving rise to an invariant symplectic area characterized by double integral periodicity. This is what qualifies the quantum interference pattern in terms of the notion of the quantum of action, expressed symplectically through Planck's constant.

In this sense, it becomes obvious that sharp specifications of the position or momentum observables, are untenable and actually irrelevant to the deciphering of the double slit interference pattern. What is actually relevant and necessary is the geometric encoding of double integral periodicity in terms of a cellular integral lattice structure of translations in position and momentum unfolding in a simply-connected way the corresponding homology basis loops of the induced torus in these variables. It is exactly this integral lattice structure that qualifies the semantics of objective indistinguishability in the case of the double slit experiment. Remarkably, this physically natural and decisive aspect of double periodicity remains only implicit in the Schrödinger representation of the Heisenberg commutation relations in the position representation.

8 Integrality of the quantum of action and the geometric phase

The signed symplectic area invariant constraining the observed interference pattern in the double slit experiment can be equivalently qualified in terms of the corresponding S^1-valued geometric phase, i.e. in terms of a holonomy group element. In this context, we recall that the notion of a connection in the total space of the Heisenberg group $\mathbb{H}$, or its universal covering group $\bar{\mathbb{H}}$ delimits an Abelian subgroup, and consequently, a commutative subalgebra of the Heisenberg algebra. In other words, the Abelian homological shadow is effectuated with respect to this commutative substructure. An area-delimiting boundary therein qualifies the third coordinate as a geometric phase in the former case, and as a signed symplectic area in the latter case.

Given the toroidal topological structure of this Abelian shadow, according to the preceding, in the geometric simply-connected manifestation of the interference pattern what is essential and decisive is the instantiation of a cellular integral lattice structure of translations in position and momentum bearing an invariant scale or measure of area. This amounts to the requirement that in the unitary representation of the Heisenberg group on the principal S^1-bundle over the space of rays, considered as a symplectic phase space via the natural symplectic structure provided by the Hermitean inner product, the curvature 2-form of this inner-product induced connection, is integral in units of area. Thus, if we consider that the quantum of action defines an invariant scale of area with respect to this lattice, the curvature form should be integral in units of Planck's constant.

In more detail, we may employ the connection 1-form A, defined according to:

$$A(\psi) \cdot \varphi = -Im\langle\psi|\varphi\rangle$$

where Im denotes the imaginary part of the Hermitian inner product. The closed and exact differential 2-form R, i.e. the pertinent curvature 2-form is given by the differential of A, that is:

$$R(\varphi, \psi) = 2Im\langle\varphi|\psi\rangle,$$

identified in terms of the standard symplectic form reduced to the space of rays.

Therefore, its surface integration gives twice the signed symplectic area of the two-dimensional region enclosed by a boundary loop in the space of rays, whence the complex S^1-valued exponential of the same gives the holonomy, or equivalently, the global geometric phase. Notice again that since $S^1 \cong \mathbb{R}/\mathbb{Z}$, symplectic areas of boundary loops differing by an integer provide the same geometric phase. This is important to realize that the Abelian shadow displayed on the interference pattern is of a homological nature, as we will describe in more detail in the sequel.

If we consider the established representation relation:

$$W(x, y)W(x', y') = e^{(i\hbar/2)\omega} W(x + x', y + y'),$$

we also conclude that the commutativity condition constituting the Abelian shadow requires that the complex exponential phase factor should be unity, i.e. that the symplectic area should be integral in units of Planck's constant. At this point, it becomes clear that the instantiation of this type of Abelian shadow on the screen involves the consideration of both the discrete Heisenberg group, and the continuous one, where the first stands for a $\mathbb{Z}$-central extension and the second for an S^1-central extension over a cellular integral lattice structure of translations in position and momentum in the space of rays delineating a maximal commutative subgroup.

It is worth reflecting once again on the peculiar type of non-commutativity encoded in the Heisenberg group via central characters. It is the nilpotent characterization of this group together with its specification via a principal bundle structure endowed with a connection that allows it to mediate between the purely non-commutative, ordered world of the free group in two generators and the Abelian shadow of this world in the manner of the homotopy/homology interrelation.

In this unique way, the type of non-commutativity imposed by the canonical commutation relations can be decoded via a bundle connection to the information of an invariant global geometric phase or of a signed symplectic area integral in units of Planck's constant, with respect to the commutative substructure delineated by the employed connection. In turn, this commutative substructure of a toroidal origin, assumes existence only as a homological Abelian shadow due to the integrality condition, rooted physically on the double periodicity character of the interference pattern on a screen.

9 Cohomological interpretation of modular variables

To gain a deeper insight on the integrality condition, we come back to the fact that $\mathbf{S}\mathcal{H}$ constitutes a principal fiber bundle over the space of rays $\mathbf{P}\mathcal{H}$ with structure group S^1,

$$pr : \mathbf{S}\mathcal{H} \to \mathbf{P}\mathcal{H},$$
$$|\psi\rangle \longmapsto pr(|\psi\rangle) = \Psi = |\psi\rangle\langle\psi|.$$

In this bundle setting, the action of the central subgroup S^1 of the Heisenberg group $\mathbb{H}$ leaves the fibers of the principal bundle $pr : \mathbf{S}\mathcal{H} \to \mathbf{P}\mathcal{H}$ invariant, and therefore, it can be naturally identified with the structure group of this principal bundle. This principal bundle is endowed with a connection induced by the Hermitian inner product.

Instead of the principal bundle structure, we may equivalently focus on the associated complex line bundle structure with fiber $\mathbb{C}$ endowed also with a connection. In this state of affairs it is well-known that the curvature of this complex line bundle assumes the status of a symplectic form if and only if the curvature is integral. More precisely, a global closed 2-form, identified as the curvature $\mathbf{R}$ differential invariant of a complex line bundle with a connection over a base space X functions as a symplectic form if and only if its two-dimensional de Rham cohomology class is integral:

$$[\mathbf{R}] \in H^2(X, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{R})$$

Since the Abelian shadow pertains to the commutative toroidal substructure, according to the preceding, the base X should be identified in this case with an even-dimensional torus in the space of rays, thought of as a compact Abelian group. We focus our attention on the case of the two-dimensional torus:

$$X := T^2 \cong S^1 \times S^1 \cong \left(\frac{\mathbb{R}}{\mathbb{Z}}\right)^2.$$

Since X is a compact Abelian group whose group of characters is $\hat{X}$, its Fourier algebra $\mathcal{A} := \mathcal{A}(X)$, namely the algebra of absolutely convergent Fourier series on X with the l^1-norm, is a complex unital commutative and self-adjoint Banach-Šilov algebra having the Banach approximation property, whose topological spectrum is (homeomorphic to) X [16]. The study of a complex line bundle over the torus from this viewpoint has been first proposed and worked out in detail by Selesnick [22,23]. In the sequel, we follow Selesnick's approach keeping in mind the symplectic rendering of this framework in relation to the role of the discrete Heisenberg group. In this manner, we obtain a cohomological interpretation of the modular variables, which leads to an irreducible unitary representation of the Weyl–Heisenberg group in terms of theta functions.

Each element ν of $\mathbb{Z}^2$ defines a character on T^2, which for Λ in T^2 and υ in $\mathbb{R}^2$, it takes the form:

$$\chi_\nu(\Lambda) = exp(2\pi i \nu \cdot \upsilon)$$

Therefore, we obtain that $\hat{T}^2 \cong \mathbb{Z}^2$. We consider the standard orthonormal basis in $\mathbb{Z}^2$ and denote the corresponding characters by χ_1, χ_2. The important issue is that the modelling of a quantum beam may be expressed in terms of the commutative observable algebra sheaf $\mathbf{A} := \mathbf{A}(T^2)$ of absolutely convergent Fourier series on T^2.

In this way, the pair $(T^2, \mathbf{A})$ constitutes the Gelfand spectrum of the algebra of observables $\mathcal{A}$. Together with the $\mathbb{C}$-algebra sheaf $\mathbf{A}$ we also consider the Abelian group sheaf of invertible elements of $\mathbf{A}$, denoted by $\tilde{\mathbf{A}}$. The set of sections of a complex line bundle L over X forms a sheaf of sections localized over X, called a line sheaf of states $\mathbf{L}$. Locally for any state $x \in X$ there exists an open set U of X such that:

$$\mathbf{L} \mid_U \cong \mathbf{A} \mid_U$$

It is also standard that the set of sections of a line sheaf $\mathbf{L}$ can be equipped locally with the structure of a Hilbert space.

In this setting, the basic result that is established by means of the Chern isomorphism is that:

$$\mathfrak{E} := \mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$$

meaning that each equivalence classes of line sheaves of states in the Picard group $\mathrm{Pic}(X)$ is in bijective correspondence with a cohomology class in the integral two-dimensional cohomology group of X [17]. An equivalence class of line sheaves equipped locally with the structure of a Hilbert space is identified as the carrier state space of a quantum beam [18,31].

Next, we would like to make use of the compact connected Abelian group structure of X. In the case of compact connected Abelian group every non vanishing element of $\mathcal{A}$ can be written as the product of an element of $\exp \mathcal{A}$ with a unique character χ. Hence we obtain:

$$H^1(X,\mathbb{Z}) \cong \frac{\tilde{\mathcal{A}}}{\exp \mathcal{A}} \cong \hat{X} \cong \mathbb{Z}^2$$

Furthermore, the Čech cohomology algebra of X with coefficients in $\mathbb{Z}$ is naturally isomorphic with the exterior algebra being generated by elements of first degree:

$$\wedge^p H^1(X,\mathbb{Z}) \cong H^p(X,\mathbb{Z})$$

Thus, we deduce that:

$$H^p(X,\mathbb{Z}) \cong \wedge^p \hat{X} \cong \wedge^p \mathbb{Z}^k$$

Henceforth, in the case of interest, we obtain:

$$H^p(X,\mathbb{Z}) \cong \wedge^p \hat{X} \cong \wedge^p \mathbb{Z}^2$$

where $p = 1, 2$. Additionally, since X is connected, $\hat{X}$ is torsion-free. Therefore, $\wedge^p \hat{X}$ is also torsion-free, and we finally obtain that $H^p(X,\mathbb{Z})$ is torsion-free. The inclusion of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces the morphism:

$$H^p(X,\mathbb{Z}) \to H^p(X,\mathbb{R}) \cong H^p(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

which is injective, since $H^p(X,\mathbb{Z})$ is torsion-free.

At the next stage, we use the fact that each equivalence class of line sheaves of states is in bijective correspondence with a cohomology class in the integral two-dimensional cohomology group of X in order to obtain the following isomorphisms of Abelian groups:

$$\mathfrak{C} \cong H^2(X,\mathbb{Z}) \cong \wedge^2 \hat{X} \cong \wedge^2 \mathbb{Z}^2 \cong \mathbb{Z}$$

Therefore, if we consider a generator $\chi \wedge \zeta$ of $\wedge^2 \hat{X} \cong \mathbb{Z}$ we obtain a line sheaf of states $\mathbf{L}_{\chi\wedge\zeta}$. Because of the above isomorphism of Abelian groups, we conclude that each line sheaf of states $\mathbf{L}$ on X of a quantum beam is equivalent to a finite tensor product of line sheaves of states of the form $\mathbf{L}_{\chi\wedge\zeta}$ over $\mathbf{A}$:

$$\mathbf{L} \cong \bigotimes_{i,j=1}^{2} \mathbf{L}_{\chi_i\wedge\zeta_j}$$

where the Chern isomorphism δ_c evaluated in $H^2(X,\mathbb{Z})$ is expressed by:

$$\delta_c(\mathbf{L}) = \sum_{i,j=1}^{2} \chi_i \wedge \zeta_j.$$

In addition, since $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$ is injective, the integral Chern cohomology class of a line sheaf of states $\delta_c(\mathbf{L})$ is completely determined by its image in $H^2(X,\mathbb{R})$, namely by its corresponding real Chern cohomology class. This injective morphism in cohomology classes is given according to the above by:

$$\wedge^2\mathbb{Z}^2 \hookrightarrow \wedge^2\mathbb{R}^2$$
$$\mathbb{Z} \hookrightarrow \mathbb{R}$$

We consider the standard orthonormal basis in $\mathbb{Z}^2$ and denote the corresponding characters by χ_1, χ_2. This basis induces a basis in $\wedge^2\mathbb{Z}^2 \cong \mathbb{Z}$, with respect to which the Abelian group $\wedge^2\mathbb{Z}^2 \cong \mathbb{Z}$ can be identified with the group of integer-valued skew-symmetric or alternating forms on $\mathbb{Z}^2$. By utilizing this basis extension by the inclusion $\mathbb{Z}^2 \to \mathbb{R}^2$, we obtain the $\mathbb{R}$-vector space of real-valued skew-symmetric or alternating bilinear forms, which is finally identified with the space $\wedge^2\mathbb{R}^2 \cong \mathbb{R}$.

Therefore, the real Chern class of a line sheaf on X is representable by a real-valued skew-symmetric bilinear form on $\mathbb{R}^2$, which is integer-valued if restricted $\mathbb{Z}^2$. Inversely, any real-valued skew-symmetric bilinear form on $\mathbb{R}^2$, which is integer-valued if restricted to $\mathbb{Z}^2$, represents the real Chern class of a line sheaf on X. Therefore, the above correspondence is a bijective correspondence, and in consonance with the symplectic semantics implicated via the discrete Heisenberg group.

Furthermore, we know that a global closed 2-form is the curvature R of a differential line sheaf, i.e. of a line sheaf equipped with a connection, if and only if its 2-dimensional de Rham cohomology class is integral, that is:

$$[R] \in H^2(X, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{R}) \hookrightarrow H^2(X, \mathbb{C})$$

Therefore, any real-valued skew-symmetric bilinear form on $\mathbb{R}^2$, which is integer-valued if restricted to $\mathbb{Z}^2$, thus qualified as a symplectic form, also represents the curvature 2-dimensional cohomology class of a gauge equivalence class of differential line sheaves on X, and conversely, it is represented in differential terms by a corresponding global closed 2-form [16,18,22].

Because of the fact that each real-valued skew-symmetric bilinear form on $\mathbb{R}^2$, which is integer-valued if restricted to $\mathbb{Z}^2$, determines a uniquely defined line sheaf on X, it is important to examine how we may obtain a line sheaf on X directly from such a form. For this purpose, we consider the standard lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. For υ in $\mathbb{R}^2$ and ν in $\mathbb{Z}^2$, the module of continuous sections of the line sheaf on X is determined by the module of continuous sections of the line sheaf on $\mathbb{R}^2$ via the condition [22]:

$$\psi(\upsilon + \nu) = u_\nu(\upsilon)\psi(\upsilon)$$

where $u_\nu(\upsilon)$ is a unitary phase. This relation expresses the indistinguishability condition in the present case. Therefore, since

$$\psi(\upsilon + \nu + \nu') = u_{\nu+\nu'}(\upsilon)\psi(\upsilon) = u_{\nu'}(\upsilon + \nu)u_\nu(\upsilon)\psi(\upsilon)$$

we obtain the cocycle condition:

$$u_{\nu+\nu'}(\upsilon) = u_{\nu'}(\upsilon + \nu)u_\nu(\upsilon)$$

Therefore, given cocycles $u_\nu(\upsilon)$ as above stalk-wise, we may erect a line sheaf, or equivalently a line bundle, on the torus $X := T^2 \cong S^1 \times S^1 \cong (\frac{\mathbb{R}}{\mathbb{Z}})^2$ by defining for each $\nu \in \mathbb{Z}^2$ the action on $\mathbb{C} \times \mathbb{R}^2$:

$$(z, \upsilon) \mapsto (u_\nu(\upsilon) \cdot z, \upsilon + \nu)$$

The quotient of the trivial complex line bundle on $\mathbb{R}^2$ by the above-defined action of $\mathbb{Z}^2$ provides a complex line bundle on X, and all line bundles on X may be effected in this manner.

We recall that any real-valued skew-symmetric bilinear form on $\mathbb{R}^2$, which is integer-valued if restricted to $\mathbb{Z}^2$, also represents the curvature 2-dimensional cohomology class of a gauge equivalence class of differential line sheaves on X, identified concomitantly as a symplectic form via the discrete Heisenberg group in our framework, and conversely. Let ω be such a symplectic form, and also let $\varsigma : \mathbb{Z}^2 \to \mathbb{R}$ be any function satisfying the following congruence relation, for $\mu, \nu \in \mathbb{Z}^2$:

$$\varsigma_{\mu\nu} := \varsigma(\mu + \nu) = \varsigma_\mu + \varsigma_\nu + \omega(\mu, \nu) \quad \text{mod } 2$$

Then, the complex line bundle on X, defined in terms $u_\nu(\upsilon)$, according to the preceding, admits the specification of the cocycle $u_\nu(\upsilon)$ according to:

$$u_\nu(\upsilon) = \exp \pi i(\varsigma_n + \omega(\mu, \nu))$$

such that, up to isomorphism, it is independent of the choice of ς, and moreover, its real Chern class is represented by ω. Thus, ω is a real symplectic form in $H^2(X, \mathbb{R}) \cong \wedge^2\mathbb{R}^2 \cong \mathbb{R}$, being identified as the image of the integral symplectic form ω in $H^2(X,\mathbb{Z}) \cong \wedge^2\mathbb{Z}^2 \cong \mathbb{Z}$. Note that it is denoted by the same symbol ω due to the injectivity of the map $H^2(X,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{R})$.

We conclude that the module of continuous sections of the line sheaf on X characterized by Chern class ω is determined and isomorphically represented by the module of continuous sections of the line sheaf on $\mathbb{R}^2$, i.e. functions $\rho : \mathbb{R}^2 \to \mathbb{C}$, which satisfy the condition:

$$\rho(\upsilon+\nu) = \rho(\upsilon)\exp\pi i(\varsigma_\nu + \omega(\mu,\nu))$$

These functions $\rho : \mathbb{R}^2 \to \mathbb{C}$ considered as analytic ones are known as theta functions [8,19,20].

Finally, we conclude that since $H^2(X,\mathbb{Z}) \cong \wedge^2\mathbb{Z}^2 \cong \mathbb{Z}$, meaning that the Abelian group of isomorphism classes of line sheaves of states over X, denoted by $\mathfrak{E}$, is isomorphic with the free Abelian group $\mathbb{Z}$, there exists a single generator ω of $\mathfrak{E}$. Therefore, the Hilbert space of sections H_ω, for each integer $\kappa \neq 0$, denoted by the ${H_\omega}^\kappa$, by virtue of the Stone-von Neumann theorem, admits an irreducible unitary representation of the Weyl–Heisenberg group with multiplier

$$\exp\pi i\kappa\frac{\Omega}{h} = \exp\frac{i}{2\hbar}\cdot\kappa\Omega,$$

where the quantum of action appears explicitly, in terms of theta functions.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Aharonov, Y., Anandan, J.: Phase change during a cyclic quantum evolution. Phys. Rev. Lett. **58**(16), 1593–1596 (1987)
2. Aharonov, Y., Bergmann, P.J., Lebowitz, J.L.: Time symmetry in the quantum process of measurement. Phys. Rev. B. **134**(6), 1410–1416 (1964)
3. Aharonov, Y., Cohen, E., Gruss, E., Landsberger, T.: Measurement and collapse within the two-state vector formalism. Quantum Stud. Math. Found. **1**(1–2), 133–146 (2014)
4. Aharonov, Y., Petersen, A., Pendleton, H.: Modular variables in quantum theory. Int. J. Theor. Phys. **2**, 213–230 (1969)
5. Aharonov, Y., Rohrlich, D.: Quantum Paradoxes: Quantum Theory For the Perplexed. Wiley-VCH, London (2003)
6. Berry, M.V.: Quantal phase factors accompanying adiabatic changes. Proc. R. Soc. Lond. A **392**, 45 (1984)
7. Brunn, K.H.: Über Verkettung. Sitzungsbericht der Bayerischen Akademie der Wissenschaft Mathematisch Naturwissenschaftliche Abteilung **22**, 77–99 (1892)
8. Cartier, P. Quantum mechanical commutation relations and theta functions. In: Borel, A., Mostow, G.D. (eds.) *Proceedings of Symposia in Pure Mathematics Volume IX: Algebraic Groups and Discontinuous Subgroups*. American Mathematical Society, Providence (1966)
9. Cromwell, P., Beltrami, E., Rampichini, M.: The Borromean rings. Math. Intell. **20**(1), 53–62 (1998)
10. Debrunner, H. Links of Brunnian type. Duke Math. J. **28**, 17–23 (1961)
11. Gromov, M.: Pseudoholomorphic curves in symplectic manifolds. Invent. Math. **82**, 307–347 (1985)
12. Hatcher, A.: Algebraic Topology. Cambridge University Press, Cambridge (2002)
13. Kawauchi, A. *A Survey of Knot Theory*. Birkhäuser, Boston (1996)
14. Lindström, B., Zetterström, H.-O.: Borromean circles are impossible. Am. Math. Mon. **98**(4), 340–341 (1991)
15. Mackey, G.W.: Unitary representations of group extensions, *I*. Acta Math. **99**, 265–311 (1958)
16. Mallios, A.: *Geometry of Vector Sheaves. An Axiomatic Approach to Differential Geometry, Volume II: Geometry Examples and Applications*. Kluwer, Dordrecht (1998)
17. Mallios, A.: On an axiomatic approach to geometric prequantization: a classification scheme à la Kostant-Souriau-Kirillov. J. Math. Sci. N. Y. **95**, 2648–2668 (1999)
18. Mallios, A., Zafiris, E.: Differential Sheaves and Connections: A Natural Approach to Physical Geometry. World Scientific, New Jersey (2016)
19. Mumford, D.: Tata lectures on theta. III. In: Progress in Mathematics, Vol. 97. Birkhäuser Boston Inc., Boston (1991)
20. Polishchuk, A.: Abelian Varieties, Theta Functions and the Fourier Transform, Cambridge Tracts in Mathematics 153. Cambridge University Press, Cambridge (2003)
21. Schwinger, J.: Quantum Kinematics and Dynamics. Westview Press, Boulder (2000)
22. Selesnick, S.A.: Line bundles and harmonic analysis on compact groups. Math. Z. **146**, 53–67 (1976)
23. Selesnick, S.A.: Rank one projective modules over certain Fourier algebras In: Fourman, M.P., Mulvey, C.J., Scott, D.S. (eds.) Applications of Sheaves. Lecture Notes in Mathematics, vol. 753. Springer, Berlin (1979)

24. Simon, B.: Holonomy, the quantum adiabatic theorem, and Berry's phase. Phys. Rev. Lett. **51**, 2167 (1983)
25. Tollaksen, J., Aharonov, Y., Casher, A., Kaufherr, T., Nussinov, S.: Quantum interference experiments, modular variables and weak measurements. N. J. Phys. **12**, 013023 (2010)
26. Varadarajan, V.S.: Geometry of Quantum Theory, 2nd edn. Springer, New York (1985)
27. von Müller, A., Zafiris, E.: Concept and Formalization of Constellatory Self-Unfolding. On Thinking. Springer, Cham (2018)
28. von Müller A.: *Die Selbstentfaltung der Welt*. Siedler, München (2020)
29. Weyl, H.: Theory of Groups and Quantum Mechanics. Dover, New York (1931)
30. Wilczek, F., Shapere, A.: *Geometric Phases in Physics*. World Scientific, Singapore (1989)
31. Zafiris, E.: The global symmetry group of quantum spectral beams and geometric phase factors. Adv. Math. Phys. Article ID **124393** (2015). https://doi.org/10.1155/2015/124393
32. Zafiris, E.: Loops. Projective invariants and the realization of the Borromean topological link in quantum mechanics. Quantum Stud. Math. Found. **3**, 337–359 (2016)
33. Zafiris, E., Müller, A.v. : On the discrete Heisenberg group and commutative modular variables in quantum mechanics: I. The Abelian symplectic shadow and integrality of area. Quantum Stud. Math. Found. (2021). https://doi.org/10.1007/s40509-021-00251-z

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.