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# Continuity of some operators arising in the theory of superoscillations

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**Abstract** The study of superoscillations naturally leads to the analysis of a large class of convolution operators acting on spaces of entire functions. In particular, the key point is often the proof of the continuity of these operators on appropriate spaces. Most papers in the current literature utilize abstract methods from functional analysis to establish such continuity. In this paper, on the other hand, we rely on some recent advances in the study of entire functions, to offer explicit proofs of the continuity of such operators. To demonstrate the applicability and the flexibility of these explicit methods, we will use them to study the important case of superoscillations associated with quadratic Hamiltonians. The paper also contains a list of interesting open problems, and we have collected as well, for the convenience of the reader, some well-known results, and their proofs, on Gamma and Mittag–Leffler functions that are often used in our computations.

**Keywords** Superoscillations · Entire functions · Infinite order differential operators

## 1 Introduction

The notion of superoscillatory behavior first appears in a series of works of Aharonov and Berry, see [1,12,13,18–20]. In this context, there are good physical reasons for such a behavior, but the discoverers pointed out the apparently paradoxical nature of such functions, thus opening the way for a more thorough mathematical analysis

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of the phenomenon. In the last years, superoscillations have been systematically studied also from the mathematical point of view, see [2–8,11,22] and the monograph [9].

The classical example of superoscillatory function is the following: let $a > 1$ be a real number, we define the sequence of complex valued functions $F_n(x, a)$ defined on $\mathbb{R}$ by

$$F_n(x,a) = \left(\cos\left(\frac{x}{n}\right) + ia\sin\left(\frac{x}{n}\right)\right)^n = \sum_{k=0}^{n} C_k(n,a)\mathrm{e}^{i(1-2k/n)x}, \tag{1}$$

where

$$C_k(n,a) = \binom{n}{k}\left(\frac{1+a}{2}\right)^{n-k}\left(\frac{1-a}{2}\right)^{k}, \tag{2}$$

and $\binom{n}{k}$ denotes the binomial coefficients. The first thing one notices is that if we fix $x \in \mathbb{R}$, and we let $n$ go to infinity, we immediately obtain that

$$\lim_{n\to\infty} F_n(x,a) = \mathrm{e}^{iax}.$$

Moreover, it is not difficult to see that such convergence is uniform on all compact sets in $\mathbb{R}$ but it is not uniform on all of $\mathbb{R}$, see [3]. The representation in terms of $\mathrm{e}^{i(1-2k/n)x}$, together with the calculation of the limit of $F_n(x, a)$ when $n$ goes to infinity, explains why such a sequence is called superoscillatory.

There are several mathematical problems associated with superoscillations and the list, far from being complete, is as follows:

(I) Since superoscillations arise naturally in the context of quantum mechanics, it is important to study the evolution of superoscillatory functions under Schrödinger equation with different potential.
(II) The creation of larger classes of superoscillating functions that extend the fundamental example we described above.
(III) The study of superoscillatory functions in several variable.
(IV) The approximation of the Schwartz test functions and distributions by bounded limited functions associated with superoscillations.
(IV) The approximation of Sato's hyperfunctions by bounded limited functions associated with superoscillations.
(V) The approximation of fractal functions by superoscillations.

The above problems have been under investigations by several authors so that the theory of superoscillations has now become also a mathematical theory.

A common denominator of the above-mentioned problems is that their understanding always relies on the study of the continuity of classes of convolution operators, which appear naturally in connection with the superoscillating functions. These convolution operators mostly operate on spaces of entire functions with growth conditions, to which we will dedicate the next section.

To be more precise, the study of the evolution of superoscillations requires to determine the continuity of operators like

$$P_\lambda(t,\partial_z) = \sum_{n=0}^{\infty} \frac{\lambda(t)^n}{n!}\partial_z^{pn},$$

where $\lambda(t)$ is a given bounded function for the parameter $t \in [0, T]$, and $p$ is a natural number. We will consider these operators as acting on the analytic extension to $\mathbb{C}$ of the functions $F_n(x, a)$.

For historical reasons, the continuity of such convolution operators has been deduced by the theory of the Fourier transform. It turned out that in several cases it is necessary to study convolution operator with coefficients that depend also on the variable $z \in \mathbb{C}$ so we had to study operators of the form

$$Q(t,z,\partial_z) = \sum_{n=0}^{\infty} a_n(t,z)\partial_z^{pn},$$

where $\{a_n(t,z)\}_{n\in\mathbb{N}_0}$ are entire functions in $z$ depending on the parameter $t \in [0,T]$. In this case, we found useful to develop a more direct method that avoids the use of the Fourier transform but uses just the theory of holomorphic functions. This fact has reduced enormously the theoretical tools also for the case of constant coefficients convolution operators that can now be more accessible to audience of non-specialists. For this reason, we compute explicitly a couple of examples to show how these techniques work. Precisely, we show explicitly the continuity of the operators $P_\lambda(t,\partial_z)$ defined above and of the operator

$$U(t,\partial_z) = \sum_{m=0}^{\infty} \frac{(-it/2)^m}{m!}(t+\partial_z)^m \partial_z^m$$

that appears in the evolution of superoscillations in uniform electric field. We conclude this introduction with some bibliographical remarks on recent applications of this theory to different potentials: while the historical development is described in [15], we refer the reader to [22] for the evolution of superoscillations in magnetic field and to [23] for the case of the centrifugal potential. As far as the relations between superoscillations and theory of distributions and hyperfunctions is concerned, the most recent progress is obtained in [24,25]. Finally, an historical introduction to superoscillatory function theory is given in [14].

## 2 Continuity of the convolution operator $P_\lambda(t,\partial_z)$

Let $f$ be a non-constant entire function of a complex variable $z$. We define

$$M_f(r) = \max_{|z|=r} |f(z)|, \quad \text{for } r \geq 0.$$

The non-negative real number $\rho$ defined by

$$\rho = \limsup_{r\to\infty} \frac{\ln \ln M_f(r)}{\ln r}$$

is called the order of $f$. If $\rho$ is finite then $f$ is said to be of finite order and if $\rho = \infty$ the function $f$ is said to be of infinite order.

In the case $f$ is of finite order we define the non-negative real number

$$\sigma = \limsup_{r\to\infty} \frac{\ln M_f(r)}{r^\rho},$$

which is called the type of $f$. If $\sigma \in (0,\infty)$ we call $f$ of normal type, while we say that $f$ is of minimal type if $\sigma = 0$ and of maximal type if $\sigma = \infty$.

**Definition 2.1** Let $p$ be a positive number. We define the class $A_1$ to be the set of entire functions such that there exists $C > 0$ and $B > 0$ for which

$$|f(z)| \leq C \exp(B|z|), \quad \forall z \in \mathbb{C}.$$

To prove our main results we need an important lemma that characterizes the coefficients of entire functions with growth conditions.

**Lemma 2.2** *The function*

$$f(z) = \sum_{j=0}^{\infty} f_j z^j$$

*belongs to* $A_1$ *if and only if there exists* $C_f > 0$ *and* $b > 0$ *such that*

$$|f_j| \leq C_f \frac{b^j}{\Gamma(j+1)}.$$

Lemma 2.2 has been proved in [16] and is a crucial fact in what follows.

We now study, for $p \in \mathbb{N}$, the following operator:

$$P_\lambda(t, \partial_z) = \sum_{n=0}^{\infty} \frac{\lambda(t)^n}{n!} \partial_z^{pn},$$

where $\lambda(t)$ is a complex valued bounded function for $t \in [0, T]$ for some $T \in (0, \infty)$ on the space of entire functions of exponential type. The main result is the following theorem.

**Theorem 2.3** *Let $\lambda(t)$ be a bounded function for $t \in [0, T]$ for some $T \in (0, \infty)$ and let $f \in A_1$. Then, for $p \in \mathbb{N}$, we have $P_\lambda(t, \partial_z) f \in A_1$ and $P_\lambda(t, \partial_z)$ is continuous on $A_1$, that is $P_\lambda(t, \partial_z) f \to 0$ as $f \to 0$.*

*Proof* Let us consider

$$\begin{aligned} P_\lambda(t, \partial_z) f(z) &= \sum_{n=0}^{\infty} \frac{\lambda(t)^n}{n!} \partial_z^{pn} f(z) \\ &= \sum_{n=0}^{\infty} \frac{\lambda(t)^n}{n!} \partial_z^{pn} \sum_{j=0}^{\infty} f_j z^j \\ &= \sum_{n=0}^{\infty} \frac{\lambda(t)^n}{n!} \sum_{j=pn}^{\infty} f_j \frac{j!}{(j-pn)!} z^{j-pn} \\ &= \sum_{n=0}^{\infty} \frac{\lambda(t)^n}{n!} \sum_{k=0}^{\infty} f_{pn+k} \frac{(pn+k)!}{k!} z^k \end{aligned}$$

and now we take the modulus

$$|P_\lambda(t, \partial_z) f(z)| \le \sum_{n=0}^{\infty} \frac{|\lambda(t)|^n}{n!} \sum_{k=0}^{\infty} |f_{pn+k}| \frac{(pn+k)!}{k!} |z|^k$$

and using Lemma 2.2 on the coefficients $f_{pn+k}$ we have the estimate

$$|f_{pn+k}| \le C_f \frac{b^{pn+k}}{\Gamma(pn+k+1)}$$

and using the gamma function estimate, see the Appendix,

$$(a+b)! \le 2^{a+b} a! b!$$

we also have

$$(pn+k)! \le 2^{pn+k} (pn)! k!$$

so we get

$$|P_\lambda(t, \partial_z) f(z)| \le \sum_{n=0}^{\infty} \frac{|\lambda(t)|^n}{n!} \sum_{k=0}^{\infty} C_f \frac{b^{pn+k}}{\Gamma(pn+k+1)} \frac{2^{pn+k}(pn)! k!}{k!} |z|^k.$$

We now use the estimate, see the Appendix,

$$\frac{1}{\Gamma(a+b+2)} \le \frac{1}{\Gamma(a+1)} \frac{1}{\Gamma(b+1)}$$

to separate the two series, so we have

$$\frac{1}{\Gamma(pn - \frac{1}{2} + k - \frac{1}{2} + 2)} \le \frac{1}{\Gamma(pn + \frac{1}{2})} \frac{1}{\Gamma(k + \frac{1}{2})}$$

and so

$$|P_\lambda(t,\partial_z)f(z)| \le C_f \sum_{n=0}^{\infty} \frac{((2b)^p|\lambda(t)|)^n}{n!}\frac{(pn)!}{\Gamma(pn+\frac{1}{2})}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+\frac{1}{2})}(2b|z|)^k.$$

Now observe that the series in $k$ satisfies the estimate

$$\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+\frac{1}{2})}(2b|z|)^k \le C\mathrm{e}^{2b|z|}$$

because of the properties of the Mittag–Lefler function, see the Appendix, for some constant $C > 0$. Now we have to show that the series in $n$ is convergent. In fact, we have that the series

$$\sum_{n=0}^{\infty} \frac{((2b)^p|\lambda(t)|)^n}{n!}\frac{(pn)!}{\Gamma(pn+\frac{1}{2})}$$

has positive terms, so we study the asymptotic behavior. Set

$$A_n := \frac{((2b)^p|\lambda(t)|)^n}{n!}\frac{(pn)!}{\Gamma(pn+\frac{1}{2})}$$

and recall the duplication formula for the Gamma function

$$\Gamma(pn)\Gamma(pn+\frac{1}{2}) = 2^{1-2pn}\sqrt{\pi}\,\Gamma(2pn)$$

we set

$$A_n := \frac{((2b)^p|\lambda(t)|)^n}{n!}\frac{(pn)!\Gamma(pn)}{2^{1-2pn}\sqrt{\pi}\,\Gamma(2pn)}$$

from the functional equation of the gamma function $z\Gamma(z) = \Gamma(z+1)$ we also have

$$A_n := \frac{((2b)^p|\lambda(t)|)^n}{n!}\frac{(pn)!}{2^{1-2pn}\sqrt{\pi}}\frac{\frac{\Gamma(pn+1)}{pn}}{\frac{\Gamma(2pn+1)}{2pn}}$$

$$A_n := \frac{((8b)^p|\lambda(t)|)^n}{n!}\frac{(pn)!}{\sqrt{\pi}}\frac{\frac{\Gamma(pn+1)}{pn}}{\frac{\Gamma(2pn+1)}{pn}}$$

$$A_n := \frac{((8b)^p|\lambda(t)|)^n}{n!}\frac{(pn)!}{\sqrt{\pi}}\frac{\Gamma(pn+1)}{\Gamma(2pn+1)}$$

and so

$$A_n := \frac{((8b)^p|\lambda(t)|)^n}{n!}\frac{(pn)!}{\sqrt{\pi}}\frac{(pn)!}{(2pn)!}$$

and using the Stirling formula $m! \sim \sqrt{2\pi m}(m/e)^m$ we get

$$\begin{aligned}
A_n &\sim \frac{((8b)^p|\lambda(t)|)^n}{n!}\frac{(pn)!}{\sqrt{\pi}}\frac{(pn)!}{(2pn)!}\\
&\sim \frac{1}{\sqrt{\pi}}\frac{((8b)^p|\lambda(t)|)^n}{n!}\frac{[\sqrt{2\pi pn}(pn/e)^{pn}]^2}{\sqrt{2\pi 2pn}(2pn/e)^{2pn}}\\
&\sim \frac{1}{\sqrt{\pi}}\frac{((8b)^p|\lambda(t)|)^n}{n!}\frac{\sqrt{p\pi n}}{4^{pn}}\\
&\sim \sqrt{p}\frac{((2b)^p|\lambda(t)|)^n}{n!}\sqrt{n},
\end{aligned}$$

so the series is convergent. So we set

$$G_\lambda(t) := \sum_{n=0}^{\infty} \frac{((2b)^p|\lambda(t)|)^n}{n!} \frac{(pn)!}{\Gamma(pn+\frac{1}{2})}$$

and we obtain the estimate

$$|P_\lambda(t,\partial_z)f(z)| \le C_f G_\lambda(t) C e^{2b|z|}.$$

This tells that $P_\lambda(t,\partial_z)$ takes $A_1$ into $A_1$ and the continuity follows from the fact that for $C_f \to 0$ we have $|P_\lambda(t,\partial_z)f(z)| \to 0$. □

## 2.1 Some applications

(I) In the case of the harmonic oscillator we have to study the continuity of the operator

$$U(t,\partial_z) := \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{2}\sin t\cos t\right)^n \frac{\partial^{2n}}{\partial z^{2n}}, \tag{3}$$

so the above results apply for $p = 2$ and $\lambda(t) = \frac{i}{2}\sin t\cos t$.

(II) Another example with time-depending coefficients is the following Cauchy problem:

$$i^{m-1}\frac{\partial}{\partial t}\psi(x,t) = \lambda'(t)\frac{\partial^m}{\partial x^m}\psi(x,t), \quad \psi(x,0) = F_n(x,a)$$

where $\lambda(0) = 0$ and $\lambda \in C^1$, using the Fourier transform method we can find the solution that is given by

$$\psi_n(x,t) = \sum_{k=0}^{n} C_k(n,a)e^{ix(1-2k/n)}e^{i\lambda(t)(1-2k/n)}.$$

The solution can be written as

$$\psi_n(z,t) = U(t,\partial_z)F_n(z,a)$$

where

$$U(t,\partial_z) = \sum_{\ell=0}^{\infty} \frac{(i\lambda(t))^\ell}{\ell!}\partial_z^{m\ell}.$$

# 3 The case of the operator of the electric field

In the paper [10], we have considered the evolution of superoscillations and as a corollary of Theorem 3.6 in [10] we have the following known result:

**Corollary 3.1** *Let $a > 1$. Then the solution of the Cauchy problem*

$$i\partial_t\psi(t,x) = -\frac{1}{2}\partial_x^2\psi(t,x) - x\psi(t,x), \quad \psi(0,x) = \sum_{k=0}^{n} C_k(n,a)e^{ix(1-2k/n)}, \tag{4}$$

*is given by*

$$\psi_n(t,x) = \sum_{k=0}^{n} C_k(n,a)e^{-it^3/6}e^{-i(1-2k/n)t((1-2k/n)+t)/2}e^{i((1-2k/n)+t)x}. \tag{5}$$

*Moreover,*

$$\lim_{n\to\infty}\psi_n(t,x)=\mathrm{e}^{-it^3/6}\,\mathrm{e}^{-iat(a+t)/2}\mathrm{e}^{i(a+t)x}.$$

To show the last part of the above theorem, that is, to compute the limit

$$\lim_{n\to\infty}\psi_n(t,x)=\mathrm{e}^{-it^3/6}\,\mathrm{e}^{-iat(a+t)/2}\mathrm{e}^{i(a+t)x}$$

one has to write the solution (5) in terms of convolution operators. Indeed, considering the series expansion

$$\mathrm{e}^{-i(1-2k/n)t((1-2k/n)+t)/2}=\sum_{m=0}^{\infty}\frac{1}{m!}(-i(1-2k/n)t((1-2k/n)+t)/2)^m,$$

we observe that the functions

$$\psi_n(t,x)=\ \mathrm{e}^{-it^3/6}\,\mathrm{e}^{itx}\sum_{k=0}^{n}C_k(n,a)\mathrm{e}^{-i(1-2k/n)t((1-2k/n)+t)/2}\mathrm{e}^{ix(1-2k/n)} \tag{6}$$

can be written in the following way:

$$U(t,\partial_z)=\sum_{m=0}^{\infty}\frac{(-it/2)^m}{m!}(t+\partial_z)^m\partial_z^m$$

(when passing to the complex variable $z$). Thus, the solution becomes

$$\psi_n(t,z)=\ \mathrm{e}^{-it^3/6}\,\mathrm{e}^{itz}U(t,\partial_z)F_n(z,a).$$

The aim of this section is to give a direct proof of the continuity of the operator $U$.

**Theorem 3.2** *The operator*

$$U(t,\partial_z)=\sum_{m=0}^{\infty}\frac{(-it/2)^m}{m!}(t+\partial_z)^m\partial_z^m$$

*acts continuously from $A_1$ into itself.*

*Proof* We have

$$U(t,\partial_z)f(z)=\sum_{m=0}^{\infty}\frac{(-it/2)^m}{m!}(t+\partial_z)^m\partial_z^m\sum_{j=0}^{\infty}f_jz^j$$

and

$$\begin{aligned}U(t,\partial_z)f(z)&=\sum_{m=0}^{\infty}\frac{(-it/2)^m}{m!}(t+\partial_z)^m\partial_z^m\sum_{j=0}^{\infty}f_jz^j\\&=\sum_{m=0}^{\infty}\frac{(-it/2)^m}{m!}\sum_{\ell=0}^{m}\binom{m}{\ell}t^{m-\ell}\partial_z^{\ell+m}\sum_{j=0}^{\infty}f_jz^j\\&=\sum_{m=0}^{\infty}\frac{(-it/2)^m}{m!}\sum_{\ell=0}^{m}\binom{m}{\ell}t^{m-\ell}\sum_{j=\ell+m}^{\infty}f_j\frac{j!}{(j-\ell-m)!}z^{j-\ell-m}\\&=\sum_{m=0}^{\infty}\frac{(-it/2)^m}{m!}\sum_{\ell=0}^{m}\binom{m}{\ell}t^{m-\ell}\sum_{k=0}^{\infty}f_{m+\ell+k}\frac{(m+\ell+k)!}{k!}z^k.\end{aligned}$$

With similar computations, as we did in Theorem 2.3, we get

$$|U(t,\partial_z)f(z)| \le C_f \sum_{m=0}^{\infty} \frac{(|t|/2)^m}{m!} \sum_{\ell=0}^{m} \binom{m}{\ell} |t|^{m-\ell} \sum_{k=0}^{\infty} \frac{b^{m+\ell+k}}{\Gamma(m+\ell+k+1)} \frac{2^{m+\ell+k}(m+\ell)!k!}{k!} |z|^k,$$

and, therefore,

$$|U(t,\partial_z)f(z)| \le C_f \sum_{m=0}^{\infty} \frac{(b|t|)^m}{m!} \sum_{\ell=0}^{m} \binom{m}{\ell} |t|^{m-\ell} (2b)^\ell \frac{(m+\ell)!}{\Gamma(m+\ell+\frac{1}{2})} \sum_{k=0}^{\infty} \frac{(2b|z|)^k}{\Gamma\left(k+\frac{1}{2}\right)}.$$

Now we observe thanks to the duplication formula

$$\frac{(m+\ell)!}{\Gamma(m+\ell+\frac{1}{2})} = 4^{m+\ell} \frac{(m+\ell)!}{2\sqrt{\pi}} \frac{\Gamma(m+\ell)}{\Gamma(2(m+\ell))}$$

and the functional equation of the gamma function $z\Gamma(z) = \Gamma(z+1)$

$$\frac{(m+\ell)!}{\Gamma(m+\ell+\frac{1}{2})} = 4^{m+\ell} \frac{(m+\ell)!}{2\sqrt{\pi}} \frac{\frac{\Gamma(m+\ell+1)}{m+\ell}}{\frac{\Gamma(2(m+\ell)+1)}{2(m+\ell)}}$$

which gives

$$\frac{(m+\ell)!}{\Gamma(m+\ell+\frac{1}{2})} = 4^{m+\ell} \frac{(m+\ell)!}{\sqrt{\pi}} \frac{(m+\ell)!}{(2(m+\ell))!}$$

but since

$$\frac{(n!)^2}{(2n)!} \le 1,$$

we get

$$\frac{(m+\ell)!}{\Gamma(m+\ell+\frac{1}{2})} \le 4^{m+\ell}.$$

So the estimate of the operator becomes

$$|U(t,\partial_z)f(z)| \le C_f \sum_{m=0}^{\infty} \frac{(b|t|)^m}{m!} \sum_{\ell=0}^{m} \binom{m}{\ell} |t|^{m-\ell} (2b)^\ell 4^{m+\ell} \sum_{k=0}^{\infty} \frac{(2b|z|)^k}{\Gamma\left(k+\frac{1}{2}\right)}$$

and since, see the Appendix,

$$\sum_{k=0}^{\infty} \frac{(2b|z|)^k}{\Gamma(k+\frac{1}{2})} \le C\mathrm{e}^{2b|z|}$$

we have

$$|U(t,\partial_z)f(z)| \le C_f \sum_{m=0}^{\infty} \frac{(4b|t|)^m}{m!} \sum_{\ell=0}^{m} \binom{m}{\ell} |t|^{m-\ell} (8b)^\ell C\mathrm{e}^{2b|z|}$$

but

$$\sum_{\ell=0}^{m} \binom{m}{\ell} |t|^{m-\ell} (8b)^\ell = (|t|+8b)^m,$$

we finally get

$$|U(t, \partial_z) f(z)| \leq \ C\, C_f \ \mathrm{e}^{4b|t|(|t|+8b)} \ \mathrm{e}^{2b|z|}$$

and so we get the statement. □

The above proof can be adapted to more general problems like the case when we have the fourth-order operator as in the following example, already considered in [10].
Let $a > 1$, then the solution of the Cauchy problem

$$i\partial_t \psi(t,x) = -\frac{1}{2}\partial_x^2 \psi(t,x) - \frac{3}{8}\partial_x^4 \psi(t,x) - x\psi(t,x), \quad \psi(0,x) = \sum_{k=0}^{n} C_k(n,a)\mathrm{e}^{ix(1-2k/n)},$$

is given by

$$\psi_n(t,x) = \sum_{k=0}^{n} C_k(n,a)\mathrm{e}^{-it^3/6}\,\mathrm{e}^{-i(1-2k/n)t((1-2k/n)+t)/2}\mathrm{e}^{i((1-2k/n)+t)x}\ \mathrm{e}^{i\frac{3}{40}[(t+a)^5-a^5]}. \tag{7}$$

Moreover, we have

$$\lim_{n\to\infty} \psi_n(t,x) = \mathrm{e}^{-it^3/6}\,\mathrm{e}^{-iat(a+t)/2}\mathrm{e}^{i(a+t)x}\mathrm{e}^{i\frac{3}{40}[(t+a)^5-a^5]}.$$

## 4 Some open problems on superoscillations

### 4.1 Approximations of the Weierstrass function

This problem is suggested by a paper of Berry and Morly-Short [17], where they propose to study the representation of fractal function by band-limited sequences of superoscillatory functions. We consider the Weierstrass fractal function

$$W(x,D,\gamma) = \sum_{m=0}^{\infty} \frac{\cos(\gamma^m x)}{\gamma^{m(2-D)}}$$

where $\gamma > 1$ and $D \in (1,2)$ is the fractal dimension of the graph of the function $W$. We will use the superoscillatory function $F_n(x,a)n$ to approximate the function $W$. We recall that uniformly on the compact sets of $\mathbb{R}$ we have

$$\lim_{n\to\infty} F_n(x,a) = \mathrm{e}^{iax}.$$

By the Euler identity we have that

$$W(x,D,\gamma) = \sum_{m=0}^{\infty} \frac{Re(\mathrm{e}^{i\gamma^m x})}{\gamma^{m(2-D)}},$$

so we consider the following problem.

**Problem 4.1** *For $\gamma > 1$ and $D \in (1,2)$, approximate uniformly on the compact sets of $\mathbb{R}$ the function*

$$w(x,D,\gamma) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{i\gamma^m x}}{\gamma^{m(2-D)}}$$

*by the band-limited sequence.*

We observe that

$$w(x,D,\gamma)=\sum_{m=0}^{\infty}\frac{e^{i\gamma^m x}}{\gamma^{m(2-D)}}=\sum_{m=0}^{\infty}\frac{1}{\gamma^{m(2-D)}}\lim_{n\to\infty}F_n(x,\gamma^m)$$

but as we will show in the next few lines, one cannot directly exchange the series and the limit. Indeed observe that

$$\tilde{w}(x,D,\gamma)=\lim_{n\to\infty}\sum_{m=0}^{\infty}\frac{1}{\gamma^{m(2-D)}}F_n(x,\gamma^m)$$

$$\tilde{w}(x,D,\gamma)=\lim_{n\to\infty}\sum_{m=0}^{\infty}\frac{1}{\gamma^{m(2-D)}}\sum_{j=0}^{n}C_j(n,\gamma^m)e^{ix\left(1-\frac{2j}{n}\right)}$$

and also

$$\tilde{w}(x,D,\gamma)=\lim_{n\to\infty}\sum_{j=0}^{n}\sum_{m=0}^{\infty}\frac{1}{\gamma^{m(2-D)}}C_j(n,\gamma^m)e^{ix\left(1-\frac{2j}{n}\right)}.$$

So we obtain

$$\tilde{w}(x,D,\gamma)=\lim_{n\to\infty}\sum_{j=0}^{n}K_j(n,\gamma,D)e^{ix\left(1-\frac{2j}{n}\right)}$$

where we set

$$K_j(n,\gamma,D):=\sum_{m=0}^{\infty}\frac{1}{\gamma^{m(2-D)}}C_j(n,\gamma^m),$$

so we have to compute

$$K_j(n,\gamma,D):=\sum_{m=0}^{\infty}\frac{1}{\gamma^{m(2-D)}}C_j(n,\gamma^m).$$

Since

$$C_j(n,\gamma^m):=\binom{n}{j}\left(\frac{1+\gamma^m}{2}\right)^{n-j}\left(\frac{1-\gamma^m}{2}\right)^{j}$$

we have

$$K_j(n,\gamma,D):=\binom{n}{j}\sum_{m=0}^{\infty}\frac{1}{\gamma^{m(2-D)}}\left(\frac{1+\gamma^m}{2}\right)^{n-j}\left(\frac{1-\gamma^m}{2}\right)^{j},$$

and one immediately sees that the series

$$\sum_{m=0}^{\infty}\frac{1}{\gamma^{m(2-D)}}\left(\frac{1+\gamma^m}{2}\right)^{n-j}\left(\frac{1-\gamma^m}{2}\right)^{j}$$

diverges. But we observe that with the new representation

$$\lim_{n\to\infty}\sum_{j=0}^{n}C_j(n,\gamma^{1/m})e^{it(1-2j/n)^{m^2}}=e^{it\gamma^m}$$

the series

$$L_j(n):=\sum_{m=0}^{\infty}\frac{1}{\gamma^{m(2-D)}}\left(\frac{1+\gamma^{1/m}}{2}\right)^{n-j}\left(\frac{1-\gamma^{1/m}}{2}\right)^{j}$$

converges. The problem is to see if it converges to the Weierstrass fractal function.

### 4.2 The case of continuous $F_n(x, a)$

Another interesting problem is to replace the discrete sequence $F_n(x, a)$ by its continuous counterpart that is obtained by replacing the index $n$ with a continuous variable $u$. In this case, the expression for $F_n$ becomes

$$\mathcal{F}_u(x, a) = \int_0^u \binom{u}{y} \left(\frac{1+a}{2}\right)^{u-y} \left(\frac{1-a}{2}\right)^{y} e^{ix(1-(2y)/u)} dy$$

where

$$\binom{u}{y} = \frac{\Gamma(u+1)}{\Gamma(u-y+1)\Gamma(y+1)},$$

and one would want to study the properties of this family of functions in the same spirit as what has been done so far.

## Appendix

We state in this section some well-known results on the gamma function and the Mittag–Leffler functions that we have used in the proofs.

**Lemma 5.1** *Let $j, k \in \mathbb{N}$, then we have*

$$(j+k)! \leq 2^{j+k} j!k!.$$

*Proof* Let $\binom{p}{j}$ be the binomial coefficients, then it is well known that from the Newton binomial formula, we have

$$2^p = \sum_{j=0}^{p} \binom{p}{j} = \sum_{j=0}^{p} \frac{p!}{j!(p-j)!},$$

so

$$\frac{p!}{j!(p-j)!} \leq 2^p$$

and setting $p - j = k$ we get the statement. □

**Lemma 5.2** *Let $n, k \in \mathbb{N}$, then we have*

$$\Gamma(n+1)\Gamma(k+1) \leq \Gamma(n+k+2).$$

*Proof* Let

$$B(p, q) := \int_0^1 t^{p-1}(1-t)^{q-1} dt$$

be the beta function $B$. Its relation with the gamma function $\Gamma$ is given by

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

This can be shown in two steps. First, with the change of variable $t = \cos^2(\vartheta)$ the beta function can be written as

$$B(p, q) = 2 \int_0^{\pi/2} \cos^{2p-1}(\vartheta) \sin^{2q-1}(\vartheta)\, d\vartheta$$

Second, we observe that

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-t} t^{p-1}\, dt \int_0^\infty e^{-s} s^{q-1}\, ds$$

and with the change of variables

$t = r^2 \cos^2(\vartheta), \quad s = r^2 \sin^2(\vartheta)$

the previous formula becomes

$$\Gamma(p)\Gamma(q) = 4 \int_0^{\pi/2} \int_0^{\infty} r^{2p+2q-1} \mathrm{e}^{-r^2} \cos^{2p-1}(\vartheta) \sin^{2q-1}(\vartheta)\, \mathrm{d}\vartheta\, \mathrm{d}r.$$

By setting $r^2 = u$ in the above relation we obtain

$$\Gamma(p)\Gamma(q) = \Gamma(p+q)B(p,q).$$

Finally, we observe that

$$\frac{\Gamma(n+1)\Gamma(k+1)}{\Gamma(n+k+2)} = B(n+1, k+1) \leq \int_0^1 t^n (1-t)^k \mathrm{d}t \leq \int_0^1 \mathrm{d}t = 1;$$

since, for $t \in [0, 1]$ it is $t^n(1-t)^k \leq 1$, and this ends the proof. □

We conclude with a useful estimate that we have not used in this paper, but it enters into several problems in convolution operators associated with superoscillations.

**Lemma 5.3** *Let* $q \in [1, \infty)$. *Then we have*

$$\Gamma\Big(\frac{n}{q} + 1\Big) \leq (n!)^{1/q}.$$

*Proof* It is a direct consequence of Hölder inequality. Consider $p$ and $q$ such that $1/p + 1/q = 1$, observe that

$$\begin{aligned}\Gamma\Big(\frac{n}{q} + 1\Big) &= \int_0^{\infty} \mathrm{e}^{-t} t^{n/q}\, \mathrm{d}t \\ &= \int_0^{\infty} \mathrm{e}^{-t(1/p+1/q)} t^{n/q}\, \mathrm{d}t,\end{aligned}$$

so we obtain

$$\begin{aligned}\Gamma\left(\frac{n}{q} + 1\right) &= \int_0^{\infty} \mathrm{e}^{-t/q} t^{n/q}\, \mathrm{e}^{-t/p}\, \mathrm{d}t \\ &\leq \left(\int_0^{\infty} \mathrm{e}^{-t} t^n\, \mathrm{d}t\right)^{1/q} \left(\int_0^{\infty} \mathrm{e}^{-t}\, \mathrm{d}t\right)^{1/p} \\ &= \left(\int_0^{\infty} \mathrm{e}^{-t} t^n\, \mathrm{d}t\right)^{1/q} \\ &= (n!)^{1/q}.\end{aligned}$$

□

On the Mittag–Leffler function

The Mittag–Leffler function is defined by its power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \quad \mathrm{Re}(\alpha) > 0.$$

The series converges in the whole complex plane for all $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$. For all $\mathrm{Re}(\alpha) < 0$ it diverges everywhere on $\mathbb{C} \setminus \{0\}$. For $\mathrm{Re}(\alpha) = 0$ the radius of convergence is $R = \mathrm{e}^{\pi |Im(\alpha)|/2}$. The most interesting fact is that for $\mathrm{Re}(\alpha) > 0$ the Mittag–Leffler function is an entire function of finite order. Indeed using Stirling's asymptotic formula

$$\Gamma(\alpha k + 1) = \sqrt{2\pi}(\alpha k)^{\alpha k + 1/2} e^{-\alpha k}(1 + o(1)), \quad \text{for } k \to \infty,$$

so that for

$$c_k = \frac{1}{\Gamma(\alpha k + 1)}$$

for $\alpha > 0$ we have

$$\limsup_{k\to\infty} \frac{k \ln k}{\ln \frac{1}{|c_k|}} = \limsup_{k\to\infty} \frac{k \ln k}{\ln |\Gamma(\alpha k + 1)|} = \frac{1}{\alpha}$$

and

$$\limsup_{k\to\infty} \left(k^{1/\rho} \sqrt[k]{|c_k|}\right) = \limsup_{k\to\infty} \left(k^{1/\rho} \sqrt[k]{\frac{1}{|\Gamma(\alpha k + 1)|}}\right) = (e/\alpha)^{\alpha}.$$

This means that:
*for each $\alpha \in \mathbb{C}$ such that $Re(\alpha) > 0$ the Mittag–Leffler function is an entire function of order $\rho = 1/Re(\alpha)$ and of type $\sigma = 1$.*
This function provides a generalization of the exponential function because we replace $k! = \Gamma(k + 1)$ by $(\alpha k)! = \Gamma(\alpha k + 1)$ in the denominator of the power terms of the exponential series. A useful generalization that we have used in the computations of this paper is the two-parametric Mittag–Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \ \beta \in \mathbb{C}, \quad \mathrm{Re}(\alpha) > 0.$$

The function $E_{\alpha,\beta}(z)$ for $\alpha, \ \beta \in \mathbb{C}$ and $Re(\alpha) > 0$ is an entire function of $\rho = 1/\mathrm{Re}(\alpha)$ and of type $\sigma = 1$ for every $\beta \in \mathbb{C}$.

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