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REGULAR PAPER

A detailed proof of the von Neumann's Quantum Ergodic Theorem

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Abstract We present a simplified proof of the von Neumann's Quantum Ergodic Theorem. This important result was initially published in German by von Neumann in 1929. We are interested here in the time evolution ψ_t, $t \geq 0$ (for large times) under the Schrodinger equation associated with a given fixed Hamiltonian $H : \mathcal{H} \to \mathcal{H}$ and a general initial condition ψ_0. The dimension of the Hilbert space $\mathcal{H}$ is finite.

Keywords von Neumann's Quantum Ergodic Theorem · Generic orthogonal decomposition · Hamiltonian

1 Introduction

Consider a fixed Hamiltonian H (a complex self-adjoint operator) acting on a complex Hilbert space $\mathcal{H}$ of dimension D, where $D \geq 3$. Then, $\mathcal{H}$ can be written as

$$\mathcal{H} = \mathcal{V}_1 \oplus ... \oplus \mathcal{V}_K,$$

where each $\mathcal{V}_a$, $a = 1, 2, ..., K$, is the subspace of eigenvectors associated with the eigenvalue λ_a, and $\lambda_1 < \lambda_2 < ... < \lambda_K$.

We fixed an initial condition ψ_0 for the dynamic Schrodinger evolution. We consider the time evolution $\psi_t = e^{-i\,t\,H}(\psi_0)$, $t \geq 0$, and we are interested in properties for most of the large times (not all large times).

Now, we consider another decomposition $\mathcal{D}$ of $\mathcal{H}$ (which has nothing to do with the previous one)

$$\mathcal{H} = \mathcal{H}_1 \oplus ... \oplus \mathcal{H}_N, \quad N \geq 2.$$

We can consider a natural probability on the set Δ of possible decompositions $\mathcal{D}$ and we are interested here in properties for most of the decompositions $\mathcal{D}$. For small $\delta > 0$, we are interested in the concept of a $(1-\delta)$ generic decomposition $\mathcal{D}$ (in the probabilistic sense).

For a given fixed subspace $\mathcal{H}_\nu$ of $\mathcal{H}$, $\nu = 1, ..., N$, the observable $P_{\mathcal{H}_\nu}$ (the orthogonal projection on $\mathcal{H}_\nu$) is such that the mean value of the state ψ_t, $t \geq 0$, is given by $E_{\psi_t}(P_{\mathcal{H}_\nu}) =< P_{\mathcal{H}_\nu}(\psi_t), \psi_t >= |P_{\mathcal{H}_\nu}(\psi_t)|^2$.

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In the first part of the paper, following the basic guidelines of the original work by von Neumann, we present lower bound conditions (in terms of δ, etc) on the dimensions d_ν, $\nu = 1, 2, .., N$, of the different values of $\mathcal{H}_\nu$ of a $(1-\delta)$-generic orthogonal decomposition $\mathcal{D}$ of the form $\mathcal{H} = \mathcal{H}_1 \oplus ... \oplus \mathcal{H}_N$, in such way that the dynamic time evolution ψ_t, $t \geq 0$, of a given ψ_0, for most of the large times t, has the property that the expected value $E_{\psi_t}(P_{\mathcal{H}_\nu})$ is almost $\frac{d_\nu}{D}$. In this way, there is an approximately uniform spreading of ψ_t among the different values of $\mathcal{H}_\nu$ of a generic decomposition $\mathcal{D}$. In this part, the main result is Theorem 15. We point out that these estimates are for a fixed initial condition ψ_0.

The von Neumann's Quantum Ergodic Theorem provides uniform estimates for all ψ_0. This result is presented in Theorem 19. This will be done in the second part of the paper which begins in Sect. 4. To get this theorem, it will be necessary to assume hypothesis on the eigenvalues of the Hamiltonian H (see hypothesis $\mathfrak{N}\,\mathfrak{R}$ just after Lemma 16).

Suppose, for instance, that $A : \mathcal{H} \to \mathcal{H}$ is an observable and this self-adjoint operator has spectral decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus ... \oplus \mathcal{H}_N,$$

where $\mathcal{H}_p$, $p = 1, ..., N$ is the subspace of eigenvectors associated with the eigenvalue β_p and $\beta_1 < \beta_2 < ... < \beta_N$. The probability that the measurement of A on the state ψ_t is β_p is given by $< P_{\mathcal{H}_p}(\psi_t), \psi_t >$. This shows the relevance of the result. The point of view here is not to look for generic observables but for generic decompositions.

We stress a point raised on [3]. What is proved is a property of the kind: for most $\mathcal{D}$, something is true for all ψ_0. In addition, not a property of the kind: for all ψ_0, something is true for most $\mathcal{D}$.

Of course, the main result can also be stated in terms of limits, when $T \to \infty$, of means $\frac{1}{T}\int E_{\psi_t}(P_{\mathcal{H}_\nu})\mathrm{d}t$, which is a more close expression to the one present in the classical Ergodic Theorem.

We present here a simplified proof (with less hypothesis in some parts) when $\dim \mathcal{H}$ is finite of this important result which was initially published in German by von Neumann in 1929 (see [6]). The paper [5] presents a translation from German to English of this work of von Neumann. This 1929 paper also considers the concept of Entropy for such setting. We will not consider this topic in our note.

Several papers with interesting discussions about this work appeared recently (see, for instance, [1–3,5] and other papers which mention these four)

Consider a general connected compact Riemannian manifold X and its volume form. When properly normalized, this procedure defines a natural probability w_X over X.

Given a compact Lie group (real) G, one can consider the associated bi-invariant Riemannian metric. If H is a closed subset of G, this metric can be considered in the quotient space $X = \frac{G}{H}$, and in this way, we get a probability on such manifold X. We will denote by π the projection.

When we consider expected values of a function f, this we will be taken with respect to the above-mentioned probability.

Lemma 1 *Given a continuous function $f : X \to \mathbb{C}$ and $\pi : G \to X$ the canonical projection, then*

(a) $vol\,(S) = \dfrac{vol\,(\pi^{-1}(S))}{vol\,(H)}$

for every Borel set $S \subset X$, and

(b) $E_X(f) = E_G(f \circ \pi)$.

The first integral is taken with respect to the volume form w_X and the second with respect to the volume form w_G.

Note that vol $(G) =$ vol (X) vol (H).

The proof is left for the reader.

Suppose $\mathcal{H}$ is a complex Hilbert space of finite dimension D with an inner product $<, >$ and a norm $|\ \ |$.

Suppose we fix a decomposition $\mathcal{D}$, that is

$$\mathcal{D} : \mathcal{H} = \mathcal{H}_1 \oplus ... \oplus \mathcal{H}_N$$

$N > 1$, is a orthogonal direct sum, where $\dim \mathcal{H}_\nu = d_\nu > 0$ for all $\nu = 1, 2, ..., N$.

Denote P_ν the orthogonal projection of $\mathcal{H}$ over $\mathcal{H}_\nu$.

Moreover, $S = \{\psi \in \mathcal{H} \,|\, |\psi| = 1\}$ denotes the unitary sphere. S has a Riemannian structure with a metric induced by the norm in $\mathcal{H}$. In the same way as before, there is an associated probability w_S is S.

Lemma 2 *For any* $\nu = 1, 2..., N$,

$$E_S(|P_\nu\,(\,.\,)|^2) = \int_S |P_\nu\,(\phi)\,|^2\, d\, w_S(\phi) = \frac{d_\nu}{D}.$$

Proof Suppose ν is fixed, then take $\psi_1, \psi_2, ..., \psi_D$, and orthogonal basis of $\mathcal{H}$, such that $\psi_1, \psi_2, ..., \psi_{d_\nu}$ is an orthogonal basis of $\mathcal{H}_\nu$.

Given $\phi = \sum_{j=1}^{D} x_j\, \psi_j \in S$, where $\sum_{j=1}^{D} |x_j|^2 = 1$, then

$$\int_S |P_\nu\,(\phi)\,|^2\, d\, w_S(\phi) = \int_S \sum_{j=1}^{d_\nu} |x_j|^2 d\, w_S(x).$$

Note that the integral $\int_S |x_j|^2 d\, w_S(x)$ is independent of j and

$$\int_S \sum_{j=1}^{D} |x_j|^2 d\, w_S(x) = \text{ vol } (S) = 1.$$

Therefore, for any j

$$\int_S |x_j|^2 d\, w_S(x) = \frac{1}{D}.$$

Therefore, it follows that

$$\int_S \sum_{j=1}^{d_\nu} |x_j|^2 d\, w_S(x) = \frac{d_\nu}{D}.$$

□

Lemma 3 *For any* $\nu = 1, 2..., N$,

$$Var_S(|P_\nu\,(\,.\,)\,|^2) = \int_S \left(|P_\nu\,(\phi)|^2 - \frac{d_\nu}{D}\right)^2 d\, w_S(\phi) = \frac{d_\nu\,(D - d_\nu)}{D^2\,(D+1)}.$$

Proof To simplify the notation we take $\nu = 1$. Then, we denote $d = d_1$ and $P = P_1$.

Take $\psi_1, \psi_2, ..., \psi_D$, and orthogonal basis of $\mathcal{H}$, such that, $\psi_1, \psi_2, ..., \psi_d$ is an orthogonal basis of $\mathcal{H}_1$.

By last Lemma, we have

$$\begin{aligned}\int_S \left(|P\,(\phi)|^2 - \frac{d}{D}\right)^2 d\, w_S(\phi) &= \int_S |P\,(\phi)|^4\, d\, w_S(\phi) - 2\,\frac{d}{D} \int_S |P\,(\phi)|^2\, d\, w_S(\phi) + \left(\frac{d}{D}\right)^2 \\ &= \int_S |P\,(\phi)|^4\, d\, w_S(\phi) - \left(\frac{d}{D}\right)^2.\end{aligned}$$

If $\phi = \sum_{j=1}^{D} x_j\, \psi_j \in S$, then $P(\phi) = \sum_{j=1}^{d} x_j\, \psi_j$.

Therefore

$$\int_S |P\,(\phi)|^4\, d\, w_S(\phi) = \frac{1}{\text{vol (S)}} \int_S (\sum_{j=1}^{d} |x_j|^2)^2\, \mathrm{d}S(x) = \frac{d^2 + d}{D\,(D+1)}.$$

The last equality follows from a standard computation (see "Appendix 1").

From this follows the claim. □

2 Changing the decomposition

$\mathcal{H}$ is fixed for the rest of the paper.

Now, we change our point of view. We fix $\phi \in \mathcal{H}$ and we consider different decompositions of $\mathcal{H}$ in direct sum. More precisely, we fix $D = \dim \mathcal{H}$ and N and we consider fixed natural positive numbers d_ν, $\nu = 1, 2, ..., N$, such that $d_1 + d_2 + ... + d_N = D$, and then, all possible choices of orthogonal decompositions with this data.

We denote by $\Delta(d_1, d_2, ..., d_N, \mathcal{H}) = \Delta$ the set of all possible $\mathcal{D}$, that is, all possible orthogonal direct sum decompositions:

$$\mathcal{D} : \mathcal{H} = \mathcal{H}_1 \oplus ... \oplus \mathcal{H}_N.$$

For fixed $\nu = 1, 2, ..., N$, then $P_\nu(\mathcal{D})$ denotes the projection on $\mathcal{H}_\nu$ associated with the decomposition $\mathcal{D}$.

Each choice of orthogonal basis $\psi_1, \psi_2, ..., \psi_D$ of $\mathcal{H}$ defines a possible choice of direct orthogonal sum decomposition:

$$\mathcal{H}_1 \text{ is generated by } \{\psi_1, ..., \psi_{d_1}\}, \ \mathcal{H}_2 \text{ is generated by } \{\psi_{d_1+1}, ..., \psi_{d_1+d_2}\},$$

and so on.

The set of all orthogonal basis is identified with the set of unitary operators $U(D)$ which defines a compact Lie group and a Haar probability structure.

In this way,

$$\Delta = \frac{U(D)}{U(d_1) \times U(d_2) \times ... \times U(d_N)}.$$

In the same way as before, we get a probability w_Δ over Δ. Therefore, it has a meaning the probability $w_\Delta(B)$ of a Borel set $B \subset \Delta$ of decompositions.

Lemma 4 *Consider a continuous function $f : \mathbb{R} \to \mathbb{R}$. Then, for fixed $\nu = 1, 2..., N$, and fixed $\tilde{\phi}$ and $\tilde{\mathcal{D}}$*

$$\int_S f(|P_\nu(\tilde{\mathcal{D}})\, \phi\,|)\, d\, w_S(\phi) = \int_\Delta f(|P_\nu(\mathcal{D})\, \tilde{\phi}\,|)\, d\, w_\Delta(\mathcal{D}).$$

This constant value is independent of $\tilde{\phi}$ and $\tilde{\mathcal{D}}$.

Proof If $U : \mathcal{H} \to \mathcal{H}$, is unitary, then $U\,\mathcal{D}$ denotes

$$U(\mathcal{H}_1) \oplus ... \oplus U(\mathcal{H}_N).$$

Then, for fixed ϕ and $\mathcal{D}$, we have

$$P_\nu(U\,\mathcal{D})\, U\,(\phi) = U\ P_\nu(\mathcal{D})\phi.$$

We prove the claim for P_1. Suppose $\psi_1, \psi_2, ..., \psi_D$, is an orthogonal basis of $\mathcal{H}$, such that $\psi_1, \psi_2, ..., \psi_{d_1}$ is an orthogonal basis of $\mathcal{H}_1$.

We can express $\phi = \sum_{j=1}^D x_j\, \psi_j$, and moreover, $U(\phi) = \sum_{j=1}^D x_j\, U(\psi_j)$.

$U(\psi_1), U(\psi_2), ..., U(\psi_D)$ is an orthogonal basis of $\mathcal{H}$ associated with $U\,\mathcal{D}$ and $U(\psi_1), U(\psi_2), ..., U(\psi_{d_1})$ is an orthogonal basis of $U(\mathcal{H}_1)$.

Then,

$$P_1(U\,\mathcal{D})\, U\,(\phi) = P_1(U\,\mathcal{D}) \left(\sum_{j=1}^{D} x_j\, U(\psi_j) \right) = \sum_{j=1}^{d_1} x_j\, U(\psi_j).$$

By the other hand

$$U\ P_1(\mathcal{D})\phi = U\ P_1(\mathcal{D}) \left(\sum_{j=1}^{D} x_j\, \psi_j \right) = U \left(\sum_{j=1}^{d_1} x_j\, \psi_j \right) = \sum_{j=1}^{d_1} x_j\, U(\psi_j),$$

and this shows the claim.

Therefore, we get

$$| P_\nu(U\,\mathcal{D})\,U\,(\phi)\,| \;= |\,U^{-1}\;P_\nu(U\,\mathcal{D})\,U\,(\phi)\,| \;= |\,U^{-1}U\;P_\nu(\mathcal{D})\phi| = |\,P_\nu(\mathcal{D})\phi|.$$

Finally, for a fixed $\mathcal{D}$ and a variable U

$$\int_S f(|P_\nu(\mathcal{D})\,\phi\,|)\,d\,w_S(\phi) \;= \int_S f(|\,P_\nu(U\,\mathcal{D})\,U\,(\phi)\,|)\,d\,w_S(\phi) = \int_S f(|\,P_\nu(U\,\mathcal{D})\;(\phi)\,|)\,d\,w_S(\phi),$$

because w_S is invariant by the action of U.

Then, the above integral on the variable ϕ is constant by the action of U in a given decomposition $\mathcal{D}$.

Now, consider a fixed ϕ_1 and another general $\phi_2 = U(\phi_1)$, where U is unitary.

As w_Δ is invariant by the action of U, the integral

$$\begin{aligned}\int_\Delta f(|P_\nu(\mathcal{D})\,\phi_2\,|)\,d\,w_\Delta(\mathcal{D}) &= \int_\Delta f(|P_\nu(U\,\mathcal{D})\,U(\phi_1)\,|)\,d\,w_\Delta(\mathcal{D}) = \int_\Delta f(|U\;\;P_\nu(\mathcal{D})\,\phi_1\,|)\,d\,w_\Delta(\mathcal{D})\\ &= \int_\Delta f(|\;\;P_\nu(\mathcal{D})\,\phi_1\,|)\,d\,w_\Delta(\mathcal{D})\end{aligned}$$

is constant and independent of ϕ.

Remember that $w_S \times w_\Delta$ is a probability.

Consider now

$$\begin{aligned}\int\int f(|P_\nu(\mathcal{D})\,\phi\,|)\,d\,w_S(\phi)d\,w_\Delta(\mathcal{D}) &= \int\left[\int f(|P_\nu(\mathcal{D})\,\phi\,|)\,d\,w_S(\phi)\right]d\,w_\Delta(\mathcal{D})\\ &= \int\left[\int f(|P_\nu(\mathcal{D})\,\phi\,|)\,d\,w_\Delta(\mathcal{D})\right]d\,w_S(\phi),\end{aligned}$$

then by Fubini, we get the claim of the Lemma (since the unitary group acts transitively on S and on Δ). □

Corollary 5 *Consider a fixed $\phi \in \mathcal{H}$, such that $|\phi| = 1$.*

Then, for $\nu = 1, 2..., N$, we get that

$$E_\Delta(|P_\nu\,(\,.\,)(\phi)\,|^2) = \frac{d_\nu}{D},$$

and

$$Var_\Delta(|P_\nu\,(\,.\,)(\phi)\,|^2) = \frac{d_\nu\,(D-d_\nu)}{D^2\,(D+1)},$$

where . denotes integration with respect to $\mathcal{D}$.

Proof This is consequence of Lemmas 2, 3, and 4. □

Definition 6 Given $\delta > 0$, a Hilbert space $\mathcal{H}$ and natural positive numbers d_j, $j = 1, 2, ..., N$, such that $d_1 + d_2 + ... + d_N = D = \dim \mathcal{H}$, we say that a property is true for $\mathcal{D} \in \Delta(d_1, .., d_N, \mathcal{H})$, in $(1-\delta)$ sense, if the property is not true only for elements $\mathcal{D}$ in a set of probability w_Δ smaller than δ.

Corollary 7 *Suppose $\epsilon > 0$ and $\delta > 0$ are given. Consider natural positive numbers d_ν, $\nu = 1, 2, ..., N$, such that $d_1 + d_2 + ... + d_N = D = \dim \mathcal{H}$, and moreover, assume that for all $\nu = 1, 2..., N$*

$$d_\nu > D - \frac{\epsilon^2\,\delta D\,(D+1)}{N^2}.$$

Consider a fixed ϕ such that $|\phi| = 1$. Then, for decompositions, $\mathcal{D} \in \Delta(d_1, .., d_N, \mathcal{H})$ in the $(1-\delta)$ sense, and $\nu = 1, 2..., N$, we have

$$|\;|P_\nu\,(\mathcal{D})(\phi)\,|^2 - \frac{d_\nu}{D}\;\;| < \epsilon\sqrt{\frac{d_\nu}{D\,N}}. \tag{1}$$

Proof By Corollary 5 and Markov inequality, we have

$$w_\Delta \left(\left[|P_\nu\,(\mathcal{D})(\phi)\,|^2 \;-\; \frac{d_\nu}{D} \right]^2 \geq \epsilon^2 \frac{d_\nu}{D\,N} \right) \leq \frac{d_\nu\,(D-d_\nu)}{D^2\,(D+1)}\,\frac{D\,N}{\epsilon^2\,d_\nu} = \frac{N\,(D-d_\nu)}{\epsilon^2\,D\,(D+1)}.$$

Then, the probability that all N inequalities do not happen is

$$1 - N\,\frac{N\,(D-d_\nu)}{\epsilon^2\,D\,(D+1)} > 1-\delta$$

by hypothesis. □

The corollary above means that for a fixed ϕ, if the d_ν are all not very small, then for a big part of the decompositions $\mathcal{D}$, we have that

$$|P_\nu\,(\mathcal{D})(\phi)\,|^2$$

is close by the mean value $\frac{d_\nu}{D}$.

Definition 8 Given a Hilbert space $\mathcal{H}$ and a fixed decomposition $\mathcal{D}$ (associated with natural positive numbers d_j, $j=1,2,...,N$, such that $d_1+d_2+...+d_N = D = \dim \mathcal{H}$, we define a semi-norm in such a way that for a linear operator $\rho : \mathcal{H} \to \mathcal{H}$, by

$$|\,\rho\,|_\infty = |\,\rho\,|_\infty^{\mathcal{D}} = \sup_{1\leq\nu\leq N} |\,\mathrm{Tr}\,(\rho\;\; P_\nu(\mathcal{D})\,|$$

The above means that if $|\,\rho\,|_\infty$ is small, then all expected values $E_{P_\nu}(\rho)$, $\nu=1,2,...,N$, are small

$|\,\phi><\phi\,|$ will denote the orthogonal projection on the unitary vector ϕ in the Hilbert space $\mathcal{H}$.

Lemma 9 *Consider a $\phi \in \mathcal{H} = \mathcal{H}_1 \oplus ... \oplus \mathcal{H}_N$, such that $|\phi| = 1$. Denote $\rho_{mc} = \frac{1}{D} I_{\mathcal{H}}$.*

Then

$$|\;|\,\phi><\phi\,| \;-\; \rho_{mc}\,|_\infty \;\;=\; \sup_{1\leq\nu\leq N} \;|\;|P_\nu(\mathcal{D})\,(\phi)\,|^2 - \frac{d_\nu}{D}\;\;|.$$

Proof Suppose $\psi_1,\psi_2,...,\psi_D$ is orthogonal basis of $\mathcal{H}$, such that $\psi_1,\psi_2,...,\psi_{d_1}$ is an orthogonal basis of $\mathcal{H}_1$.

If $\phi = \sum_{j=1}^D x_j\phi_j$, then for $i = 1,2,...,d_1$

$$|\,\phi><\phi\,|\,|P_1(\phi_i)>=|\,\phi><\phi\,|\,|\phi_i>=\sum_{j=1}^{D} \overline{x_i}\,x_j\,\phi_j$$

and

$$|\,\phi><\phi\,|\,|P_1(\phi_i)>=0$$

for $i > d_1$.

Therefore

$$\mathrm{Tr}\;[\,|\,\phi><\phi\,|\,|P_1\,(.)\;>\,] = \sum_{j=1}^{d_1}\;|x_j|^2 \;=\; |P_1(\phi)|^2.$$

In an analogous way, we have that for any ν

$$\mathrm{Tr}\;[\,|\,\phi><\phi\,|\,|P_\nu\,(.)\;>\,] = \;|P_\nu(\phi)|^2.$$

From this follows the claim. □

From the above, it follows:

Corollary 10 *Under the hypothesis of Corollary 7, we get that for decompositions $\mathcal{D} \in \Delta(d_1,..,d_N,\mathcal{H})$ in the $(1-\delta)$ sense*

$$|\;|\,\phi><\phi\,| \;-\; \rho_{mc}\,|_\infty \;\;\leq\; \sup_{1\leq\nu\leq N} \;\epsilon\sqrt{\frac{d_\nu}{N\,D}}.$$

□

3 Estimations on time

Definition 11 Given $\delta > 0$, we say that a property for the parameters $t \in \mathbb{R}$ is true for $(1-\delta)$-most of the large times, if

$$\liminf_{T\to\infty} \frac{1}{T}\mu(A_T) \;>\; 1-\delta,$$

where A_T is the set of $t \in [0, T]$, where the property is verified and μ is the Lebesge measure on $\mathbb{R}$.

Lemma 12 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and non- negative. Consider a certain $\gamma > 0$.*

Suppose ρ is such that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T f(t)\,\mathrm{d}t \;<\; \rho.$$

Then, $f(t) < \gamma$ for $1-\frac{\rho}{\gamma}$-most of the large times.

Proof

$$\int_0^T f(t)\,\mathrm{d}t \ge \int_{f(t)\ge\gamma}^T f(t)\,\mathrm{d}t \ge \gamma\,\mu(\{t \in [0,T] \,|\, f(t) \ge \gamma\}).$$

Therefore

$$\limsup_{T\to\infty} \frac{1}{T}\mu(\{\,t \in [0,T]\; f(t) \ge \gamma\,\} \;<\; \frac{\rho}{\gamma},$$

and finally

$$\liminf_{T\to\infty} \frac{1}{T}\mu(\{\,t \in [0,T]\; f(t) < \gamma\,\} \;>\; 1-\frac{\rho}{\gamma}.$$

□

Suppose $\mathcal{H}$ is Hilbert space, and d_j, $j = 1, 2, ..., N$ are such that $d_1 + d_2 + ... + d_N = D = \dim \mathcal{H}$, and $H : \mathcal{H} \to \mathcal{H}$ a self-adjoint operator. Consider a fixed $\phi_0 \in \mathcal{H}$, with $|\phi_0| = 1$, and $\psi_t = e^{-i\,t\,H}\,\phi_0$, $t \ge 0$, a solution of the associated Schrodinger equation.

Lemma 13 *For fixed T and $\nu = 1, 2, ..., N$, consider the function*

$f_{\nu,T} : \Delta(d_1, d_2, ..., d_N, \mathcal{H}) \times\, S \to \mathbb{R}$,

given by

$$f_{\nu,T}(\mathcal{D},\phi) \;=\; \frac{1}{T}\int_0^T \left(|P_\nu(\mathcal{D})\,\psi_t\,|^2 - \frac{d_\nu}{D}\right)^2 \mathrm{d}t.$$

Then, $f_{\nu,T}$ converges uniformly on $(\mathcal{D},\phi) \in \Delta(d_1, d_2, ..., d_N, \mathcal{H}) \times S$ when $T \to \infty$, for any $\nu = 1, 2, ..., N$.

Proof Suppose $\phi_1, \phi_2, ..., \phi_D$ is a set of eigenvectors of H which is an orthonormal basis of $\mathcal{H}$.

Assume that $\phi_0 = \sum_{j=1}^{D} x_j \phi_j$. Then

$$\psi_t = \sum_{j=1}^{D} x_j\, e^{-i\,t\,E_j}\,\phi_j,$$

where E_j, $j = 1, 2, .., D$ are the corresponding eigenvalues.

Then, for a given ν

$$|P_\nu(\mathcal{D})\,\psi_t\,|^2 =< \psi_t,\, P_\nu(\mathcal{D})\,(\psi_t) >= \sum_{\alpha,\beta} x_\alpha \overline{x_\beta} e^{-i\,t\,(E_\alpha - E_\beta)} \phi_j < \phi_\alpha,\, P_\nu(\mathcal{D})\,(\phi_\beta) > .$$

Therefore

$$\left(|P_\nu(\mathcal{D})\,\psi_t\,|^2 - \frac{d_\nu}{D}\right)^2 = \sum_{w=1}^{M} L_{w,\nu}(\mathcal{D},\phi)\, e^{i\,u_w t},$$

where $M \in \mathbb{N}$, $u_1, .., u_M$ are real constants and $|\, L_{w,\nu}(\mathcal{D},\phi)\,| \leq 2$.

Then

$$f_{\nu,T}(\mathcal{D},\phi) = \sum_{u_w=0}^{M} L_{w,\nu}(\mathcal{D},\phi) + \frac{1}{T}\sum_{u_w\neq 0}^{M} L_{w,\nu}(\mathcal{D},\phi)\left(\frac{e^{i\,u_w\,T}}{i\,u_w} - \frac{1}{i\,u_w}\right).$$

Finally, we get

$$|\, f_{\nu,T}(\mathcal{D},\phi) - \sum_{u_w=0}^{M} L_{w,\nu}(\mathcal{D},\phi)\,| \leq \frac{1}{T}\,\frac{4\,M}{\inf_{u_w\neq 0}\,|u_w|}.$$

As M is fixed, the claim follows from this. □

Corollary 14

$$\int_\Delta \left(\lim_{T\to\infty}\frac{1}{T}\int_0^T \left(|P_\nu(\mathcal{D})\,\psi_t\,|^2 - \frac{d_\nu}{D}\right)^2 \mathrm{d}t\right) d\mathrm{w}_\Delta(\mathcal{D}) = \frac{d_\nu\,(D-d_\nu)}{D^2\,(D+1)},$$

for any $\nu = 1, 2, .., N$.

Proof By Lemma 13 and Corollary 5, we have that

$$\begin{aligned}
&\int_\Delta \left[\lim_{T\to\infty}\frac{1}{T}\int_0^T \left(|P_\nu(\mathcal{D})\,\psi_t\,|^2 - \frac{d_\nu}{D}\right)^2 \mathrm{d}t\right] d\mathrm{w}_\Delta(\mathcal{D})\\
&= \lim_{T\to\infty}\frac{1}{T}\int_\Delta d\mathrm{w}_\Delta(\mathcal{D})\left(\int_0^T \left(|P_\nu(\mathcal{D})\,\psi_t\,|^2 - \frac{d_\nu}{D}\right)^2 \mathrm{d}t\right)\\
&\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t \int_\Delta \left(|P_\nu(\mathcal{D})\,\psi_t\,|^2 - \frac{d_\nu}{D}\right)^2 d\mathrm{w}_\Delta(\mathcal{D}) = \frac{d_\nu\,(D-d_\nu)}{D^2\,(D+1)}.
\end{aligned}$$

□

Theorem 15 *Suppose* $\epsilon > 0$, $\delta > 0$ *and* $\delta' > 0$ *are given. Consider natural positive numbers* d_ν, $\nu = 1, 2, ..., N$, *such that* $d_1 + d_2 + ... + d_N = D = dim\,\mathcal{H}$, *and, moreover, assume that, for all* $\nu = 1, 2..., N$,

$$d_\nu > D - \frac{\epsilon^2\,\delta\,\delta'\,D\,(D+1)}{N^3}.$$

Suppose $H : \mathcal{H} \to \mathcal{H}$ *is self-adjoint,* **the unitary vector** $\psi_0 \in \mathcal{H}$ **is fixed**, *and* $\psi_t = e^{-i\,t\,H}(\psi_0)$, $t \geq 0$.

Then, for $(1-\delta)$*-most of the decompositions* $\mathcal{D} \in \Delta(d_1, d_2, ..., d_N, \mathcal{H})$, *the inequalities*

$$|\,|E_{\psi_t}(P_{\mathcal{H}_\nu}) - \frac{d_\nu}{D}\,| = |\,|P_\nu(\mathcal{D})\,\psi_t\,|^2 - \frac{d_\nu}{D}\,| < \epsilon\sqrt{\frac{d_\nu}{N\,D}} \qquad (\nu = 1, 2, ..., N)$$

are true for $(1-\delta')$*-most of the large times.*

The estimates depend on the initial condition ψ_0.

Proof We denote

$$f_\nu(\mathcal{D}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(|P_\nu(\mathcal{D})\,\psi_t\,|^2 - \frac{d_\nu}{D} \right)^2 \mathrm{d}t.$$

From Corollary 14, for each ν

$$w_\Delta\left(\left\{\mathcal{D} \in \Delta \;:\; f_\nu(\mathcal{D}) \geq \frac{\epsilon^2\,\delta'\,d_\nu}{D\,N^2}\right\}\right) \leq \frac{d_\nu\,(D-d_\nu)}{D^2\,(D+1)}\,\frac{D\,N^2}{\epsilon^2\delta'\,d_\nu} = \frac{N^2\,(D-d_\nu)}{D\,(D+1)\,\epsilon^2\,\delta'}.$$

Therefore, there exists a set $S \subset \Delta$, such that

$$w_\Delta(S) \geq 1 - \frac{N^3\,(D-d_\nu)}{D\,(D+1)\,\epsilon^2\,\delta'} > 1 - \delta,$$

and, at the same time $f_\nu(\mathcal{D}) < \frac{\epsilon^2\,\delta'\,d_\nu}{D\,N^2}$, for all $\mathcal{D} \in S$ and all $\nu = 1, 2..., N$.

Now, taking in Lemma 12 $\rho = \frac{\epsilon^2\,\delta'\,d_\nu}{D\,N^2}$, and $\gamma = \frac{\epsilon^2\,d_\nu}{D\,N}$, we get for all $\mathcal{D} \in S$ and all $\nu = 1, 2, ..., N$

$$|\;|P_\nu(\mathcal{D})\,\psi_t\,|^2 - \frac{d_\nu}{D}\;| < \epsilon\sqrt{\frac{d_\nu}{N\,D}} \qquad (\nu = 1, 2, ..., N),$$

for $(1-\frac{\delta'}{N})$-most of the large times.

Therefore, the above inequalities for all $\nu = 1, 2, .., N$ are true for $(1-\delta')$ most of the large times. □

Note that the mean value $f_\nu(\mathcal{D})$ depends of the Hamiltonian H but the bounds of last theorem does not depend on H.

4 Uniform estimates

In this section, we will refine the last result considering uniform estimates which are independent of the initial condition ψ_0 (for the time evolution associated with the fixed Hamiltonian $H : \mathcal{H} \to \mathcal{H}$).

Suppose $\epsilon > 0$, $\delta > 0$, and $\delta' > 0$ are given. Consider natural positive numbers d_ν, $\nu = 1, 2, ..., N$, such that $d_1 + d_2 + ... + d_N = D = \dim \mathcal{H}$

We denote for each $\psi_0 \in \mathcal{H}$, where $|\psi_0| = 1$, and $\mathcal{D} \in \Delta = \Delta(d_1, ..., d_N; \mathcal{H})$

$$f_\nu(\psi_0, \mathcal{D}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(|P_\nu(\mathcal{D})\,\psi_t\,|^2 - \frac{d_\nu}{D} \right)^2 \mathrm{d}t,$$

where $\psi_t = e^{-i\,t\,H}(\psi_0)$ (see Lemma 13).

Lemma 16 *Suppose are given $\epsilon > 0$ and $\delta' > 0$. Assume that there exists non-negative continuous functions $g_\nu : \Delta \to \mathbb{R}$, $\nu = 1, 2, .., N$, and $K > 0$, such that*

$$\text{(a)}\, f_\nu(\psi_0, \mathcal{D}) \leq g_\nu,\;\; \textit{for all}\;\; \mathcal{D} \in \Delta \;\; \textit{and for all}\;\; \psi_0 \in \mathcal{H} \;\textit{with}\; |\psi_0| = 1, \tag{2}$$

$$\text{(b)} \int_\Delta g_\nu(\mathcal{D})\, d\mathrm{w}_\Delta(\mathcal{D}) \; < K. \tag{3}$$

Suppose δ is such that

$$1 \;>\; \delta \geq \frac{K\,D\,N^3}{\epsilon^2\,\delta'\,d_\nu},\; \nu = 1, 2, .., N. \tag{4}$$

Then, for $(1-\delta)$*-most of the* $\mathcal{D} \in \Delta$*, we have*

$$| \, |P_\nu(\mathcal{D}) \, \psi_t \,|^2 - \frac{d_\nu}{D} \,| \leq \epsilon \sqrt{\frac{d_\nu}{N \, D}}, \ \ \nu = 1, 2, .., N, \tag{5}$$

for $(1-\delta')$*-most of the large times and for any* $\psi_0 \in \mathcal{H}$ *with* $|\psi_0| = 1$.

Proof Note that

$$w_\Delta(\{\mathcal{D} \in \Delta \ : \ g_\nu(\Delta) \geq \delta' \, \epsilon^2 \, \frac{d_\nu}{N^2 \, D}\} \ < K \, \frac{N^2 \, D}{\delta' \, \epsilon^2 d_\nu} \ < \ \frac{\delta}{N}, \ \nu = 1, 2, .., N.$$

Therefore, there exists a subset $E \subset \Delta$, such that $w_\Delta(E) < 1 - \delta$ and $g_\nu(\Delta) < \delta' \, \epsilon^2 \, \frac{d_\nu}{N^2 \, D}$, for all $\Delta \in E$ and all $\nu = 1, 2..., N$.

The conclusion is: if $\Delta \in E$, then $f_\nu(\psi_0, \mathcal{D}) \ < \delta' \, \epsilon^2 \, \frac{d_\nu}{N^2 \, D}$, for all $\nu = 1, 2..., N$, and all ψ_0 with norm 1.

The proof of the claim now follows from the reasoning of Theorem 15 and Lemma 12. □

Note that to have δ in expression (4) small, it is necessary that all d_ν are large.

We assume now several hypothesis on H. Consider a certain orthogonal basis of eigenvectors $\phi_1, \phi_2, ..., \phi_D$ of H. We denote by E_j, $j = 1, 2, .., D$ the corresponding eigenvalues.

We assume hypothesis $\mathfrak{N}\,\mathfrak{R}$ which says

a) H is not degenerate, that is, $E_\alpha \neq E_\beta$, for $\alpha \neq \beta$,
   and
b) H has no resonances, that is, $E_\alpha - E_\beta \neq E_{\alpha'} - E_{\beta'}$, unless $\alpha = \alpha'$ and $\beta = \beta'$, or, $\alpha = \beta$ and $\alpha' = \beta'$.

Lemma 17

$$f_\nu(\psi_0, \mathcal{D}) \leq \max_{1 \leq \alpha \neq \beta \leq D} | < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\beta > |^2 + \max_{1 \leq \alpha \leq D} \left(< \phi_\alpha, P_\nu(\mathcal{D}) \, \phi_\alpha \ > - \frac{d_\nu}{D} \right)^2,$$

for all $\psi_0 \in \mathcal{H}$*, such that* $|\psi_0| = 1$*, and for all* $\mathcal{D} \in \Delta(d_1, ..., d_N; \mathcal{H})$ *and all* $\nu = 1, 2..., N$.

Proof Suppose $\psi_0 = \sum_{\alpha=1}^D c_\alpha \, \phi_\alpha$. Then

$$\psi_t = \sum_{\alpha=1}^{D} c_\alpha \, e^{-i \, t \, E_\alpha} \phi_\alpha, \, t \geq 0,$$

and

$$| \, P_\nu(\mathcal{D}) \psi_t|^2 = < \psi_t, \, P_\nu(\mathcal{D}) \psi_t > = \sum_{1 \leq \alpha, \beta \leq D} c_\alpha \, \overline{c}_\beta e^{-it \, (E_\alpha - E_\beta)} \quad < \phi_\alpha, \, P_\nu(\mathcal{D}) \phi_\beta > .$$

Therefore

$$\left(| \, P_\nu(\mathcal{D}) \psi_t|^2 \, - \frac{d_\nu}{D} \right)^2 = \sum_{1 \leq \alpha, \beta, \gamma, \delta \leq D}^{D} c_\alpha \, \overline{c}_\beta \, c_\gamma \overline{c}_\delta \, e^{-it \, [\, (E_\alpha - E_\beta) - (E_\delta - E_\gamma) \,]} < \phi_\alpha, \, P_\nu(\mathcal{D}) \phi_\beta > < \phi_\gamma, \, P_\nu(\mathcal{D}) \phi_\delta >$$
$$- 2 \, \frac{d_\nu}{D} \sum_{1 \leq \alpha, \beta \leq D} c_\alpha \, \overline{c}_\beta e^{-it \, (E_\alpha - E_\beta)} \ < \phi_\alpha, \, P_\nu(\mathcal{D}) \phi_\beta > + \frac{d_\nu^2}{D^2}.$$

Using the above expression in the computation of integral $f_\nu(\psi_0, \mathcal{D})$ will remain just the terms, where the coefficient of t is zero. By hypothesis, this will happen just when $\alpha = \delta$ and $\beta = \gamma$, or, $\alpha = \beta$ and $\gamma = \delta$.

Note that the case $\alpha = \beta = \gamma = \delta$ is counted twice in the estimation.

Therefore

$$
\begin{aligned}
f_\nu(\psi_0, \mathcal{D}) = & \sum_{1 \le \alpha, \beta \le D} |c_\alpha|^2 \, |c_\beta|^2 \ \ | < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\beta > |^2 \\
& + \sum_{1 \le \alpha, \gamma \le D} |c_\alpha|^2 \, |c_\gamma|^2 \ < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\alpha > < \phi_\gamma, P_\nu(\mathcal{D}) \phi_\gamma > \\
& - \sum_{1 \le \alpha \le D} |c_\alpha|^4 \, | < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\alpha > |^2 - 2 \frac{d_\nu}{D} \sum_{1 \le \alpha \le D} |c_\alpha|^2 \ < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\alpha > \ + \frac{d_\nu^2}{D^2},
\end{aligned}
$$

because $< \phi_\gamma, P_\nu(\mathcal{D}) \phi_\delta > = \overline{< \phi_\delta, P_\nu(\mathcal{D}) \phi_\gamma >}$.

Finally, putting together the first and third terms:

$$
f_\nu(\psi_0, \mathcal{D}) = \sum_{1 \le \alpha \ne \beta \le D} |c_\alpha|^2 \, |c_\beta|^2 \ \ | < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\beta > |^2 + \left(\sum_{1 \le \alpha \le D} |c_\alpha|^2 \ < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\alpha > \ - \frac{d_\nu}{D} \right)^2 .
$$

By the other hand

$$
\begin{aligned}
\sum_{1 \le \alpha \ne \beta \le D} |c_\alpha|^2 \, |c_\beta|^2 \ \ | < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\beta > |^2 & \le \max_{1 \le \alpha \ne \beta \le D} | < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\beta > |^2 \sum_{1 \le \alpha, \beta \le D} |c_\alpha|^2 \, |c_\beta|^2 \\
& = \max_{1 \le \alpha \ne \beta \le D} | < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\beta > |^2 \left(\sum_{1 \le \alpha \le D} |c_\alpha|^2 \right)^2 \\
& = \max_{1 \le \alpha \ne \beta \le D} | < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\beta > |^2,
\end{aligned}
$$

because $|\psi_0| = 1$.

By the same reason

$$
\begin{aligned}
| \sum_{1 \le \alpha \le D} |c_\alpha|^2 \ < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\alpha > \ - \frac{d_\nu}{D} | & = \left| \sum_{1 \le \alpha \le D} |c_\alpha|^2 \left(< \phi_\alpha, P_\nu(\mathcal{D}) \phi_\alpha > \ - \frac{d_\nu}{D} \right) \right| \\
& \le \max_{1 \le \alpha \le D} \left| < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\alpha > \ - \frac{d_\nu}{D} \right| .
\end{aligned}
$$

□

Now, we define for each $\nu = 1, 2, ..., N$, the continuous function $g_\nu(\mathcal{D}) : \Delta(d_1, ..., d_N; \mathcal{H}) = \Delta \to \mathbb{R}$ given by

$$
g_\nu(\mathcal{D}) = \max_{1 \le \alpha \ne \beta \le D} | < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\beta > |^2 + \max_{1 \le \alpha \le D} \left| < \phi_\alpha, P_\nu(\mathcal{D}) \phi_\alpha > \ - \frac{d_\nu}{D} \right|^2 . \tag{6}
$$

We point out that for each $\mathcal{D}$, the expression $g_\nu(\mathcal{D})$ depends just on H because as E_α are all different, the eigenvector basis is unique up to a changing in order and multiplication by scalar of modulus one.

Now, we need a fundamental technical Lemma.

Lemma 18 *There exist a constant $C_1 > 0$, such that*

$$
\int_\Delta g_\nu(\mathcal{D}) w_\Delta(\mathcal{D}) < \frac{10 \log D}{D}, \ \ \nu = 1, 2, ..., N,
$$

if, $C_1 \log D \ < \ d_\nu \ < \ \frac{D}{C_1}$.

Note that if D is large, there is a lot of room for the values d_ν to be able to satisfy last inequality. We will prove this fundamental lemma in the next sections.

If we assume the Lemma is true, then:

Theorem 19 *Given $\epsilon, \delta > 0$ and $\delta' > 0$, take $d_1, d_2, ..., d_N$, such that, if $D = d_1 + ... + d_N$, $N > 0$, then the following inequalities are true*

$$\max\left(C_1, \frac{10N^3}{\epsilon\,\delta\,\delta'}\right)\log D < d_\nu < \frac{D}{C_1}, \ \nu = 1, 2, .., N,$$

where C_1 comes from Lemma 18.

Assume that $\mathcal{H}$ is a Hilbert space of dimension D and $H : \mathcal{H} \to \mathcal{H}$ is a self-adjoint Hamiltonian without resonances and degeneracies, then for $(1-\delta)$ most of the decompositions $\mathcal{D} \in \Delta(d_1, ..., d_N; \mathcal{H})$ the system of inequalities

$$|\ |P_\nu(\mathcal{D})\psi_t|^2 - \frac{d_\nu}{D}| < \epsilon\sqrt{\frac{d_\nu}{N\,D}}, \ \nu = 1, 2, ..., N$$

are true for most of the $(1-\delta')$ large times and for any initial condition, $\psi_0 \in \mathcal{H}$, $|\psi_0| = 1$.

Proof By hypothesis and Lemma 18, we get

$$\int_\Delta g_\nu(\mathcal{D})w_\Delta(\mathcal{D}) < \frac{10\log D}{D}, \ \nu = 1, 2, ..., N.$$

The claim follows from Lemma 16 by taking $K = \frac{10\log D}{D}$. □

Main conclusion:

As we said before, for a given fixed subspace $\mathcal{H}_\nu$ of $\mathcal{H}$, the observable $P_{\mathcal{H}_\nu}$ (the orthogonal projection on $\mathcal{H}_\nu$) is such that the mean value $E_{\psi_t}(P_{\mathcal{H}_\nu})$ of the state ψ_t is $< P_{\mathcal{H}_\nu}(\psi_t), \psi_t >= |P_{\mathcal{H}_\nu}(\psi_t)|^2$.

For a fixed Hamiltonian H acting on a Hilbert space $\mathcal{H}$ of dimension D, the main theorem gives lower bound conditions on the dimensions d_ν, $\nu = 1, 2, .., N$, of the different $\mathcal{H}_\nu$ values of a $(1-\delta)$-generic orthogonal decomposition $\mathcal{D}$ of the form $\mathcal{H} = \mathcal{H}_1 \oplus ... \oplus \mathcal{H}_N$, in such a way that the dynamic time evolution ψ_t, obtained from any fixed initial condition ψ_0, for most of the large times t, has the property that the projected component $P_\nu(\mathcal{D})(\psi_t) = P_{\mathcal{H}_\nu}(\psi_t)$ is almost uniformly distributed (in terms of expected value) with respect to the relative dimension size $\frac{d_\nu}{D}$ of $\mathcal{H}_\nu$. In this way, there is an approximately uniform spreading of ψ_t among the different values of $\mathcal{H}_\nu$ of the decomposition $\mathcal{D}$.

5 Proof of Lemma 18

The Lemmas 22 and 23 will permit to reduce the integration problem from the unitary group to a problem in the real line.

We will need first an auxiliary lemma. We denote by S^k the unitary sphere in $\mathbb{R}^{k+1}$ and S^k_r the sphere of radius $r > 0$ in $\mathbb{R}^{k+1}$. We consider the usual metric on them.

The next lemma is a classical result on Integral Geometry (see [4]). We will provide a simple proof in "Appendix 2".

Lemma 20 *Suppose X is a Riemannian compact manifold, $f : X \to \mathbb{R}$ a C^∞-function and $g : \mathbb{R} \to \mathbb{R}$ a continuous function. We define*

$$G(v) = \int_{f \le v} (g \circ f)\,\lambda,$$

where λ is the volume form on X. Suppose that $a \in \mathbb{R}$ is a regular value of f. Then, G is differentiable at $v = a$ and

$$\frac{\mathrm{d}G}{\mathrm{d}v}(a) = g(a) \int_{X_a} \frac{\lambda_a}{|\ grad\ f\ |},$$

where X_a is the level manifold $f = a$ and λ_a is the induced volume form in X_a.

Corollary 21 *Given positive integers d, D, where $1 < d < D - 1$, denote by S the unitary sphere on $\mathbb{R}^{2D}$ with the usual metric. Define*

$f(x) = x_1^2 + \ldots + x_{2d}^2$, *where* $x \in S$ *and* $g : \mathbb{R} \to \mathbb{R}$ *is a continuous function.*

Suppose

$$G(v) = \int_{f \le v} (g \circ f)\, d\lambda,$$

then G is of class C^1 and

$$\frac{\mathrm{d}G}{\mathrm{d}v}(v) = \frac{2\,\pi^D}{(d-1)!\,(D-d-1)!}\, g(v)\, v^{d-1}\,(1-v)^{D-d-1}, \quad \textit{if } 0 \le v \le 1,$$

and $\frac{\mathrm{d}G}{\mathrm{d}v}(v) = 0$, *if* $v < 0$ *or* $v > 1$.

Proof For $x_1^2 + \ldots + x_{2d}^2 = v$, we have

grad $f(x) = 2\,((1-v)x_1, \ldots, (1-v)x_{2d}, -v\,x_{2d+1}, \ldots, -v\,x_{2D})$.

Then, $|\text{grad } f(x)| = 2\sqrt{v\,(v-1)}$, which is constant over $S_v = \{f = v\}$. Note that

$$S_v = S^{2d-1}_{\sqrt{v}} \times S^{2\,(D-d)\,-1}_{\sqrt{1-v}}, \quad 0 < v < 1.$$

From last Lemma and from the above expression, it follows that (remember that vol $(S_r^{2n-1}) = \frac{2\,\pi^n}{(n-1)!}\, r^{2n-1}$)

$$\frac{\mathrm{d}G}{\mathrm{d}v}(v) = g(v)\, \frac{1}{2\sqrt{v\,(1-v)}}\, \frac{2\,\pi^d\,(\sqrt{v})^{2d-1}}{(d-1)!}\, \frac{2\,\pi^{D-d}(\sqrt{(1-v)})^{2\,(D-d)\,-1}}{(D-d-1)!}$$
$$= \frac{2\,\pi^D\, v^{d-1}\,(1-v)^{D-d-1}}{(d-1)!\ \ (D-d-1)!}, \qquad 0 < v < 1.$$

In the case $v < 0$ or $v > 1$, we have that G is constant. Finally, as S_0 and S_1 are submanifolds of S, we have that G is continuous for $v = 0$ and $v = 1$. □

From now on, we fix ν, where $1 \le \nu \le N$, and we define

$$e_{\alpha,\beta}(\mathcal{D}) = <\ \phi_\alpha, P_\nu(\mathcal{D})\,\phi_\beta\ >, \quad \mathcal{D} \in \Delta, \quad 1 \le \alpha,\ \beta \le D, \quad e_{\alpha,\beta} : \Delta \to \mathbb{C},$$

where $\phi_1, \ldots, \phi_D$ is the orthonormal basis for $\mathcal{H}$ which were fixed in Sect. 4.

Lemma 22 *Suppose $1 < d_\nu < D - 1$. Let $a \ge 0$ be such $\sqrt{a} < \frac{d_\nu}{D}$ and $\sqrt{a} + \frac{d_\nu}{D} < 1$. Then, the probability, such that $(e_{\alpha,\beta} - \frac{d_\nu}{D})^2 \ge \alpha$ is*

$$\frac{(D-1)!}{(d_\nu-1)!\,(D-d_\nu-1)!} \int_{[0,\,\frac{d_\nu}{D}-\sqrt{a}]\cup[\frac{d_\nu}{D}+\sqrt{a},\,1]} u^{d_\nu-1}\,(1-u)^{D-d_\nu-1}\,\mathrm{d}u.$$

Lemma 23 *Suppose $1 < d_\nu < D - 1$. Let $\alpha \ne \beta$ and $0 \le a \le 1/4$. Then, the probability such that $|\,e_{\alpha,\beta}\,|^2 \ge a$ is*

$$\frac{(D-1)!}{(d_\nu-1)!\,(D-d_\nu-1)!} \int_{1/2-\sqrt{1/4-a}}^{1/2+\sqrt{1/4-a}} \frac{(w\,(1-w)-a)^{D-2}}{w^{D-d_\nu-1}\,(1-w)^{d_\nu-1}}\mathrm{d}w.$$

Proof of Lemma 22 We just have to consider the case $\nu = 1$. We write $d = d_1$ and denote by P the orthogonal projection of $\mathcal{H}$ over $\mathbb{C}\phi_1 + ... + \mathbb{C}\phi_d$.

We denote by $p : \mathbb{U} \to \Delta$ the projection defined in the beginning of Sect. 2, where $\mathbb{U}$ denotes the group of unitary transformations of $\mathcal{H}$.

If $U \in \mathbb{U}$, then

$e_{\alpha,\alpha}(p(U)) =< \phi_\alpha$, orthogonal projection of ϕ_α in $\mathbb{C}U(\phi_1) + ... + \mathbb{C}\,U(\phi_d) >=$
$< U^{-1}(\phi_\alpha), P(U^{-1}\phi_\alpha) >$.

Denote $q : \mathbb{U} \to S$, where $q(U) = U(\phi_\alpha)$, $U \in \mathbb{U}$ and $\sigma : S \to \mathbb{R}$, where $\sigma(\phi) =< \phi, P(\phi) >$, $\phi \in S$, and where S is the unitary sphere of $\mathcal{H}$.

Then, we get the following commutative diagram:

$$\begin{tikzcd} \mathbb{U} \arrow[rr, "\text{inverse}"] \arrow[d, "p"'] & & \mathbb{U} \arrow[d, "q"] \\ \Delta \arrow[dr, "e_{\alpha,\alpha}"'] & & S \arrow[dl, "\sigma"] \\ & \mathbb{R} & \end{tikzcd}$$

As the inverse preserves the metric, it follows from Lemma 1 a) that the probability of $e_{\alpha,\alpha} \leq b$ is equal to the probability that $\sigma \leq b$. Note that the metric on S as quotient of $\mathbb{U}$ is the same as the induced by $\mathcal{H}$, because $\mathbb{U}$ acts transitively on S.

It will be more easy to make the computations via the right hand side of the diagram.

We identify $\mathcal{H}$ with $\mathbb{C}^D = \mathbb{R}^{2D}$, via $\phi_1, \phi_2, ..., \phi_D$. Then, S is identified with the unitary sphere in $\mathbb{R}^{2D}$, also denoted by S, and

$$\sigma : S \to \mathbb{R}, \quad \sigma(x) = x_1^2 + ... + x_{2d}^2, \quad x \in S.$$

Therefore, by Corollary 21 with $g = 1$, we get

$$\frac{d\,(\mathrm{Vol}\,(\sigma \leq v))}{d\,v} = \frac{2\,\pi^D}{(d-1)\,!\,(D-d-1)\,!}\,v^{d-1}\,(1-v)^{D-d-1}, \quad \text{if } 0 \leq v \leq 1,$$

and

$$\frac{d\,(\mathrm{Vol}\,(\sigma \leq v))}{d\,v} = 0,$$

if $v < 0$ or $v > 1$.

Now, we normalize dividing by vol $S = \frac{2\,\pi^D}{(D-1)\,!}$ and we get

$$\frac{d\,(\mathrm{prob}\,(\sigma \leq v))}{d\,v} = \frac{(D-1)\,!}{(d-1)\,!\,(D-d-1)\,!}\,v^{d-1}\,(1-v)^{D-d-1}, \quad \text{if } 0 \leq v \leq 1.$$

As $(e_{\alpha,\alpha} - \frac{d}{D})^2 \geq a$ is equivalent to

$$e_{\alpha,\alpha} \geq \frac{d}{D} + \sqrt{a}, \quad \text{or} \quad e_{\alpha,\alpha} \leq \frac{d}{D} - \sqrt{a},$$

we get that the probability of $(e_{\alpha,\alpha} - \frac{d}{D})^2 \geq a$ is equal to the probability of $\sigma \geq \frac{d}{D} + \sqrt{a}$ or $\sigma \leq \frac{d}{D} - \sqrt{a}$. From this follows that the probability of $(e_{\alpha,\alpha} - \frac{d}{D})^2 \geq a$ is equal to

$$\frac{(D-1)\,!}{(d-1)\,!\,(D-d-1)\,!}\left[\int_{\frac{d}{D}+\sqrt{a}}^{1} v^{d-1}\,(1-v)^{D-d-1}\mathrm{d}v + \int_0^{\frac{d}{D}-\sqrt{a}} v^{d-1}\,(1-v)^{D-d-1}\mathrm{d}v\right].$$

Observe that $\sigma =$ constant is an analytic subset of S, and therefore, the associated probability is zero. The case $a = 0$ is trivial. □

Proof of Lemma 23 We just have to consider the case $\nu = 1$. Take $d = d_1$ and as before, we denote by P the orthogonal projection of $\mathcal{H}$ over $\mathbb{C}\phi_1 + ... + \mathbb{C}\phi_d$. Once more we denote by $p : \mathbb{U} \to \Delta$ the projection defined in the beginning of Sect. 2.

If $U \in \mathbb{U}$, then
$e_{\alpha,\beta}(p(U)) =< \phi_\alpha$, orthogonal projection of ϕ_β in $\mathbb{C}U(\phi_1) + ... + \mathbb{C}\, U(\phi_d) >=$
$< U^{-1}(\phi_\alpha),\, P(U^{-1}\phi_\beta) >$.

Denote $q_{\alpha,\beta} : \mathbb{U} \to S \times S$, where $q_{\alpha,\beta}(U) = (U(\phi_\alpha), U(\phi_\beta))$, $U \in \mathbb{U}$, and S is the unitary sphere of $\mathcal{H}$.
Denote by $M = q_{\alpha,\beta}(\mathbb{U}) = \{(\phi, \psi) \in S \times S \,|\, \phi$ is orthogonal to $\psi\}$.
Let $H_{\alpha,\beta} \subset \mathbb{U}$ the closed subgroup of the U, such that $U(\phi_\alpha) = \phi_\alpha$ and $U(\phi_\beta) = \phi_\beta$.
Then, $M = \mathbb{U}/H_{\alpha,\beta}$ and $q_{\alpha,\beta} : \mathbb{U} \to M$ is the canonical projection.
The quotient metric on M is the induced by $S \times S$, because $\mathbb{U}$ acts transitively on M.
Let $f : M \to \mathbb{C}$ given by $f(\phi, \psi) =< \phi, P(\psi) >$. Then, we get the following commutative diagram:

$$\begin{tikzcd}
\mathbb{U} \arrow[r, "\text{inverse}"] \arrow[d, "p"'] & \mathbb{U} \arrow[d, "q_{\alpha,\beta}"] \\
\Delta \arrow[dr, "e_{\alpha,\beta}"'] & M \arrow[d, "f"] \\
 & \mathbb{C}
\end{tikzcd}$$

As the inverse preserves the metric of $\mathbb{U}$, it follows that the probability of $|e_{\alpha,\alpha}|^2 \le a$ is equal to the probability that $|f|^2 \le a$ by Lemma 1 a).

Now, consider $\varphi : M \to S$, such that $\varphi(\phi, \psi) = \psi$. This defines a C^∞ locally trivial fiber bundle with fiber S^{2D-3}. Indeed, $E_\psi = \varphi^{-1}(\psi)$ is the unitary sphere of the subspace $\mathcal{H}_\psi$ which is the orthogonal set to ψ in $\mathcal{H}$.
Given $u \in \mathbb{R}$ denote:

$$F_u(\psi) = E_\psi \cap \{|f|^2 \le u\}, \quad \psi \in S.$$

Then

$$\mathrm{Vol}\,(\{|f|^2 \le u\}) = \int_S \mathrm{vol}_{E_\psi}(F_u(\psi))\ \mathrm{d}S\,(\psi).$$

For each ψ, we get $\psi' \in \mathcal{H}$ via

$$P(\psi) = c\psi + \psi', \text{ where } c \in \mathbb{C} \text{ and } \psi' \text{ is orthogonal to } \psi.$$

Note that $\psi' \in \mathcal{H}_\psi$. Then

$$f(\phi, \psi) =< \phi, P(\psi) >=< \phi, \psi' >,$$

and it follows that

$$F_u(\psi) = \{\phi \in E_\psi \ :\ |< \phi, \psi' >|^2 \le u\}, \ u \in \mathbb{R}, \ \psi \in S.$$

There exist an isomorphism identifying $\mathcal{H}_\psi = \mathbb{C}^{D-1} = \mathbb{R}^{2D-2}$ between Hilbert spaces which transform ψ' in $(|\psi'|, 0, ..., 0)$. This isomorphism identifies E_ψ with the unitary sphere E on $\mathbb{R}^{2D-2}$ and $F_u(\psi)$ with the set:

$$\{x \in E \ :\ |\psi'|^2(x_1^2 + x_2^2) \le u\}.$$

Now, applying Corollary 21 with $D - 1$ instead of D, $d = 1$, $g = 1$, and $v = \frac{u}{|\psi'|^2}$, we get

$$\frac{d\,\mathrm{Vol}_{E_\psi} F_u(\psi)}{\mathrm{d}u} = \frac{2\pi^{D-1}}{(D-3)\,!}\left(1 - \frac{u}{|\psi'|^2}\right)^{D-3}\frac{1}{|\psi'|^2} = \frac{2\pi^{D-1}}{(D-3)\,!}\,\frac{(|\psi'|^2 - u)^{D-3}}{|\psi'|^{2(D-2)}},$$

for all $\psi \in S$ and $0 < u \leq |\psi'|^2$, and

$$\frac{d\,\mathrm{Vol}_{E_\psi} F_u(\psi)}{\mathrm{d}u} = 0$$

if $|\psi'|^2 \leq u \leq 1$, for any $\psi \in S$.

Then, we get that $\frac{d\,\mathrm{Vol}_{E_\psi} F_u(\psi)}{\mathrm{d}u}$ is a continuous function of (u, ψ) for $0 < u \leq 1$ and $\psi \in S$. As S is compact, we can take derivative inside the integral and we get

$$\frac{d\,\mathrm{Vol}(|f|^2 \leq u)}{\mathrm{d}u} = \int_S \frac{d\,\mathrm{Vol}_{E_\psi} F_u(\psi)}{\mathrm{d}u}\,\mathrm{d}S(\psi)$$

for any $0 < u \leq 1$.

By the definition of ψ', it is easy to see that $|\psi'|^2 = |P(\psi)|^2\,(1 - |P(\psi)|^2)$.

Now, we consider $g_u : \mathbb{R} \to \mathbb{R}$, where

$$g_u(w) = \frac{(w\,(1-w) - u)^{D-3}}{(w\,(1-w))^{D-2}}$$

if $u \leq w\,(1-w)$, and $g_u(w) = 0$ in the other case.

$g_u(w)$ is a continuous function of u and w when $0 < u \leq 1$, $0 \leq w \leq 1$.

From this follows that

$$\frac{d\,\mathrm{Vol}(|f|^2 \leq u)}{\mathrm{d}u} = \frac{2\,\pi^{D-1}}{(D-3)\,!} \int_S (g_u \circ |P(\psi)|^2)\,\mathrm{d}S(\psi)$$

for any $0 < u \leq 1$.

Now, we normalize dividing by $\mathrm{Vol}\,(M) = \frac{2\,\pi^{D-1}}{(D-2)\,!}\,\frac{2\,\pi^{D}}{(D-1)\,!}$ and we get

$$\frac{d\,\mathrm{Prob}(|f|^2 \leq u)}{\mathrm{d}u} = \frac{(D-1)\,!\,(D-2)}{(2\,\pi^D)} \int_S (g_u \circ |P(\psi)|^2)\,\mathrm{d}S(\psi) \tag{7}$$

for any $0 < u \leq 1$.

Denote

$$A(u, w) = \int_{|P(\psi)|^2 \leq w} (g_u \circ |P(\psi)|^2)\,\mathrm{d}S(\psi),$$

for any $0 < u \leq 1$, $0 \leq w \leq 1$.

By Corollary 21, we get

$$\int_S (\,g_u \circ |P(\psi)|^2)\,\mathrm{d}S(\psi) \;= A(u, 1) = A(u, 1) - A(u, 0) = \int_0^1 \frac{\partial A}{\partial w}(u, w)\,\mathrm{d}w,$$

for any $0 < u \leq 1$.

Estimating $\frac{\partial A}{\partial w}$ by Corollary 21 and substituting in (7), we finally get

$$\frac{d\,\mathrm{Prob}(|f|^2 \leq u)}{\mathrm{d}u} = \frac{(D-1)\,!\,(D-2)}{(d-1)\,!\,(D-d-1)\,!} \int_{u \leq w\,(1-w)} \frac{(w\,(1-w) - u)^{D-3}}{w^{D-d-1}\,(1-w)^{d-1}}\,d\mathrm{w}$$

for any $0 < u \leq 1$.

If $u > 1/4$, $w\,(1-w) < u$ for all w and the integral is zero.

If $0 < u \leq 1/4$, $u \leq w(1-w)$ is equivalent to

$$1/2 - \sqrt{1/4 - u} \leq w \leq 1/2 + \sqrt{1/4 - u}.$$

Then

$$\frac{d\,\mathrm{Prob}(|f|^2 \leq u)}{\mathrm{d}u} = \frac{(D-1)\,!\,(D-2)}{(d-1)\,!\,(D-d-1)\,!} \int_{1/2-\sqrt{1/4-u}}^{1/2+\sqrt{1/4-u}} \frac{(w\,(1-w) - u)^{D-3}}{w^{D-d-1}\,(1-w)^{d-1}}\,d\mathrm{w},$$

if $0 < u \leq 1/4$, and

$$\frac{d\,\mathrm{Prob}(|f|^2 \leq u)}{\mathrm{d}u} = 0$$

if $1/4 \leq u \leq 1$.

Finally, for $0 < a \leq 1/4$

$$\mathrm{Prob}(|f|^2 \geq a) = \frac{(D-1)\,!\,(D-2)}{(d-1)\,!\,(D-d-1)\,!} \int_a^{1/4} \mathrm{d}u \int_{1/2-\sqrt{1/4-u}}^{1/2+\sqrt{1/4-u}} \frac{(w\,(1-w)-u)^{D-3}}{w^{D-d-1}\,(1-w)^{d-1}}\,d\mathrm{w}.$$

Considering the double integral in the region $a \leq u \leq w\,(1-w)$, we get

$$\begin{aligned}\mathrm{Prob}(|f|^2 \geq a) &= \frac{(D-1)\,!\,(D-2)}{(d-1)\,!\,(D-d-1)\,!} \int_{1/2-\sqrt{1/4-u}}^{1/2+\sqrt{1/4-a}} \mathrm{d}w \int_a^{w\,(1-w)} \frac{(w\,(1-w)-u)^{D-3}}{w^{D-d-1}\,(1-w)^{d-1}}\,\mathrm{d}u \\ &= \frac{(D-1)\,!}{(d-1)\,!\,(D-d-1)\,!} \int_{1/2-\sqrt{1/4-a}}^{1/2+\sqrt{1/4-a}} \frac{(w\,(1-w)-a)^{D-2}}{w^{D-d-1}\,(1-w)^{d-1}}\,\mathrm{d}w.\end{aligned}$$

The case $a = 0$ is trivial. □

Remark Note that if $g : \Delta \to \mathbb{R}$ is a continuous function such that $0 \leq g(\mathcal{D}) \leq r$, for all $\mathcal{D} \in \Delta$, then we get the estimate

$$\int_\Delta g(\mathcal{D})\,w_\Delta(\mathcal{D}) = \int_{g \geq a} g(\mathcal{D})\,w_\Delta(\mathcal{D}) + \int_{g<a} g(\mathcal{D})\,w_\Delta(\mathcal{D}) \leq r\ \mathrm{Prob}\ (g \geq a) + a,$$

for $0 \leq a \leq 1$.

Given positive integer numbers d, D and $a \in \mathbb{R}$, such that

$$1 < d < D-1, \quad 0 \leq a \leq \frac{d^2}{D^2} \text{ and } \frac{d}{D} + \sqrt{a} \leq 1$$

we define

$$I(d, D, a) = \frac{(D-1)\,!}{(d-1)\,!\,(D-d-1)\,!} \int_{[0,\,\frac{d}{D}-\sqrt{a}] \cup [\frac{d}{D}+\sqrt{a},\,1]} u^{d-1}\,(1-u)^{D-d-1}\,\mathrm{d}u.$$

In the following, we will use the estimate $\theta = 11/12$.

Lemma 24 *There exists a constant $C > 4$, such that if $a \geq 0$, $d \geq 1$ $C\,\log D < d < \frac{D}{C}$ and $\frac{1}{D} < \sqrt{a} < \frac{d}{8\,D}$, then*

$$I(d, D, a) < \frac{D}{\sqrt{d}}\,e^{-\frac{\theta\,a\,D^2}{2\,d}}.$$

Proof Note that our hypothesis implies that $1 < d < D-1$, $a^2 < \frac{d^2}{D^2}$ and $\frac{d}{D} + \sqrt{a} < 1$.

a) By Stirling formula, when $D \to \infty$, $d \to \infty$, $D/d \to \infty$, we get that

$$\frac{(D-1)\,!}{(d-1)\,!\,(D-d-1)\,!} \sim \frac{1}{e}\sqrt{\frac{d}{2\pi}} \left(\frac{d}{D}\right)^{-d} \left(1 - \frac{d}{D}\right)^{d-D}.$$

As $\sqrt{\frac{1}{2\pi}} < 1$, there exists a constant A such that if $D > A$, $d > A$ and $D/d > A$, we get

$$\frac{(D-1)\,!}{(d-1)\,!\,(D-d-1)\,!} < \frac{\sqrt{d}}{2} \left(\frac{d}{D}\right)^{-d} \left(1 - \frac{d}{D}\right)^{d-D}.$$

If we take $C > A+1$, it follows from the hypothesis of the Lemma that $D > d\,C > d\,A$, $d > C\log D > C > A$ and $D - d > d\,C - d = d(C-1) > d\,A > A$.

b) The derivative of $u^{d-1}\,(1-u)^{D-d-1}$ with respect to u in $(0,1)$ is zero only on the point $u=\frac{d-1}{D-1}$ which is smaller than d/D.

Moreover

$$\frac{d}{D}-\sqrt{a}<\frac{d}{D}-\frac{1}{D}=\frac{d-1}{D}<\frac{d-1}{D-1}.$$

Then, $\frac{d-1}{D-1}\in(\frac{d}{D}-\sqrt{a},\,\frac{d}{D})\subset(\frac{d}{D}-\sqrt{a},\,\frac{d}{D}+\sqrt{a})$.

From this it follows that $u^{d-1}\,(1-u)^{D-d-1}$ takes its maximal values on the set $[0,\frac{d}{D}-\sqrt{a}]\cup[\frac{d}{D}+\sqrt{a},1]$ on the point $\frac{d}{D}-\sqrt{a}$ or on the point $\frac{d}{D}+\sqrt{a}$.

Under our hypothesis, if $C>A+1$, we get that for $\epsilon=1$ or -1:

$$\begin{aligned}I(d,D,a)&<\frac{\sqrt{d}}{2}\left(\frac{d}{D}\right)^{-d}\left(1-\frac{d}{D}\right)^{d-D}\left(\frac{d}{D}+\epsilon\sqrt{a}\right)^{d-1}(1-\frac{d}{D}-\epsilon\sqrt{a})^{D-d-1}\\&=\frac{\sqrt{d}}{2}\,\frac{(1+\epsilon\frac{D}{d}\sqrt{a})^{d}\,(1-\epsilon\frac{D}{D-d}\sqrt{a})^{D-d}}{(\frac{d}{D}+\epsilon\sqrt{a})\,\left(1-\frac{d}{D}-\epsilon\sqrt{a}\right)}.\end{aligned}$$

c) If $\epsilon=1$ with $C>4$, $C>A+1$, we get

$$(\frac{d}{D}+\epsilon\sqrt{a})\left(1-\frac{d}{D}-\epsilon\sqrt{a}\right)=\frac{d}{D}+\sqrt{a}-\frac{d^2}{D^2}-2\frac{d}{D}\sqrt{a}-a>$$

$$\frac{d}{D}-\frac{d^2}{D^2}-2\frac{d}{D}\sqrt{a}>\frac{d}{D}-\frac{d^2}{D^2}-2\frac{d^2}{8\,D^2}=\frac{d}{D}-\frac{5\,d^2}{4\,D^2}>\frac{d}{D}\left(1-\frac{5\,d}{4\,D}\right)>\frac{d}{2\,D}.$$

If $\epsilon=1$ with $C>4$, $C>A+1$, one can show in the same way that

$$\left(\frac{d}{D}+\epsilon\sqrt{a}\right)\left(1-\frac{d}{D}-\epsilon\sqrt{a}\right)>\frac{d}{2\,D}.$$

In this way, we finally get that for $\epsilon=1$ or $\epsilon=-1$

$$\begin{aligned}I(d,D,a)&<\frac{\sqrt{d}}{2}\frac{2\,D}{d}\left(1+\epsilon\,\frac{D}{d}\sqrt{a}\right)^{d}(1-\epsilon\frac{D}{D-d}\sqrt{a})^{D-d}\\&=\frac{D}{\sqrt{d}}\left(1+\epsilon\,\frac{D}{d}\sqrt{a}\right)^{d}\left(1-\epsilon\frac{D}{D-d}\sqrt{a}\right)^{D-d}.\end{aligned}$$

Note that

$$\begin{aligned}&\frac{D}{\sqrt{d}}(1+\epsilon\,\frac{D}{d}\sqrt{a})^{d}\left(1-\epsilon\frac{D}{D-d}\sqrt{a}\right)^{D-d}\\&=\frac{D}{\sqrt{d}}\exp\left[d\,\log(1+\epsilon\,\frac{D}{d}\sqrt{a})+(D-d)\log\left(1-\epsilon\frac{D}{D-d}\sqrt{a}\right)\right]\\&<\frac{D}{\sqrt{d}}\exp\left[d\left(\epsilon\,\frac{D}{d}\sqrt{a}-\frac{1}{2}\frac{D^2}{d^2}\,a+\frac{\epsilon}{3}\frac{D^3}{d^3}\,a^{3/2}\right)+(D-d)\left(-\epsilon\frac{D}{D-d}\sqrt{a}\right)\right].\end{aligned}$$

This is so because $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+...$, for $|x|<1$, $\frac{D}{d}\sqrt{a}<1/8$, and $\frac{D}{D-d}\sqrt{a}<\frac{1}{24}$.

Therefore, if $C>4$ and $C>A+1$, then

$$I(d,D,a)<\frac{D}{\sqrt{d}}\exp\left[-\frac{1}{2}\,\frac{D^2}{d}\,a+\frac{\epsilon}{3}\,\frac{D^3}{d^2}\,a^{3/2}\right],$$

for $\epsilon=1$ or $\epsilon=-1$.

Note that

$$\frac{|\frac{\epsilon}{3}\,\frac{D^3}{d^2}\,a^{3/2}|}{|-\frac{1}{2}\,\frac{D^2}{d}\,a|}=\frac{2}{3}\,\frac{D}{d}\,a^{1/2}<\frac{2}{3}\,\frac{D}{d}\,\frac{d}{8\,D}=\frac{1}{12}.$$

Therefore, if $C > 4$ and $C > A + 1$, we finally get

$$I(d, D, a) < \frac{D}{\sqrt{d}} e^{-\frac{\theta}{2} \frac{D^2}{d} a}.$$

$\square$

Motivated by the Remark before Lemma 24, we will choose a convenient choice of a.

Corollary 25 *There exist $C_0 > 4$, such that if d and D are such that $C_0 \log D < d < \frac{D}{C_0}$, then*

$$I(d, D, a) < \frac{1}{D^3 \sqrt{d}},$$

where $a = \frac{8\, d\, \log D}{\theta\, D^2}$.

Proof Take $C_0 > C$ (of Lemma 24) and $C_0 > 24^2$. Then

$$\sqrt{a} = \sqrt{\frac{8}{\theta}}\, \frac{\sqrt{d\, \log D}}{D} < 3\, \frac{\sqrt{d\, \frac{d}{C_0}}}{D} = \frac{3\, d}{D\, \sqrt{C_0}} < \frac{3\, d}{D\, 24} = \frac{d}{8\, D},$$

because $\frac{8}{\theta} < 9$.

Moreover, $\sqrt{a} > \frac{\sqrt{d\, \log D}}{D} > \frac{1}{D}$.

By Lemma 24, we get that

$$I(d, D, a) < \frac{D}{\sqrt{d}}\, e^{-\frac{8\, \theta\, D^2}{2\, d}\, \frac{8}{\theta}\, \frac{d\, \log D}{D^2}} = \frac{D}{\sqrt{d}}\, e^{-4\, \log D} = \frac{1}{D^3 \sqrt{d}}.$$

$\square$

Lemma 26 *Suppose C_0 is the constant of Corollary 25. Given $1 \leq \nu \leq N$, suppose that $C_0 \log D < d_\nu < \frac{D}{C_0}$, then*

$$\int_\Delta \max_{1 \leq \alpha \leq D} \left(< \phi_\alpha, P_\nu(\mathcal{D})\, \phi_\alpha > - \frac{d_\nu}{D} \right)^2 w_\Delta(\mathcal{D}) \; < \; \frac{9\, d_\nu\, \log D}{D^2}.$$

Proof Suppose $a = \frac{8\, d_\nu\, \log D}{\theta\, D^2}$.

By Corollary 25 and Lemma 22 (see also the beginning of the proof of Lemma 24), we get that the probability of the above integrand to be great or equal to a is smaller than $D\, \frac{1}{D^3\, \sqrt{d_\nu}} = \frac{1}{D^2 \sqrt{d_\nu}}$.

As we point out in the Remark before Lemma 24, the integral is smaller than

$$\frac{1}{D^2 \sqrt{d_\nu}} + \frac{8}{\theta}\, \frac{d_\nu\, \log D}{D^2}.$$

Note that

$$\frac{\frac{1}{D^2\, \sqrt{d_\nu}}}{\frac{d_\nu\, \log D}{D^2}} = \frac{1}{d_\nu^{3/2}\, \log D} < 9 - \frac{8}{\theta} = \frac{3}{11},$$

because $d_\nu^{3/2}\, \log D > C_0^{3/2}\, (\log D)^{5/2} > C_0^{3/2} > 8 > \frac{11}{3}$.

Therefore

$$\frac{1}{D^2 \sqrt{d_\nu}} + \frac{8}{\theta}\, \frac{d_\nu\, \log D}{D^2} < (9 - \frac{8}{\theta})\, \frac{d_\nu\, \log D}{D^2} + \frac{8}{\theta}\, \frac{d_\nu\, \log D}{D^2} = \frac{9\, d_\nu\, \log D}{D^2}.$$

$\square$

In Lemma 18, the function g_ν is defined as the sum of two terms (see expression (6). The Lemma 26 takes care of the upper bound of the integral of the second term. Now, we will estimate the upper bound for the first term (using the Remark done before Lemma 24). First, we need two lemmas.

Lemma 27 *Suppose ϕ and ψ are orthonormal and $E \subset \mathcal{H}$ is a subspace. Denote by P the orthogonal projection of $\mathcal{H}$ over E.*

Then, $| < \phi,\ P(\psi) > |^2 \leq 1/4$.

Proof If ψ is orthogonal to E or $\psi \in E$, we have that $< \phi, P(\psi) >= 0$.

Suppose ψ is not on E and is also not orthogonal to E. Suppose $\psi = \psi_1 + \psi_2$, where ψ_1 is orthogonal to E and $\psi_2 \in E$.

Let $\lambda = |\psi_1|$ and $\mu = |\psi_2|$, then $\psi_1 = \lambda e_1$, $\psi_2 = \mu\, e_2$, where e_1 and e_2 are orthonormal.

Denote by θ the orthogonal projection of ϕ over $\mathbb{C}\, e_1 + \mathbb{C}\, e_2$. Then

$$|\theta| \leq 1 \ \text{ and } \ \alpha \ = < \phi, P(\psi) >=< \phi, \psi_2 >=< \theta, \psi_2 > .$$

Now, $< \phi,\ \psi >= 0$ implies that

$$0 =< \phi, \psi_1 > + < \phi, \psi_2 >=< \theta, \psi_1 > + < \theta, \psi_2 > .$$

Suppose $\theta = a\, e_1 + b\, e_2$, then $|a|^2 + |b|^2 \leq 1$. By the other hand, $1 = |\psi| = |\psi_1 + \psi_2| = |\lambda|^2 + |\mu|^2$ and

$$\alpha =< \theta, \psi_2 >= b\,\overline{\mu}, \quad < \theta, \psi_1 >= a\,\overline{\lambda}, \quad a\,\overline{\lambda} = -b\,\overline{\mu} = -\alpha.$$

From this, it follows that $|\frac{-\alpha}{a}|^2 + |\frac{\alpha}{b}|^2 = 1$, that is, $|\alpha|^2 = \frac{|a|^2 \| b|^2}{|a|^2+|b|^2} < \frac{1}{4}$.

Note that if $a\,b = 0$, then $\alpha = 0$. $\square$

Lemma 28 *Given positive integers d, D, where $1 < d$ and $D > 2d + 2$ denote*

$$f(t) = (1-t)^{d+1-D}\,(1+t)^{1-d} + (1+t)^{d+1-D}\,(1-t)^{1-d}.$$

then, $f(t)$ is increasing on the interval $(0, 1)$.

Proof For any $t \in (0, 1)$, we have

$$f'(t) \ = (1-t)^{d+1-D}\,(1+t)^{1-d}\left[\frac{1-d}{1+t} - \frac{d+1-D}{1-t}\right] + (1+t)^{d+1-D}\,(1-t)^{1-d}\left[\frac{d+1-D}{1+t} - \frac{1-d}{1-t}\right].$$

Taking $z = \frac{1+t}{1-t} > 1$, we get

$$(1+t)^{D-1}\,f'(t) = z^{D-d-1}\,[(D-d-1)\,z - (d-1)] + z^{d-1}\,[(d-1)\,z - (D-d-1)] >$$

$$z^{D-d-1}\,[(D-d-1) - (d-1)] + z^{d-1}\,[\,(d-1) - (D-d-1)] =$$

$$(z^{D-d-1} - z^{d-1})\,(D-2\,d) > 0,$$

because $z > 1$, $\square$

Suppose $0 \leq a < 1/4$ and d, D positive integers, such that $1 < d < D-1$. Define

$$J(d, D, a) \ = \frac{(D-1)\,!}{(d-1)\,!\,(D-d-1)\,!}\int_{1/2-\sqrt{1/4-a}}^{1/2+\sqrt{1/4-a}} \frac{(w\,(1-w)-a)^{D-2}}{w^{D-d-1}\,(1-w)^{d-1}}\,\mathrm{d}w.$$

Lemma 29 *Suppose d, D are positive integers $1 < d$, $2\,d + 2 < D$. Then*

$$0 \leq J(d, D, a) < e^{-4\,a(D-3/2)}, \quad \textit{where } \ 0 \leq a < 1/4.$$

Proof Note that $J(d, D, a)$ is positive.

In the integration, we divide the integral in two parts: $[1/2 - \sqrt{1/4 - a},\ 1/2]$ and $[1/2,\ 1/2 + \sqrt{1/4 - a}]$.

We make a change of variable $w = 1/2 - \sqrt{1/4 - x}$ on the first interval and $w = 1/2 + \sqrt{1/4 - x}$ on the second interval. On both cases, we get $x = w(1 - w)$ and $a \le x \le 1/4$.

From this, it follows

$$J(d, D, a) = \frac{(D-1)!}{(d-1)!\,(D-d-1)!} \int_a^{1/4} (x-a)^{D-2} \left[\left(\frac{1}{2} - \sqrt{\frac{1}{4} - x} \right)^{d+1-D} \left(\frac{1}{2} + \sqrt{\frac{1}{4} - x} \right)^{1-d} + \left(\frac{1}{2} + \sqrt{\frac{1}{4} - x} \right)^{d+1-D} \left(\frac{1}{2} - \sqrt{\frac{1}{4} - x} \right)^{1-d} \right] \frac{1}{2\sqrt{1/4 - x}}\, \mathrm{d}x$$

$$= \frac{2^{D-2}\,(D-1)!}{(d-1)!\,(D-d-1)!} \int_a^{1/4} (x-a)^{D-2} [(1 - \sqrt{1-4x})^{d+1-D}\,(1 + \sqrt{1-4x})^{1-d} + (1 + \sqrt{1-4x})^{d+1-D}\,(1 - \sqrt{1-4x})^{1-d}\,] \frac{1}{\sqrt{1-4x}}\, \mathrm{d}x.$$

Now, we consider $y = \frac{x-a}{1/4-a}$. In this case $(1 - 4x) = (1 - 4a)(1 - y)$.

Then

$$J(d, D, a) =$$

$$\frac{(1-4a)^{3/2}\,(D-1)!}{2^D\,(d-1)!\,(D-d-1)!} \int_0^1 y^{D-2} [\,(1 - \sqrt{1-4a}\,\sqrt{1-y})^{d+1-D}$$

$$(1 + \sqrt{1-4a}\,\sqrt{1-y})^{1-d} + (1 + \sqrt{1-4a}\,\sqrt{1-y})^{d+1-D}$$

$$(1 - \sqrt{1-4a}\,\sqrt{1-y})^{1-d}\,] \frac{1}{\sqrt{1-y}}\, dy.$$

Note that just the expression under [ ] depends on a. For each $y \in (0, 1)$, we have $\sqrt{1-4a}\,\sqrt{1-y} \in (0, 1)$ is an decreasing function of a. It follows from Lemma 28 that for each $y \in (0, 1)$, the integrand is a decreasing function of a.

Therefore, $\frac{J(d,D,a)}{(1-4a)^{D-3/2}}$ is a decreasing function of a. As $J(d, D, 0) = 1$ (see Lemma 23), it follows that

$$J(d, D, a) \le (1 - 4\,a)^{D-3/2}, \quad 0 \le a < 4.$$

Finally, note that $(1 - 4\,a)^{D-3/2} \le e^{-4\,a\,(D-3/2)}$ □

Corollary 30 *If* $1 < d,\ D > 2\,d + 2$ *and* $\frac{\log D}{D} < \frac{1}{3}$, *then*

$$J(d, D, a) < D^{-3} e^{\frac{9 \log D}{2D}}, \quad \textit{where } a = \frac{3}{4} \frac{\log D}{D}.$$

Proof It follows from Lemma 29, because $0 < \frac{3}{4}\frac{\log D}{D} < \frac{1}{4}$. □

Lemma 31 *Suppose* $1 \le \nu \le N$, $3 < d_\nu$, $D > 2\,d_\nu + 2$, *and* $\frac{\log D}{D} < \frac{1}{5}$.

Then

$$\int_\Delta \max_{1 \le \alpha \ne \beta \le D} |<\phi_\alpha, P_\nu(\mathcal{D})\phi_\beta>|^2\, w_\Delta(\mathcal{D}) < \frac{\log D}{D} \tag{8}$$

where $\phi_1, .., \phi_D$ *is an orthonormal basis of eigenvectors for* H *(without resonances).*

Proof By Lemma 23 and Corollary 30, the probability that the integrand is bigger than a is smaller than

$$\frac{D\,(D-1)}{2}\,D^{-3}e^{\frac{9\,\log D}{2\,D}}, \quad a=\frac{3\,\log D}{4\,D},$$

because as $e_{\alpha,\beta}=\overline{e_{\beta,\alpha}}$, we just have to take $\alpha<\beta$.

By the Remark before Lemma 24, the integral is smaller than

$$\frac{3}{4}\,\frac{\log D}{D}+\frac{D\,(D-1)}{8}D^{-3}\,e^{\frac{9\,\log D}{2\,D}},$$

because by Lemma 27 $|e_{\alpha,\beta}|<1/4$.

As $D-1<D$, we have

$$\frac{\frac{D\,(D-1)}{8}D^{-3}\,e^{\frac{9\,\log D}{2\,D}}}{\frac{\log D}{D}}<\frac{1}{8\,D}e^{\frac{9\,\log D}{2\,D}}\,\frac{D}{\log D}=e^{\frac{9\,\log D}{2\,D}}\,\frac{1}{8\,\log D}.$$

Now, as $D\geq 9$, $\log D\geq 2$, we get

$$\frac{1}{8\,\log D}e^{\frac{9\,\log D}{2\,D}}<\frac{1}{16}e^{9/10}<\frac{e}{16}<1/4.$$

Now, we put the two estimates together $\frac{3}{4}\,\frac{\log D}{D}+\frac{1}{4}\,\frac{\log D}{D}$ and we get the claim of the Lemma. □

The Lemma 18 follows from Lemmas 26 and 31. In this way, we get the claim of the Quantum Ergodic Theorem of von Neumann.

Appendix 1

In this Appendix, we will show that

$$\frac{1}{\mathrm{vol}\,(\mathrm{S})}\int_S\left(\sum_{j=1}^{d}|x_j|^2\right)^2\,\mathrm{d}S(x)=\frac{d^2+d}{D\,(D+1)}. \tag{9}$$

First, we will show that when S is the unitary sphere in $\mathbb{R}^n$, $m\geq 1$, and $n\geq m$, then

$$\int_S\left(\sum_{j=1}^{m}|x_j|^2\right)^2\,\mathrm{d}S(x)=\mathrm{vol}\,(\mathrm{S})\frac{m^2+2m}{n\,(n+2)}. \tag{10}$$

It is easy to see that (9) follows from (10).

1) $\int x_j^2\mathrm{d}S(x)=\frac{\mathrm{vol}(S)}{n}$ for $j=1,2,...,n$, because the integral does not depend of j.
2) Suppose B is the unitary ball in $\mathbb{R}^n$. Consider in polar coordinates

   $T:S\times[0,1]\to B,$

   where $T(x,\rho)=\rho\,x$.
   Then, $T^*(\mathrm{d}x_1\wedge...\wedge\mathrm{d}x_n)=\rho^{n-1}dS(x)\wedge d\rho$.
   Therefore

   $$\int_B(x_1^2+..+x_n^2)\mathrm{d}x_1...\mathrm{d}x_n=\int_{S\times[0,1]}T^*((x_1^2+..+x_n^2)\mathrm{d}x_1\wedge...\wedge\mathrm{d}x_n)=$$

   $$\int_{S\times[0,1]}\rho^{n+1}dS(x)\wedge d\rho=\mathrm{vol}(S)\int_0^1\rho^{n+1}d\rho=\frac{\mathrm{vol}(S)}{n+2}.$$

   Finally, $\int_B x_j^2\mathrm{d}x_1...\mathrm{d}x_n=\frac{\mathrm{vol}(S)}{n\,(n+2)}$, because it is independent of $j=1,2...,n$.

3) For $j = 1, 2..., n$, we have
$$\int_S x_j^4 \, dS(x) = 3 \int_B x_j^2 \, \mathrm{d}x_1 ... \mathrm{d}x_n = \frac{3\mathrm{vol}(S)}{n\,(n+2)}$$
by the divergent theorem and by 2) above.
4) If $\leq i < j \leq n$, then
$$\int_S x_i^2 \, x_j^2 \, dS(x) = \int_B x_j^2 \, \mathrm{d}x_1 ... \mathrm{d}x_n = \frac{\mathrm{vol}(S)}{n\,(n+2)}.$$
by the divergent theorem and by 2) above.

The integral
$$\int_S \left(\sum_{j=1}^{d} |x_j|^2 \right)^2 \, \mathrm{d}S(x)$$
is a sum of terms of the kind $\int_S x_i^2 \, x_j^2 \, dS(x)$, $i \neq j$, and $\int_S x_j^4 \, dS(x)$, $j = 1, 2, ..n$.

Just collecting the different terms and using the estimates above, we get the initial claim (10).

Appendix 2: Proof of Lemma 20

Suppose $\epsilon > 0$ is small enough, consider

$f\,|_{f^{-1}(a-\epsilon,\, a+\epsilon)} : f^{-1}(a-\epsilon,\, a+\epsilon) \to (a-\epsilon,\, a+\epsilon)$.

Given $h \in \mathbb{R}$, $0 < |h| < \epsilon$, then integrating $(g \circ f)\,\lambda$, we get

$$G(a+h) - G(a) = \int_0^h g(a+t)\,\mathrm{d}t \int_{X_{a+t}} \frac{\lambda_{a+t}}{|\ \mathrm{grad}\, f\ |}.$$

(where λ_v is the volume form on $X_v = f^{-1}(v)$ for $v \in (a-\epsilon,\, a+\epsilon)$), because $\mathrm{d}f(\mathrm{grad} f) = |\ \mathrm{grad}\, f\ |^2$. From this follows that for some $0 \leq \theta \leq 1$, we have

$$G(a+h) - G(a) = \ h\, g(a+\ \theta\, h) \int_{X_{a+\theta\, h}} \frac{\lambda_{a+\theta\, h}}{|\ \mathrm{grad}\, f\ |},$$

Now, we divide the above expression by h and we take the limit when $h \to 0$ □

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