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RESEARCH ARTICLE

On ordered equilibria in games with increasing best responses

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Abstract
This paper investigates conditions under which games have totally ordered equilibria, which has implications for the stability as well as the Pareto-ranking of equilibria. We first show that when best responses are strongly increasing, all games with up to three players as well as games with symmetric best responses have totally ordered equilibria. Furthermore, the same results hold when best responses are non-decreasing and equilibria are strict. Non-decreasingness and strong increasingness of best responses are implied by payoffs satisfying the single-crossing and strict single-crossing properties, and hence the former assumptions are weaker than what is assumed in games of strategic complements. Nevertheless, we show that even when the stronger assumption of increasing differences is satisfied, non-symmetric games with more than three players generally do not have totally ordered equilibria.

Keywords Strategic complements · Monotone games · Ordered equilibria · Equilibrium chain

JEL Classification C60 · C70 · C72

1 Introduction

One useful feature of games of strategic complements (GSC) is the structure of the set of Nash equilibria. Milgrom and Shannon (1994) show that GSC have largest and smallest equilibria. Zhou (1994) shows that in general supermodular games, the set of Nash equilibria is a complete lattice, which is further generalized in Sabarwal (2023a, b). Moreover, Echenique (2003) proves a stronger result in two player games

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with complementarities and totally ordered strategy spaces, showing that the equilibrium set in such games is a sublattice. Such structure facilitates comparative statics analysis as well as equilibrium selection via global games, among other applications.

Even more can be said when the set of equilibria is not only a lattice, but also totally ordered, where for any two equilibria x and y of the game, we either have that x is greater or equal to y or vice versa. Barthel and Hoffmann (2023) show that when the set of equilibria in a GSC is totally ordered, then under mild conditions on the strict increasingness of best responses, at least one equilibrium is guaranteed to be stable. Milgrom and Roberts (1990) provide conditions under which we can Pareto-rank totally ordered equilibria in supermodular games.

This paper provides conditions under which the set of equilibria in a game can be guaranteed to be totally ordered. Related to our results are those of Vives (1985), who shows that the equilibrium set in strictly supermodular games with two or three players and totally ordered action spaces is totally ordered. We show that conditions strictly weaker than those required of GSC and supermodular games are enough to generate a wider range of results. In particular, we show that when best responses are strongly increasing, so that the smallest best response to some action by opponents is larger than the largest best response to a lower action, then the set of equilibria is totally ordered in games of up to three players, as well as in N player games where best responses are symmetric. We also show that the same results hold when best response correspondences are non-decreasing, as long as equilibria are strict. We illustrate the fact that games which exhibit strongly increasing best responses may fail to satisfy increasing differences or the single-crossing property, but also that games where payoffs satisfy the strict single-crossing property exhibit strongly increasing best responses.

Finally, we show that even under the strongest version of GSC, where payoff functions satisfy increasing differences, games with more than three players, as well as games with multi-dimensional action spaces, generally do not have totally ordered equilibria. Examples of each case are given.

2 Definitions and results

We will make use of lattice concepts, which can be found in Topkis (1998). For any partially ordered set $(X, \geq)$, and any non-empty subset $S \subseteq X$, we let $\vee S$ and $\wedge S$ denote the least upper bound and greatest lower bound of S, respectfully. A correspondence $f : X \rightrightarrows Y$ between partially ordered sets is complete lattice valued if for each $x \in X$, $\vee f(x), \wedge f(x) \in f(x)$.

Definition 1 A game $\mathcal{G} = \{\mathcal{I}, (\mathcal{A}_i, \pi_i)_{i \in \mathcal{I}}, \geq\}$ has the following elements:

1. $\mathcal{I}$ is a finite set of players, $\mathcal{I} = \{1, 2, \ldots, N\}$.
2. Each player $i \in \mathcal{I}$ has a totally ordered action space $(\mathcal{A}_i, \geq)$. We denote products of players' action spaces by $\mathcal{A} = \prod_{i \in \mathcal{I}} \mathcal{A}_i$, which is endowed with the product order. We will abuse notation and let $\geq$ denote all product orders in their respective context.

3. Each player $i \in \mathcal{I}$ has a payoff function $\pi_i : \mathcal{A}_{-i} \to \mathbb{R}$, and a corresponding best response correspondence $\phi_i : \mathcal{A}_{-i} \rightrightarrows \mathcal{A}_i$ defined as

$$\phi_i(a_{-i}) = \underset{a_i \in \mathcal{A}_i}{arg\,max}\ \pi_i(a_i, a_{-i}).$$

which we assume is non-empty and complete lattice valued for all $a_{-i} \in \mathcal{A}_{-i}$.

We will say that $\phi_i : \mathcal{A}_{-i} \rightrightarrows \mathcal{A}_i$ is non-decreasing if it is complete lattice valued, and for all $a'_{-i} > a_{-i}$, $\vee\phi_i(a'_{-i}) \geq \vee\phi_i(a_{-i})$ and $\wedge\phi_i(a'_{-i}) \geq \wedge\phi_i(a_{-i})$. Echenique (2002) introduces the slightly stronger notion of ϕ_i being strongly increasing, which is satisfied if, for any $a'_{-i}, a_{-i} \in \mathcal{A}_{-i}$ such that $a'_{-i} > a_{-i}$, we have that $\wedge\phi_i(a'_{-i}) \geq \vee\phi_i(a_{-i})$. Note that ϕ_i being strongly increasing is implied by ϕ_i being non-decreasing and singleton valued. Lastly, we say that $\mathcal{G}$ is a game with symmetric best responses if, in addition to Definition 1, we have that for each $i, j \in \mathcal{I}$, $\mathcal{A}_i = \mathcal{A}_j$, and for each $a_{-i,j} \in \mathcal{A}_{-i,j}$ and $a \in \mathcal{A}_i = \mathcal{A}_j$,

$$\phi_i(a_{-i,j}, a_j = a) = \phi_j(a_{-i,j}, a_i = a).$$

Amir et al. (2008) define a standard notion of symmetric games which requires that any two players i and j receive the same utility over all player-wise permutations of a given strategy profile. This definition implies our definition of symmetric best responses when one considers the permutation which interchanges indices i and j, while keeping all other indices the same. In particular, symmetric best responses allow for utility to differ at equivalent strategy profiles for each player, as long as the best responses are the same. Given these concepts, we have the following result:

Theorem 1 *Let $\mathcal{G}$ be a game with non-decreasing best responses. Then the set of Nash equilibria is totally ordered if it is non-empty and either of the following is satisfied:*

1. *$N = 2$, and for at least one player $i \in \mathcal{I}$, ϕ_i is strongly increasing.*
2. *$N = 3$, and for each player $i \in \mathcal{I}$, ϕ_i is strongly increasing.*
3. *$\mathcal{G}$ is a game with symmetric best responses, and for each player $i \in \mathcal{I}$, ϕ_i is strongly increasing.*

Proof Suppose that $N = 2$, and let a^* and a^{**} be two equilibria such that $a^* \neq a^{**}$. Suppose WLOG that $a_1^{**} > a_1^*$ and $a_2^{**} < a_2^*$, and that ϕ_1 is strongly increasing. Then

$$a_1^* \geq \wedge\phi_1(a_2^*) \geq \vee\phi_1(a_2^{**}) \geq a_1^{**},$$

which is a contradiction. If, on the other hand, ϕ_2 is strongly increasing, then

$$a_2^{**} \geq \wedge\phi_2(a_1^{**}) \geq \vee\phi_2(a_1^*) \geq a_2^*,$$

which is also a contradiction. Hence, $a^* \neq a^{**}$ implies either $a^{**} \geq a^*$ or $a^* \geq a^{**}$. Now suppose $N = 3$, that each ϕ_i is strongly increasing, and suppose a^* and a^{**} are two equilibria such that $a^* \neq a^{**}$. Suppose WLOG that $a_1^{**} > a_1^*$. Notice then that $a_2^{**} \geq a_2^*$ and $a_3^* > a_3^{**}$ results in a contradiction, since ϕ_3 strongly increasing implies

$$a_3^{**} \geq \wedge\phi_3(a_{-3}^{**}) \geq \vee\phi_3(a_{-3}^*) \geq a_3^*.$$

A similar contradiction arises if $a_3^{**} \geq a_3^*$ and $a_2^* > a_2^{**}$. Lastly, if $a_2^* > a_2^{**}$ and $a_3^* > a_3^{**}$, then

$$a_1^* \geq \wedge\phi_1(a_{-1}^*) \geq \vee\phi_1(a_{-1}^{**}) \geq a_1^*,$$

which is also a contradiction. Hence, $a^{**} \geq a^*$. Lastly, suppose that $\mathcal{G}$ is symmetric, each ϕ_i is strongly increasing, and consider some equilibrium a^*. Suppose that for two players $i, j \in \mathcal{I}$, $a_i^* > a_j^*$. Then

$$\begin{aligned}\vee\phi_i(a_{-i}^*) \geq a_i^* > a_j^* \geq \wedge\phi_j(a_{-j}^*) &= \wedge\phi_j(a_{-i,j}^*, a_i = a_i^*)\\ = \wedge\phi_i(a_{-i,j}^*, a_j = a_i^*) &\geq \vee\phi_i(a_{-i,j}^*, a_j = a_j^*) = \vee\phi_i(a_{-i}^*),\end{aligned}$$

so that $\vee\phi_i(a_{-i}^*) > \vee\phi_i(a_{-i}^*)$, a contradiction. Hence each equilibrium is symmetric, and since strategy spaces are totally ordered, it follows that all equilibria are totally ordered. □

It is natural to ask whether the assumption of strongly increasing best responses can be relaxed. Theorem 2 shows that the results of Theorem 1 hold for non-decreasing best responses as long as equilibria are strict, where each player best responds uniquely to the equilibrium actions of opponents. Because the proof is similar to that of Theorem 1, it is given in the Appendix.

Theorem 2 *Let $\mathcal{G}$ be a game with non-decreasing best responses. Then the set of Nash equilibria is totally ordered if it is non-empty, all equilibria are strict, and either of the following is satisfied:*

1. *$N = 2$ or $N = 3$.*
2. *$\mathcal{G}$ is a game with symmetric best responses.*

Proof See Appendix. □

By Milgrom and Shannon (1994) and (Shannon 1995), games of (strict) strategic complements (GSC), where payoffs satisfy the (strict) single-crossing property, have non-decreasing (strongly increasing) best responses, respectively. Vives (1985) shows that the equilibrium set in two or three player strict GSC is totally ordered. The results of Theorems 1 and 2 improve upon these results, since games with non-decreasing or strongly increasing best responses need not satisfy the single-crossing property. The example below illustrates a game where the single-crossing property is violated, but, according to our results, has ordered equilibria due to having increasing best responses.

Example 1 Consider the following payoff matrix:

		Player 2		
		1	2	3
Player 1	1	**2,2**	1,1	8,1
	2	1,1	5,5	7,6
	3	1,8	6,7	**9,9**

Suppose for each player actions are ordered as $3 > 2 > 1$. Then ϕ_1 is non-decreasing, as $\phi_1(1) = \{1\}$, $\phi_1(2) = \{3\}$, and $\phi_1(3) = \{3\}$, yet $\pi_1(2, 2) > \pi_1(2, 1)$ and $\pi_1(1, 3) > \pi_1(2, 3)$, so that the single-crossing property is violated. Nevertheless, because ϕ_2 is also strongly increasing, we have ordered equilibria $(1, 1)$ and $(3, 3)$, as Theorem 1 suggests. Conversely, by Theorem 4 in Milgrom and Shannon (1994), the single-crossing property implies non-decreasing best responses, and by Theorem 4 in Shannon (1995), the strict single-crossing property implies strongly increasing best responses. Hence, having non-decreasing (strongly increasing) best responses is weaker than payoffs satisfying the (strict) single-crossing property.

Applications of these results include Theorem 2 in Barthel and Hoffmann (2023), which shows that when the set of equilibria in a GSC is totally ordered, then under mild conditions on the strict increasingness of best responses, at least one equilibrium is guaranteed to be stable. Furthermore, Theorem 7 in Milgrom and Roberts (1990) provides conditions under which one can Pareto-rank totally ordered equilibria in GSC when payoffs satisfy increasing differences, which can be extended to the case of the single-crossing property. Lastly, we examine to what extent the conditions in Theorems 1 and 2 can be relaxed. The example below shows that even in GSC where payoff functions satisfy increasing differences, and best responses are strongly increasing, games where $N > 3$ do not generically have totally ordered equilibria.

Example 2 Consider the following four player game where each player i's action space is given by $\mathcal{A}_i = \{1, 2\}$, where $2 > 1$, and payoffs are as given in the table below:

		P3			
		1		**2**	
		P2		P2	
P4	P1	1	2	1	2
1	1	**5,5,5,5**	2,0,0,2	2,0,0,2	**1,1,1,1**
1	2	0,2,2,0	0,0,0,0	0,0,0,0	0,2,2,0
2	1	0,2,2,0	0,0,0,0	0,0,0,0	0,2,2,0
2	2	**1,1,1,1**	2,0,0,2	2,0,0,2	**4,4,4,4**

Best responses are unique and strongly increasing. Notice that this game has four Nash equilibria, $(1, 1, 1, 1)$, $(1, 2, 2, 1)$, $(2, 1, 1, 2)$ and $(2, 2, 2, 2)$. As in the previous example, this game has a largest equilibrium $(2, 2, 2, 2)$ and smallest equilibrium $(1, 1, 1, 1)$, but the two middle equilibria are unordered. Hence, strongly increasing best responses cannot guarantee totally ordered equilibria when $N > 3$.

Finally, we examine whether strongly increasing best responses are enough to guarantee totally ordered equilibria when action spaces are multi-dimensional. Once again, the example below shows that the answer is unfortunately no.

Example 3 Consider the following two player game in which each player i's strategy space is given by $\mathcal{A}_i = \{1, 2\}^2$, where payoffs are summarized below:

Player 2

Player 1	(1,1)	(1,2)	(2,1)	(2,2)
(1,1)	**2,2**	-1,1	1,0	-2,-1
(1,2)	1,-1	0,0	**2,2**	1,1
(2,1)	0,1	**2,2**	0,0	1,0
(2,2)	-1,-2	1,1	0,1	**2,2**

As in Example 2, each payoff function π_i satisfies increasing differences, and each ϕ_i is strongly increasing. Nevertheless, we see that this game possesses unordered equilibria $((2, 1), (1, 2))$ and $((1, 2), (2, 1))$. Hence, strongly increasing best responses is not sufficient to guarantee totally ordered equilibria when strategy spaces are not themselves totally ordered.

Appendix

Proof of Theorem 2

Proof Suppose that $N = 2$, and let a^* and a^{**} be two equilibria such that $a^* \neq a^{**}$. Suppose WLOG that $a_1^{**} > a_1^*$ and $a_2^{**} < a_2^*$, and that ϕ_1 is strongly increasing. Then

$$a_1^* \geq \wedge\phi_1(a_2^*) \geq \vee\phi_1(a_2^{**}) \geq a_1^{**},$$

which is a contradiction. If, on the other hand, ϕ_2 is strongly increasing, then

$$a_2^{**} \geq \wedge\phi_2(a_1^{**}) \geq \vee\phi_2(a_1^*) \geq a_2^*,$$

which is also a contradiction. Hence, $a^* \neq a^{**}$ implies either $a^{**} \geq a^*$ or $a^* \geq a^{**}$. Now suppose $N = 3$, that each ϕ_i is strongly increasing, and suppose a^* and a^{**} are two equilibria such that $a^* \neq a^{**}$. Suppose WLOG that $a_1^{**} > a_1^*$. Notice then that $a_2^{**} \geq a_2^*$ and $a_3^* > a_3^{**}$ results in a contradiction, since ϕ_3 strongly increasing implies

$$a_3^{**} \geq \wedge\phi_3(a_{-3}^{**}) \geq \vee\phi_3(a_{-3}^*) \geq a_3^*.$$

A similar contradiction arises if $a_3^{**} \geq a_3^*$ and $a_2^* > a_2^{**}$. Lastly, if $a_2^* > a_2^{**}$ and $a_3^* > a_3^{**}$, then

$$a_1^* \geq \wedge\phi_1(a_{-1}^*) \geq \vee\phi_1(a_{-1}^{**}) \geq a_1^*,$$

which is also a contradiction. Hence, $a^{**} \geq a^*$. Lastly, suppose that $\mathcal{G}$ is symmetric, each ϕ_i is strongly increasing, and consider some equilibrium a^*. Suppose that for two players $i, j \in \mathcal{I}$, $a_i^* > a_j^*$. Then

$$\begin{aligned}\vee\phi_i(a_{-i}^*) \geq a_i^* > a_j^* \geq \wedge\phi_j(a_{-j}^*) &= \wedge\phi_j(a_{-i,j}^*, a_i = a_i^*)\\ &= \wedge\phi_i(a_{-i,j}^*, a_j = a_i^*) \geq \vee\phi_i(a_{-i,j}^*, a_j = a_j^*) = \vee\phi_i(a_{-i}^*),\end{aligned}$$

so that $\vee\phi_i(a_{-i}^*) > \vee\phi_i(a_{-i}^*)$, a contradiction. Hence each equilibrium is symmetric, and since strategy spaces are totally ordered, it follows that all equilibria are totally ordered. □

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