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RESEARCH ARTICLE

# Generalization of the social coalitional equilibrium structure

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**Abstract**

We generalize the notion of Ichiishi (Econometrica 49(2):369–377, 1981)'s *social coalitional equilibrium* to a *multi-layered coalition structure* with *parameters*, in which agents can incorporate simultaneously multiple coalition structures with multiple independent coalition-deviation opportunities. For each opportunity, agents play a social coalitional equilibrium (SCE) game, called a *sub-parametric SCE game*, constrained by external environment (parameters and joint decisions of all other sub-parametric SCE games). The generalized social coalitional equilibrium (GSCE) concept is, therefore, considered to be a synthesis of the Nash equilibrium concept and the cooperative solution concept. We provide the definition of GSCE and give the proof of existence theorem. Through some applications to general equilibrium models, the GSCE concept provides a conceptual framework for describing coexisting different industries having independent investment opportunities and their simultaneously determined *industrial organizations*.

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## 1 Introduction

The social coalitional equilibrium (SCE) by Ichiishi (1981) is a significant concept that provides us a unified perspective on economic (abstract market equilibrium) settings and cooperative game-theoretic arguments for finding stable coalition structures in society. He utilized his social coalitional equilibrium concept to characterize the formation of firms as a hybrid cooperative nature in non-cooperative market settings (Ichiishi 1993).

From a general equilibrium framework, however, Ichiishi's hybrid equilibrium concept has a serious restriction: the *admissible coalition structure* is a partition of the set of agents. His characterization of firms, therefore, is typically a case where each agent cannot be an owner of two or more firms, like the labor-owned company in Ichiishi (1993). Needless to say, in a standard private ownership general equilibrium setting, it is clearly insufficient to restrict the agents' coalitional structures to the class of partitions. Many kinds of coalitions exist for different purposes and benefits, and an agent is allowed to belong simultaneously to several types with different purposes. The formation of firms should also be characterized under such settings.

In this paper, we generalize the concept of Ichiishi's social coalitional framework as a *generalized social coalitional equilibrium* (GSCE) model with parameters so that agents can incorporate multiple *admissible coalition structures* and take into consideration their budgetary constraints by cooperating simultaneously in multiple independent coalition-deviation opportunities. In each opportunity, as in an SCE game, agents form a coalition and jointly determine a cooperative strategy depending on the complementary decision of all outside existing coalitions. The agent's behavior in each opportunity (sub-parametric SCE game) is thus constrained by the external environment (parameters and joint decisions of outside coalitions) and his/her own actions in other opportunities.[1] A generalized social coalitional equilibrium is obtained if and only if no coalition can guarantee higher utility for its members in each opportunity. This generalized social coalitional equilibrium concept, therefore, formulates a *multi-layered core* driven by intertwined multiple sub-parametric SCE games. Sufficient condition, the balancedness condition parameterized by parameters and complementary cooperative strategies, is introduced for the GSCE framework to have *generalized social coalitional equilibrium*. Accordingly, the main result, the existence of a *generalized social coalitional equilibrium*, has been proven in this paper.

[1] The generalized social coalitional equilibrium concept, especially the multi-layered coalition structure, is intimately involved in the hybrid solution concept introduced by Zhao (1992). In contrast to his concept, however, each agent is allowed to participate simultaneously in multiple sub-parametric SCE games (coalition-deviation opportunity). The sub-parametric SCE game here means a sub-SCE game in which the agent's feasible strategy and utility are parameterized by the strategies of all other sub-SCE games, see Zhao (1992).

Applying the main result, we address coalition production economies to determine the formation of firms and their *share-holding rates* as a result of our equilibrium multi-layered investment purposes with the coalition deviation possibilities of agents. We consider technology, a correspondence, that depends on the type of coalition and investments for it. Two classes of economies are studied in this paper: one is the called Ichiishi-Quinzii economy, in which consumer's investments are restricted to the amounts of his/her initial holdings; the other is called the Arrow-Debreu economy, in which consumer's investments beyond the amounts of his/her initial holdings are available. We also focus on the *optimality* properties of the equilibria with multi-layered coalition structures. For every Ichiishi-Quinzii production economy, there exists a *generalized social coalition equilibrium* state that is Pareto optimum. In particular, in an economy with increasing returns under a single-layer coalition structure, the existence of equilibrium and an optimality property, as stated in Ichiishi and Quinzii (1983), are considered special consequences of our results. In contrast, we show that every equilibrium allocation involving its equilibrium multi-layered coalition structure is Pareto optimum for every Arrow-Debreu production economy with firm formation.

The generalized social coalitional equilibrium framework and its application to a production economy are also inspired by Boehm (1974), Greenberg (1979), Ichiishi (1977), and Bonnisseau and Iehle (2007). We present a general framework by investigating agents' strategies in multi-layered cooperative opportunities where payoffs depend on the type of coalition and parameters like prices that can be used to represent the budgetary constraints of investments. The generalized social coalitional equilibrium concept is also based on a standard NTU cooperative solution concept, developed by Aumann and Peleg (1960), Aumann (1961), Scarf (1967), and Scarf (1971). Our concept generalized the social coalitional equilibrium concept of Ichiishi (1981) and provides a method for extending NTU games to multi-layered coalition structures or exogenous environments as well as a mathematical tool for formulating its core.[2]

The generalized social coalitional equilibrium concept is applicable to many fields in the real world. People make homes in sociology and form corporate in labor economics. In the international economy, nations participate in international organizations related to trade, such as the WTO, as well as in economic agreements, such as FTAs and EPAs. In both cases, the social structure is close to a multi-layered coalition structure, in which agents are partitioned into groups by the social sector, simultaneously making joint decisions according to their behavior elsewhere. Therefore, the generalized social coalitional equilibrium concept can be assumed to be an appropriate theoretical model for interpreting real-world situations.

The outline of this paper is organized as follows. In Sect. 2, we generalize Ichiishi's social coalitional equilibrium existence lemma to utilize it for describing a society in which multiple coalition structures are naturally characterized as a hybrid one-shot equilibrium. Section 3 confirms the meaning and validity of our *balancedness condition* that plays an essential role in our existence result. The proof of our existence

[2] The existence result of the ($\alpha$-)core was proven in exchange economy with non-ordered preference (Yannelis 1991; Scalzo 2022) and in NTU games with non-ordered preferences (Kajii 1992; Martins-da Rocha and Yannelis 2011), with infinite player game (Yang 2017), and with infinite strategies (Yang and Zhang 2019).

of the equilibrium theorem is treated in Sect. 4. Section 5 extends the production economy formulated by Ichiishi and Quinzii (1983) by applying the generalized social coalitional equilibrium concept and provides the existence theorem of firm-formation general equilibrium.

We use $R$ as the set of real numbers. For finite set $A$, denote by $\sharp A$ the number of elements of $A$. We write $R^K$ to denote the $R^{\sharp K}$, $\sharp K$-dimensional vector space. The order relations on $R^K$, $\geqq$ and $>$ are defined respectively as $(x_k)_{k\in K} \geqq (y_k)_{k\in K}$ iff $x_k \geqq y_k$ for all $k$, and $(x_k)_{k\in K} > (y_k)_{k\in K}$ iff $(x_k)_{k\in K} \geqq (y_k)_{k\in K}$ and $(x_k)_{k\in K} \neq (y_k)_{k\in K}$. We also define relation $\gg$ as $(x_k)_{k\in K} \gg (y_k)_{k\in K}$ iff $x_k > y_k$ for all $k$. By $R^K_+$ and $R^K_{++}$, we represent sets $\{x \in R^K \mid x \geqq 0\}$ and $\{x \in R^K \mid x \gg 0\}$. For $n$-dimensional Euclidean space $R^n$, notation $e^1 = (1, 0, \ldots, 0)$, $e^2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e^n = (0, \ldots, 0, 1)$ represents the standard base elements.

## 2 Generalized social coalitional equilibrium

In this section, we extend the *social coalitional equilibrium* (Ichiishi 1981) and its framework so that each agent can cooperate with others by forming coalitions. First, to treat *multiple cooperate opportunities*, $t = 1, \ldots, \lambda$, we generalize Ichiishi's single coalition structure model to the case where multiple coalition structures are formed. Second, to treat messages as given parameters for each agent, the cooperative game is parameterized by an element of a set (message space).

### 2.1 SCE under multiple coalition structures

Let $N = \{1, 2, \ldots, n\}$ be a non-empty finite index set of all the agents and let $\mathfrak{N}$ be the set of all non-empty subsets of $N$ (or all *coalitions*). Assume $\lambda$ kinds of *cooperate opportunities* (or *coalition types*) and denote $\Lambda = \{1, 2, \ldots, \lambda\}$. For each coalition type $t \in \Lambda$, agents form a coalition, and a *coalition structure* is identified with a sequence of $\lambda$ partitions of $N$, $\mathfrak{T} = (\mathfrak{T}^1, \ldots, \mathfrak{T}^\lambda)$, where $\mathfrak{T}^t$ is a partition of $N$. Let $\bar{\mathfrak{T}}$ be the set of all the coalition structures, the set of all *admissible* coalition structures which consists of all the possible sequences of $\lambda$ coalition partitions, $\mathfrak{T} \subset \bar{\mathfrak{T}}$, is fixed and defined as a non-empty finite set. The finiteness of $\mathfrak{T}$, derived from the finiteness of $\Lambda = \{1, \ldots, \lambda\}$, and the independency of $\lambda$ types of cooperate opportunities are two important assumptions of our model for describing firm formation.

Each agent $i \in N$ has a strategy set, $X_i$, which is a subset of Euclidean space $R^k$.[3] Denote by $X_S$ the product $\prod_{i\in S} X_i$ for each $S \in \mathfrak{N}$. We also denote by $x_S = (x_i)_{i\in S} \in X_S$. In the following, without any additional notation, we do not distinguish between $x$ and $x_N$ where $x_N$ is an element of $X_N = \prod_{i\in N} X_i$.

Suppose that for each social coalition structure $\mathfrak{T} = (\mathfrak{T}^1, \ldots, \mathfrak{T}^\lambda) \in \mathfrak{T}$, coalition type $t \in \Lambda$, and coalition $S \in \mathfrak{N}$, there is a correspondence:

$$K^{\mathfrak{T},t,S} : X_N \to X_S, \tag{1}$$

[3] For example, one can define $X_i$ by $X_i \equiv X_i^0 \times X_i^1 \times \cdots \times X_i^\lambda$ where $X_i^0$ is an action directly affecting agent $i$'s utility function and $X_i^k$, $k = 1, \ldots, \lambda$, represents agent $i$'s action set in the opportunity $k$.

which is a *feasible-strategy constraint correspondence* of coalition $S$ for coalition type $t$ under coalition structure $\mathfrak{T}$.[4] Adding to the constraint correspondences, we also assume that each agent $i$ in coalition $S$ for type $t$ has preference $\precsim_i$ on strategy set $X_i$ that can be represented by a *utility function*:

$$u_i^{\mathfrak{T},t,S} : X_N \times X_S \to R_+. \tag{2}$$

Now a *society* is described by the following list:

$$\big((X_i)_{i\in N}, (K^{\mathfrak{T},t,S}, (u_i^{\mathfrak{T},t,S})_{i\in S})_{(\mathfrak{T},t,S)\in \mathfrak{T}\times\Lambda\times\mathfrak{N}}\big).$$

A *social coalitional equilibrium* (SCE) is a pair $(x^*, \mathfrak{T}^*)$ of strategy profile $x_N^* \in X_N$ and admissible coalition structure $\mathfrak{T}^* = (\mathfrak{T}^{1^*}, \ldots, \mathfrak{T}^{\lambda^*}) \in \mathfrak{T}$ satisfying the following two conditions:

(SCE1: Feasibility) For each $t \in \Lambda$ and $S \in \mathfrak{T}^{t*}$, $x_S^* \in K^{\mathfrak{T}^*,t,S}(x^*)$.

(SCE2: Stability) There are no $s \in \Lambda$, $D \in \mathfrak{N}$, and $y_D \in K^{\mathfrak{T}^*,s,D}(x^*)$ such that for all $i \in D$,

$$u_i^{\mathfrak{T}^*,s,D}(x^*, y_D) > u_i^{\mathfrak{T}^*,s,S^s(i)}(x^*, x^*_{S^s(i)}),$$

where $S^s(i)$ is the unique coalition in opportunity $s$ such that $i \in S^s(i) \in \mathfrak{T}^{s*}$.

In the above, we defined SCE as a concept based on multiple coalition structures, $t = 1, \ldots, \lambda$. If we assume that for each coalition type $t$ and coalition $S$, feasible-strategy constraint correspondences do not depend on the coalition structure, i.e., $K^{\mathfrak{T},t,S}$ does not depend on $\mathfrak{T}$ for each $t \in \Lambda$, and if we consider special case $\lambda = 1$, then our framework coincides with the setting of Ichiishi (1981).[5]

### 2.2 Generalized SCE with parameters

In this paper, we further generalize the above SCE framework as a social coalitional equilibrium model with parameters. Suppose an additional information or message structure that parametrically defines an SCE setting. Let $X_0 \subset R^\ell$, $\ell \geq 1$, be a set of the parameters and element $x_0 \in X_0$ parametrically defines an SCE setting

[4] It should be emphasized that the members of coalition $S$ jointly determines cooperative strategies at the opportunity $t$ giving the complementary decisions of every existing coalition $D \in \mathfrak{T}^k$ for all $k \in \Lambda$ when $(x_N, \mathfrak{T})$ prevails. Such complementary decisions involve existing actions of the members of coalition $S$ in all other opportunities $k \neq t$, so that the feasible cooperative strategies of the coalition $S$ will also be constrained by the joint strategies of the existing coalition in every opportunity $k \neq t$ in which its members participate. If we define $X_i \equiv X_i^0 \times X_i^1 \times \cdots \times X_i^\lambda$ following footnote 3, then given strategies $x_N = (x_j^0, x_j^1, \ldots, x_j^\lambda)_{j\in N}$, the feasible-strategy constraint correspondence can be considered as $(\hat{x}_i^0, , x_i^1, \ldots, \hat{x}_i^t, \ldots, x_i^\lambda)_{i\in S} \in K^{\mathfrak{T},t,S}(x_N)$.

[5] Such a feasible-strategy dependency on the constraints of the multi-layered coalition structure is essential for characterizing the correlation of strategies among all cooperate investment opportunities. The problem of the robustness of such a multi-layered coalition structure cannot be treated as a straightforward extension of the single-layered SCE argument.

through feasible-strategy constraint correspondence $K$ for each $(\mathfrak{T}, t, S)$, where $S$ is an arbitrary element of $\mathfrak{N}$:

(Constraint for Strategy: $K$): $K^{\mathfrak{T},t,S} : X_0 \times X_N \to X_S$.

Based on these parameterized constraint correspondences, conditions (SCE1) and (SCE2) are generalized for each parameter $x_0 \in X_0$:

(GSCE1: Parameterized $K$ Feasibility under $x_0$) For each $t \in \Lambda$ and $S \in \mathfrak{T}^{t*}$, we have $x_S^* \in K^{\mathfrak{T}^*,t,S}(x_0, x^*)$.
(GSCE2: Parameterized $K$ Stability under $x_0$) There are no $s \in \Lambda$, $D \in \mathfrak{N}$, and $y_D \in K^{\mathfrak{T}^*,s,D}(x_0, x^*)$ such that for all $i \in D$,

$$u_i^{\mathfrak{T}^*,s,D}(x^*, y_D) > u_i^{\mathfrak{T}^*,s,S^s(i)}(x^*, x^*_{S^s(i)}),$$

where $S^s(i)$ is the unique coalition such that $i \in S^s(i) \in \mathfrak{T}^{s*}$.

It is also assumed that parameters are restricted by a correspondence, $G_0 : X_0 \times X_N \to X_0$. Hence, the *generalized sense of society* is a list:

$$\left(X_0, (X_i)_{i \in N}, G_0, \left(K^{\mathfrak{T},t,S}, (u_i^{\mathfrak{T},t,S})_{i \in S}\right)_{(\mathfrak{T},t,S) \in \mathfrak{T} \times \Lambda \times \mathfrak{N}}\right).$$

A *generalized social coalitional equilibrium* (GSCE) is a triplet, $(x_0^*, x^*, \mathfrak{T}^*)$, of parameter $x_0^* \in X_0$, strategy profile $x_N^* \in X_N$, and admissible coalition structure $\mathfrak{T}^* = (\mathfrak{T}^{1^*}, \ldots, \mathfrak{T}^{\lambda^*}) \in \mathfrak{T}$, satisfying (GSCE1) under $x_0^*$, (GSCE2) under $x_0^*$, and the following (GSCE3):

(GSCE3: Fixed Point Parameter) $x_0^* \in X_0$ satisfies $x_0^* \in G_0(x_0^*, x^*)$.

For the generalized social coalitional equilibrium model, we have the following equilibrium existence theorem, which is an extension of the SCE existence lemma of Ichiishi and Quinzii (1983). The proof and a rigorous prediction for condition (v) will be given in Sect. 4.

**Proposition 1** *For society* $\left(X_0, (X_i)_{i \in N}, G_0, \left(K^{\mathfrak{T},t,S}, (u_i^{\mathfrak{T},t,S})_{i \in S}\right)_{(\mathfrak{T},t,S) \in \mathfrak{T} \times \Lambda \times \mathfrak{N}}\right)$, *generalized social coalitional equilibrium* $(x_0^*, x^*, \mathfrak{T}^*) \in X_0 \times X_N \times \mathfrak{T}$ *exists if the following conditions are satisfied:*

*(i)* $X_0 \subset R^\ell$ *and* $X_i \subset X^k$, $i \in N$, *are non-empty, compact, and convex subsets.*
*(ii) For each* $S \in \mathfrak{N}$, $t \in \Lambda$, *and* $\mathfrak{T} \in \mathfrak{T}$, $K^{\mathfrak{T},t,S} : X_0 \times X_N \to X_S$ *is a continuous correspondence that is closed and non-empty valued.*
*(iii) For each* $i \in N$, $S \in \mathfrak{N}$, $t \in \Lambda$ *and* $\mathfrak{T} \in \mathfrak{T}$, $u_i^{\mathfrak{T},t,S} : X_N \times X_S \to R_+$ *is a continuous function.*
*(iv) The society is balanced. (Correspondence K satisfies the balancedness condition described in Sect. 3.)*
*(v) For each* $x \in X_N$ *and* $c \in R^N$, *socially feasible upper-contour set at x for c (described in Sect. 4) is convex.*
*(vi)* $G_0$ *is an upper-semicontinuous non-empty convex valued correspondence.*

## 3 Balancedness condition for GSCE framework

To show the existence of GSCE, we extend the notion of a *balanced game*. Given the set of all coalitions, $\mathcal{N} = \{A \subset N | \, A \neq \emptyset\}$, a finite family, $\{B_s\}_{s=1}^m$, of elements of $\mathcal{N}$, is *balanced* if there are non-negative real numbers, $\alpha_1, \alpha_2, \ldots, \alpha_m$, such that for each $i \in N$, $\sum_{B_s \ni i} \alpha_s = 1$. [6] In the literature of cooperative game theory, a coalitional-form game without side payments, $V : \mathcal{N} \to R^N$, is *balanced* if any utility allocation $(c_i)_{i \in N} \in R^N$ with a balanced family $\{B_s\}_{s=1}^m$ such that $(c_i)_{i \in B_s} \in V(B_s)$ for each $s = 1, \ldots, m$, satisfies $(c_i)_{i \in N} \in V(N)$. (A utility allocation attainable for all coalitions in a certainly balanced sub-family is also attainable in society.) Ichiishi (1981) generalizes such conditions to the SCE framework. In the following, we further extend the notion of balancedness to the GSCE structure.

As we formalized in Sect. 3, for each $\mathfrak{T} \in \mathfrak{T}$, $t \in \Lambda$, parameter $x_0 \in X_0$, and arbitrary strategy profile $(x) \in X_N$, coalition $S \in \mathcal{N}$ defines feasible strategy allocations and utility allocations for deviation as $K^{\mathfrak{T},t,S}(x_0, x) \subset X_S$ and $\{(u_i^{\mathfrak{T},t,S}(x, y_S))_{i \in S} | \, y_S \in K^{\mathfrak{T},t,S}(x_0, x)\}$. Therefore, for each $t \in \Lambda$ and $(x_0, x) \in X_0 \times X_N$, we can define a *generalized coalitional-form game without side payments*, $V_{x_0,x}^t : \mathcal{N} \to R^N$ as

$$V_{x_0,x}^t(S) = \{(c_i)_{i \in N} | \, \exists \mathfrak{T} \in \mathfrak{T}, \exists y_S \in K^{\mathfrak{T},t,S}(x_0, x), \forall i \in S, c_i \leqq u_i^{\mathfrak{T},t,S}(x, y_S)\} \subset R^N. \tag{3}$$

A generalized SCE game parameterized by elements of $X_0$ is balanced if the following condition is satisfied:

(Balanced GSCE) Given $(x_0, x) \in X_0 \times X_N$, if for each $t \in \Lambda$, a utility allocation, $(c_i^t)_{i \in N} \in R^N$, is such that we have a balanced family, $\{B_s^t\}_{s=1}^{m(t)}$, satisfying that $c_{B_s^t} \in V_{x_0,x}^t(B_s^t)$ for all $s = 1, \ldots, m(t)$, then there exist strategy profile $y \in X_N$ and coalition structure $\mathfrak{T}^* = (\mathfrak{T}^{1*}, \ldots, \mathfrak{T}^{\lambda*}) \in \mathfrak{T}$ such that $y_S \in K^{\mathfrak{T}^*,t,S}(x_0, x)$ for each $S \in \mathfrak{T}^{t*}$ and $t \in \Lambda$ ($y$ is feasible at $(x_0, x)$ under $\mathfrak{T}^*$) and $c_i^t \leqq u_i^{\mathfrak{T}^*,t,S}(x, y_S)$ for all $i \in S$, $S \in \mathfrak{T}^{t*}$ and $t \in \Lambda$ (utility allocation $(c_i^t)_{i \in N}$ attainable for balanced family $\{B_s^t\}_{s=1}^{m(t)}$ is also attainable under $y$ for all $t \in \Lambda$).

## 4 Existence of equilibrium

For parameter $x_0 \in X_0$, strategy profile $x = (x_i)_{i \in N} \in X_N$, and utility profile $c = (c_i)_{i \in N} \in R^N$, consider the sets of strategy profiles that are feasible and seem as good as level $c = (c_i)_{i \in N}$ at $(x_0, x)$ for all the members of each coalition in a certain admissible social coalition structure $\mathfrak{T} \in \mathfrak{T}$. We call set $U(x_0, x, c) = \{(y_i)_{i \in N} \in$

[6] In other words, by using $\sharp N - 1$ dimensional standard simplex $\Delta = \mathrm{co}\{e^i | \, i \in N\}$, if we identify each $B_s \subset N$ with barycenter $b_s$ of its $\sharp B_s - 1$ dimensional face $\mathrm{co}\{e^i | \, i \in B_s\}$, then the balancedness condition is equivalent to saying that there is a convex combination among points $b_s$, $s = 1, 2, \ldots, m$, such that $\sum_{s=1}^m a_s b_s$ is the barycenter of $\Delta$.

$X_N \mid \exists \mathfrak{T} = (\mathfrak{T}^1, \ldots, \mathfrak{T}^\lambda) \in \mathfrak{T}, \forall t \in \Lambda, \forall S \in \mathfrak{T}^t, (y_i)_{i \in S} \in K^{\mathfrak{T},t,S}(x_0, x)$ and $\forall i \in S, u_i^{\mathfrak{T},t,S}(x, y_S) \geqq c_i\}$ the *socially feasible upper-contour set* at $(x_0, x)$ for $c$. We also denote by $K(x_0, x)$, set $\{(y_i)_{i \in N} \in X_N \mid \exists \mathfrak{T} = (\mathfrak{T}^1, \ldots, \mathfrak{T}^\lambda) \in \mathfrak{T}, \forall t \in \Lambda, \forall S \in \mathfrak{T}^t, (y_i)_{i \in S} \in K^{\mathfrak{T},t,S}(x_0, x)\}$, which is the *socially feasible set* at $(x_0, x)$. Now we have a rigorous description of condition (v).

*(v′) For each $x \in X_N$ and $c \in R^N$, $U(x_0, x, c)$ is convex.*

**Theorem 1** *For society $\left(X_0, (X_i)_{i \in N}, G_0, \left(K^{\mathfrak{T},t,S}, (u_i^{\mathfrak{T},t,S})_{i \in S}\right)_{(\mathfrak{T},t,S) \in \mathfrak{T} \times \Lambda \times \mathcal{N}}\right)$, generalized social coalitional equilibrium $(x_0^*, x^*, \mathfrak{T}^*) \in X_0 \times X_N \times \mathfrak{T}$ exists if conditions (i), (ii), (iii), (iv), (v'), and (vi) are satisfied.*

***Proof*** Let $M$ be a positive real number greater than $u_i^{\mathfrak{T},t,S}(x, y_S)$ for all $i \in N, x \in X_N$, $y_S \in X_S$, $S \in \mathcal{N}$, $t \in \Lambda$, and $\mathfrak{T} \in \mathfrak{T}$. Such number $M$ exists since $N$, $\mathcal{N}$, $\Lambda$, and $\mathfrak{T}$ are finite, all strategy sets are compact, and all utility functions are continuous. Given the base of $R^N = R^{\sharp N} = R^n$, $(e^i)_{i \in N}$, let $D^N$ be simplex $\overline{(-(Mn)e^i)_{i \in N}}$ in non-positive orthant $-R_+^N$. Then for each $(x_0, x) \in X_0 \times X_N$ and $t \in \Lambda$, we obtain continuous function $r_{x_0,x}^t : D^N \to R_+$, such that for each $a \in D^N$,

$$r_{x_0,x}^t(a) = \max\{r \in R \mid a + re \in V_{x_0,x}^t(S), S \in \mathcal{N}\}, \tag{4}$$

where $e = \sum_{i \in N} e^i = (1, 1, \ldots, 1) \in R^N$. One can assure the continuity of $r_{x_0,x}^t$ by the routine method through Berge's maximum theorem. Let us define a function, $f_{x_0,x}^t : D^N \to R^N$, for each $t \in \Lambda$ as

$$f_{x_0,x}^t(a) = a + r_{x_0,x}^t(a)e, \tag{5}$$

for each $a \in D^N$. Function $f_{x_0,x}^t$ is also continuous.

For each $t \in \Lambda$, $(x_0, x) \in X_0 \times X_N$, and $S \in \mathcal{N}$, define $C_S^t(x_0, x) \subset D^N$ as

$$C_S^t(x_0, x) = \{b \in D^N \mid f_{x_0,x}^t(b) \in V_{x_0,x}^t(S)\}. \tag{6}$$

Note that for each $t$ and $S \in \mathcal{N}$, the graph of correspondence $C_S^t : X_0 \times X_N \to D^N$ is closed since the graph of correspondence $V_{(\cdot)}^t(S) : X_0 \times X_N \ni (x_0, x) \mapsto V_{x_0,x}^t(S) \subset R^N$ is closed under the finiteness of $\mathfrak{T}$. Moreover, for each $t \in \Lambda$ and $(x_0, x) \in X_0 \times X_N$, we can verify that class $\{C_S^t(x_0, x) \mid S \in \mathcal{N}\}$ satisfies the following KKMS-condition:

$$\begin{aligned} &\forall T \in \mathcal{N}, (\sharp T - 1)\text{-dimensional face } D^T \\ &= \overline{(-(Mn)e^i)_{i \in T}} \text{ of } D^N \text{ is a subset of } \bigcup_{S \subset T} C_S^t(x_0, x). \end{aligned} \tag{7}$$

Indeed, class $\{C_S^t(x_0, x) \mid S \in \mathcal{N}\}$ clearly covers $D^N$. So if $b = (b_i)_{i \in N} \in D^T$ exists such that $b \notin C_S^t(x_0, x)$ for all $S \subset T$, then since $T \neq N$, we can take $S'$ and $j \in S'$ such that $b \in C_{S'}^t(x_0, x)$ and $j \in S' \setminus T$. Since $b_j = 0$, and since at $b$,

$b + r^t_{x_0,x}(b)e$ must be an element of $R^{S'}_+ = \{(c_i)_{i\in N} | \forall i \in S', c_i \geqq 0\}$, $j$-th coordinate of $b + r^t_{x_0,x}(b)e$ must be greater than the distance between $D^T$ and $R^N_+$. Hence, $j$-th coordinate of $b + r^t_{x_0,x}(b)e = f^t_{x_0,x}(b) \in V^t_{x_0,x}(S')$ must be greater than $M$, a contradiction. Therefore, by KKMS-Theorem (Shapley 1973, Theorem 3.1.2), for each $t \in \Lambda$ and $(x_0, x) \in X_0 \times X_N$, balanced family $\mathfrak{B}^t_{x_0,x} \subset \mathfrak{N}$ exists such that $\bigcap_{B\in\mathfrak{B}^t_{x_0,x}} C^t_B(x_0, x) \neq \emptyset$.

Under the balancedness condition for the society, for $\lambda$ types of elements $a^t \in \bigcap_{B\in\mathfrak{B}^t_x} C^t_B(x_0, x)$, $t \in \Lambda$, there exist a feasible strategy profile, $y = (y_i)_{i\in N} \in X_N$ for an admissible social coalition structure, $\mathfrak{T} = (\mathfrak{T}^1, \ldots, \mathfrak{T}^\lambda) \in \mathfrak{X}$, (i.e., $(y_i)_{i\in T} \in K^{\mathfrak{T},t,T}(x_0, x)$ for each $T$ in $\mathfrak{T}^t$ for each $t \in \Lambda$,) such that for each $t \in \Lambda$, $(c^t_j)_{j\in N} = f^t_{x_0,x}(a^t)$ satisfies $\forall T \in \mathfrak{T}^t$, $(c^t_j)_{j\in T} \leqq (u^{\mathfrak{T},t,T}_j(x, (y_i)_{i\in T}))_{j\in T}$. It follows that

$$\forall t \in \Lambda, y \in U(x_0, x, f^t_{x_0,x}(a^t)) \subset K(x_0, x), \tag{8}$$

i.e., feasible strategy profile $y$ belongs to the socially feasible upper contour set at $(x_0, x)$ for $f^t_{x_0,x}(a^t)$ for each $t \in \Lambda$. This, especially, means that for each $(x_0, x)$ closed set $K(x_0, x)$ is non-empty.

Denote by $(D^N)^\lambda$ the $\lambda$-times product of $D^N$. Now, we can define two mappings on $X_0 \times X_N \times (D^N)^\lambda$ to itself. Let $b_T$ be the barycenter of $D^T$ for each $T \in \mathfrak{N}$ and consider mapping $F : X_0 \times X_N \times (D^N)^\lambda \to X_0 \times X_N \times (D^N)^\lambda$ as follows:

$$\begin{aligned} F(x_0, x, a^1, \ldots, a^\lambda) = \{(x_0, x)\} &\times \mathrm{co}\{b_T | a^1 \in C^1_T(x_0, x)\} \\ &\times \cdots \times \mathrm{co}\{b_T | a^\lambda \in C^\lambda_T(x_0, x)\}, \end{aligned} \tag{9}$$

where co$A$ denotes the convex hull of set $A$. $F$ is non-empty valued correspondence having closed graph (since every $C^t_T$ has). Furthermore, for each $(x_0, x) \in X_0 \times X_N$ and $(a^t)_{t\in\Lambda} \in (D^N)^\lambda$, consider a distance between the set of socially attainable utility allocations and $f^1_{x_0,x}(a^1), \ldots, f^\lambda_{x_0,x}(a^\lambda)$ as follows:

$$\begin{aligned} V(x_0, x, (a^t)_{t\in\Lambda}) = \operatorname*{argmin}_v\{\|v\| \,|\, &\exists y \in K(x_0, x), \forall t \in \Lambda, f^t_{x_0,x}(a^t) \\ &-v \leqq (u^{\mathfrak{T}(y),t,T(i)}_i(x, y_{T(i)}))_{i\in N}\}, \end{aligned} \tag{10}$$

where $\mathfrak{T}(y) = (\mathfrak{T}^1, \ldots, \mathfrak{T}^\lambda) \in \mathfrak{X}$ denotes a social coalition structure under which $y$ is feasible, $T(i)$ denotes the unique coalition in $\mathfrak{T}^t$ that includes $i$, and $y_{T(i)} = (y_j)_{j\in T(i)}$ for $y = (y_i)_{i\in N}$. Mapping $V : (x_0, x, (a^t)_{t\in\Lambda}) \mapsto R$ has a closed graph since $K : (x_0, x) \mapsto K(x_0, x)$ has. Define mapping $G : X_0 \times X_N \times (D^N)^\lambda \to X_0 \times X_N \times (D^N)^\lambda$ as

$$\begin{aligned} G(x_0, x, a) = \mathrm{co}&\left(G_0(x_0, x) \times \bigcup_{v\in V(x_0,x,a)} \left(\bigcap_{t\in\Lambda} U(x_0, x, f^t_{x_0,x}(a^t) - v)\right)\right) \\ &\times\{b_N\} \times \cdots \times \{b_N\}, \end{aligned} \tag{11}$$

where $a = (a^1, \ldots, a^\lambda) \in (D^N)^\lambda$ and $\{b_N\} \times \cdots \times \{b_N\}$ denotes the $\lambda$ times product of $\{b_N\}$. Since we define $V$ so as to ensure the non-emptiness for the intersection among $U(x_0, x, f^t_{x_0,x}(a^t) - v)$'s, $G$ is non-empty and convex valued. $G$ has a closed graph since $U$ and $V$ have. (Correspondence $U$ has a closed graph since $K$ is continuous.) Remember that $X_0 \times X_N$ and $D^N$ are subsets of vector spaces, $R^\ell \times (R^k)^n$ and $R^n$, respectively. Note that for each $(x_0, x, a) \in X_0 \times X_N \times (D^N)^\lambda$, $(x_0, x, a) + (G(x_0, x, a) - F(x_0, x, a))$ is a subset of $X_0 \times X_N \times (D^N)^\lambda$. Moreover, at each $(x_0, x, a)$ such that $0 \notin G(x_0, x, a) - F(x_0, x, a)$, a closed hyperplane $H(x_0, x, a) \subset R^\ell \times (R^k)^n \times (R^n)^\lambda$ (a continuous linear form on $R^\ell \times (R^k)^n \times (R^n)^\lambda$) exists such that $F(x_0, x, a)$ and $G(x_0, x, a)$ are strictly separated by $H(x_0, x, a)$. Therefore, if we define mapping $\varphi$ on $X_0 \times X_N \times (D^N)^\lambda$ to itself as

$$\varphi(x_0, x, a) = (x_0, x, a) + (G(x_0, x, a) - F(x_0, x, a)), \tag{12}$$

correspondence $\varphi$ satisfies condition (K1) of fixed-point theorem in Urai (2000, Theorem 1) (see also Urai (2010, p.36, Theorem 2.1.10)). Hence, $\varphi$ has a fixed point, $(x_0^*, x^*, a^*)$, where $a^* = (a^{1*}, \ldots, a^{\lambda*})$, so $F$ and $G$ has a coincidence point, $(x_0^*, x^*, a^*)$, in $F(x_0^*, x^*, a^*) \cap G(x_0^*, x^*, a^*)$.

By (9) and (11), family of $T \subset N$ satisfying $a^{t*} \in C^t_T(x_0^*, x^*)$ is balanced for all $t \in \Lambda$. It follows that as we see at (8), socially feasible strategy profile $y$ and $\mathfrak{T} \in \mathfrak{T}$ exist such that $y \in U(x_0^*, x^*, f^t_{x_0^*,x^*}(a^{t*}))$ for each $t \in \Lambda$. This especially means, however, by definitions of $V$ (see (10)), $V(x_0^*, x^*, a^*) = \{0\}$. Therefore, by (9) and (11), since each $U(x_0, x, c)$ is convex by (v), we have

$$x^* \in \bigcap_{t \in \Lambda} U(x_0^*, x^*, f^t_{x_0^*,x^*}(a^*)). \tag{13}$$

This also means under the balancedness condition that $x^*$ is socially feasible under a certain $\mathfrak{T}^* = (\mathfrak{T}^{1*}, \ldots, \mathfrak{T}_\lambda{}^*) \in \mathfrak{T}$ (GSCE1: Feasibility). Furthermore, condition that $\forall t \in \Lambda, \forall T \in \mathfrak{T}^{t*}, \forall j \in T, u_j^{\mathfrak{T}^*,t,T}(x^*, (x_j^*)_{j \in T}) \geqq c^t_j$, where $c^t_j$ is the $j$-th coordinate of $f^t_{x_0^*,x^*}(a^*)$, means (through definitions (4) and (5)) that no coalition of any type can improve the utility allocation under $(x_0^*, x^*)$ (GSCE2: Stability). By the fixed point property, (GSCE3) is automatically satisfied. □

## 5 Production economy

Sections 2–4 analyzed the existence of the generalized social coalitional equilibrium. An important application of the generalized social coalitional equilibrium framework is firm formation or the industrial organization problem for production economies with multiple investment opportunities compatible with the standard general equilibrium setting. In this paper, we consider two classes of production economy. One is a generalization of the production economy provided by Ichiishi and Quinzii (1983, Case 1) in which the market is divided into investment commodities and output commodities markets, where investment commodities are non-desirable for every consumer and investment levels are restricted by endowments. The other is a generalization of the Arrow-Debreu production economy where consumers' investment levels are not restricted by their endowments. Hence, as an extension of the standard general equilibrium setting in Debreu (1959) as well as the setting of the Ichiishi-Quinzii production economy, we obtain existence and strong optimality results for the firm formation (multi-industrial organization) general equilibrium problem.

Without any particular notation, all the mathematical symbols below are identical to the settings in Sects. 2–4.

### 5.1 Ichiishi-Quinzii production economy

Let the commodity space be $R^{\bar{\ell}+\bar{\kappa}}$ containing $\bar{\ell} \geq 1$ investment commodities and $\bar{\kappa} \geq 1$ output commodities, and denote for each element $a \in R^{\bar{\ell}+\bar{\kappa}}$, its investment commodity coordinates as $a_I = (a_1, \ldots, a_{\bar{\ell}})$, and output commodity coordinates as $a_O = (a_{\bar{\ell}+1}, \ldots, a_{\bar{\ell}+\bar{\kappa}})$. The consumption set of consumer $i$, $\bar{X}_i$, is a compact subset of $R_+^{\bar{\ell}+\bar{\kappa}}$. Let $t \in \Lambda = \{1, 2, \ldots, \lambda\}$ be the investment opportunities, firm $S \subset N$ in opportunity $t$ with a compact and convex-valued technology correspondence $Y^{t,S}$ : $R^{\bar{\ell}+\bar{\kappa}} \ni \bar{z}^t = (\bar{z}_I^t, 0) \mapsto \bar{y} = (0, \bar{y}_O^{t,S}) \in Y^{t,S}(\bar{z}_I^t) \subset R^{\bar{\ell}+\bar{\kappa}}$ is formed by consumers in $S$ with a total cooperation investment $\bar{z}^{t,S} = \sum_{i \in S} \bar{z}_i^t$. Therefore, an admissible multi-layered coalition structure $\mathfrak{T} = (\mathfrak{T}^1, \ldots, \mathfrak{T}^\lambda) \in \mathfrak{T}$ can be regarded as firm structures for $\lambda$-types industries. Let $\Delta^{\bar{k}-1}$ be the $(\bar{k}-1)$-dimensional standard simplex. The price vectors and the cooperative share-holding rates are denoted by $p \in P = \{(p_I, p_O) \in R_+^{\bar{\ell}+\bar{\kappa}} | p_I \in \Delta^{\bar{\ell}-1}, p_O \in \Delta^{\bar{\kappa}-1}\}$ and $\bar{\theta} \in \bar{\Theta} = \{(\bar{\theta}^{t,S})_{(t,S) \in \Lambda \times \mathfrak{N}} | \forall (t, S) \in \Lambda \times \mathfrak{N}, \bar{\theta}^{t,S} = (\bar{\theta}_i^{t,S})_{i \in S}, \bar{\theta}^{t,S} \in \Delta^{\#S-1}\}$. Given price vector $p$ and investment level $\bar{z}$, we denote the supply correspondence of firm $S$:

$$\psi^{t,S}(p, (\bar{z}_j^t)_{j \in S}) \equiv \{\bar{y}^{t,S} \in Y^{t,S}((\bar{z}_j^t)_{j \in S}) | \bar{y}^{t,S} \in \operatorname{argmax} \pi^{t,S}(p, (\bar{z}_j^t)_{j \in S}) \equiv p \cdot \bar{y}^{t,S}\}.$$

Regarding the consumer side, let $\omega_i = (\omega_{I,i}, 0) \in \bar{X}_i$ be consumer $i$'s endowment. Consumer $i$ selects consumption vector $\bar{x}_i = (\bar{x}_{I,i}, \bar{x}_{O,i})$ to maximize his utility function $u_i : \bar{X}_i \to R_+$. Utility function $u_i$ is assumed to be continuous and quasi-concave for every $i \in N$. Then given a multi-layered industrial structure, $\mathfrak{T} \in \mathfrak{T}$, and the levels of investments and share-holding rates, $(((\bar{z}_i^t)_{t \in \Lambda})_{i \in N}, \bar{\theta})$, consumer $i$'s utility optimization problem is specified:

$$\begin{aligned} \max_{\bar{x}_i} \quad & u_i(\bar{x}_i) \\ \text{subject to} \quad & p_I \cdot \bar{x}_{I,i} \leq p_I \cdot (\omega_{I,i} - \sum_{t=1}^{\lambda} \bar{z}_{I,i}^t), \\ & p_O \cdot \bar{x}_{O,i} \leq \sum_{t=1}^{\lambda} \bar{\theta}_i^{t,S^t(i)} \pi^{t,S^t(i)}(p, (\bar{z}_j^t)_{j \in S^t(i)}), \qquad (14) \end{aligned}$$

where $S^t(i)$ is the unique coalition in opportunity $t$ such that $i \in S^t(i) \in \mathfrak{T}^t$.

So far, we have extended the Ichiishi-Quinzii production framework to a multi-layered industrial structure. The next concern is the existence of a generalized social coalitional equilibrium for such a production economy,[7]

**Proposition 2** *For Ichiishi-Quinzii production economy* $\mathcal{E} = \big((\bar{X}_i, u_i, \omega_i)_{i\in N}, (Y^{t,S})_{(t,S)\in\Lambda\times\mathcal{N}}, \mathfrak{T}\big)$, *let* $\big(X_0, (X_i)_{i\in N}, G_0, \big(K^{\mathfrak{T},t,S}, (u_i^{\mathfrak{T},t,S})_{i\in S}\big)_{(\mathfrak{T},t,S)\in\mathfrak{T}\times\Lambda\times\mathcal{N}}\big)$ *be a society representing the Ichiishi-Quinzii production economy* $\mathcal{E}$*. Suppose that this society satisfies conditions (iv), (v) in Proposition 1, then there exists a generalized social coalitional equilibrium state* $(p^*, (\bar{x}_i^*, (\bar{z}_i^{t*})_{t\in\Lambda})_{i\in N}, \bar{\theta}^*, \mathfrak{T}^*)$ *such that the following conditions are satisfied:*

*(Budget Feasibility)* $(\bar{x}_i^*, (\bar{z}_i^{t*})_{t\in\Lambda})_{i\in N}$ *is budgetary feasible under* $\mathfrak{T}^*$, $p^*$ *and* $\bar{\theta}^*$ *such that for every* $i \in N$,

$$p_I^* \cdot \bar{x}_{I,i}^* \leq p_I^* \cdot \left(\omega_{I,i} - \sum_{t=1}^{\lambda} \bar{z}_{I,i}^{t*}\right)$$

$$p_O^* \cdot \bar{x}_{O,i}^* \leq \sum_{t=1}^{\lambda} \bar{\theta}_i^{t,S^t(i)*} \pi^{t,S^t(i)}(p^*, (\bar{z}_j^{t*})_{j\in S^t(i)}),$$

*where* $S^t(i)$ *is the unique coalition in opportunity* $t$ *such that* $i \in S^t(i) \in \mathfrak{T}^{t*}$.
*(Market Feasibility)*

$$\sum_{i=1}^{n} \bar{x}_{I,i}^* \in \sum_{i=1}^{n} \left(\omega_{I,i} - \sum_{t=1}^{\lambda} \bar{z}_{I,i}^{t*}\right)$$

$$\sum_{i=i}^{n} \bar{x}_{O,i}^* \in \sum_{t=1}^{\lambda} \sum_{S\in\mathfrak{T}^{t*}} \psi^{t,S}(p^*, (\bar{z}_j^{t*})_{j\in S}).$$

*(Price Stability)* *For all* $p \in P$, $t \in \Lambda$ *and* $S \in \mathfrak{T}^{*t}$, $\bar{y}^{t,S} \in \psi^{t,S}(p^*, (\bar{z}_j^{t*})_{j\in S})$, *we have*

$$p_I \cdot \left(\sum_{i=1}^{n} \bar{x}_{I,i}^* - \sum_{i=1}^{n} (\omega_{I,i} - \sum_{t=1}^{\lambda} \bar{z}_{I,i}^{t*})\right) \leqq p_I^* \cdot \left(\sum_{i=1}^{n} \bar{x}_{I,i}^* - \sum_{i=1}^{n} (\omega_{I,i} - \sum_{t=1}^{\lambda} \bar{z}_{I,i}^{t*})\right),$$

$$p_O \cdot \left(\sum_{i=1}^{n} \bar{x}_{O,i}^* - \sum_{t=1}^{\lambda} \sum_{S\in\mathfrak{T}^{t*}} \bar{y}_O^{t,S}\right) \leq p_O^* \cdot \left(\sum_{i=1}^{n} \bar{x}_{O,i}^* - \sum_{t=1}^{\lambda} \sum_{S\in\mathfrak{T}^{t*}} \bar{y}_O^{t,S}\right).$$

*(Market Stability)* *There are no investment type* $s \in \Lambda$, *coalition* $D \in \mathcal{N}$, *consumption* $(\hat{x}_i)_{i\in D}$, *investment* $(\hat{z}_i^s)_{i\in D}$, *and share-holding rates* $\hat{\theta}^{s,D}$ *such that*

[7] The detailed definition of a society $\big(X_0, (X_i)_{i\in N} G_0, \big(K^{\mathfrak{T},t,S}, (u_i^{\mathfrak{T},t,S})_{i\in S}\big)_{(\mathfrak{T},t,S)\in\mathfrak{T}\times\Lambda\times\mathcal{N}}\big)$ will be given in Appendix A.

$u_i(\hat{x}_i) > u_i(\bar{x}_i^*)$ *for all* $i \in D$ *satisfying the following condition:*

$$p_I^* \cdot \hat{x}_{I,i} \leqq p_I^* \cdot \left(\omega_{I,i} - \sum_{t \neq s} \bar{z}_{I,i}^{t*} - \hat{z}_{I,i}^{s}\right),$$

$$p_O^* \cdot \hat{x}_{O,i} \leqq \sum_{t \neq s} \bar{\theta}_i^{t,S^t(i)*} \pi^{t,S^t(i)}(p^*, (\bar{z}_j^{t*})_{j \in S^t(i)}) + \hat{\theta}_i^{s,D} \pi^{s,D}(p^*, (\hat{z}_j^{s})_{j \in D}).$$

***Proof*** Appendix A. □

The existence theorem was proven in Proposition 2. The question of the optimality of a generalized social coalitional equilibrium arises.

**Definition 2** A *Pareto optimum* is a list $((\bar{x}_i, (\bar{z}_i^t)_{t \in \Lambda})_{i \in N}, (\bar{y}^{t,S}((\bar{z}_j)_{j \in S}))_{(t,S) \in \Lambda \times \mathfrak{T}^t}, \mathfrak{T})$ for which there exists no other list $((\hat{x}_i, (\hat{z}_i^t)_{t \in \Lambda})_{i \in N}, (\hat{y}^{t,S}((\hat{z}_j^t)_{j \in S}))_{(t,S) \in \Lambda \times \mathfrak{T}^t}, \hat{\mathfrak{T}})$ such that $u_i(\hat{x}_i) \geq u_i(\bar{x}_i)$ for all $i \in N$ and $u_j(\hat{x}_j) > u_j(\bar{x}_j)$ for at least one $j \in N$.

Unfortunately, under the Ichiishi-Quinzii economy with a multi-layered industrial structure, a competitive equilibrium allocation involving its equilibrium multi-layered industrial structure may not be Pareto optimal.[8] However, the following result shows that the set of Pareto optimum equilibria is not empty:

**Proposition 3** *Suppose that generalized social coalitional equilibrium exists for production economy* $\mathcal{E}$. *Then*

(i) *if* $\#\Lambda = 1$, *every generalized social coalitional equilibrium is Pareto optimum;*
(ii) *if* $\#\Lambda \geq 2$, *there exists an equilibrium list* $((\bar{x}_i^*, (\bar{z}_i^{t*})_{t \in \Lambda})_{i \in N}, (\bar{y}^{t,S*}((\bar{z}_j^{t*})_{j \in S}))_{(t,S) \in \Lambda \times \mathfrak{T}^{t*}}, \mathfrak{T}^*)$ *that is Pareto optimum.*

***Proof*** Appendix B. □

***Remark*** We provided a generalization of Ichiishi and Quinzii (1983)'s result. In particular, if $\#\Lambda = 1$, the existence of a general equilibrium and its optimality stated in Ichiishi and Quinzii (1983) are the consequences of our results even by defining a production set with increasing returns.[9] Our equilibrium concept is also suitable for a general environment where consumers simultaneously incorporate investments for several different firms based on their budgetary constraints. Such generalized solutions were not explored in Boehm (1974) or Greenberg (1979) who considered such environments where a consumer may work in multiple firms.

[8] Society maximizes profit under investment set $\prod_{i=1}^n \bar{Z}_i^t((\sum_{s \neq t} \bar{z}_j^s)_{j \in N})$ associated with multi-layered coalition structure $\mathfrak{T}^{t*}$ in every $t \in \Lambda$, although it may not generate the largest total profit through admissible investment set $\prod_{i=i}^n \bar{Z}_i$ and $\mathfrak{T}$; see the proof of Lemma 1.

[9] By the definition of $Y^{t,S}$, the compactness of consumption set $\bar{X}_i$ implies that $Y^{t,S}((\omega_i)_{i \in S})$ is bounded even with the property of increasing returns, so, $\pi(p^*, \bar{z}^*) = \pi^{\mathfrak{T}^*}(p^*, \bar{z}^*)$; see the proof of Lemma 2.

### 5.2 Arrow-Debreu production economy

In this subsection, we further extend the Ichiishi-Quinzii framework to the Arrow-Debreu production economy. The major difference between the Ichiishi-Quinzii and Arrow-Debreu production economies is whether consumers can purchase investment commodities through the market using the profit distribution from the firms they joined. Similarly, we address the existence of social coalitional equilibrium and its optimality property, and follow the notation for the Ichiishi-Quinzii production economy defined in the previous subsection, although we assume that price $p \in P$ is a vector in simplex $\Delta^{\bar{\ell}+\bar{\kappa}-1}$.[10] We also generalize technology $Y^{t,S}$ to a compact, convex-valued correspondece $Y^{t,S} : R^{\bar{\ell}+\bar{\kappa}} \ni \bar{z}^t = (\bar{z}^t_I, \bar{z}^t_O) \mapsto \bar{y} = (\bar{y}^{t,S}_I, \bar{y}^{t,S}_O) \in Y^{t,S}(\bar{z}^t_I) \subset R^{\bar{\ell}+\bar{\kappa}}$. Given investments and share-holding rates $(((\bar{z}^t_i)_{t\in\Lambda})_{i\in N}, \bar{\theta})$, consumer $i$'s utility optimization problem under a multi-layered industrial structure $\mathfrak{T} \in \mathfrak{T}$ is then specified:

$$\max_{\bar{x}_i} \quad u_i(\bar{x}_i)$$

$$\text{subject to} \quad p \cdot \bar{x}_i \leq p \cdot (\omega_i - \bar{z}_i) + \sum_{t=1}^{\lambda} \bar{\theta}_i^{t,S^t(i)} \pi^{t,S^t(i)}(p, (\bar{z}^t_j)_{j\in S^t(i)}), \qquad (15)$$

where $S^t(i)$ is the unique coalition in opportunity $t$ such that $i \in S^t(i) \in \mathfrak{T}^t$.

We present the existence result[11]:

**Proposition 4** *For an Arrow-Debreu production economy, $\mathcal{E} = \big((\bar{X}_i, u_i, \omega_i)_{i\in N}, (Y^{t,S})_{(t,S)\in\Lambda\times\mathfrak{N}}, \mathfrak{T}\big)$, let $\big(X_0, (X_i)_{i\in N}, G_0, \big(K^{\mathfrak{T},t,S}, (u_i^{\mathfrak{T},t,S})_{i\in S}\big)_{(\mathfrak{T},t,S)\in\mathfrak{T}\times\Lambda\times\mathfrak{N}}\big)$ be a society representing the Arrow-Debreu production economy $\mathcal{E}$. Suppose that this society satisfies conditions (iv), (v) in Proposition 1, then there exists a generalized social coalitional equilibrium state $(p^*, (\bar{x}_i^*, (\bar{z}_i^{t*})_{t\in\Lambda})_{i\in N}, \bar{\theta}^*, \mathfrak{T}^*)$ such that the following conditions are satisfied:*

*(Budget Feasibility) $(\bar{x}_i^*, (\bar{z}_i^{t*})_{t\in\Lambda})_{i\in N}$ is budgetary feasible under $\mathfrak{T}^*$, $p^*$ and $\bar{\theta}^*$ such that for every $i \in N$,*

$$p^* \cdot \bar{x}_i^* \leq p^* \cdot (\omega_i - \bar{z}_i^*) + \sum_{t=1}^{\lambda} \bar{\theta}_i^{t,S^t(i)*} \pi^{t,S^t(i)}(p^*, (\bar{z}_j^{t*})_{j\in S^t(i)}),$$

*where $S^t(i)$ is the unique coalition in opportunity $t$ such that $i \in S^t(i) \in \mathfrak{T}^{t*}$.*

[10] One can extend the share-holding rates defined in the Ichiishi-Quinzii production economy to $\bar{\theta}^{t,S} = (\bar{\theta}_i^{t,S})_{i\in N} \in \Delta^{n-1}$ for every $(t,S) \in \Lambda \times \mathfrak{N}$, which is the share-holding rates defined in the Arrow-Debreu general equilibrium economy. However, it is easy to verify that the GSCE state in Proposition 4 is still a GSCE state under such an extended framework because no cooperator in any $S \in \mathfrak{T}^{t*}$ can benefit from the deviation.

[11] The detailed definition of a society representing the Arrow-Debreu production economy $\mathcal{E}$ will be given in Appendix C.

*(Market Feasibility)*

$$\sum_{i=1}^{n} \bar{x}_i^* \in \sum_{i=1}^{n}(\omega_i - \sum_{t=1}^{\lambda} \bar{z}_i^{t*}) + \sum_{t=1}^{\lambda} \sum_{S \in \mathfrak{T}^{t*}} \psi^{t,S}(p^*, (\bar{z}_i^{t*})_{i \in S}).$$

*(Price Stability) For all* $p \in P$, $t \in \Lambda$ *and* $S \in \mathfrak{T}^{t*}$, $\bar{y}^{t,S} \in \psi^{t,S}(p^*, (\bar{z}_j^{t*})_{j \in S})$, *we have*

$$p \cdot \left( \sum_{i=1}^{n} \bar{x}_i^* - \sum_{i=1}^{n}(\omega_i - \sum_{t=1}^{\lambda} \bar{z}_i^{t*}) - \sum_{t=1}^{\lambda} \sum_{S \in \mathfrak{T}^{t*}} \bar{y}^{t,S} \right)$$
$$\leqq p^* \cdot \left( \sum_{i=1}^{n} \bar{x}_i^* - \sum_{i=1}^{n}(\omega_i - \sum_{t=1}^{\lambda} \bar{z}_i^{t*}) - \sum_{t=1}^{\lambda} \sum_{S \in \mathfrak{T}^{t*}} \bar{y}^{t,S} \right).$$

*(Market Stability) There are no investment type* $s \in \Lambda$, *coalition* $D \in \mathcal{N}$, *consumption* $(\hat{x}_i)_{i \in D}$, *investment* $(\hat{z}_i^s)_{i \in D}$, *and share-holding rates* $\hat{\theta}^{s,D}$ *such that* $u_i(\hat{x}_i) > u_i(\bar{x}_i^*)$ *for all* $i \in D$ *satisfying the following condition:*

$$p^* \cdot \hat{x}_i \leqq p^* \cdot (\omega_i - \sum_{t \neq s} \bar{z}_i^{t*} - \hat{z}_i^s) + \sum_{t \neq s} \bar{\theta}_i^{t,S^t(i)*} \pi^{t,S^t(i)}(p^*, (\bar{z}_j^{t*})_{j \in S^t(i)})$$
$$+ \hat{\theta}_i^{s,D} \pi^{s,D}(p^*, (\hat{z}_j^s)_{j \in D}).$$

***Proof*** Appendix C. □

In contrast to the Ichiishi-Quinzii production economy, the extension to the complete investment commodity markets can be regarded as markets for capitalists. The following result shows that every generalized competitive equilibrium allocation involving the equilibrium multi-layered industrial structure is Pareto optimum.

**Proposition 5** *Every generalized social coalitional equilibrium state obtained in Proposition 4 is Pareto optimum.*

***Proof*** Appendix D. □

## 6 Conclusion

This paper generalizes the social coalitional structures by Ichiishi (1981) and his framework where agents cooperate under multi-layered coalition structures. We introduce the notion of generalized social coalitional equilibrium and provide an existence theorem under the balancedness condition. Applications to the generalized social coalitional equilibrium framework are both the Ichiishi-Quinzii and Arrow-Debreu production economies with a multi-layered industrial structure. Using the balancedness condition, we prove the existence of a generalized social coalitional equilibrium under each production economy and argue that in the Ichiishi-Quinzii production economy, there always exists an equilibrium that is Pareto optimum. In contrast, every equilibrium allocation involving its equilibrium multi-layered coalition structure is Pareto optimum in the Arrow-Debreu production economy. This generalized social coalitional equilibrium concept elucidates the issues of firm formations and industrial structures through general equilibrium analyses.

**Data Availability** This manuscript does not contain any data, so there should be no need for a data availability statement.

## Appendix A: Proof of Proposition 2

We follow the proof of Theorem in Ichiishi and Quinzii (1983) (p. 406 $\sim$ 412). We first reduce the Ichiishi-Quinzii production economy to an $n$-person cooperative game. Then the existence of general equilibrium results from the generalized social coalitional equilibrium theorem is presented in this paper. For this purpose, we define consumer $i$'s cooperative strategy set as a product of the consumption set, the set of investment amounts, and the set of share-holding rates:

$$X_i = \bar{X}_i \times \bar{Z}_i \times \bar{\Theta}_i \subset R^{\bar{\ell}+\bar{\kappa}} \times (R_+^{\bar{\ell}+\bar{\kappa}})^{\lambda} \times \bar{\Theta}_i,$$

where $(\bar{z}_i^t)_{t\in\Lambda} \in \bar{Z}_i \subset (R^{\bar{\ell}+\bar{\kappa}})^{\lambda}$ and $\theta_i \in \bar{\Theta}_i = \{(\bar{\theta}_i^{t,S^t(i)})_{(t,S^t(i))\in\Lambda\times\mathfrak{N}} | \forall(t, S^t(i)) \in \Lambda\times\mathfrak{N}, i \in S^t(i), \bar{\theta}_i^{t,S^t(i)} = pr_i\bar{\theta}^{t,S^t(i)} \in \Delta^{\#S-1}\}$. It is clearly that $\bar{\Theta} = \prod_{i=1}^n \bar{\Theta}_i$. By definition, the strategy set $X_i$ is non-empty, compact, and convex. For each $S \in \mathfrak{N}$, we denote by $\prod_{i\in S} X_i$ the product of the strategy sets for consumers $i \in S$. The feasible-strategy correspondences are specified as $K^{\mathfrak{T},t,S} : X_0 \times X_N \to X_S$. Let $X_0 \equiv P$ be the parameter set. Taking any $\mathfrak{T} = (\mathfrak{T}^1, \ldots, \mathfrak{T}^{\lambda}) \in \mathfrak{T}$, $t \in \Lambda$, and $S \in \mathfrak{N}$, for each parameter $p \in X_0$ and any $(\bar{x}_i, \bar{z}_i, \bar{\theta}_i)_{i\in N} \in X_N$, let $K^{\mathfrak{T},t,S}(p, (\bar{x}_i, \bar{z}_i, \bar{\theta}_i)_{i\in N})$ be the set of all $(\hat{x}_i, \hat{z}_i, \hat{\theta}_i)_{i\in S} \in X_S$ such that the following conditions are satisfied:

$$((\hat{z}_i^s)_{s\in\Lambda})_{i\in S} \in \bar{Z}_S = \prod_{i\in S} \bar{Z}_i, \hat{z}_i^s = \bar{z}_i^s \text{ except for } s = t,$$
$$((\hat{\theta}_i^{s,D})_{(s,D)\in\Lambda\times\mathfrak{N}})_{i\in S} \in \bar{\Theta}_S, \hat{\theta}_i^{s,D} = \bar{\theta}_i^{s,D^s(i)} \text{ except for } (s, D) = (t, S),$$
$$p_I \cdot \hat{x}_{I,i} \leqq p_I \cdot (\omega_{I,i} - \sum_{s=1}^{\lambda} \hat{z}_{I,i}^s),$$
$$p_O \cdot \hat{x}_{O,i} \leqq \sum_{s\neq t} \hat{\theta}_i^{s,D^s(i)} \pi^{s,D^s(i)}(p, (\hat{z}_j^s)_{j\in D^s(i)}) + \hat{\theta}_i^{t,S} \pi^{t,S}(p, (\hat{z}_j^t)_{j\in S}),$$

where $D^s(i)$ is the unique coalition in the opportunity $s$ such that $i \in D^s(i) \in \mathfrak{T}^s$.

Given such correspondences, consumer $i$'s utility function in $n$-person cooperative game representing his own preference is now defined as follows: for each $\mathfrak{T} \in \mathfrak{T}$, $t \in \Lambda$, and $i \in N$, such that $i \in S \in \mathfrak{T}^t$, we take a continuous function $u_i^{\mathfrak{T},t,S}$ :

$X_N \times X_S \to R_+$ representing the utility function $u_i$ as

$$u_i^{\mathfrak{T},t,S}(x,(x_j)_{j\in S}) = u_i^{\mathfrak{T},t,S}((\bar{x},\bar{z},\bar{\theta}),(\hat{x}_j,\hat{z}_j,\hat{\theta}_j)_{j\in S}).$$

Finally, we define the parameter restriction correspondence, $G_0 : X_0 \times X_N \ni (p,x) \mapsto G_0(p,x) \subset P$, as follows:

(**Definition of** $G_0$) Given $\bar{z} = ((\bar{z}_i^t)_{t\in\Lambda})_{i\in N}$, let $Y^{\mathfrak{T}}(\bar{z})$ be the union of sets, $\bigcup_{\mathfrak{T}\in\mathfrak{T}} Y^{\mathfrak{T}}(\bar{z})$. Define $\pi(p,\bar{z})$ as $\pi(p,\bar{z}) = \max\{p_O \cdot \bar{y} | \, \bar{y} \in Y^{\mathfrak{T}}(\bar{z})\}$, and $\psi(p,\bar{z})$ as $\psi(p,\bar{z}) = \{y \in Y^{\mathfrak{T}}(\bar{z}) | \, p_O \cdot \bar{y} = \pi(p,\bar{z})\}$. Then, $G_0(p,(\bar{x}_i,\bar{z}_i,\bar{\theta}_i)_{i\in N})$ is the set $\operatorname{argmax}_{\hat{p}\in P}\{\hat{p}_I\big(\sum_{i=1}^n \bar{x}_{I,i} - \sum_{i=1}^n(\omega_{I,i} - \sum_{t=1}^{\lambda} \bar{z}_{I,i}^t)\big)$ and $\hat{p}_O \cdot \bar{y} | \, \bar{y} \in \psi(p,\bar{z})\}$.

With the notation of the generalized social coalitional equilibrium framework, we then show that there exists a generalized social coalitional equilibrium under the Ichiishi-Quinzii production economy. To do so, we first define consumer $i$'s gain at opportunity $t$ in a multi-layered coalition structure $\mathfrak{T} \in \bar{\mathfrak{T}}$ as a real-valued function such that given a parameter $x_0 = p$,

$$g_i^{\mathfrak{T},t}(p,\bar{z}_i^t,\bar{\theta}_i^t) = \bar{\theta}_i^{t,S^t(i)} \pi^{t,S^t(i)}(p,(\bar{z}_j^t)_{j\in S^t(i)}),$$

where $((\bar{z}_i^t,\bar{\theta}_i^t)_{t\in\Lambda})_{i\in N} \in \prod_{i=1}^n(\bar{Z}_i \times \bar{\Theta}_i)$. Therefore, every consumer $i$ plays his strategy $(p\cdot z_i, g_i)$ in cooperative game. Let $\bar{x}_i(p,\omega_i,(\bar{z}_i^t,\bar{\theta}_i^t)_{t\in\Lambda})$ be a solution of (14), we denote consumer $i$'s indirect utility fuction as $v_i(\bar{x}_i(p,\omega_i,(\bar{z}_i^t,\bar{\theta}_i^t)_{t\in\Lambda}))$. Using $g_i^{\mathfrak{T},t}$, we write $v_i(p,\omega_i,(g_i^{\mathfrak{T},t}(p,\bar{z}_i^t,\bar{\theta}_i^t))_{t\in\Lambda})$ to stand for $v_i(\bar{x}_i(p,\omega_i,(\bar{z}_i^t,\bar{\theta}_i^t)_{t\in\Lambda}))$.[12] By the definition of utility, $v_i(p,\omega_i,(g_i^{\mathfrak{T},t}(p,\bar{z}_i^t,\bar{\theta}_i^t))_{t\in\Lambda})$ is also continuous and non-decreasing. Using $(p\cdot z_i, g_i)$ and $v_i$, we then prove the following existence lemma:

**Lemma 1** *For society* $\big(X_0,(X_i)_{i\in N},G_0,\big(K^{\mathfrak{T},t,S},(u_i^{\mathfrak{T},t,S})_{i\in S}\big)_{(\mathfrak{T},t,S)\in\mathfrak{T}\times\Lambda\times\mathcal{N}}\big)$ *representing the Ichiishi-Quinzii production economy* $\mathcal{E}$*, there exists a generalized social coalitional equilibrium* $(x_0^*,x^*,\mathfrak{T}^*) \in X_0 \times X_N \times \mathfrak{T}$.

***Proof*** We will prove this lemma using the logic similar to the proof of Theorem in Ichiishi and Quinzii (1983). We need to check that there is a list $(x_0^*,x^*,\mathfrak{T}^*)$ that satisfies all the conditions described in Proposition 1. By definition, it is easy to verify that the conditions (i) , (ii) , (iii) , and (vi) are satisfied. To establish the condition (iv) , we denote the non-side-payment game by:

$$V_{p,x}^t(S) = \{(c_i)_{i\in N} \subset R^N | \, \exists \mathfrak{T} \in \mathfrak{T}, \exists y_S \in K^{\mathfrak{T},t,S}(x_0,x), \forall i \in S, c_i \leqq u_i^{\mathfrak{T},t,S}(x,y_S)\}. \tag{16}$$

[12] Let $\xi^i \equiv p\cdot(\omega_i - z_i)$, $\eta^i \equiv g_i$ and $(p,q) \equiv (p_I,p_O)$, each consumer $i$' feasible strategy $\sigma^i = (\xi^i,\eta^i)$ and his utility $v_i(p,q,\xi^i,\eta^i)$ defined in Ichiishi and Quinzii (1983) can be represented by $(p\cdot z_i, g_i)$ and $v_i(p,\omega_i,(g_i^{\mathfrak{T},t}(p,\bar{z}_i^t,\bar{\theta}_i^t))_{t\in\Lambda})$, respectively.

From the assumption in Proposition 2, this non-side-payment game is balanced. Consequently, all the conditions of Proposition 1 are fulfilled, and there exists a generalized social coalitional equilibrium $(x_0^*, x^*, \mathfrak{T}^*)$ by Theorem 1. $\square$

As far as there exists a generalized social coalitional equilibrium in the Ichiishi-Quinzii production economy $\mathcal{E}$, the result cannot directly extend to prove the existence of general equilibrium since the price manipulation will lose its sense for a coalition structure $\mathfrak{T}^* \in \mathfrak{T}$ such that $\pi^{\mathfrak{T}^*}(p, \bar{z}) \neq \pi(p, \bar{z})$, where $\psi^{\mathfrak{T}^*}(p, \bar{z})$ is a proper subset of $\psi(p, \bar{z})$ by the definition of $G_0$. However, the following lemma can be proved and completes the proof of Proposition 2.

**Lemma 2** *Let us consider an Ichiishi-Qienzii production economy and let $(p^*, (\bar{x}_i^*, (\bar{z}_i^{t*})_{t\in\Lambda}, \bar{\theta}_i^*)_{i\in N}, \mathfrak{T}^*)$ be a GSCE, then condition $\pi(p^*, \bar{z}^*) = \pi^{\mathfrak{T}^*}(p^*, \bar{z}^*)$ is satisfied, and we have (Market-Stability) condition together with the following (Price Stability) condition. Moreover, in such a case, we have (Market-Feasibility) condition and hence, $(p^*, (\bar{x}_i^*, (\bar{z}_i^{t*})_{t\in\Lambda})_{i\in N}, \bar{\theta}^*, \mathfrak{T}^*)$ with an defined $\bar{\theta}^{t,S*} = (\bar{\theta}_i^{t,S^t(i)*})_{i\in S^t(i)}$ for each $S^t(i) = S \in \mathfrak{T}^{t*}$ and $t \in \Lambda$, forms an equilibrium of production economy.*

***Proof*** By the proof of Lemma 1, it is easy to verify that the definition of $G_0$ assures the (Price Stability) as long as $\pi(p^*, \bar{z}^*) = \pi^{\mathfrak{T}^*}(p^*, \bar{z}^*)$.[13] Condition (Budget Feasibility) follows from (GSCE1). It is also straightforward that from condition (GSCE1) together with the definitions of $\bar{Z}_i$ and $\bar{X}_N$, we have (Investment Feasibility). Since the budget feasibility is satisfied for every $S \in \mathfrak{T}^{t*}$, $t = 1, 2, \ldots, \lambda$, we have $p_I^* \cdot \bar{x}_{I,i}^* \leqq p_I^* \cdot (\omega_{I,i} - \sum_{t=1}^{\lambda} \bar{z}_{I,i}^{t*})$ and $p_O^* \cdot \bar{x}_{O,i}^* \leqq \sum_{t=1}^{\lambda} \bar{\theta}_i^{t,S^t(i)*} \pi^{t,S^t(i)}(p^*, (\bar{z}_j^*)_{j\in S^t(i)})$. Moreover, since the investment commodities are non-desirable for all the consumers, the above $\leqq$ must hold as $=$ and $\bar{x}_{I,i}^* = (0, \cdots, 0)$ for all $i \in N$. Also, it assures that $p^*$ is strictly positive. Then for every $p_I \in \Delta^{\bar{\ell}-1}$,

$$
\begin{aligned}
p_I \cdot \left( \sum_{i=1}^{n} \omega_{I,i} - \sum_{i=1}^{n} \sum_{t=1}^{\lambda} \bar{z}_{I,i}^{t*} \right) &\leq p_I^* \cdot \left( \sum_{i=1}^{n} \omega_{I,i} - \sum_{i=1}^{n} \sum_{t=1}^{\lambda} \bar{z}_{I,i}^{t*} \right) \\
&\leq \sum_{i=1}^{n} p_I^* \cdot \left( \omega_{I,i} - \sum_{t=1}^{\lambda} \bar{z}_{I,i}^{t*} \right) \\
&\leq 0
\end{aligned}
$$

The above inequality implies that $\sum_{i=1}^{n} \bar{z}_{I,i} \leq \sum_{i=1}^{n} \omega_{I,i}$. Besides, since for each $i \in N, t \in \Lambda$, and $S \in \mathfrak{T}^{t*}$, there has $\bar{y}^{t,S} \in \psi^{t,S}(p^*, (\bar{z}_j^{t*})_{j\in S})$, so for all $p_O \in \Delta^{\bar{\kappa}-1}$, the following inequalities are hold:

$$
\begin{aligned}
p_O \cdot \left( \sum_{i=1}^{n} \bar{x}_{O,i}^* - \sum_{t=1}^{\lambda} \sum_{S\in\mathfrak{T}^{t*}} \bar{y}_O^{t,S} \right) &\leq p_O^* \cdot \left( \sum_{i=1}^{n} \bar{x}_{O,i}^* - \sum_{t=1}^{\lambda} \sum_{S\in\mathfrak{T}^{t*}} \bar{y}_O^{t,S} \right) \\
&= \sum_{i=1}^{n} p_O^* \cdot \bar{x}_{O,i}^* - \sum_{t=1}^{\lambda} \sum_{S\in\mathfrak{T}^{t*}} \sum_{i\in S} \bar{\theta}_i^{t,S*} p_O^* \cdot \bar{y}_O^{t,S}
\end{aligned}
$$

[13] One can check that under the condition of Balancedness Condition, for all $\bar{z}$, $\bigcup_{\mathfrak{T}\in\mathfrak{T}} Y^{\mathfrak{T}}(\bar{z}) = Y^{\mathfrak{T}^*}(\bar{z})$.

$$
\begin{aligned}
&= \sum_{i=1}^{n} \left( p_O^* \cdot \bar{x}_{O,i}^* - \sum_{t=1}^{\lambda} \bar{\theta}_i^{t,S^t(i)*} p_O^* \cdot \bar{y}_O^{t,S^t(i)} \right) \\
&\leq 0
\end{aligned}
$$

where for each $(t, S)$ and $i \in S = S^t(i)$, $\bar{\theta}_i^{t,S*} = \bar{\theta}_i^{t,S^t(i)*}$. Consequently, $p_O^* \in R_{++}^{\bar{\kappa}}$ and $\sum_{i=1}^{n} \bar{x}_{O,i}^* \leq \sum_{t=1}^{\lambda} \sum_{S \in \mathfrak{T}^{t*}} \bar{y}_O^{t,S}$ are held from $Y^{t,S}$ is compact- and convex-value. The above inequality ensures that the price stability condition together with $p^* \in R_{++}^{\bar{\ell}} \times R_{++}^{\bar{\kappa}}$ implies the condition (Market Feasibility). □

## Appendix B: Proof of Proposition 3

The proof is divided into two parts, one is $\#\Lambda = 1$, and the other is $\#\Lambda \geq 2$

($\#\Lambda = 1$) Suppose not. Then there exists one equilibrium allocation involving its equilibrium industrial structure that is Pareto dominated by an allocation $((\hat{x}_i, (\hat{z}_i^t)_{t \in \Lambda})_{i \in N}, (\hat{y}^{t,S}((\hat{z}_j^t)_{j \in S}))_{(t,S) \in \Lambda \times \hat{\mathfrak{T}}})$ with its industrial structure $\hat{\mathfrak{T}}$. Since $\Lambda = \{1\}$, then we simplify the symbols $(\hat{z}_i^t)_{t \in \Lambda}$, $(\hat{y}^{t,S})_{(t,S) \in \Lambda \times \hat{\mathfrak{T}}}$ and $(g_i^t)_{t \in \Lambda}$ as $\hat{z}_i$, $(\hat{y}^S)_{S \in \hat{\mathfrak{T}}}$ and $g_i$ for all $i, j \in N$ and $S \subset N$, respectively. Given the price vector $p^*$, Definition 2 together with indirect utility function $v_i$ defined in Appendix A implies that there must exists $D \in \hat{\mathfrak{T}}$ such as for every $i \in D$, $g_i^{\hat{\mathfrak{T}}}(p^*, \hat{z}_i, \hat{\theta}_i) \geq g_i^{\mathfrak{T}}(p^*, \bar{z}_i, \bar{\theta}_i)$ and at least one $j \in D$, $g_j^{\hat{\mathfrak{T}}}(p^*, \hat{z}_i, \hat{\theta}_i) > g_i^{\mathfrak{T}}(p^*, \bar{z}_j, \bar{\theta}_j)$. Also, since $((\hat{x}_i, \hat{z}_i)_{i \in N}, (\hat{y}^{t,S}((\hat{z}_j^t)_{j \in S}))_{(t,S) \in \Lambda \times \hat{\mathfrak{T}}})$ is an admissible allocation under industrial structure $\hat{\mathfrak{T}}$, we have

$$
p_I^* \cdot \sum_{i \in D} \hat{z}_{I,i} \leq p_I^* \cdot \sum_{i \in D} \omega_{I,i}.
$$

That is, $(\hat{z}_i)_{i \in D}$ is also admissible under every equilibrium structure $\mathfrak{T}^*$ since $\bar{x}_{I,i}^* = 0$ for every $i \in N$.[14] Combine the above facts, all the consumers, $i \in D$, will deviate cooperatively from $\mathfrak{T}^*$ by submitting the investments $(\hat{z}_i)_{i \in D}$, which is contradiction to the Market Stability.

($\#\Lambda \geq 2$) Suppose not. Then for every equilibria defined in Proposition 2, there exists an admissble list $((\hat{x}_i, (\hat{z}_i)_{t \in \Lambda})_{i \in N}, (\hat{y}^{t,D}((\hat{z}_j^t)_{j \in D}))_{(t,D) \in \Lambda \times \hat{\mathfrak{T}}^t}, \hat{\mathfrak{T}})$ such that for every $i \in N$, $u_i(\hat{x}_i) \geq u_i(x_i^*)$ and there exists at least one $j \in N$ that $u_i(\hat{x}_j) > u_i(x_j^*)$. Since every equilibrium $p^*$ is non-negative, by definition 2, for every $i \in N$, and $D \in \hat{\mathfrak{T}}^t$, we have

$$
\sum_{i=1}^{n} p_I^* \cdot \hat{x}_{I,i} = \sum_{i=1}^{n} p_I^* \cdot (\omega_{I,i} - \sum_{t=1}^{\lambda} \hat{z}_{I,i}^t),
$$

[14] The investment commodities are non-desirable for every consumer.

$$\sum_{i=1}^{n} p_O^* \cdot \hat{x}_{O,i} = \sum_{t=1}^{\lambda} \sum_{D \in \hat{\mathfrak{T}}^t} p_O^* \cdot \hat{y}_O^{t,D}((\hat{z}_j^t)_{j \in S}).$$

Recall the definition of Balancedness Condition, by adjusting investments and share holdings rates, there always exists an equilibrium price $p^*$ and an equilibrium investments $((\bar{z}_i^{t*})_{t \in \Lambda})_{i \in N}$ attainable under the equilibrium multi-layered coalition structure $\mathfrak{T}^*$ such that the following condition holds:

$$\begin{aligned} p_O^* \cdot \sum_{t=1}^{\lambda} \sum_{S \in \mathfrak{T}^{t*}} \bar{y}_O^{t,S*} &= \sum_{t=1}^{\lambda} \sum_{S \in \mathfrak{T}^{t*}} \pi^{t,S}(p^*, (z_j^{t*})_{j \in S}) \geq \sum_{t=1}^{\lambda} \sum_{D \in \mathfrak{T}^t} \pi^{t,D}(p^*, (z_j^t)_{j \in D}) \\ &= p_O^* \cdot \sum_{t=1}^{\lambda} \sum_{D \in \mathfrak{T}^t} y_O^{t,D} \end{aligned}$$

for any investments $(((z_i^t)_{t \in \Lambda})_{i \in N}, (y^{t,D}((z_j^t)_{j \in D}))_{(t,D) \in \Lambda \times \mathfrak{T}^t})$ attainable under $\mathfrak{T} \in \mathfrak{T}$. Since the preference of each consumer is local nonsatiated, the list which is Pareto optimum must satisfy the following conditions:

$$\begin{aligned} 0 = \sum_{i=1}^{n} p_I^* \cdot \left(\omega_{I,i} - \sum_{t=1}^{\lambda} \bar{z}_{I,i}^{t*}\right) &\leq \sum_{i=1}^{n} p_I^* \cdot \hat{x}_{I,i} = \sum_{i=1}^{n} p_I^* \cdot (\omega_{I,i} - \sum_{t=1}^{\lambda} \hat{z}_{I,i}^t), \\ \sum_{i=1}^{n} p_O^* \cdot \hat{x}_{O,i} = \sum_{t=1}^{\lambda} \sum_{D \in \hat{\mathfrak{T}}^t} p_O^* \cdot \hat{y}_O^{t,D}((\hat{z}_j^t)_{j \in S}) &\leq \sum_{t=1}^{\lambda} \sum_{S \in \mathfrak{T}^{t*}} p_O^* \cdot \bar{y}_O^{t,S*}((\bar{z}_j^{t*})_{j \in S}). \end{aligned}$$

That is, the list is Pareto optimal only if the following conditions hold:

$$\begin{aligned} 0 = \sum_{i=1}^{n} p_I^* \cdot \hat{x}_{I,i} &= \sum_{i=1}^{n} p_I^* \cdot \left(\omega_{I,i} - \sum_{t=1}^{\lambda} \bar{z}_{I,i}^{t*}\right), \\ \sum_{i=1}^{n} p_O^* \cdot \hat{x}_{O,i} &= \sum_{t=1}^{\lambda} \sum_{S \in \mathfrak{T}^{t*}} p_O^* \cdot \bar{y}_O^{t,S*}((\bar{z}_j^{t*})_{j \in S}) = \sum_{i=1}^{n} p_O^* \cdot \bar{x}_{O,i}^*. \end{aligned}$$

From the above condistions, we have that the Pareto optimal allocation $(\hat{x}_i)_{i \in N}$ is also attainable under the equilibrium multi-layered coalition structure $\mathfrak{T}^*$ associated with the equilibrium investments $((z_i^*)_{t \in \Lambda})_{i \in N}$. And hence, we get a contradiction.

## Appendix C: Proof of Proposition 4

We follow the logic in Appendix A to show the existence theorem. We reduce an Arrow-Debreu production economy to an $n$-person cooperative game by defining consumer

$i$'s strategy as the set

$$X_i = \bar{X}_i \times \bar{Z}_i \times \bar{\Theta}_i \subset R^{\bar{\ell}+\bar{\kappa}} \times (R_+^{\bar{\ell}+\bar{\kappa}})^\lambda \times \bar{\Theta}_i.$$

The feasible-strategy correspondence $K^{\mathfrak{T},t,S} : X_0 \times X_N \to X_S$ therefore are denoted as follows:
Let $X_0 \equiv P$ be the parameter set, taking any $\mathfrak{T} = (\mathfrak{T}^1, \ldots, \mathfrak{T}^\lambda) \in \mathfrak{T}$, $t \in \Lambda$, and $S \in \mathfrak{N}$, for each parameter $p \in X_0$ and any $(\bar{x}_i, \bar{z}_i, \bar{\theta}_i)_{i\in N} \in X_N$, let $K^{\mathfrak{T},t,S}(p, (\bar{x}_i, \bar{z}_i, \bar{\theta}_i)_{i\in N})$ be the set of all $(\hat{x}_i, \hat{z}_i, \hat{\theta}_i)_{i\in S} \in X_S$ such that the following conditions are satisfied:

$$((\hat{z}_i^s)_{s\in\Lambda})_{i\in S} \in \bar{Z}_S = \prod_{i\in S} \bar{Z}_i, \hat{z}_i^s = \bar{z}_i^s \text{ except for } s = t,$$

$$((\hat{\theta}_i^{s,D})_{(s,D)\in\Lambda\times\mathfrak{N}})_{i\in S} \in \bar{\Theta}_S, \hat{\theta}_i^{s,D} = \bar{\theta}_i^{s,D^s(i)} \text{ except for } (s, D) = (t, S),$$

$$\begin{aligned} p \cdot \hat{x}_i \leqq\ & p \cdot (\omega_i - \sum_{s=1}^{\lambda} \hat{z}_i^s) + \sum_{s\neq t} \hat{\theta}_i^{s,D^s(i)} \pi^{s,D^s(i)}(p, (\hat{z}_j^s)_{j\in D^s(i)}) \\ & + \hat{\theta}_i^{t,S} \pi^{t,S}(p, (\hat{z}_j^t)_{j\in S}), \end{aligned}$$

where $D^s(i)$ is the unique coalition in the opportunity $s$ such that $i \in D^s(i) \in \mathfrak{T}^s$.

Given such correspondences, consumer $i$'s utility function in $n$-person cooperative game is now defined as a continuous function $u_i^{\mathfrak{T},t,S} : X_N \times X_S \to R_+$ such that for each $\mathfrak{T} \in \mathfrak{T}$, $t \in \Lambda$, and $i \in N$, such that $i \in S \in \mathfrak{T}$,

$$u_i^{\mathfrak{T},t,S}(x, (x_j)_{j\in S}) = u_i^{\mathfrak{T},t,S}((\bar{x}, \bar{z}, \bar{\theta}), (\hat{x}_j, \hat{z}_j, \hat{\theta}_j)_{j\in S}).$$

Finally, defining the parameter restriction correspondence, $G_0 : X_0 \times X_N \ni (p, x) \mapsto G_0(p, x) \subset P$ as follows:

(**Definition of** $G_0$) Given $\bar{z} = ((\bar{z}_i^t)_{t\in\Lambda})_{i\in N}$, let $Y^{\mathfrak{T}}(\bar{z})$ be the union of sets, $\bigcup_{\mathfrak{T}\in\mathfrak{T}} Y^{\mathfrak{T}}(\bar{z})$. Define $\pi(p, \bar{z})$ as $\pi(p, \bar{z}) = \max\{p \cdot \bar{y} | \bar{y} \in Y^{\mathfrak{T}}(\bar{z})\}$, and $\psi(p, \bar{z})$ as $\psi(p, \bar{z}) = \{y \in Y^{\mathfrak{T}}(\bar{z}) | p \cdot \bar{y} = \pi(p, \bar{z})\}$. Then, $G_0(p, (\bar{x}_i, \bar{z}_i, \bar{\theta}_i)_{i\in N})$ is the set $\operatorname{argmax}_{\hat{p}\in P}\{\hat{p} \cdot \left(\sum_{i=1}^n \bar{x}_i - \sum_{i=1}^n(\omega_i - \sum_{k=1}^\lambda \bar{z}_i^k) - \bar{y}\right) | \bar{y} \in \psi(p, \bar{z})\}$.

In contrast to Ichiishi-Quinzii production economy, since consumers can purchase by using the distribution from the firms they join, so that we will prove the existence lemma through the consumer $i$'s net-gain real-valued function $g_i^{\mathfrak{T},t}$ in a multi-layered coalition structure $\mathfrak{T} \in \bar{\mathfrak{T}}$ such that given a parameter $x_0 = p$,

$$g_i^{\mathfrak{T},t}(p, \bar{z}_i^t, \bar{\theta}_i^t) = \bar{\theta}_i^{t,S^t(i)} \pi^{t,S^t(i)}(p, (\bar{z}_j^t)_{j\in S^t(i)}) - p \cdot z_i^t,$$

where $((\bar{z}_i^t, \bar{\theta}_i^t)_{t\in\Lambda})_{i\in N} \in \prod_{i=1}^n (\bar{Z}_i \times \bar{\Theta}_i)$. Similar to the definition of indirect utility in Appendix A, the consumer $i$'s indirect utility is denoted as a continuous, non-decreasing, and real-valued function $v_i(\bar{x}_i(p, \omega_i, (\bar{z}_i^t, \bar{\theta}_i^t)_{t\in\Lambda}))$ where

$\bar{x}_i(p, \omega_i, (\bar{z}_i^t, \bar{\theta}_i^t)_{t\in\Lambda})$ solves (15). Using the net-gain function $g_i^{\mathfrak{T},t}$ defined above, we also write $v_i(p, \omega_i, (g_i^{\mathfrak{T},t}(p, \bar{z}_i^t, \bar{\theta}_i^t))_{t\in\Lambda})$ to stand for $v_i(\bar{x}_i(p, \omega_i, (\bar{z}_i^t, \bar{\theta}_i^t)_{t\in\Lambda}))$.

**Lemma 3** *Given an parameter $x_0 = p$, there always exist $\bar{\mathfrak{T}} \in \bar{\mathfrak{X}}$, and $((\bar{z}_i^t, \bar{\theta}_i^t)_{t\in\Lambda})_{i\in N} \in \prod_{i=1}^n(\bar{Z}_i \times \bar{\Theta}_i)$ such that for every $t \in \Lambda$, for every $\hat{\mathfrak{T}} \in \bar{\mathfrak{X}}$ and for every $((\hat{z}_i^t, \hat{\theta}_i^t)_{t\in\Lambda})_{i\in N} \in \prod_{i=1}^n(\bar{Z}_i \times \bar{\Theta}_i)$,*

$$\sum_{i=1}^n g_i^{\bar{\mathfrak{T}},t}(p, \bar{z}_i^t, \bar{\theta}_i^t) \geq \sum_{i=1}^n g_i^{\hat{\mathfrak{T}},t}(p, \hat{z}_i^t, \hat{\theta}_i^t).$$

*Furthermore, for every $s \in \Lambda$ and for every admissible coalition sturcture $\hat{\mathfrak{T}}^s \neq \bar{\mathfrak{T}}^s$, if $\sum_{i=1}^n g_i^{\bar{\mathfrak{T}},s}(p, \bar{z}_i^s, \bar{\theta}_i^s) > \sum_{i=1}^n g_i^{\hat{\mathfrak{T}},s}(p, \hat{z}_i^s, \hat{\theta}_i^s)$, then there always exists $S \in \bar{\mathfrak{T}}^s$ together with $((\bar{z}_i^t, \bar{\theta}_i^t)_{t\in\Lambda})_{i\in N}$ such that for every $i \in S$, $g_i^{\bar{\mathfrak{T}},s}(p, \bar{z}_i^s, \bar{\theta}_i^s) \geq g_i^{\hat{\mathfrak{T}},s}(p, \hat{z}_i^s, \hat{\theta}_i^s)$ and at least one $j \in S$, $g_j^{\bar{\mathfrak{T}},s}(p, \bar{z}_j^s, \bar{\theta}_j^s) > g_j^{\hat{\mathfrak{T}},s}(p, \hat{z}_j^s, \hat{\theta}_j^s)$ for any $((\hat{z}_i^t, \hat{\theta}_i^t)_{t\in\Lambda})_{i\in N} \in \prod_{i=1}^n(\bar{Z}_i \times \bar{\Theta}_i)$.*

***Proof*** The statement of first part that, there always exist $\bar{\mathfrak{T}} \in \bar{\mathfrak{X}}$, and $((\bar{z}_i^t, \bar{\theta}_i^t)_{t\in\Lambda})_{i\in N} \in \prod_{i=1}^n(\bar{Z}_i \times \bar{\Theta}_i)$ such that for every $t \in \Lambda$, for every $\hat{\mathfrak{T}} \in \bar{\mathfrak{X}}$ and for every $((\hat{z}_i^t, \hat{\theta}_i^t)_{t\in\Lambda})_{i\in N} \in \prod_{i=1}^n(\bar{Z}_i \times \bar{\Theta}_i)$,

$$\sum_{i=1}^n g_i^{\bar{\mathfrak{T}},t}(p, \bar{z}_i^t, \bar{\theta}_i^t) \geq \sum_{i=1}^n g_i^{\hat{\mathfrak{T}},t}(p, \hat{z}_i^t, \hat{\theta}_i^t),$$

is always holding by the definition of $g_i^{\mathfrak{T},t}$.

So it is sufficient to verify the last part. Supposing not. Defining a set $A$ as follows:

$$A \equiv \{S \in \bar{\mathfrak{T}}^s \mid \sum_{i\in S} g_i^{\hat{\mathfrak{T}},s}(p, \hat{z}_i^s, \hat{\theta}_i^s) \geq \sum_{i\in S} g_i^{\bar{\mathfrak{T}},s}(p, \bar{z}_i^s, \bar{\theta}_i^s)\}.$$

By the hypothesis, it is easy to see that the complement set $A^C = \emptyset$. Then, $A = \bar{\mathfrak{T}}^s$ and

$$\sum_{S\in A}\sum_{i\in S} g_i^{A,s}(p, \bar{z}_i^s, \bar{\theta}_i^s) \leq \sum_{S\in\hat{\mathfrak{T}}^s}\sum_{i\in S} g_i^{\hat{\mathfrak{T}},s}(p, \hat{z}_i^s, \hat{\theta}_i^s)$$

for every $((\hat{z}_i^t, \hat{\theta}_i^t)_{t\in\Lambda})_{i\in N} \in \prod_{i=1}^n(\bar{Z}_i \times \bar{\Theta}_i)$. Hence we get a contradiction to the statement of the first part except for the case such that

$$\sum_{i=1}^n g_i^{\mathfrak{T},s}(p, \bar{z}_i^s, \bar{\theta}_i^s) = \sum_{i=1}^n g_i^{\hat{\mathfrak{T}},s}(p, \hat{z}_i^s, \hat{\theta}_i^s).$$

$\square$

Lemma 3 together with $K^{\mathfrak{T},t,S}$ shows that once the price $p$ prevail, consumers' coalition-deviation strategies in opportunity $t$ are independent of those in any other opportunity $s \neq t$, and only the multi-layered coalition structure $\bar{\mathfrak{T}}$, that realizes the maximum of the total net-profits in each $t \in \Lambda$, may be a stable (non-deviation) one. Those facts imply that for the Arrow-Debreu economy $\mathcal{E}$, GSCE exists only if its equilibrium multi-layered coalition structure $\mathfrak{T}^*$ satisfies Lemma 3, i.e., $\mathfrak{T}^* = \bar{\mathfrak{T}}$. However, $\bar{\mathfrak{T}}$ may not be an element of $\mathfrak{T}$, so we use the condition of Balancedness Condition defined in subsection 5.1 to have the next existence lemma.

**Lemma 4** *For society* $\left(X_0, (X_i)_{i \in N}, G_0, \left(K^{\mathfrak{T},t,S}, (u_i^{\mathfrak{T},t,S})_{i \in S}\right)_{(\mathfrak{T},t,S) \in \mathfrak{T} \times \Lambda \times \mathcal{N}}\right)$ *representing the Arrow-Debreu production economy* $\mathcal{E}$*, there exists a generalized social coalitional equilibrium* $(x_0^*, x^*, \mathfrak{T}^*) \in X_0 \times X_N \times \mathfrak{T}$.

***Proof*** The logic of this proof follows the argument in Lemma 1. By definition, the conditions (i) , (ii) , (iii) , and (vi) defined in Proposition 1 are always satisfied. Also, we denote the non-side-payment game by:

$$V_{p,x}^t(S) = \{(c_i)_{i \in N} \subset R^N \mid \exists \mathfrak{T} \in \mathfrak{T}, \exists y_S \in K^{\mathfrak{T},t,S}(x_0, x), \forall i \in S, c_i \leqq u_i^{\mathfrak{T},t,S}(x, y_S)\}. \tag{17}$$

and this game is balanced by the assumption in Proposition 4. Consequently, all the conditions of Proposition 1 are fulfilled, and there exists a generalized social coalitional equilibrium $(x_0^*, x^*, \mathfrak{T}^*)$ by Theorem 1. □

We have shown that there exists GSCE in the Arrow-Debreu production framework. The next lemma shows that the GSCE list satisfies all the conditions in Proposition 4.

**Lemma 5** *Let us consider an Arrow-Debreu production economy and let* $(p^*, (\bar{x}_i^*, (\bar{z}_i^{t*})_{t \in \Lambda}, \bar{\theta}_i^*)_{i \in N}, \mathfrak{T}^*)$ *be a GSCE, then condition* $\pi(p^*, \bar{z}^*) = \pi^{\mathfrak{T}^*}(p^*, \bar{z}^*)$ *is satisfied, and we have (Market-Stability) condition together with the following (Price Stability) condition. Moreover, in such a case, we have (Market-Feasibility) condition and hence,* $(p^*, (\bar{x}_i^*, (\bar{z}_i^{t*})_{t \in \Lambda})_{i \in N}, \bar{\theta}^*, \mathfrak{T}^*)$ *with an defined* $\bar{\theta}^{t,S*} = (\bar{\theta}_i^{t,S^t(i)*})_{i \in S^t(i)}$ *for each* $S^t(i) = S \in \mathfrak{T}^{t*}$ *and* $t \in \Lambda$*, forms an equilibrium of production economy.*

***Proof*** The proof also follows Lemma 2 in Appendix A. We need to check two conditions: one is $\pi(p^*, \bar{z}^*) = \pi^{\mathfrak{T}^*}(p^*, \bar{z}^*)$, and the other is (Market Feasibility) in Proposition 4.

The first condition is always holding by Lemma 3. To verify the second condition, let $p \in \Delta^{\bar{\ell}+\bar{\kappa}-1}$, we have:

$$\begin{aligned} & p \cdot \left( \sum_{i=1}^{n} \bar{x}_i^* - \sum_{i=1}^{n} \omega_i - \sum_{i=1}^{n} \sum_{t=1}^{\lambda} \bar{z}_i^{t*} - \sum_{t=1}^{\lambda} \sum_{S \in \mathfrak{T}^{t*}} \phi^{t,S}(p^*, (\bar{z}_j^*)_{j \in S}) \right) \\ & \leq p^* \cdot \left( \sum_{i=1}^{n} \bar{x}_i^* - \sum_{i=1}^{n} \omega_i - \sum_{i=1}^{n} \sum_{t=1}^{\lambda} \bar{z}_i^{t*} \right) - p^* \cdot \sum_{t=1}^{\lambda} \sum_{S \in \mathfrak{T}^{t*}} \phi^{t,S}(p^*, (\bar{z}_j^*)_{j \in S}) \end{aligned}$$

$$
\begin{aligned}
&= p^* \cdot \left( \sum_{i=1}^{n} \bar{x}_i^* - \sum_{i=1}^{n} \omega_i - \sum_{i=1}^{n} \sum_{t=1}^{\lambda} \bar{z}_i^{t*} \right) - \sum_{t=1}^{\lambda} \sum_{S \in \mathfrak{T}^{t*}} \pi^{\mathfrak{T}^*, t, S}(p^*, (\bar{z}_j^*)_{j \in S}) \\
&= p^* \cdot \left( \sum_{i=1}^{n} \bar{x}_i^* - \sum_{i=1}^{n} \omega_i - \sum_{i=1}^{n} \sum_{t=1}^{\lambda} \bar{z}_i^{t*} \right) - \sum_{i=1}^{n} \sum_{t=1}^{\lambda} \theta_i^{t, S*} \pi^{\mathfrak{T}^*, t, S}(p^*, (\bar{z}_j^*)_{j \in S}) \\
&= \sum_{i=1}^{n} \left( p^* \cdot (\bar{x}_i^* - \omega_i - \sum_{t=1}^{\lambda} \bar{z}_i^{t*} \right) - \sum_{t=1}^{\lambda} \theta_i^{t, S^t(i)*} \pi^{\mathfrak{T}^*, t, S^t(i)}(p^*, (\bar{z}_j^*)_{j \in S^t(i)}) \\
&\leq 0.
\end{aligned}
$$

Therefore,

$$
\sum_{i=1}^{n} \bar{x}_i^* \in \sum_{i=1}^{n} \left( \omega_i - \sum_{t=1}^{\lambda} \bar{z}_i^{t*} \right) + \sum_{t=1}^{\lambda} \sum_{S \in \mathfrak{T}^{t*}} \psi^{t, S}(p^*, (\bar{z}_i^{t*})_{i \in S}).
$$

Hence, the proof is completed. □

## Appendix D: Proof of Proposition 5

Suppose not. Then there exists one equilibrium allocation with its certain equilibrium multi-layered industrial structure $\mathfrak{T}^*$ is Pareto dominated by some allocation $((\hat{x}_i, (\hat{z}_i^t)_{t \in \Lambda})_{i \in N}, (\hat{y}^{t,S}((\hat{z}_j^t)_{j \in S}))_{(t,S) \in \Lambda \times \hat{\mathfrak{T}}})$ with its multi-layered industrial structure $\hat{\mathfrak{T}}$. Given the equilibrium price vector $p^*$, Definition 2 together with Lemma 3 in Appendix C implies that:

$$
\begin{aligned}
p^* \cdot \sum_{i=1}^{n} \bar{x}_i^* &\leq p^* \cdot \sum_{i=1}^{n} \hat{x}_i \\
&= p^* \cdot \left( \sum_{i=1}^{n} (\omega_i - \sum_{t=1}^{\lambda} \hat{z}_i^t) + \sum_{t=1}^{\lambda} \sum_{S \in \hat{\mathfrak{T}}^t} \hat{y}^{t,S}((\hat{z}_j^t)_{j \in S}) \right) \\
&\leq p^* \cdot \sum_{i=1}^{n} \omega_i + \sum_{i=1}^{n} \sum_{t=1}^{\lambda} g_i^{\mathfrak{T}^*, t}(p^*, \bar{z}_i^{t*}, \bar{\theta}_i^{t*})
\end{aligned}
$$

Budget Feasibility implies that the above condition holds only if $p^* \cdot \sum_{i=1}^{n} \bar{x}_i^* = p^* \cdot \sum_{i=1}^{n} \hat{x}_i$ which contradicts to the fact that there exists at least one $j \in N$ such that $u_j(\bar{x}_j^*) < u_j(\hat{x}_j)$. The proof is completed.

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