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RESEARCH ARTICLE

Existence of alpha-core allocations in economies with non-ordered and discontinuous preferences

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Abstract
We study exchange economies where the preference relation of each consumer depends also on the consumptions of the other consumers. In the setting of economies with a finite number $n \geq 2$ of consumers and non-ordered and discontinuous preferences, we give sufficient and necessary conditions for the existence of alpha-core allocations in the sense of Yannelis (Equilibrium theory in infinite dimensional spaces, 102–123, 1991). The result has been obtained by means of the Ky Fan minimax inequality.

Keywords Exchange economies with interdependent preferences · Non-ordered preferences · Alpha-core allocations · Ky Fan minimax inequality

Mathematics Subject Classification C71 · D51

1 Introduction

The paper deals with exchange economies where the well-being of each consumer is affected by the consumption of other consumers. So, we focus on economies where the preference relations of consumers are *interdependent* (see Veblen 1934; Duesenberry 1949; Ok and Koçkesen 2000; Reiter 2001; Drakopoulos 2012, among the others). Examples of interdependent preferences are given by economies with externalities (see Borglin 1973; Zhao 1996; Dupor and Liu 2003; Tian and Yang 2009). In this setting, we are interested in feasible allocations that cannot be refused by coalitions of consumers. More precisely, we consider the alpha-core allocations introduced by Yannelis (1991). If a feasible allocation is alpha-core in the sense of Yannelis, then no coalition of consumers can allocate own total endowment between the members so that all of them get the best regardless the reactions of the opponents. Here, the reactions

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of consumers outside a coalition are all possible allocations between themselves of the sum of their initial endowments. So, the alpha-core in the sense of Yannelis is different from the alpha-core of Aumann (1961), where, instead, there are no feasibility constraints.

The existence of Yannelis's alpha-core allocations has been proved for exchange economies with 2 consumers and non-necessarily complete and transitive preference relations (Yannelis 1991). Holly (1994) showed that Yannelis's result cannot be extended to economies with more than 2 consumers. So, it would appear that conditions which guarantee the existence of Yannelis's alpha-core allocations in economies with more than 2 consumers have to be stronger than those by Yannelis (1991). At the present, no results on the non-emptiness of Yannelis's alpha-core other than Yannelis (1991) have been given.

The aim of the present paper is to investigate on the possibility to obtain a general result on the existence of Yannelis's alpha-core allocations in exchange economies with a number $n \geq 2$ of consumers and conditions no stronger than those considered by Yannelis (1991). In line with recent papers on the existence of Nash (and strong) equilibria and non-emptiness of Aumann's alpha-core,[1] our approach to Yannelis's alpha-core allocations existence problem is based on the Ky Fan minimax inequality (Fan 1972). More precisely, we consider exchange economies where consumers' preferences are not necessarily complete and transitive (*non-ordered preferences*). Given an economy $\mathcal{E}$, we define a real-valued function $\Theta_{\mathcal{E}}$ such that a feasible allocation x^* belongs to Yannelis's alpha-core of $\mathcal{E}$ if and only if $\Theta_{\mathcal{E}}(\widetilde{x}, x^*) \leq 0$ for each vector $\widetilde{x} = (x_S)_S$, where, for each coalition S, the vector $x_S = \left(x_{S,i}\right)_{i \in S}$ is a reallocation of the total endowment of S. The element x^* is a solution to the Ky Fan minimax inequality corresponding to the function $\Theta_{\mathcal{E}}$. More generally, given a function $\phi : Y \times Z \longrightarrow \mathbb{R}$, the Ky Fan minimax inequality corresponding to ϕ is the problem of finding $z^* \in Z$ such that $\phi(y, z^*) \leq 0$ for all $y \in Y$. So, sufficient (and necessary) conditions for the existence of solution to the Ky Fan minimax inequality allow to identify conditions on the primitives of $\mathcal{E}$ which guarantee the non-emptiness of Yannelis's alpha-core. In this way, we find two properties, called *coalitional deviation property* and *coalitional transfer quasi-convexity*. For exchange economies with n consumers ($n \geq 2$), we prove that, when coalitional deviation property is satisfied, Yannelis's alpha-core is non-empty if and only if the coalitional transfer quasi-convexity holds true. Moreover, we give an example of exchange economy with 3 consumers where: (i) Yannelis's alpha-core is non-empty; (ii) the coalitional deviation property and coalitional transfer quasi-convexity hold true; (iii) the assumptions on preferences given by Yannelis (1991) are not satisfied.

Economies with interdependent preferences include, as special case, economies where the preferences of consumers do not depend on the consumptions of the other consumers. In this case, Yannelis's alpha-core coincides with the standard core. So, coalitional deviation property and coalitional transfer quasi-convexity allow to characterize the non-emptiness of the core in economies where the preferences are not complete and transitive. The result that we obtain in this setting improves the previous ones by Scarf (Scarf 1967, 1971), Border (1984) and Yannelis (1991).

[1] Scalzo (2019; 2020), Basile and Scalzo (2020).

The paper is organized as follows. Section 2 recalls the Ky Fan minimax inequality and gives a solution existence result. Section 3 introduces the setting of exchange economies with interdependent preferences. Section 4 presents the coalitional deviation property and coalitional transfer quasi-convexity and gives the main result of the paper; moreover, examples are provided to illustrate our properties. The special case of economies with non-interdependent preferences is investigated in Sect. 5. Comparisons between our results and the previous ones are given in Sect. 6. Section 7 concludes the paper.

2 Recalls on the Ky Fan minimax inequality

Let Y and Z be non-empty sets and let ϕ be a real-valued function defined on $Y \times Z$. The problem:

$$(\text{KF}) : \begin{cases} \text{find } z^* \in Z \text{ such that} \\ \phi(y, z^*) \leq 0 \quad \forall\, y \in Y \end{cases}$$

is the so-called *Ky Fan minimax inequality* (Fan 1972). An element z^* satisfying the inequality (KF) is said to be a *solution*. The set of solutions to (KF) is denoted by S_ϕ. The Ky Fan minimax inequality plays a significative role in Game Theory because it allows to identify classes of discontinuous games where Nash equilibria and strong equilibria exist: see Baye et al. (1993) and Scalzo (2013; 2019; 2020). Moreover, necessary and sufficient conditions for the non-emptiness of Aumann's alpha-core have been obtained by means of (KF) (see Basile and Scalzo 2020). Among the properties assumed on the function ϕ in the mentioned literature, we focus on the following fundamental ones (see Baye et al. 1993; Scalzo 2019):

- (A1) ϕ is *slightly diagonally transfer continuous*, that is: $\phi(y, z) > 0$ implies that there exists an open neighborhood O_z of z and $y' \in Y$ such that $\phi(y', z') > 0$ for all $z' \in O_z \backslash S_\phi$;
- (A2) ϕ is *diagonally transfer quasi-concave*, that is: for every $\{y^1, ..., y^k\} \subset Y$ there exists $\{z^1, ..., z^k\} \subset Z$, with $y^h \mapsto z^h$ for $h = 1, ..., k$, such that if $\{z^{h_1}, ..., z^{h_l}\} \subseteq \{z^1, ..., z^k\}$ and $z \in \text{sco}\{z^{h_1}, ..., z^{h_l}\}$, we get $\phi(y, z) \leq 0$ for at least one $y \in \{y^{h_1}, ..., y^{h_l}\}$.

Note that linear and topological structures in properties (A1) and (A2) are necessary only on the space which includes Z. Moreover, (A2) is a necessary condition for $S_\phi \neq \emptyset$. When $Y = Z$ and Z is a convex and compact subset of a Hausdorff topological vector space, we have that the slight diagonal transfer continuity and diagonal transfer quasi-concavity allow the existence of solutions to (KF): see Proposition 1 in Scalzo (2019). This result is based on the following generalization of the KKM-lemma due to Tian (1993):

Lemma 1 *Let Y be a topological space and let Z be a non-empty, compact and convex subset of a Hausdorff topological vector space. Suppose that the correspondence $G : Y \rightrightarrows Z$ satisfies the following properties:*

(B1) $z \notin G(y)$ *implies that there exists* $y' \in Y$ *such that* $z \notin \mathrm{cl}G(y')$;
(B2) *for every* $\{y^1, ..., y^k\} \subset Y$, *there exists* $\{z^1, ..., z^k\} \subset Z$, *with* $y^h \mapsto z^h$ *for* $h = 1, ..., k$, *such that* $\mathrm{co}\{z^{h_1}, ..., z^{h_l}\} \subseteq \bigcup_{j=1}^{l} G(y^{h_j})$ *for each* $\{z^{h_1}, ..., z^{h_l}\} \subseteq \{z^1, ..., z^k\}$.

Then, $\bigcap_{y \in Y} G(y)$ *is non-empty and compact.*[2]

Properties (B1) and (B2) can be stated also for a generic set Y. Moreover, the proof of Lemma 1 - see pages 953 and 954 in Tian (1993) - does not need that Y is a topological space. So, Lemma 1 holds true even for a generic non-empty set Y. In the light of this remark, using the arguments of the proof of Proposition 1 by Scalzo (2019), we obtain:

Lemma 2 *Let* Y *be a non-empty set and let* Z *be a non-empty, convex and compact subset of a Hausdorff topological vector space. Assume that* ϕ *is a real-valued function defined on* $Y \times Z$ *that satisfies the properties* (A1) *and* (A2). *So, the solution set of* (KF) *is non-empty.*

3 Exchange economies with interdependent preferences

Denote by $\mathcal{E}$ an exchange economy where the number of consumers is finite (N denotes the set of consumers) and the bundles of goods are included in a subset of a Hausdorff topological vector space. For each $i \in N$, let X_i be the consumption set of consumer i and let $e_i \in X_i$ be her/his initial endowment. All consumption sets are assumed to be convex and closed. The elements of $X = \prod_{j \in N} X_j$ are called *allocations*. An allocation x is said to be *feasible* if $\sum_{i \in N} x_i = \sum_{i \in N} e_i$; the set of feasible allocations is denoted by $\mathcal{F}$. For every non-empty subset S of N (we call S *coalition*), we set: $X_S = \prod_{i \in S} X_i$; $X_{-S} = \prod_{j \notin S} X_j$; $\mathcal{F}_S = \left\{x_S \in X_S : \sum_{i \in S} x_i = \sum_{i \in S} e_i\right\}$; $\mathcal{F}_{-S} = \left\{x_{-S} \in X_{-S} : \sum_{j \notin S} x_j = \sum_{j \notin S} e_j\right\}$. The set of coalitions is denoted by $\mathcal{N}$. In the following, we suppose that $\mathcal{F}_S$ is compact for every coalition S.

We assume that the well-being of each consumer is affected by the consumptions of other consumers. So, the preference relations of consumers are defined on the set of allocations. Moreover, we do not require that the preferences are complete or transitive. For each $i \in N$, let $P_i : X \rightrightarrows X$ be the mapping where, for every $x \in X$, $P_i(x)$ is the set of allocations that consumer i strictly prefers to x, that is: $z \in P_i(x)$ if and only if (z, x) belongs to the asymmetric part of preference relation of consumer i. Finally, we set $\mathcal{E} = \langle X_i, P_i, e_i \rangle_{i \in N}$.

[2] Tian called *transfer closed-valued* a correspondence satisfying property (B1) and *transfer FS-convex* a correspondence with property (B2).

4 Non-emptiness of the alpha-core: sufficient and necessary conditions

Assume that $\mathcal{E} = \langle X_i, P_i, e_i \rangle_{i \in N}$ is an exchange economy as introduced in the previous section. Our focus is on the following alpha-core concept introduced by Yannelis (1991):

Definition 1 Let z be a feasible allocation and let S be a coalition. We say that S *Y-blocks* z if there exists $x_S \in \mathcal{F}_S$ such that $\{x_S\} \times \mathcal{F}_{-S} \subseteq \bigcap_{i \in S} P_i(z)$. We call *Yannelis's alpha-core* the set of feasible allocations that cannot Y-blocked by coalitions.

Yannelis's alpha-core of $\mathcal{E}$ is denoted by $\mathcal{C}^Y(\mathcal{E})$. We aim to introduce necessary and sufficient conditions for the non-emptiness of Yannelis's alpha-core. We consider economies $\mathcal{E}$ where the lower sections of each mapping P_i are not necessarily open sets (*discontinuous preferences*, see Section 5).

For each P_i and for each x and z belonging to X, we define $\widehat{P_i}(x, z) = 1$ if $x \in P_i(z)$ and $\widehat{P_i}(x, z) = 0$ otherwise. Let $\widetilde{\mathcal{F}} = \prod_{S \in \mathcal{N}} \mathcal{F}_S$; the generic element of $\widetilde{\mathcal{F}}$ is $\widetilde{x} = (x_S)_{S \in \mathcal{N}}$. We consider the function $\Theta_{\mathcal{E}} : \widetilde{\mathcal{F}} \times \mathcal{F} \longrightarrow \mathbb{R}$ defined as follows:

$$\Theta_{\mathcal{E}}(\widetilde{x}, z) = \sum_{S \in \mathcal{N}} \min_{i \in S} \min_{y_{-S} \in \mathcal{F}_{-S}} \widehat{P_i}\left((x_S, y_{-S}), z\right) \quad \forall\, (\widetilde{x}, z) \in \widetilde{\mathcal{F}} \times \mathcal{F}\,. \tag{1}$$

A feasible allocation x^* belongs to Yannelis's alpha-core of $\mathcal{E}$ if and only if $\Theta_{\mathcal{E}}(\widetilde{x}, x^*) = 0$ for all $\widetilde{x} \in \widetilde{\mathcal{F}}$. Indeed, if $x^* \in \mathcal{C}^Y(\mathcal{E})$, for every coalition S and for every $x_S \in \mathcal{F}_S$, we get $(x_S, y_{-S}) \notin P_i(x^*)$ for some player $i \in S$ and some $y_{-S} \in \mathcal{F}_{-S}$. This implies that $\Theta_{\mathcal{E}}(\widetilde{x}, x^*) = 0$ for each $\widetilde{x}$. Similarly, one can prove the converse.

So, since the function $\Theta_{\mathcal{E}}$ defined by (1) has non-negative vales, the non-emptiness of Yannelis's alpha-core can be investigated by using the Ky-Fan minimax inequality. In order to do this, let us introduce, in the setting of the present paper, properties considered by Scalzo (2020) and Basile and Scalzo (2020).

Definition 2 We say that $\mathcal{E}$ satisfies the *coalitional deviation property* if $z \in \mathcal{F} \backslash \mathcal{C}^Y(\mathcal{E})$ implies that there exists an open neighborhood O_z of z and $\widetilde{x}' = (x'_S)_{S \in \mathcal{N}} \in \widetilde{\mathcal{F}}$ such that, for each $z' \in \left[O_z \cap \mathcal{F}\right] \backslash \mathcal{C}^Y(\mathcal{E})$, there exists a coalition S for whom $\{x'_S\} \times \mathcal{F}_{-S} \subseteq \bigcap_{i \in S} P_i(z')$.[3]

Remark 1 The condition introduced in Definition 2 is satisfied when the preference relations of consumers allows mappings P_i with open graph[4]: see the following Proposition 1. For example, this is the case when the preferences are represented by continuous utility functions. However, the coalitional deviation property holds in economies with discontinuous preferences: see Example 1.

[3] This property is given in the spirit of the *single deviation property* introduced by Nessah and Tian (2008) - under the name of *weak transfer quasi-continuity* - and Reny (2009); see also Nessah and Tian (2016) and Reny (2016).

[4] We recall that the graph of a mapping $P : Y \rightrightarrows Z$ is defined by $\mathrm{Gr}(P) = \{(y, z) \in Y \times Z : z \in P(y)\}$. If Y and Z are topological spaces, we say that P has open graph if $\mathrm{Gr}(P)$ is an open subset of $Y \times Z$ with respect to the product topology.

Proposition 1 *Assume that the economy $\mathcal{E} = \langle X_i, P_i, e_i \rangle_{i \in N}$ is such that P_i has open graph for all $i \in N$. Then, $\mathcal{E}$ satisfies the coalitional deviation property.*

Proof We recall that $\mathcal{F}_S$ is compact for all coalitions S. Let $z \notin \mathcal{C}^Y(\mathcal{E})$. So, there exists a coalition S and $x_S \in \mathcal{F}_S$ such that $(x_S, y_{-S}) \in P_i(z)$ for all $y_{-S} \in \mathcal{F}_{-S}$ and for all $i \in S$. Since $\mathrm{Gr}(P_i)$ is open for each $i \in S$, we have an open covering $\left\{U^i_{y_{-S}} : y_{-S} \in \mathcal{F}_{-S}\right\}$ of $\mathcal{F}_{-S}$, where $y_{-S} \in U^i_{y_{-S}}$, and a family $\left\{O^{i,y_{-S}}_z : y_{-S} \in \mathcal{F}_{-S}\right\}$ of open neighborhood of z such that $O^{i,y_{-S}}_z \times \left[\{x_S\} \times U^i_{y_{-S}}\right] \subseteq \mathrm{Gr}(P_i)$ for every $y_{-S} \in \mathcal{F}_{-S}$. Since $\mathcal{F}_{-S}$ is compact, we obtain $\mathcal{F}_{-S} \subseteq \bigcup_{h=1}^k U^i_{y^h_{-S}}$ where $\left\{y^1_{-S}, ..., y^k_{-S}\right\} \subset \mathcal{F}_{-S}$. We set $O^i_z = \bigcap_{h=1}^k O^{i,y^h_{-S}}_z$. If $z' \in O^i_z$, given $y_{-S} \in \mathcal{F}_{-S}$, y_{-S} belongs to $U^i_{y^h_{-S}}$ for some $h \in \{1, ..., k\}$, and we get $\left(z', (x_S, y_{-S})\right) \in \mathrm{Gr}(P_i)$. So, $(x_S, y_{-S}) \in P_i(z')$ for all $y_{-S} \in \mathcal{F}_{-S}$ and for all $z' \in O^i_z$. Now, apply the arguments above to every $i \in S$: define $O_z = \bigcap_{i \in S} O^i_z$ and $\widetilde{x}' \in \widetilde{\mathcal{F}}$ such that $x'_S = x_S$. Finally, one can see that the coalitional deviation property is satisfied. □

Definition 3 We say that $\mathcal{E}$ is *coalitional transfer quasi-convex* if, for every $\{\widetilde{x}^1, ..., \widetilde{x}^k\} \subset \widetilde{\mathcal{F}}$, there exists $\{z^1, ..., z^k\} \subset \mathcal{F}$ - where $\widetilde{x}^h \mapsto z^h$ for $h = 1, ..., k$ - such that, for each $\{z^{h_1}, ..., z^{h_l}\} \subseteq \{z^1, ..., z^k\}$ and for each $z \in \mathrm{sco}\{z^{h_1}, ..., z^{h_l}\}$, there exists $\widetilde{x} = (x_S)_{S \in \mathcal{N}} \in \{\widetilde{x}^{h_1}, ..., \widetilde{x}^{h_l}\}$ so that no coalition S can Y-block z by using x_S.[5]

Remark 2 Note that coalitional transfer quasi-convexity is a necessary condition for the existence of Yannelis's alpha-core allocations. Indeed, if $\mathcal{C}^Y(\mathcal{E})$ is non-empty, given $\{\widetilde{x}^1, ..., \widetilde{x}^k\} \subset \widetilde{\mathcal{F}}$, it is sufficient to choose $\{z^1, ..., z^k\} \subseteq \mathcal{C}^Y(\mathcal{E})$. Nevertheless, coalitional transfer quasi-convexity is not a sufficient condition for the non-emptiness of $\mathcal{C}^Y(\mathcal{E})$: see Example 2.
The coalitional transfer quasi-convexity finds inspiration from the following situation. Assume that each consumer i has an ordinal preference relation $\succsim_i$ which is convex, that is: $y^j \succsim_i x$ for $j = 1, ..., l$ and $z \in \mathrm{co}\{y^1, ..., y^l\}$ imply $z \succsim_i x$. If $\{\widetilde{x}^1, ..., \widetilde{x}^k\} \subset \widetilde{\mathcal{F}}$, we can define $x^h = x^h_N$ for $h = 1, ..., k$, where x^h_N is the allocation that the grand coalition gets in $\widetilde{x}^h$. If all preferences have the same minimal element on $\{x^1, ..., x^k\}$, that we denote by x, we get $z \succsim_i x$ for all $i \in N$ and for all $z \in \mathrm{sco}\{x^1, ..., x^k\}$. So, for each $z \in \mathrm{sco}\{x^1, ..., x^k\}$, there exists $\widetilde{x} \in \{\widetilde{x}^1, ..., \widetilde{x}^k\}$ such that the grand coalition cannot use own allocation in $\widetilde{x}$ in order to Y-blocks z. In the coalitional transfer quasi-convexity we assume an uniform behaviour of coalitions in choosing $\widetilde{x}$

[5] If $\{a^1, ..., a^k\}$ is a subset of a vector space, we set:

$$\mathrm{co}\{a^1, ..., a^k\} = \left\{\sum_{h=1}^k \lambda_h a^h : \sum_{h=1}^k \lambda_h = 1 \text{ and } \lambda_h \geq 0 \text{ for } h = 1, ..., k\right\} \text{ and}$$

$$\mathrm{sco}\{a^1, ..., a^k\} = \left\{\sum_{h=1}^k \lambda_h a^h : \sum_{h=1}^k \lambda_h = 1 \text{ and } \lambda_h > 0 \text{ for } h = 1, ..., k\right\}.$$

so that z cannot be Y-blocked. Moreover, in the coalitional transfer quasi-convexity, the set of allocation $\{z^1, ..., z^k\}$ does not need to coincide with $\{x^1, ..., x^k\}$ previously defined.
Because of the uniform behavior described above, the coalitional transfer quasi-convexity is not connected with the standard convexity assumption for non-ordinal preference relations, that is: (ii) $x \notin \mathrm{co}P_i(x)$ for all $x \in X$ and for all $i \in N$. More precisely, coalitional transfer quasi-convexity does not imply (ii) and (ii) does not imply coalitional transfer quasi-convexity: see Sect. 6.

The following example introduces an exchange economy where the preferences satisfy the properties introduced in Definitions 2 and 3.

Example 1 Let $\mathcal{E}$ be the exchange economy with one good and three consumers - $N = \{1, 2, 3\}$ - such that: the consumption set of each consumer i is $X_i = [0, 1]$ and her/his preference relation is given by means of the utility function u_i defined on $X = \prod_{i \in N} X_i$ as below:

$$\begin{aligned}
\text{if } i, j \in \{1,2\} \ (i \neq j) \quad & u_i(x) = 1 \quad \text{if } x_i > x_j \\
& u_i(x) = 0 \quad \text{if } x_i < x_j \\
& u_i(x) = t \quad \text{if } x_i = x_j = t \\
\text{and} \quad & u_3(x) = 1 \quad \text{if } x_3 = \max\{x_1, x_2\} \\
& u_3(x) = 0 \quad \text{otherwise} .
\end{aligned}$$

We set $P_i(x) = \{z \in X : u_i(z) > u_i(x)\}$ for each $i \in N$ and for each $x \in X$. The initial endowments of consumers are $e_1 = e_2 = \frac{1}{4}$ and $e_3 = \frac{1}{2}$.

One can look at consumer 1 as an individual who is jealous of consumer 2, and viceversa, while consumer 3 gets a benefit if she/he consumes an amount of good equal to the highest consumption among the other individuals (this phenomena is known as *keeping up with the Joneses*: see, for example, Ok and Koçkesen 2000). In this situation, each consumer obtains a social status that depends on the relative consumptions of the good. So, the economy has negative consumption externalities (among the others, see Dupor and Liu 2003; Tian and Yang 2009).

Let us prove that $\mathcal{C}^Y(\mathcal{E}) = \{x \in \mathcal{F} : x_1 = x_3 \geq x_2\} \cup \{x \in \mathcal{F} : x_2 = x_3 \geq x_1\}$. If $x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, we get $u_1(x) = u_2(x) = \frac{1}{3}$ and $u_3(x) = 1$; so, a Y-blocking coalition cannot include consumer 3. On the other hand, neither consumer 1 nor consumer 2 can Y-block x by themselves: for examples, if $\{1\}$ tries to Y-block x, she/he gets the possible outcome $u_1\left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right) = 0$ (consumer 3 reallocate a share of own endowment to consumer 2). Similarly if $\{2\}$ tries to Y-block x. Coalition $\{1, 2\}$ cannot Y-blocks x because consumers 1 and 2 obtain positive outcomes only if they consume the same share of resource $e_1 + e_2$, that is: $u_i\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4} < u_i(x) = \frac{1}{3}$ with $i = 1, 2$. If $x \in \mathcal{F}$ and $x_1 = x_3 > x_2$, consumers 1 and 3 get the highest outcome; so, a coalition including consumers 1 or 3 cannot Y-blocks x. Moreover, $\{2\}$ does not Y-block x. The same arguments apply on feasible allocations such that $x_2 = x_3 > x_1$. Now, assume that $z \in \mathcal{F}$ and $z \notin \{x \in \mathcal{F} : x_1 = x_3 \geq x_2\} \cup \{x \in \mathcal{F} : x_2 = x_3 \geq x_1\}$. Let $z_1 > z_2$. Since $z_3 \neq z_1$, one has $u_i(z) = 0$ for $i \in \{2, 3\}$. So, coalition $\{2, 3\}$ Y-blocks z by using $x_{\{2,3\}} = \left(\frac{3}{8}, \frac{3}{8}\right)$ ($e_1 = \frac{1}{4}$ is the only one amount of good feasible to consumer

1). Similarly, if $z_2 > z_1$, coalition $\{1,3\}$ Y-blocks z. Finally, suppose that $z_1 = z_2$. In this case, each coalition $\{i,3\}$, where $i \in \{1,2\}$, Y-blocks z through $x_{\{i,3\}} = \left(\frac{3}{8}, \frac{3}{8}\right)$.

The economy satisfies the coalitional deviation property. In fact, assume that $z \notin \mathcal{C}^Y(\mathcal{E})$ and O_z is an open neighborhood of z. If $z_1 = z_2$, define $\widetilde{x}' = \left(x'_S\right)_{S\in\mathcal{N}}$ as below:

$$x'_S = \begin{cases} e_i & \text{if } S = \{i\} \text{ and } i \in N \\ \left(\frac{3}{8}, \frac{3}{8}\right) & \text{if } S = \{i,3\} \text{ and } i \in \{1,2\} \\ (e_1, e_2) & \text{if } S = \{1,2\} \\ (e_1, e_2, e_3) & \text{if } S = N \end{cases}$$

Let $z' \in \left[O_z \cap \mathcal{F}\right] \setminus \mathcal{C}^Y(\mathcal{E})$. If $z'_1 > z'_2$, coalition $\{2,3\}$ Y-blocks z' by using $x'_{\{2,3\}}$, while coalition $\{1,3\}$ uses $x'_{\{1,3\}}$ to Y-block z' if $z'_1 < z'_2$. When $z'_1 = z'_2$, both coalitions $\{1,3\}$ and $\{2,3\}$ Y-block z' through their allocations in $\widetilde{x}$. If $z_1 \neq z_2$, the property is satisfied with $\widetilde{x}'$ defined as above.

Finally, $\mathcal{E}$ is coalitional transfer quasi-convex. This is quite simple to see because $\mathcal{C}^Y(\mathcal{E})$ is non-empty.

Example 2 Consider the economy $\mathcal{E}$ where $N = \{1,2\}$, $X_1 = X_2 = [0,1]$, $(e_1, e_2) = (1,0)$ and the preference relations are defined as below:

$$P_1(x) = \begin{cases}]x_1, 1[\times]0,1[& \text{if } x_1 < 1 \\ ]0,1[\times]0,1[& \text{otherwise} \end{cases}$$

$$P_2(x) = \begin{cases}]0,1[\times]0, \min\{1-x_1, x_2\}[& \text{if } x_1 < 1 \\ ]0,1[\times]0,1[& \text{otherwise.} \end{cases}$$

If x is a feasible allocation such that $x_1 < 1$, coalition $\{1,2\}$ Y-blocks x because one has $(1-\varepsilon, \varepsilon) \in \mathcal{F} \bigcap [P_1(x) \cap P_2(x)]$ for each $\varepsilon \in]0, \min\{1-x_1, x_2\}[$. Moreover, $\{1,2\}$ Y-blocks $(1,0)$ through each feasible allocation z with $z_1 < 1$. So, Yannelis's alpha-core is empty.

Now, if $\widetilde{x} \in \widetilde{\mathcal{F}}$, one has $x_{\{1\}} = 1$, $x_{\{2\}} = 0$ and $x_{\{1,2\}} = x \in \mathcal{F}$. Let $\left\{\widetilde{x}^1, ..., \widetilde{x}^k\right\} \subset \widetilde{\mathcal{F}}$. Suppose that $x^h_{\{1,2\}} = x^h$ is such that $x^h_1 < 1$ for $h = 1, ..., k$. Consider a feasible allocation z such that $z_1 \in \left]\max_{1\leq h\leq k} x^h_1, 1\right[$ and set $\{z^1, ..., z^k\} = \{z\}$. So, for every $\widetilde{x} = (x_S)_S \in \left\{\widetilde{x}^1, ..., \widetilde{x}^k\right\}$, there are no coalitions S that Y-block z by using x_S. Moreover, if $\widetilde{x}^1 = (1, 0, (1,0))$ and $\widetilde{x}^h$ is given as above for $h = 2, ..., k$, defining z as given in the previous case, we get that each coalition cannot Y-block z through every $\widetilde{x} \in \left\{\widetilde{x}^1, ..., \widetilde{x}^k\right\}$. Finally, we deduce that $\mathcal{E}$ is coalitional transfer quasi-convex.

It is obvious that:

Proposition 2 *Given an economy $\mathcal{E}$, let $\Theta_{\mathcal{E}}$ be the function defined by (1). One has:*

i) $\mathcal{E}$ satisfies the coalitional deviation property if and only if $\Theta_{\mathcal{E}}$ is slightly diagonal transfer continuous.
ii) $\mathcal{E}$ is coalitional transfer quasi-convex if and only if $\Theta_{\mathcal{E}}$ is diagonal transfer quasi-concave.

Finally, since Yannelis's alpha-core coincides with the solution set to Ky Fan minimax inequality corresponding to the function defined by (1), in the light of Proposition 2 and Lemma 2, we obtain the following characterization of the non-emptiness of Yannelis's alpha-core in a class of economies with discontinuous and non-ordered preference relations.

Theorem 1 *Assume that the economy $\mathcal{E} = \langle X_i, P_i, e_i \rangle_{i \in N}$ satisfies the coalitional deviation property. Then, Yannelis's alpha-core is non-empty if and only if $\mathcal{E}$ is coalitional transfer quasi-convex.*

5 The case of non-interdependent preferences

The result given in the previous section allows to obtain new conditions for the existence of core allocations in economies where the well-being of each consumer does not depend on the consumptions of the other individuals. In this case, the economy $\mathcal{E} = \langle X_i, P_i, e_i \rangle_{i \in N}$ is characterized by consumers for which $P_i : X_i \rightrightarrows X_i$ for every $i \in N$; in this case, we say that consumers' preferences are *selfish*. Then, the core is defined as follows (see Border (1984) and Yannelis (1991)):

Definition 4 Let z be a feasible allocation and let S be a coalition. We say that S *blocks* z if there exists $x_S \in \mathcal{F}_S$ such that $x_i \in P_i(z_i)$ for all $i \in S$. The *core* is the set of unblocked feasible allocations.

Given an exchange economy $\mathcal{E}$ with selfish preferences, we denote by $\mathcal{C}(\mathcal{E})$ the core of $\mathcal{E}$. Define $\overline{\mathcal{E}} = \left\langle X_i, \overline{P}_i, e_i \right\rangle_{i \in N}$, where $\overline{P}_i(x) = P_i(x_i) \times X_{-i}$ for each $x \in X$ and for each $i \in N$. It is clear that $\mathcal{C}(\mathcal{E}) = \mathcal{C}^Y(\overline{\mathcal{E}})$. The properties introduced in Definitions 2 and 3 can be considered also for economies with selfish preferences:

- $\mathcal{E}$ satisfies the *coalitional deviation property* if $z \in \mathcal{F} \backslash \mathcal{C}(\mathcal{E})$ implies that there exists an open neighborhood O_z of z and $\widetilde{x}' = (x'_S)_{S \in \mathcal{N}} \in \widetilde{\mathcal{F}}$ such that, for each $z' \in \left[O_z \cap \mathcal{F}\right] \backslash \mathcal{C}(\mathcal{E})$, there exists a coalition S such that $x'_{S,i} \in P_i(z'_i)$ for all $i \in S$, where $x'_S = \left(x'_{S,i}\right)_{i \in S}$.
- $\mathcal{E}$ is *coalitional transfer quasi-convex* if, for every $\{\widetilde{x}^1, ..., \widetilde{x}^k\} \subset \widetilde{\mathcal{F}}$, there exists $\{z^1, ..., z^k\} \subset \mathcal{F}$ - where $\widetilde{x}^h \mapsto z^h$ for $h = 1, ..., k$ - such that, for each $\{z^{h_1}, ..., z^{h_l}\} \subseteq \{z^1, ..., z^k\}$ and for each $z \in \text{sco}\{z^{h_1}, ..., z^{h_l}\}$, there exists $\widetilde{x} = (x_S)_{S \in \mathcal{N}} \in \{\widetilde{x}^{h_1}, ..., \widetilde{x}^{h_l}\}$ so that no coalition S can block z by using x_S.

The economy $\mathcal{E}$ satisfies the coalitional deviation property if and only if $\overline{\mathcal{E}}$ satisfies it, and similarly for coalitional transfer quasi-convexity. So, from Theorem 1 we obtain the following result:

Theorem 2 *Let $\mathcal{E} = \langle X_i, P_i, e_i \rangle_{i \in N}$ be an economy with selfish preferences. Assume that $\mathcal{E}$ satisfies the coalitional deviation property. Then, the core is non-empty if and only if $\mathcal{E}$ is coalitional transfer quasi-convex.*

6 Comparison with the previous literature

The early paper by Yannelis (1991) provides sufficient conditions for the existence of Yannelis's alpha-core allocations in economies with 2 consumers where, for each $i \in \{1, 2\}$: i) X_i is a compact and convex subset of a Hausdorff topological vector space; ii) $x \notin \text{co}P_i(x)$ for each $x \in X$; iii) $P_i^{-1}(z)$ is open for each $z \in X$ (in this case we say that P_i has open lower sections).

Conditions of Theorem 1 allow the non-emptiness of Yannelis's alpha-core in economies with $n \geq 2$ consumers, while Holly (1994) proves that assumptions i), ii) and iii) above do not guarantee that $\mathcal{C}^Y(\mathcal{E}) \neq \emptyset$ in economies with more than 2 consumers.

However, assumptions of Theorem 1 are not stronger than (i), (ii) and (iii). In fact, consider the economy introduced in Example 1, that satisfies the assumptions of Theorem 1. The preference relations P_1 and P_2 do not have open lower sections. Indeed, $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) \in P_i^{-1}\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ for $i = 1, 2$ and there are allocations y' and y'' such that $y_1' > y_2'$ and $y_1'' < y_2''$ in every open neighborhood of $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$. So, $y' \notin P_1^{-1}\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and $y'' \notin P_2^{-1}\left(\frac{1}{2}, \frac{1}{2}, 0\right)$. Moreover, P_3 does not satisfy property ii) above. In fact, ii) is equivalent to quasi-concavity of the function u_i, but u_3 is not quasi-concave: indeed, if $x \in \text{sco}\left\{\left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2}\right)\right\}$, we have $u_3(x) = 0$ and $u_3\left(\frac{1}{2}, 0, \frac{1}{2}\right) = u_3\left(0, \frac{1}{2}, \frac{1}{2}\right) = 1$. This proves that coalitional transfer quasi-convexity does not imply the property: $x \notin \text{co}P_i(x)$ for each $x \in X$ and for each $i \in N$.

Remark 3 Example 3.1 by Holly (1994) introduces a 3-consumer exchange economy $\mathcal{E}$ where the preference relations are represented by linear utility functions and the set of allocations is a compact and convex subset of $\mathbb{R}^3$. Theorem 3.2 by Holly (1994) proves that $\mathcal{C}^Y(\mathcal{E}) = \emptyset$, even if $\mathcal{E}$ satisfies the assumptions i), ii) and iii) above. In the light of Proposition 1, we have that $\mathcal{E}$ satisfies the coalitional deviation property. So, from Theorem 1, we have that $\mathcal{E}$ fails to verify the coalitional transfer quasi-convexity. This proves that condition ii) does not imply the coalitional transfer quasi-convexity.

Now, let us consider economies with selfish preferences. The previous results on the existence of core allocations, at our knowledge, deal with continuous preference relations, that are preferences where the mappings P_i have open-graph: see Scarf (1969, 1971), that assumes preference relations represented by continuous utility functions; Border (1984); Yannelis (1991). Theorem 2 improves the previous results because it gives a characterization of the non-emptiness of the core in a class of economies which satisfy a condition (the coalitional deviation property) more general than continuity of preferences (see Proposition 3 and Example 3).

Proposition 3 *Let $\mathcal{E} = \langle X_i, P_i, e_i \rangle_{i \in N}$ be an exchange economy with selfish preferences and mappings P_i having open-graph. Then, $\mathcal{E}$ satisfies the coalitional deviation property.*

Proof If $z \notin \mathcal{C}(\mathcal{E})$, for at least one coalition S and $x_S = \left(x_{S,i}\right)_{i \in S} \in \mathcal{F}_S$, we get $x_{S,i} \in P_i(z_i)$ for all $i \in S$. Since P_i has open-graph, for each $i \in S$, there exists an open neighborhood O_{z_i} of z_i such that $x_{S,i} \in P_i(z_i')$ for all $z_i' \in O_{z_i}$ and for all $i \in S$. Set $O_z = \prod_{i \in S} O_{z_i} \times X_{-S}$ and defined $\widetilde{x}' \in \widetilde{\mathcal{F}}$ by $x_{S,i}' = x_{S,i}$ for each $i \in S$, one can see that the coalitional deviation property holds true. □

Remark 4 Differently to Proposition 1, in order to prove Proposition 3, one does not need compactness on the set of feasible allocations. We note that the arguments of the proof above allow to prove the following more general result: *an exchange economy with a finite number of selfish consumers satisfies the coalitional deviation property if the mappings P_i have open lower sections.*

Example 3 Let $\mathcal{E}$ be the exchange economy with one good and two consumers such that: $X_1 = X_2 = [0, 1]$; $e_1 = 1$; $e_2 = 0$; $u_1(x_1) = 1$ if $x_1 \in \left]0, \frac{1}{2}\right[$ and $u_1(x_1) = 0$ otherwise; $u_2(x_2) = x_2$ if $x_2 \in \left[0, \frac{1}{3}\right]$ and $u_2(x_2) = 1$ otherwise; $P_i(x_i) = \{z_i \in X_i : u_i(z_i) > u_i(x_i)\}$ with $i = 1, 2$. It is easy to see that P_1 does not have open lower sections; so, the graph of P_1 is not open. We have that $\mathcal{C}(\mathcal{E}) = \left\{(1-t, t) : t \in \left[\frac{1}{3}, 1\right]\right\}$. If z is a feasible allocation and $z \notin \mathcal{C}(\mathcal{E})$, one can easily find an open neighborhood O_z of z such that the coalition $\{1, 2\}$ blocks every $z' \in O_z \cap \mathcal{F}$ through the feasible allocation $\left(\frac{1}{3}, \frac{2}{3}\right)$. Let $\widetilde{x}'$ be such that $x'_{\{1,2\}} = \left(\frac{1}{3}, \frac{2}{3}\right)$. So, one can see that $\mathcal{E}$ satisfies the coalitional deviation property.

7 Conclusion

In this paper, we have considered exchange economies where the preference relation of each consumer is defined on the set of allocations and does not need to be complete and transitive. In this setting, we have given sufficient and necessary conditions for the existence of alpha-core allocations in the sense of Yannelis (1991) when the number n of consumers is greater or equal than 2. When $n = 2$, our result is not connected with a previous one by Yannelis (1991) and Holly (1994) proved that assumptions by Yannelis (1991) are not sufficient to guarantee the existence of alpha-core allocation when $n > 2$. The result has been obtained by means of the Ky Fan minimax inequality (Fan 1972). More precisely, for every exchange economy, we have identified a Ky Fan minimax inequality whose solution set coincides with Yannelis's alpha-core. So, we have applied a result on the existence of solutions to Ky Fan minimax inequality. As a byproduct of our result, we have obtained sufficient and necessary conditions for the existence of core allocations when the preferences of consumers are selfish. Examples have been given to illustrate the conditions and to compare the results with the previous literature.

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References

Aumann, R.J.: The core of a cooperative game without side payments. Trans. Am. Math. Soc. **98**, 539–552 (1961)

Basile, A., Scalzo, V.: Non-emptiness of the alpha-core: sufficient and necessary conditions. Internat. J. Game Theory **49**(4), 1143–1153 (2020)

Baye, M.R., Tian, G., Zhou, J.: Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs. Rev. Econ. Stud. **60**, 935–948 (1993)

Border, K.C.: A core existence theorem for games without ordered preferences. Econometrica **52**(6), 1537–1542 (1984)
Borglin, A.: Price characterization of stable allocations in exchange economies with externalities. J. Econ. Theory **6**(5), 483–494 (1973)
Drakopoulos, S.A.: The history of attitudes towards interdependent preferences. J. Hist. Econ. Thought **34**(4), 541–557 (2012)
Duesenberry, J.S.: Income, Saving, and the Theory of Consumer Behavior. Harvard University Press, Harvard (1949)
Dupor, B., Liu, W.F.: Jealousy and equilibrium overconsumption. Am. Econ. Rev. **93**(1), 423–428 (2003)
Fan, K.: A minimax inequality and applications. In: Shisha, O. (ed.) Inequalities, vol. 3. Academic Press, New York (1972)
Holly, C.: An exchange economy can have an empty alpha-core. Econ. Theor. **4**, 453–461 (1994)
Martins-da-Rocha, F., Nicholas C. Yannelis, N.C.: Non-emptiness of the alpha-core, mimeo Rio de Janeiro (2011)
Nassah, R., Tian, G.: Existence of Nash equilibrium in discontinuous games. MPRA Paper No. 41206, online at https://mpra.ub.uni-muenchen.de/41206/ (2008)
Nassah, R., Tian, G.: On the existence of Nash equilibrium in discontinuous games. Econ. Theory **61**(3), 515–540 (2016)
Ok, E.A., Koçkesen, L.: Negatively interdependent preferences. Soc. Choice Welf. **17**, 533–558 (2000)
Reiter, S.: Interdependent preferences and groups of agents. J. Public Econ. Theory **3**(1), 27–67 (2001)
Reny, P.J.: Further Results on the Existence of Nash Equilibria in Discontinuous Games. University of Chicago, Mimeo (2009)
Reny, P.J.: Nash equilibrium in discontinuous games. Econ. Theory **61**(3), 553–569 (2016)
Scalzo, V.: Essential equilibria of discontinuous games. Econ. Theory **54**(1), 27–44 (2013)
Scalzo, V.: Equilibrium existence in games: slight single deviation property and Ky Fan minimax inequality. J. Math. Econ. **82**, 197–201 (2019)
Scalzo, V.: Existence of strong equilibrium in discontinuous games. J. Math. Anal. Appl. **491**(1), Article 124247 (2020)
Scarf, H.: The core of an N-person game. Econometrica **54**, 1039–1054 (1967)
Scarf, H.: On the existence of a cooperative solutions for a general class of n-person games. J. Econ. Theory **3**, 169–181 (1971)
Tian, G.: Necessary and sufficient conditions for maximization of a class of preference relations. Rev. Econ. Stud. **60**, 949–958 (1993)
Tian, G., Yang, L.: Theory of negative consumption externalities with applications to the economics of happiness. Econ. Theor. **39**(3), 399–424 (2009)
Veblen, T.: The Theory of the Leisure Class: An Economic Study of Institutions. New York: The Modern Library (1934) [1899]
Yannelis, N.C.: The core of an economy without ordered preferences. In "Equilibrium Theory in Infinite Dimensional Spaces", M.A. Khan and N. Yannelis, Eds., 102 - 123 (1991)
Zhao, J.: The hybrid equilibria and core selection in exchange economies with externalities. J. Math. Econ. **26**(4), 387–407 (1996)