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RESEARCH ARTICLE

The risk-neutral non-additive probability with market frictions

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Abstract

The fundamental theory of asset pricing has been developed under the two main assumptions that markets are frictionless and have no arbitrage opportunities. In this case the market enforces that replicable assets are valued by a linear function of their payoffs, or as the discounted expectation with respect to the so-called risk-neutral probability. Important evidence of the presence of frictions in financial markets has led to study market pricing rules in such a framework. Recently, Cerreia-Vioglio et al. (J Econ Theory 157:730–762, 2015) have extended the Fundamental Theorem of Finance by showing that, with markets frictions, requiring the put–call parity to hold, together with the mild assumption of translation invariance, is equivalent to the market pricing rule being represented as a discounted Choquet expectation with respect to a risk-neutral nonadditive probability. This paper continues this study by characterizing important properties of the (unique) risk-neutral nonadditive probability v_f associated with a Choquet pricing rule f, when it is not assumed to be subadditive. First, we show that the observed violation of the call–put parity, a condition considered by Chateauneuf et al. (Math Financ 6:323–330, 1996) similar to the put–call parity in Cerreia-Vioglio et al. (2015), is consistent with the existence of bid-ask spreads. Second, the balancedness of v_f—or equivalently the non-vacuity of its core—is characterized by an arbitrage-free condition that eliminates all the arbitrage opportunities that can be obtained by splitting payoffs in parts; moreover the (nonempty) core of v_f consists of *additive* probabilities below v_f whose associated (standard) expectations are all below the Choquet pricing rule f. Third, by strengthening again the previous arbitrage-free condition, we show the existence of a *strictly positive* risk-neutral probability below v_f, which allows to recover the standard formulation of the Fundamental Theorem of Finance for frictionless markets.

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1 Introduction

This paper[1] considers a two-date stochastic model, where today ($t = 0$) is known and tomorrow ($t = 1$) is uncertain; the uncertainty is represented by a set Ω of states of nature, one (and only one) of which will be disclosed tomorrow. The set Ω will be finite in the whole paper. A (stream of) payoff is a random variable $x : \Omega \to \mathbb{R}$ or a vector $x \in \mathbb{R}^\Omega$, where $x(\omega)$ is the payoff (money) at $t = 1$ if state ω prevails. We adopt the convention that if $x(\omega) < 0$ (resp. > 0), then $|x(\omega)|$ is paid (resp. gained). A financial market is defined as a family of securities that allows investors to generate payoffs by choosing adequately the quantities of the securities that are bought or sold. The hedging price associated with this market is then a function $f : \mathbb{R}^\Omega \to \mathbb{R}$ that associates to every payoff $x \in \mathbb{R}^\Omega$ the price (or the cost) $f(x)$ to be paid today for the delivery of x at $t = 1$; here, by convention, $|f(x)|$ is paid (resp. gained) if $f(x) > 0$ (resp. < 0), in other words, a negative cost is a gain. We refer to Cerreia-Vioglio et al. (2015a) and Chateauneuf and Cornet (2022) for the detailed presentation of a market and the way to derive its hedging price $f : \mathbb{R}^\Omega \to \mathbb{R}$ from underlying given securities. Here, the hedging price f, taken as the primitive concept, is supposed to be exogenously given.

The study of markets with frictions has led to consider Choquet pricing rules that are subadditive (instead of linear in the frictionless case) as in Chateauneuf et al. (1996) and Chateauneuf and Cornet (2022). In this paper pricing rules are not assumed to be subadditive and we recall the following important generalization of the Fundamental Theorem of Finance.

Theorem 1 (Cerreia-Vioglio et al. 2015a) *Let* $f : \mathbb{R}^\Omega \to \mathbb{R}$ *be a pricing rule such that* $f \neq 0$. (a) *The following statements are equivalent*:

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[1] We recall some notations used throughout the paper. Let Ω be a finite set, we let $\mathbb{R}^\Omega$ be the vector space of functions $x : \Omega \to \mathbb{R}$. We say that $x' \geq x$ (resp. $x' > x$, resp. $x' \gg x$) if, for all $\omega \in \Omega$, $x'(\omega) \geq x(\omega)$ (resp. $x' \geq x$ and $x' \neq x$, resp. for all $\omega \in \Omega$, $x'(\omega) > x(\omega)$); moreover, $x \leq x'$ means that $x' \geq x$ and similarly for the other two relations. The lattice operations $\wedge$ and $\vee$ in $\mathbb{R}^\Omega$ are defined by $(x \wedge x')(\omega) := \min\{x(\omega), x'(\omega)\}$, $(x \vee x')(\omega) := \max\{x(\omega), x'(\omega)\}$ for all $\omega \in \Omega$. Then $\mathbb{R}^\Omega_+ := \{x \in \mathbb{R}^\Omega : x \geq 0\}$) denotes the set of non-negative functions and $\mathbb{R}^\Omega_{++} := \{x \in \mathbb{R}^\Omega : x \gg 0\}$. For $A \subseteq \Omega$, we denote by A^c the complement set of A and $\mathbf{1}_A$ the indicator (or characteristic) function of A, i.e., $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$, and $\mathbf{1}_A(\omega) = 0$ otherwise, and, by convention, $\mathbf{1}_\omega = \mathbf{1}_{\{\omega\}}$ for all ω, and $\mathbf{1}_\emptyset = 0$. When $\Omega = \{1, \ldots, n\}$, we can identify $\mathbb{R}^\Omega$ with $\mathbb{R}^n$, thus a function $x : \Omega \to \mathbb{R}$ can also be viewed as the n-tuple $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The previously defined order $\geq$ is then identified with the coordinate-wise order of $\mathbb{R}^n$, i.e., $x' = (x'_1, \ldots, x'_n) \geq x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ means $x'_i \geq x_i$ for every $i = 1, \ldots, n$. With the previous identification, for $A \subseteq \{1, \ldots, n\}$, $\mathbf{1}_A$ will now be the vector in $\mathbb{R}^n$ such that $x_i = 1$ if $i \in A$ and $x_i = 0$ otherwise. Thus we denote by $\mathbf{1}_i := \mathbf{1}_{\{i\}}$ (resp. $\mathbf{1}_\Omega$) the vector with all coordinates equal to zero, but the i-th equal to 1 (resp. with all coordinates equal to 1) so that $x = (x_1, \ldots, x_n) = x_1\mathbf{1}_1 + \cdots + x_n\mathbf{1}_n$. Without any risk of confusion, we will use indifferently the same notation μ to represent the function $\mu : \Omega \to \mathbb{R}$, the vector in $\mathbb{R}^\Omega$, the associated linear function $x \to x \cdot \mu$, or the associated set-function $A \to \mu(A) := \mathbf{1}_A \cdot \mu = \sum_{\omega \in \Omega} \mu(\omega)$ for all $A \subseteq \Omega$.

(i) *f satisfies*:
[*Monotonicity*] $f(x) \leq f(x')$ *for all* $x \leq x'$,
[*Translation Invariance*] $f(x + t\mathbf{1}_\Omega) = f(x) + tf(\mathbf{1}_\Omega)\ \forall x, \forall t \in \mathbb{R}$,
[*Constant Modularity*] $f(x \vee t\mathbf{1}_\Omega) + f(x \wedge t\mathbf{1}_\Omega) = f(x) + tf(\mathbf{1}_\Omega)\ \forall x, \forall t \geq 0$;
(ii) *f satisfies Monotonicity, Translation Invariance, and Put–Call Parity*;
(iii) *there exists a nonadditive probability v and a risk-free rate $r > -1$ such that*

$f(x) = \frac{1}{1+r}\int_\Omega x dv$ *for all* $x \in \mathbb{R}^\Omega$ [*where* $\int_\Omega$ *is the Choquet integral*].

(b) *Moreover, if v is balanced, then f satisfies*
[*Positive Bid-Ask Spreads*] $-f(-x) \leq f(x)$ *for all* x.

The first two assumptions in (i) are quite standard in the literature. The last one of Constant Modularity, introduced in financial markets by Cerreia-Vioglio et al. (2015a), can be equivalently formulated in financial terms by the put–call parity; see the PCP Condition defined and discussed hereafter. Further generalizations are provided by Cerreia-Vioglio et al. (2015b) to deal with pricing rules that may not be monotone or may not be defined on the whole space $\mathbb{R}^\Omega$, a framework that goes beyond the scope of our paper.

We recall that a *nonadditive probability* is a set function $v : 2^\Omega \to \mathbb{R}$ which satisfies $v(\emptyset) = 0$, $v(\Omega) = 1$, and is monotone, i.e., $v(A) \leq v(B)$ for all $A \subseteq B \subseteq \Omega$; for the definition and properties of the Choquet integral, we refer to Denneberg (1994) and Marinacci and Montrucchio (2004). Let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be monotone, and translation invariant (hence in particular when f is a monotone Choquet pricing rule), one checks that $f(\mathbf{1}_\Omega) > 0$ if and only if $f \neq 0$. Thus under the assumptions of Theorem 1, we define $r_f \in \mathbb{R}$ and the set function $v_f : 2^\Omega \to \mathbb{R}$ by:

$$r_f = -1 + 1/f(\mathbf{1}_\Omega), \quad v_f(A) := f(\mathbf{1}_A)/f(\mathbf{1}_\Omega) \text{ for all } A \subseteq \Omega. \quad (*)$$

As noticed in Cerreia-Vioglio et al. (2015a), the interest rate r and the nonadditive probability v, associated with f by Theorem 1 are uniquely defined and coincide respectively with r_f and v_f (as defined above), i.e., $r = r_f$ and $v = v_f$. Hereafter, r_f will be called the *risk-free interest rate* associated with f, and v_f the *risk-neutral nonadditive probability* associated with f.

We can now present the main results of our paper, which studies the properties of the risk-neutral nonadditive probability v_f, when f is a Choquet pricing rule. First, we notice that the converse of the above Assertion (b) of Theorem 1 may not be true (unless f is assumed to be subadditive) as shown by Example 1 in the Appendix. Second, Theorem 2, characterizes the Positive Bid-Ask Spread Condition, i.e., $f(-x)+f(x) \geq 0$ for all x, by a weak form of the Call–Put Parity Condition introduced by Chateauneuf et al. (1996). The Positive Bid-Ask Spread Condition with $x \geq 0$ also implies the absence of “buy and sell” arbitrage opportunities, that is, there is no payoff $x \geq 0$ such that $f(x) < -f(-x)$, thus the cost $f(x) \geq 0$ of buying x is smaller than the gain $-f(-x) \geq 0$ of selling the same payoff x. Equivalently, for $K = 2$, there is no $x_k \in \mathbb{R}^\Omega_+ \cup -\mathbb{R}^\Omega_+$ ($k = 1, \ldots, K$) such that $x_1 + \cdots + x_K \geq 0$ and $f(x_1) + \cdots + f(x_K) < 0$.

Then Theorem 3, shows that the elimination of all “buy and sell” arbitrage opportunities at *any* order $K \in \mathbb{N}$—as defined above—characterizes the nonemptyness of the core of v_f. Note that, under the additional assumption that f is subadditive, then v_f is submodular, hence the core of v_f is nonempty (Shapley 1971) but under the assumption made by Cerreia-Vioglio et al. (2015a), v_f may have an empty core as shown by Example 1.

Moreover, Theorem 3 characterizes the existence of a *strictly positive* (additive) probability P in the core of v_f, and thus extend the Fundamental Theorem of Asset Pricing in the standard frictionless case. For this purpose, more arbitrage opportunities than before need to be eliminating, by assuming that, for all integer K, there is no $x_k \in \mathbb{R}^{\Omega}_{+} \cup -\mathbb{R}^{\Omega}_{+}$ $(k = 1, \ldots, K)$ such that $x_1 + \cdots + x_K > 0$ and $f(x_1) + \cdots + f(x_K) \leq 0$. Finally, Theorem 3 also characterizes the existence of a *strictly positive* probability P in the core of v_f (resp. the nonemptyness of the core) by the strict positivity (resp. nonnegativity) of the exact cover v^e_f of v_f. This condition cannot be weakened by only assuming $v_f > 0$ (resp. $v_f \geq 0$) as shown by Example 1.

The main results discussed previously are formally presented in the next Sect. 2, the proofs are given in Sect. 3, and the Appendix provides Example 1, together with some conclusions.

2 The main results

2.1 Characterizing the absence of buy and sell arbitrage opportunities

We first recall the notion of Call–Put Parity introduced by Chateauneuf et al. (1996):

$$\text{CPP}: \ f([x - k\mathbf{1}_{\Omega}]_{+}) + f(-x) + kf(\mathbf{1}_{\Omega}) = f([k\mathbf{1}_{\Omega} - x]_{+}) \quad \text{for all } x, \ k \geq 0,$$

and we notice that, for a frictionless market, that is f is linear, it is equivalent to the notion of Put–Call Parity introduced by Cerreia-Vioglio et al. (2015a):

$$\text{PCP}: \ f([x - k\mathbf{1}_{\Omega}]_{+}) + f(-[k\mathbf{1}_{\Omega} - x]_{+}) = f(x) - kf(\mathbf{1}_{\Omega})] \quad \text{for all } x, \ k \geq 0.$$

Our first result states that, for Choquet pricing rules, the Call–Put Parity Condition CPP is equivalent to the No-Spread Condition, i.e., $f(x) + f(-x) = 0$ for all $x \geq 0$ (or for all x). Moreover, a weaker version of the Call–Put Parity (in which the equality is replaced by an inequality) is equivalent to the Positive Bid-Ask Spread Condition or the absence of buy and sell arbitrage opportunities of order 2. It is worth pointing out that the weaker form of CPP has been confirmed by empirical research, see e.g. Gould and Galai (1974), Klemkosky and Resnick (1979), and Sternberg (1994).

Theorem 2 *Let $f : \mathbb{R}^{\Omega} \to \mathbb{R}$ be a nonzero, monotone, Choquet pricing rule.*
(a) *The following assertions are equivalent*:

- *CPP*: $f([x - k\mathbf{1}_{\Omega}]_{+}) + f(-x) + kf(\mathbf{1}_{\Omega}) = f([k\mathbf{1}_{\Omega} - x]_{+})$ *for all* $x, k \geq 0$;
- $f(x) + f(-x) = 0$ *for all* $x \in \mathbb{R}^{\Omega}$;
- $f(x) + f(-x) = 0$ *for all* $x \in \mathbb{R}^{\Omega}_{+}$;
- $v_f(A) + v_f(A^c) - 1 = 0$ *for all* $A \subseteq \Omega$.

(b) *The following assertions are equivalent*:

- $f([x-k\mathbf{1}_\Omega]_+)+f(-x)+kf(\mathbf{1}_\Omega)\geq f([k\mathbf{1}_\Omega-x]_+)$ *for all* x, *all* $k\geq 0$;
- $f(x)+f(-x)\geq 0$ *for all* $x\in\mathbb{R}^\Omega$;
- $f(x)+f(-x)\geq 0$ *for all* $x\in\mathbb{R}_+^\Omega$;
- $v_f(A)+v_f(A^c)-1\geq 0$ *for all* $A\subseteq\Omega$.

The proof of Theorem 2 is given in Sect. 3.1.

We point out that the Call–Put Parity Condition CPP (or its equivalent form of No-Spread) does not guarantee in general that v_f has a nonempty core, as shown in Example 1 in the Appendix. As proved hereafter in Theorem 3, the elimination of arbitrage opportunities at any order $K\geq 1$ will be required for v_f to have a nonempty core.

We end the section with the following remark.

Remark 1 (CPP Choquet pricing rules with a nonempty core) When the Choquet pricing rule f satisfies the Call–Put Parity CPP and $\text{core}(v_f)\neq\emptyset$, then f is linear and v_f is additive.

Indeed, the Choquet pricing rule f is linear if and only if v_f is additive. We choose $\mu\in\text{core}(v_f)\neq\emptyset$, then, for all $A\subseteq\Omega$, $\mu(A)\leq v_f(A)$, $\mu(A^c)\leq v_f(A^c)$, and $\mu(\Omega)=v_f(\Omega)=1$. But $1=\mu(\Omega)=\mu(A)+\mu(A^c)\leq v_f(A)+v_f(A^c)=1$ by Theorem 2. Thus, $v_f(A)=\mu(A)$ for all $A\subseteq\Omega$, which proves that v_f is additive and f is linear. □

2.2 Absence of "split" arbitrage opportunities

We have defined previously the following Arbitrage-free Conditions:

$$\begin{aligned}\mathrm{AF}_{++}:\ &\text{For all integer } K,\text{ for all } x_k\in\mathbb{R}_+^\Omega\cup-\mathbb{R}_+^\Omega (k=1,\ldots,K),\\ &x_1+\cdots+x_K>0\Rightarrow f(x_1)+\cdots+f(x_K)>0,\\ \mathrm{AF}_{+}:\ &\text{For all integer } K,\text{ for all } x_k\in\mathbb{R}_+^\Omega\cup-\mathbb{R}_+^\Omega (k=1,\ldots,K),\\ &x_1+\cdots+x_K\geq 0\Rightarrow f(x_1)+\cdots+f(x_K)\geq 0,\end{aligned}$$

that strengthen standard notions encountered in the finance literature, by ruling out the standard (order 1) arbitrage opportunities, the order 2 buy and sell arbitrage opportunities together with the arbitrage opportunities at any order K. See Example 1 in the Appendix for the importance of eliminating arbitrage opportunities of order $K>2$.

The next proposition shows that AF_{++} is stronger than AF_+ and, whenever f is subadditive there is no need to eliminate arbitrage opportunities of order $K>1$.

Proposition 1 *Let* $f:\mathbb{R}^\Omega\to\mathbb{R}$, $f(0)=0$. (a) *If f is monotone, then*:

- $AF_{++}\Rightarrow AF_+$.

(b) *If f is monotone and subadditive, then*:

- $AF_+\iff f(x)\geq 0$ *for all* $x\geq 0$;
- $AF_{++}\iff f(x)>0$ *for all* $x>0$.

Proof ($\mathrm{AF}_{++}\Rightarrow\mathrm{AF}_+$) Let $K\in\mathbb{N}$, let $x_k\in\mathbb{R}_+^\Omega\cup-\mathbb{R}_+^\Omega(k=1,\ldots,K)$, such that $x_1+\cdots+x_K\geq 0$. Then, $n(x_1+\cdots+x_K)+\mathbf{1}_\Omega>0$ for all $n\in\mathbb{N}$. Hence,

$n(f(x_1) + \cdots + f(x_K)) + f(\mathbf{1}_\Omega) > 0$, from AF_{++}. Dividing by n, and letting $n \to +\infty$, at the limit we get $f(x_1) + \cdots + f(x_K) \geq 0$.

The proofs of the remaining assertions are straightforward. □

The next result shows that the Arbitrage-free Conditions AF_{++} and AF_{+} (that consider multiple buy and sell strategies with either $x_k \geq 0$ or $x_k \leq 0$ for every k) have equivalent formulations also of interest, when x_k can be taken in the whole space $\mathbb{R}^\Omega$ for all k. We assume hereafter that f only satisfies $f([x]_+) + f(-[x]_-) = f(x)$ for all x in the whole space $\mathbb{R}^\Omega$, a property that holds when f is a Choquet pricing rule since it is constant modular by Theorem 1 (take $t = 0$). Note that this assumption has a clear financial meaning, that is, splitting $x = [x]_+ - [x]_-$ in two (buying and selling) parts, the cost $f(x)$ is equal to the difference between the payment $f([x]_+) \geq 0$ to get the purchase $[x]_+$ and the gain $-f(-[x]_-) \geq 0$ from the sale of $[x]_-$.

Proposition 2 *Let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be monotone and satisfy*

$$f([x]_+) + f(-[x]_-) = f(x) \quad \text{for all } x \in \mathbb{R}^\Omega.$$

Then the three following assertions are equivalent:

AF_{++} : *For all integer K, for all $x_k \in \mathbb{R}^\Omega_+ \cup -\mathbb{R}^\Omega_+ (k = 1, \ldots, K)$*

$$x_1 + \cdots + x_K > 0 \Rightarrow f(x_1) + \cdots + f(x_K) > 0;$$

(i_{++}) *For all integer K, for all $x_k \in \mathbb{R}^\Omega (k = 1, \ldots, K)$*

$$x_1 + \cdots + x_K > 0 \Rightarrow f([x_1]_+) + f(-[x_1]_-) + \cdots + f([x_K]_+) + f(-[x_K]_-) > 0;$$

(ii_{++}) *For all integer K, for all $x_k \in \mathbb{R}^\Omega (k = 1, \ldots, K)$*

$$x_1 + \cdots + x_K > 0 \Rightarrow f(x_1) + \cdots + f(x_K) > 0.$$

Similarly, the three following assertions are equivalent:

AF_{+} : *For all integer K, for all $x_k \in \mathbb{R}^\Omega_+ \cup -\mathbb{R}^\Omega_+ (k = 1, \ldots, K)$*

$$x_1 + \cdots + x_K \geq 0 \Rightarrow f(x_1) + \cdots + f(x_K) \geq 0;$$

(i_{+}) *For all integer K, for all $x_k \in \mathbb{R}^\Omega (k = 1, \ldots, K)$*

$$x_1 + \cdots + x_K \geq 0 \Rightarrow f([x_1]_+) + f(-[x_1]_-) + \cdots + f([x_K]_+) + f(-[x_K]_-) \geq 0;$$

(ii_{+}) *For all integer K, for all $x_k \in \mathbb{R}^\Omega (k = 1, \ldots, K)$*

$$x_1 + \cdots + x_K \geq 0 \Rightarrow f(x_1) + \cdots + f(x_K) \geq 0.$$

Proof $[(ii_{++}) \Rightarrow (i_{++}) \Rightarrow AF_{++}]$. The proof of the first implication is immediate using the decomposition $x_k = [x_k]_+ - [x_k]_-$. For the second implication, notice that $f([x_k]_+) + f(-[x_k]_-) = f(x_k)$ when $x_k \in \mathbb{R}^\Omega_+ \cup -\mathbb{R}^\Omega_+$.

$[AF_{++} \Rightarrow (ii_{++})]$. Let K be an integer, and let $x_k \in \mathbb{R}^\Omega (k = 1, \ldots, K)$ such that $\sum_{k=1}^K [x_k]_+ - [x_k]_- = \sum_{k=1}^K x_k > 0$. Since $f([x_k]_+) + f(-[x_k]_-) = f(x_k)$, from AF_{++} we get:

$\sum_{k=1}^K f(x_k) = \sum_{k=1}^K f([x_k]_+) + f(-[x_k]_-)) > 0$.

$[(ii_+) \iff (i_+) \iff AF_+]$. The proof is similar to the previous one. □

2.3 Characterizing arbitrage-free Choquet pricing rules

The core of a nonzero monotone Choquet pricing rule $f : \mathbb{R}^\Omega \to \mathbb{R}$ is defined as the core of its associated risk-neutral *nonadditive* probability v_f, that is:

$$\text{core}\,(f) := \{\mu \in \mathbb{R}^\Omega \ : \ \forall A \subseteq \Omega,\ \mu(A) \leq v_f(A), \mu(\Omega) = v_f(\Omega) = 1\}.$$

We let $\mathcal{P}(\Omega) := \{P \in \mathbb{R}^\Omega_+ \ : \ P \cdot \mathbf{1}_\Omega = 1\}$ be the set of (additive) probability on Ω, and we notice that $\text{core}\,(f) \subseteq \mathbb{R}^\Omega_+$ since f is monotone and all elements of the core are *additive* probabilities, thus one has:

$$\text{core}\,(f) = \{P \in \mathcal{P}(\Omega) \ : \ \forall A \subseteq \Omega,\ P(A) \leq v_f(A)\}.$$

The next result shows that f satisfies AF_{++} if and only if there exists a *strictly positive* probability P in the core of v_f and if and only if the exact cover v^e_f of v_f is strictly positive (i.e., positive for all nonempty events). Similarly, f satisfies the weaker condition AF_+ if and only if its core is nonempty and if and only if the exact cover v^e_f of v_f is nonnegative (i.e., nonnegative for all events). The exact cover $v^e_f : 2^\Omega \to \mathbb{R}$ of v_f is defined by:

$$v^e_f(A) := \sup \{\mu(A) : \mu \in \text{core}\,(f)\} \in [-\infty, +\infty) \text{ for } A \subseteq \Omega.$$ [2]

Theorem 3 *Let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be a nonzero, monotone, Choquet pricing rule. Then the following assertions are equivalent*:

(i)	f *satisfies*	AF_+	(*resp.* AF_{++})
(ii)	$\forall A \subseteq \Omega, A \neq \emptyset$,	$v^e_f(A) \geq 0$	(*resp.* $v^e_f(A) > 0$);
(iii)	*one has*:	$\text{core}\,(f) \neq \emptyset$	(*resp.* $\text{core}\,(f) \cap \mathbb{R}^\Omega_{++} \neq \emptyset$);
(iv)	$\exists P \in \mathcal{P}(\Omega)$,	$\frac{1}{1+r_f} P \leq f$	(*resp. and* $P \in \mathbb{R}^\Omega_{++}$).

The proof of Theorem 3 is given in Sect. 3.2.

We end the section with a remark.

Remark 2 (Subadditive Pricing Rules) When f is additionally subadditive, the Choquet pricing rule f is submodular, thus v_f is also submodular, hence exact, that is, $v^e_f = v_f$. Consequently, f satisfies AF_{++} (resp. AF_+) if and only if v_f is strictly positive (resp. nonnegative). We point out that this equivalence may not hold when f is no longer assumed to be subadditive as shown in Example 1 in the Appendix. □

3 Proofs

3.1 Proof of Theorem 2

Let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be a nonzero, monotone, Choquet pricing rule, we show that the following four conditions are equivalent.

(1) $f(-x) + f(x) \geq 0$ (resp. $= 0$) for all $x \in \mathbb{R}^\Omega$;

[2] The value $+\infty$ is excluded. Indeed, we use the convention that $v^e_f(A) = -\infty$ if $\text{core}\,(f) = \emptyset$. Thus, if $\text{core}\,(f)$ is nonempty, noticing that it is also compact, we have $v^e_f(A) \neq +\infty$.

(2) $f([x-k\mathbf{1}_\Omega]_+) + f(-x) + kf(\mathbf{1}_\Omega) - f([-x+k\mathbf{1}_\Omega]_+) \geq 0$ (resp. $=0$) for all $x \in \mathbb{R}^\Omega$, all $k \geq 0$;
(3) $f(-x) + f(x) \geq 0$ (resp. $=0$) for all $x \in \mathbb{R}^\Omega_+$;
(4) $v_f(A) + v_f(A^c) - 1 \geq 0$ (resp. $=0$) for all $A \subseteq \Omega$.

Proof • [(1) $\Rightarrow$ (2)] Since f is a Choquet pricing rule, it is constant modular by Theorem 1, thus (taking $t=0$) we have $f(y) = f([y]_+) + f(-[-y]_+)$ for all $y \in \mathbb{R}^\Omega$.

Consequently, from (1), we get

$$f(y) - f([y]_+) + f([-y]_+) = f(-[-y]_+) + f([-y]_+) \geq 0 \text{ (resp. } = 0).$$

Now, let $x \in \mathbb{R}^\Omega$, let $k \geq 0$, and let $y := -x + k\mathbf{1}_\Omega$. From above, we get:

$$\begin{aligned} 0 &\leq \text{(resp. } =) \; f(-x+k\mathbf{1}_\Omega) - f([-x+k\mathbf{1}_\Omega]_+) + f([x-k\mathbf{1}_\Omega]_+) \\ &= f(-x) + kf(\mathbf{1}_\Omega) - f([-x+k\mathbf{1}_\Omega]_+) + f([x-k\mathbf{1}_\Omega]_+) \end{aligned}$$

since the Choquet pricing rule is translation invariant. □

• [(2) $\Rightarrow$ (3)] Let $x \geq 0$, taking $k=0$ in (2) we get

$$f(x) + f(-x) \geq 0 \text{ (resp. } = 0) \text{ [since } f(0) = 0].$$ □

• [(3) $\Rightarrow$ (4)] Let $x := \mathbf{1}_A \geq 0$, then $-x = \mathbf{1}_{A^c} - \mathbf{1}_\Omega$ and we have:

$$\begin{aligned} 0 &\leq \text{(resp. } =) \; f(-x) + f(x) = f(\mathbf{1}_{A^c} - \mathbf{1}_\Omega) + f(\mathbf{1}_A) \text{ [by (3)]} \\ &= f(\mathbf{1}_{A^c}) - f(\mathbf{1}_\Omega) + f(\mathbf{1}_A) \text{ [since } f \text{ is translation invariant]} \\ &= f(\mathbf{1}_\Omega)[v_f(A^c) - 1 + v_f(A)]. \end{aligned}$$

Thus (4) holds since $f(\mathbf{1}_\Omega) > 0$. □

• [(4) $\Rightarrow$ (1)] Without any loss of generality we assume that $f(\mathbf{1}_\Omega) = 1$ so that $v_f(A) = f(\mathbf{1}_A)$ for $A \subseteq \Omega$. Let $x \in \mathbb{R}^\Omega$, whose set of values $\{x(\omega) : \omega \in \Omega\} = \{x_1, \ldots x_K\}$ is ordered decreasingly as $x_1 > \cdots > x_k > \cdots > x_K$. Letting $A_k := \{\omega \in \Omega : x(\omega) = x_k\}$ and $B_k := \{\omega \in \Omega : -x(\omega) = -x_k\} = A_k$ for all $k = 1, \ldots, K$, we have:

$$\begin{aligned} f(x) \;=\; \int_\Omega x \, dv_f &:= (x_1 - x_2)v_f(A_1) + \cdots + (x_k - x_{k+1})v_f(A_1 \cup \cdots \cup A_k) \\ &\quad + \cdots + (x_{K-1} - x_K)v_f(A_1 \cup \cdots \cup A_{K-1}) + x_K v_f(\Omega), \\ f(-x) &:= (-x_K + x_{K-1})v_f(B_K) + \cdots + (-x_k + x_{k+1})v_f(B_K \cup \cdots \cup B_k) \\ &\quad + \cdots + (-x_2 + x_1)v_f(B_K \cup \cdots \cup B_2) + (-x_1)v_f(\Omega). \end{aligned}$$

Hence, from (4) we deduce that:

$$\begin{aligned} f(x) + f(-x) &= (x_1 - x_2)[v_f(A_1) + v_f(A_1^c)] \\ &\quad + \cdots + (x_{K-1} - x_K)[v_f(A_K^c) + v_f(A_K)] + x_K v_f(\Omega) - x_1 v_f(\Omega) \\ &\geq \text{(resp. } =)(x_1 - x_2 + \cdots + x_{K-1} - x_K + x_K - x_1)v_f(\Omega) = 0. \end{aligned}$$ □

3.2 Proof of Theorem 3

Let f be a monotone, nonzero, Choquet pricing rule, without any loss of generality we assume hereafter that $f(\mathbf{1}_\Omega) = 1$, thus $r_f = 0$. We define:

$$c(x) := \sup\left\{x \cdot \mu \;:\; \mu \in \text{core}(f)\right\} \in [-\infty, +\infty)^{3} \text{ for } x \in \mathbb{R}^\Omega,$$

[3] The value $+\infty$ is excluded. Indeed, either $c(x) = -\infty$ if core $(f) = \emptyset$ (by convention), or core $(f) \neq \emptyset$, but since it is also compact $c(x) \neq +\infty$. The same argument applies to $v_f^e(A)$.

$v_f^e(A) := c(\mathbf{1}_A) = \sup\{\mu(A) : \mu \in \mathrm{core}(f)\} \in [-\infty, +\infty)$ for $A \subseteq \Omega$, with the convention that $c(x) := -\infty$ and $v_f^e(A) = -\infty$ if $\mathrm{core}(f) = \emptyset$.

We consider the following assertions:

(1) f satisfies AF_+ (resp. AF_{++})
(2) [Balancedness$_+$] for all $\theta_A \geq 0$
$$\sum_{A\subseteq\Omega} \theta_A \mathbf{1}_A \geq \mathbf{1}_\Omega \Rightarrow \sum_{A\subseteq\Omega} \theta_A v_f(A) \geq v_f(\Omega),$$
$$\text{(resp. } \sum_{A\subseteq\Omega} \theta_A \mathbf{1}_A > \mathbf{1}_\Omega \Rightarrow \sum_{A\subseteq\Omega} \theta_A v_f(A) > v_f(\Omega)\text{)},$$
(3) for all $x \geq 0$, $c(x) \geq 0$ (resp. for all $x > 0$, $c(x) > 0$),
(4) for all $A \subseteq \Omega$, $v_f^e(A) \geq 0$ (resp. for all $A \neq \emptyset$, $v_f^e(A) > 0$),
(5) $\mathrm{core}(f) \neq \emptyset$ (resp. $\mathrm{core}(f) \cap \mathbb{R}^\Omega_{++} \neq \emptyset$),
(6) $\exists P \in \mathcal{P}(\Omega)$ (resp. $\mathcal{P}_{++}(\Omega)$) $P \leq f$, i.e., $x \cdot P \leq f(x)$ for all $x \in \mathbb{R}^\Omega$,
and we prove the following implications:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (1).$$

Proof • [(1) $\Rightarrow$ (2)] Let $\theta_A \geq 0$ ($A \subseteq \Omega$) such that $\sum_{A\subseteq\Omega} \theta_A \mathbf{1}_A \geq \mathbf{1}_\Omega$. From AF_+, using the fact that f is positively homogeneous and $f(-\mathbf{1}_\Omega) = -f(\mathbf{1}_\Omega)$ since f is translation invariant, one gets
$$0 \leq \sum_{A\subseteq\Omega} f(\theta_A \mathbf{1}_A) + f(-\mathbf{1}_\Omega) = \sum_{A\subseteq\Omega} \theta_A f(\mathbf{1}_A) - f(\mathbf{1}_\Omega)$$
$$= \sum_{A\subseteq\Omega} \theta_A v_f(A) - v_f(\Omega).$$
When f satisfies AF_{++}, the proof is similar. □

• [(2) $\Rightarrow$ (3)] ⋆ [$c(x) \geq 0$ for all $x \geq 0$] For all $x \in \mathbb{R}^\Omega_+$, we notice that:
$$c(x) := \sup\{x \cdot \mu \ :\ \mu \geq 0,\ \forall A \subsetneq \Omega, \mathbf{1}_A \cdot \mu \leq v_f(A),\ \mathbf{1}_\Omega \cdot \mu = v_f(\Omega)\}$$
is the value of a linear programming problem whose dual is defined by:
$$c^*(x) := \inf\Big\{ \sum_{A\subseteq\Omega} \theta_A v_f(A) \ :\ \forall A \subsetneq \Omega, \theta_A \geq 0, \theta_\Omega \in \mathbb{R}, \sum_{A\subseteq\Omega} \theta_A \mathbf{1}_A \geq x\Big\}.$$
We claim that $0 \leq c^*(x) < +\infty$ for all $x \geq 0$. Indeed, first we have:
$c^*(x) \leq \overline{\theta}_\Omega v_f(\Omega) < +\infty$, with $\overline{\theta}_\Omega := \max\{x(\omega) : \omega \in \Omega\}$ since $x \leq \overline{\theta}_\Omega \mathbf{1}_\Omega$.

We now show that $c^*(x) \geq 0$. Indeed, let $\theta_A \geq 0$ for $A \subsetneq \Omega$, and let $\theta_\Omega \in \mathbb{R}$ such that $\sum_{A\subseteq\Omega} \theta_A \mathbf{1}_A \geq x \geq 0$. Let $t > \max\{0, -\theta_\Omega\}$ one has $\sum_{A\subseteq\Omega} \theta_A \mathbf{1}_A + t\mathbf{1}_\Omega \geq t\mathbf{1}_\Omega$, thus $\sum_{A\subsetneq\Omega} (\theta_A/t)\mathbf{1}_A + (\theta_\Omega/t+1)\mathbf{1}_\Omega \geq \mathbf{1}_\Omega$. Thus the Balancedness Condition implies
$$\sum_{A\subsetneq\Omega} (\theta_A/t) v_f(A) + (\theta_\Omega/t + 1) v_f(\Omega) \geq v_f(\Omega).$$
Thus, $\sum_{A\subseteq\Omega} \theta_A v_f(A) \geq 0$, hence $c^*(x) \geq 0$ and the claim is proved.

Since the value $c^*(x)$ of the Dual Problem is finite, by the Strong Duality Theorem of Linear Programming, the values of the primal and the dual problems are equal. Hence $c(x) = c^*(x) \geq 0$. □

⋆ [$c(x) > 0$ for all $x > 0$] Let $x > 0$. We have proved previously that $c(x) = c^*(x)$ and is finite, thus the dual problem has a solution, by the Strong Duality Theorem of Linear Programming. Hence, there exist $\theta_A \geq 0$ ($A \subsetneq \Omega$), $\theta_\Omega \in \mathbb{R}$ such that $\sum_{A\subseteq\Omega} \theta_A \mathbf{1}_A \geq x > 0$ and $c^*(x) = \sum_{A\subseteq\Omega} \theta_A v_f(A)$. Let $t > \max\{0, -\theta_\Omega\}$ one has $\sum_{A\subsetneq\Omega} (\theta_A/t)\mathbf{1}_A + (\theta_\Omega/t + 1)\mathbf{1}_\Omega > \mathbf{1}_\Omega$. Thus the Strong Balancedness Condition implies
$$\sum_{A\subsetneq\Omega} (\theta_A/t) v_f(A) + (\theta_\Omega/t + 1) v_f(\Omega) > v_f(\Omega).$$
Hence, $c(x) = c^*(x) = \sum_{A\subseteq\Omega} \theta_A v_f(A) > 0$. □

• [(3) $\Rightarrow$ (4)]. It follows from (3) by taking $x := \mathbf{1}_A$ for $A \subsetneq \Omega$. □

• [(4) $\Rightarrow$ (5)]. ⋆ [$\mathrm{core}(f) \neq \emptyset$] From (4), taking $A = \Omega$, one has:

$0 \le v^e_f(\Omega) := \sup\{\mu(\Omega) \,:\, \mu \in \text{core}(f)\}$.

Thus, $v^e_f(\Omega) \neq -\infty$, that is, $\text{core}(f) \neq \emptyset$. □

⋆ [$\text{core}(f) \cap \mathbb{R}^{\Omega}_{++} \neq \emptyset$] From (4), we deduce that $v^e_f(\{\omega\}) > 0$ for all $\omega \in \Omega$. Thus, there exists $\mu^{\omega} \in \text{core}(f)$, which is nonempty and compact, such that:

$\mathbf{1}_{\omega} \cdot \mu^{\omega} = \sup\left\{\mathbf{1}_{\omega} \cdot \mu : \mu \in \text{core}(f)\right\} = v^e_f(\{\omega\} > 0$ for all ω.

We end the proof by showing that $\mu := (1/\#\Omega)\sum_{\omega\in\Omega}\mu^{\omega} \in \text{core}(f) \cap \mathbb{R}^{\Omega}_{++}$.

Indeed, first $\mu \in \text{core}(f)$ since μ is a convex combination of $\mu^{\omega} \in \text{core}(f)$ ($\omega \in \Omega$) and $\text{core}(f)$ is clearly convex. Second, $\mu \gg 0$ since, for all ω, $\mathbf{1}_{\omega} \cdot \mu \ge (1/\#\Omega)\mathbf{1}_{\omega} \cdot \mu^{\omega} > 0$ recalling that $\mu^{\omega'} \in \text{core}(f) \subseteq \mathbb{R}^{\Omega}_{+}$ for all ω'. □

• [(5) ⇒ (6)] Let $P \in \text{core}(f) \subseteq \mathcal{P}(\Omega)$, then, for all $A \subseteq \Omega$, $P(A) \le v_f(A) := f(\mathbf{1}_A)$, and $P(\Omega) = v_f(\Omega) = 1$. Let $x : \Omega \to \mathbb{R}$ and assume that its set of values $\{x(\omega) : \omega \in \Omega\} = \{x_1, \ldots, x_K\}$ is ranked decreasingly, that is, $x_1 > \cdots > x_k > \cdots > x_K$. We let $A_k := \{\omega \in \Omega : x(\omega) = x_k\}$, and we have:

$$
\begin{aligned}
f(x) &= \textstyle\int^C_{\Omega} x dv_f := \sum_{k=1}^{K-1}(x_k - x_{k+1})v_f(A_1 \cup \cdots \cup A_k) + x_K v_f(\Omega)\\
&\ge \textstyle\sum_{k=1}^{K-1}(x_k - x_{k+1})P(A_1 \cup \cdots \cup A_k) + x_K P(\Omega) \text{ [since } P \in \text{core}(f)]\\
&= x \cdot P \text{ [since } P \text{ is additive]}.
\end{aligned}
$$

Thus $P \in \mathcal{P}(\Omega)$ and $P \le f$, which ends the first part of the proof.

We now let $P \in \text{core}(f) \cap \mathbb{R}^{\Omega}_{++} \neq \emptyset$ by (5). Then clearly $P \in \mathcal{P}(\Omega) \cap \mathbb{R}^{\Omega}_{++}$ and $P \le f$, from the first part of the proof since $P \in \text{core}(f)$. □

• [(6) ⇒ (1)] From (6), there exists $P \in \mathcal{P}(\Omega)$ (resp. $\mathcal{P}(\Omega) \cap \mathbb{R}^{\Omega}_{++}$) such that $f(x) \ge x \cdot P$ for all x. Let $K \in \mathbb{N}$, and let $x_k \in \mathbb{R}^{\Omega}_{+} \cup -\mathbb{R}^{\Omega}_{+}$ ($k = 1, \ldots, K$) such that $x_1 + \cdots + x_K \ge 0$ (resp. > 0). Then $f(x_k) \ge x_k \cdot P$ for all $k = 1, \ldots, K$ and summing up we get:

$f(x_1) + \cdots + f(x_K) \ge (x_1 + \cdots + x_K) \cdot P \ge 0$ (resp. > 0 since $P \gg 0$). □

We end the section with a remark.

Remark 3 (Arbitrage-free and Balancedness Conditions) In the proof of Theorem 3 we have also shown that the following Arbitrage-free and Balancedness Conditions are equivalent:

- f satisfies the Arbitrage-free Condition AF_{+};
- (Balancedness) $\forall\theta_A \ge 0$, $\sum_{A\subseteq\Omega}\theta_A\mathbf{1}_A = \mathbf{1}_{\Omega} \Rightarrow \sum_{A\subseteq\Omega}\theta_A v_f(A) \ge v_f(\Omega)$;
- (Balancedness$_{+}$) $\forall\theta_A \ge 0$, $\sum_{A\subseteq\Omega}\theta_A\mathbf{1}_A \ge \mathbf{1}_{\Omega} \Rightarrow \sum_{A\subseteq\Omega}\theta_A v_f(A) \ge v_f(\Omega)$.

The two Balancedness Conditions, which are equivalent since f is monotone, characterize the nonemptyness of $\text{core}(v_f)$ from Bondareva (1963), and Shapley (1967). and they can be interpreted as follows. If the bond $\mathbf{1}_{\Omega}$ is sold as a whole and bought as parts that are fractions of event payoffs, $\theta_A\mathbf{1}_A$, then the aggregate cost of buying the parts cannot be smaller than the gain from selling $\mathbf{1}_{\Omega}$, that is, $v_f(\Omega) = f(\mathbf{1}_{\Omega}) = -f(-\mathbf{1}_{\Omega})$.

Similarly, the two following conditions are also equivalent:

- f satisfies the Arbitrage-free Condition AF_{++};
- (Balancedness$_{++}$) $\forall\theta_A \ge 0$, $\sum_{A\subseteq\Omega}\theta_A\mathbf{1}_A > \mathbf{1}_{\Omega} \Rightarrow \sum_{A\subseteq\Omega}\theta_A v_f(A) > v_f(\Omega)$. □

4 Conclusion and appendix

This paper studies the properties of the risk-neutral *nonadditive* probability v_f associated with a Choquet pricing rule f following Cerreia-Vioglio et al. (2015a) (Theorem 1) in the absence of subadditivity assumption on the pricing rule. In this framework, the usual conditions of absence of arbitrage opportunities do not guarantee the nonemptyness of the core of the risk-neutral *nonadditive* probability v_f associated with the pricing rule f as shown in Example 1 hereafter.

Second, Theorem 2, characterizes the Positive Bid-Ask Spread Condition, i.e., $f(-x) + f(x) \geq 0$ for all x, by a weak form of the Call–Put Parity Condition introduced by Chateauneuf et al. (1996). The Positive Bid-Ask Spread Condition can be re-interpreted as the absence of "buy and sell" arbitrage opportunities of order $K = 2$, that is, there is no $x_k \in \mathbb{R}_+^\Omega \cup -\mathbb{R}_+^\Omega$ $(k = 1, \ldots, K)$ such that $x_1 + \cdots + x_K \geq 0$ and $f(x_1) + \cdots + f(x_K) < 0$.

Then Theorem 3, shows that the elimination of all "buy and sell" arbitrage opportunities for *any* order $K \in \mathbb{N}$—as defined previously—characterizes the nonemptyness of the core of v_f. Note that, under the additional assumption that the Choquet pricing rule f is subadditive, then v_f is submodular, hence the core of v_f is nonempty (Shapley 1971) but under the assumption made by Cerreia-Vioglio et al. (2015a), v_f may have an empty core as shown by Example 1.

We then define a risk-neutral probability P as an element of the core of v_f whenever it is nonempty. Moreover, the (standard) expectation of every payoff x with respect to the risk-neutral probability P is below its non-additive expectation with respect to v_f, hence

$$\frac{1}{1+r_f}\int_\Omega x dP \leq \frac{1}{1+r_f}\int_\Omega x dv_f = f(x) \quad \text{for all } x,$$

and the inequality is an equality when f is linear.

Moreover, Theorem 3 characterizes Condition AF_{++}, by the existence of a *strictly positive* risk-neutral (additive) probability P in the core of f. Clearly this latter condition implies that v_f is strictly positive (i.e., positive for all nonempty events) but the converse assertion may not be true (unless f is subadditive), as shown by the following Example 1 below. Finally, Theorem 3 also characterizes Condition AF_{++} (resp. AF_{+}) by the strict positivity (resp. nonnegativity) of the exact cover v_f^e of v_f. In particular, if f is subadditive, then the Choquet pricing rule f is submodular, thus v_f is also submodular, hence exact, that is, $v_f^e = v_f$; consequently, f satisfies AF_{++} (resp. AF_{+}) if and only if v_f is strictly positive (resp. nonnegative).

Finally the following Example 1 exhibits a monotone Choquet pricing rule f such that $\mathrm{core}(f) = \emptyset$, while f satisfies all the Arbitrage-free Conditions of order 1 and 2, together with the Call–Put Parity Condition CPP, and the Put–Call Parity Condition PCP.

Example 1 Let $\Omega := \{1, 2, 3\}$, $v : 2^\Omega \to \mathbb{R}$ defined by $v(\emptyset) = 0$, $v(\Omega) = 1$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 1/4$, $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 3/4$, and let

$f : \mathbb{R}^\Omega \to \mathbb{R}$ be the Choquet pricing rule defined by $f(x) := \int_\Omega x dv$. Then, the following holds:

(i) f is monotone.
(ii) $\text{core}(f) = \emptyset$, hence f does not satisfy AF_+ and AF_{++}.
(iii) f does not have arbitrage opportunities of order 1 and 2, that is
  ⋆ $[f(x) \geq 0$ for all $x \geq 0]$ and $[f(x) > 0$ for all $x > 0]$;
  ⋆ f satisfies $f(x) + f(-x) = 0$ for all x;
  ⋆ $x_1 + x_2 \geq 0 \Rightarrow f(x_1) + f(x_2) \geq 0$.
(iv) f satisfies PCP.
(v) f satisfies CPP.
(vi) f is not subadditive (hence f is not linear).

Proof (i) f is monotone since v is monotone.
(ii) $\text{core}(f) = \emptyset$ since $\mu \in \text{core}(f)$ must satisfy $\mu_i \leq 1/4$ $(i = 1, 2, 3)$ and $\mu_1 + \mu_2 + \mu_3 = 1$ which is impossible. The second part follows from Theorem 3 but can be proved directly, taking $x_1 := \mathbf{1}_1$, $x_2 := \mathbf{1}_2$, $x_3 := \mathbf{1}_3$ and $x_4 := -\mathbf{1}_\Omega$.
(iii) The first assertion is clear. The second assertion, that is, $f(x) + f(-x) = 0$ for all x, is a consequence of Theorem 2, since $v(A) + v(A^c) = v(\Omega) = 1$ for all $A \subseteq \Omega$. To prove the third assertion, let x_1, x_2 such that $x := x_1 + x_2 \geq 0$, then $x_1 + x_2 - x = 0$ implies $f(x_1) + f(x_2) \geq f(x_1) + f(x_2 - x) \geq 0$.
(iv) It follows from Theorem 1 since f is a monotone Choquet pricing rule.
(v) It follows from Theorem 2 since f is a monotone Choquet pricing rule and satisfies $v(A) + v(A^c) = v(\Omega) = 1$ for all $A \subseteq \Omega$.
(vi) Since $f(x_1 + x_2) > f(x_1) + f(x_2)$ with $x_1 := \mathbf{1}_1$ and $x_2 := \mathbf{1}_2$. □

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