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RESEARCH ARTICLE

The Gale–Nikaido–Debreu lemma with discontinuous excess demand

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Abstract
We provide a generalization of the Gale–Nikaido–Debreu's lemma for discontinuous excess demand in the light of recent work on the existence of equilibria in games with discontinuous payoffs. The standard upper hemicontinuity property of the excess demand is replaced by the weaker concept of "continuous inclusion property" introduced by He and Yannelis (J Math Anal Appl 450(2):1421–1433, 2017) and we allow for the cone P of admissible prices to be general enough to cover cases for which commodities cannot be freely disposed of.

Keywords Existence of equilibrium · Debreu–Gale–Nikaido's lemma · Fixed-point

JEL Classification C62 · D50 · D53

1 Introduction

Following the pioneer work by Dasgupta and Maskin (1986) and Reny (1999) on the existence of equilibria in games with discontinuous payoffs, the existence of equilibria has been proved in several models of Social Sciences, in the case of discontinuouss payoffs, preferences, or excess demand functions. He and Yannelis (2017) provide an extensive study of such existence results for a class of correspondences satisfying the "continuous inclusion property," which encompasses the standard notions of upper hemicontinuity and lower hemicontinuity.

The existence of an economic equilibrium was proved by Arrow and Debreu (1954) with the help of Kakutani's fixed-point theorem. A well-known reformulation of the fixed-point argument, with more transparent economic interpretation, has then been provided by Debreu (1956), Gale (1955), and Nikaido (1956). This

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paper provides a generalization of the Gale–Nikaido–Debreu's lemma for excess demand satisfying the "continuous inclusion property" introduced by He and Yannelis (2017) and we allow for the cone P of admissible prices to be a non-degenerate closed, convex, cone of vertex 0 in the finite dimensional commodity space $\mathbb{R}^\ell$. This allows to go beyond the standard free-disposal assumption and it is general enough to cover other cases for which commodities cannot be freely disposed of.

The paper is organized as follows. In Sect. 2, we introduce the "continuous inclusion property" that will be used throughout the paper. The main existence results, Theorems 1 and 2, are stated. Altogether, they cover the cases of all possible non-degenerate, closed, convex, cones P of vertex 0 in the finite dimensional commodity space $\mathbb{R}^\ell$. The first result (Theorem 1), proves the existence of an equilibrium price, whose excess demand meets P^0, the negative polar cone of P. However, the formulation of Theorem 1, differs from the Gale–Nikaido–Debreu since the equilibrium price is proved to exist in $\text{co}\,[P \cap S]$ (the convex hull of $P \cap S$, instead of $P \cap S$), thus may be zero. Our first result provides a transparent economic interpretation of the use of a fixed-point theorem, and it explicits the role of the assumptions made on the cone P of admissible prices that is needed to guarantee that the equilibrium price is nonzero. The proof of Theorem 1 relies on Kakutani's theorem and is in fact equivalent to it, as shown in Remark 2. We also provide an example showing that, under the assumptions of Theorem 1, there may not exist any equilibrium price in $P \cap S$ (as in Theorem 2).

Our second result (Theorem 2) will then show that, if we rule out the case where P is a vector space, the equilibrium price belongs to $P \cap S$, hence is nonzero. Thus Theorem 2 is in the same spirit of and generalizes the standard result by Debreu (1956), Gale (1955) and Nikaido (1956), and further generalizations by Florenzano (1982), Florenzano and Levan (1986), and Krasa and Yannelis (1994), all dealing with upper hemicontinuous excess demand. Our result is not directly comparable with a previous result by He and Yannelis (2017) for excess demand correspondences satisfying the "continuous inclusion property;" they assume the stronger assumption that the cone P is pointed but they allow for infinite dimensional spaces. Finally, Maskin and Roberts (2008) and Tian (2016) provides also extensions of the Gale–Nikaido–Debreu's lemma but for classes of continuous and discontinuous correspondences that are not directly comparable to the one considered here.

In Sect. 3, we first provide the proof of Theorem 1. Then, we deduce from Theorem 1, the proof of Theorem 2 after showing that $P \cap S$ is the continuous retraction of $\text{co}\,[P \cap S]$. The paper ends with some concluding remarks in Sect. 4.

2 Definitions and main results

2.1 General notations and definitions

Hereafter, we shall use the following notations. In $\mathbb{R}^\ell$, let $x = (x_1, \dots, x_\ell)$, $y = (y_1, \dots, y_\ell)$, we denote by $x \cdot y := \sum_{i=1}^{\ell} x_i y_i$, the dot product, by $\|x\| := (x \cdot x)^{1/2}$ the Euclidean norm, by $B = \{x \in \mathbb{R}^\ell \mid \|x\| \leq 1\}$, the closed unit ball and by $S = \{x \in \mathbb{R}^\ell \mid \|x\| = 1\}$, the unit sphere. The notation $x \geq y$ means that $x_i \geq y_i$ for every coordinate i and we let $\mathbb{R}^\ell_+ = \{x \in \mathbb{R}^\ell \mid x \geq 0\}$. For $P \subseteq \mathbb{R}^\ell$ we denote by int P, cl P, co P, and P^0, respectively the interior, the closure, the convex hull, and the negative polar cone of P, that is, $P^0 = \{x \in \mathbb{R}^\ell \mid x \cdot p \leq 0 \text{ for all } p \in P\}$.

Let Z be a correspondence, from $C \subseteq \mathbb{R}^\ell$ to $\mathbb{R}^\ell$, that is, a mapping from C to the set of all subsets of $\mathbb{R}^m$. Then Z is said to be upper hemicontinuous if the upper inverse $\{p \in C : Z(p) \subseteq O\}$ is open in C (for its relative topology) for all open set $O \subseteq \mathbb{R}^\ell$, and Z is said to be lower hemicontinuous if the lower inverse $\{p \in C : Z(p) \cap O \neq \emptyset\}$ is open in C, for all open set $O \subseteq \mathbb{R}^\ell$.

2.2 Continuous inclusion property

Following He and Yannelis (2017), the notions of upper hemicontinuity and lower hemicontinuity are weakened as follows. A correspondence Z, from $C \subseteq \mathbb{R}^\ell$ to $\mathbb{R}^\ell$, is said to satisfy the *continuous inclusion property* if there exists a family of correspondences Z_i ($i \in I$), from $O_i \subseteq C$ to $\mathbb{R}^\ell$, which are upper hemicontinuous local selections of Z, and the family of their domains $(O_i)_i$ is an open covering of C. Formally we assume that

- Z_i is upper hemicontinuous with nonempty values, for all $i \in I$;
- $Z_i(p) \subseteq Z(p)$ for all $p \in O_i$, and all $i \in I$;
- O_i is open in C (for its relative topology) for all $i \in I$, and $C = \cup_{i \in I} O_i$.

We now provide three examples of correspondences Z, from $C \subseteq \mathbb{R}^\ell$ to $\mathbb{R}^\ell$, satisfying the continuous inclusion property:

- Z is upper hemicontinuous on C with nonempty values;
- Z admits local continuous selections $z_i : O_i \to \mathbb{R}^\ell$ ($i \in I$) with $(O_i)_{i \in I}$ being an open covering of C;
- Z is lower hemicontinuous on C, with nonempty, convex values.

In the first case, we can choose $Z_i = Z$, and $O_i = C$ for all $i \in I$, thus the continuous inclusion property is satisfied. The second example is a particular case of the first one, though an important one. The latter case follows from Michael's selection theorem (Michael 1956), which states that, in a finite dimensional space, every lower hemicontinuous correspondence Z, from $C \subseteq \mathbb{R}^\ell$ to $\mathbb{R}^\ell$, with nonempty convex values has a continuous selection; hence the continuous inclusion property is satisfied.

For a thorough study of correspondences satisfying the Continuous Inclusion Property, we refer to He and Yannelis (2017).

2.3 The main results

In equilibrium theory, equilibrium prices are required to belong to a given set P, assumed to be a closed convex cone of vertex 0, which is non-degenerate, i.e.,

$P \neq \{0\}$.[1] The excess demand correspondence Z is defined on the set $C := \text{co}\,[P \cap S]$ in our first result (and later on $P \cap S$ in our second result), the correspondence Z is assumed to satisfy the continuous inclusion property for the family $(Z_i)_{i \in I}$, and Z is assume to satisfy Walras' law on $P \cap S$, in the sense that each Z_i satisfies Walras' law on $P \cap S \cap O_i$:

$$\forall i \in I,\ \forall p \in P \cap S \cap O_i,\ \exists z_i \in Z_i(p),\ p \cdot z_i \leq 0.$$

We note that, when $Z_i = Z$ and $O_i = C$ for all i, then the above definition coincides with the following Walras' law:

$$\forall p \in P \cap S,\ \exists z \in Z(p),\ p \cdot z \leq 0,$$

for which the above inequality is asked to hold for some $z \in Z(p)$ (instead of all $z \in Z(p)$) and we refer to Section 4 for further discussion on Walras' law.

We can now state our first result, which differs from the Gale–Nikaido–Debreu' lemma, since the equilibrium price is proved to exist in $\text{co}\,[P \cap S]$ (instead of $P \cap S$), hence may be zero. Our second result will then show that, under additional assumption on the cone P of prices, the equilibrium price belongs to $P \cap S$, hence is nonzero.

Theorem 1 *Let $P \subseteq \mathbb{R}^\ell$ be a nondegenerate,[2] closed, convex, cone, with vertex* 0. *Let Z be a correspondence, from* $\text{co}\,[P \cap S]$ *to $\mathbb{R}^\ell$, with nonempty, convex, compact values, satisfying the continuous inclusion property and Walras' law on $P \cap S$. Then there exists $p^* \in$* $\text{co}\,[P \cap S]$ *such that $Z(p^*) \cap P^0 \neq \emptyset$.*

The proof of Theorem 1 is given in Sect. 3.1. It is a consequence of Kakutani's theorem, and we will also show (see Remark 2) that Kakutani's theorem is a consequence of Theorem 1. We also point out that, under the assumptions of Theorem 1, there may not exist any equilibrium price in $P \cap S$ and some (or all) equilibria may be zero.[3]

Our next result will make an additional assumption on the cone P to guarantee the existence of a nonzero equilibrium price. As in Debreu (1956), we now assume that P is not a vector space. This covers two important cases usually considered in the literature, e.g., $P = \mathbb{R}^\ell_+$, or P is a nondegenerate, closed, convex cone of $\mathbb{R}^\ell$, which is pointed, i.e., $P \cap -P = \{0\}$.

Theorem 2 *Let $P \subseteq \mathbb{R}^\ell$ be a closed convex cone with vertex* 0, *which is not a vector space. Let Z be a correspondence from $P \cap S$ to $\mathbb{R}^\ell$ with nonempty, convex, compact*

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[1] Assume, for example, that the total production set $Y \subseteq \mathbb{R}^\ell$ satisfies the free disposal assumption $Y - P^0 \subseteq Y$, for some closed convex cone $P \subseteq \mathbb{R}^\ell$; then profit maximization leads to prices belonging to the given cone P.

[2] When P is degenerate, that is, $P = \{0\}$, then $\text{co}\,[P \cap S] = \emptyset$ and the result cannot hold.

[3] Take $P = \mathbb{R}^\ell$ and consider the (single-valued) correspondence $Z : B \to \mathbb{R}^\ell$ defined by $Z(p) := \{-p\}$. Then $\emptyset \neq Z(p^*) \cap P^0$ if and only if $p^* = 0$, which is the unique equilibrium point of Z in the unit ball $B = \text{co}\,[P \cap S]$.

values, satisfying the continuous inclusion property and Walras' law on $P \cap S$. Then there exists $p^* \in P \cap S$ such that $Z(p^*) \cap P^0 \neq \emptyset$.

The proof of Theorem 2 is given in Sect. 3.2 as a consequence of Theorem 1. In order to use Theorem 1, we will show that the set $P \cap S$ is the continuous retraction of the convex compact set co $[P \cap S]$.

Theorem 2 generalizes the standard result by Debreu (1956), Gale (1955) and Nikaido (1956), together with further generalizations by Florenzano (1982), Florenzano and Levan (1986), and Krasa and Yannelis (1994), all dealing with upper hemicontinuous excess demand. For excess demand satisfying the continuous inclusion property the above theorem is not directly comparable with the result by He and Yannelis (2017) since they consider an infinite dimensional commodity space but the stronger assumption that the cone P of prices is pointed, whereas, in Theorem 2 we only assume that the cone P is not a vector space but belongs to a finite dimensional space $\mathbb{R}^\ell$. We also refer to Maskin and Roberts (2008) and Tian (2016) who provide also extensions of the Gale–Nikaido–Debreu's lemma but for classes of continuous and discontinuous correspondences that are not directly comparable to the one considered here.

3 Proofs

3.1 Proof of Theorem 1

Step 1: A selection result

We first prove a lemma that will be used in both proofs of Theorems 1 and 2. It shows that every correspondence Z, defined on a compact set C and satisfying the Continuous Inclusion Property, admits an upper hemicontinuous (global) selection with nonempty values, that satisfies Walras' law if Z satisfies Walras' law.[4]

Lemma 1 *Let $C \subseteq \mathbb{R}^\ell$ be compact and let Z be a correspondence, from C to $\mathbb{R}^\ell$, satisfying the continuous inclusion property for the family $(Z_i)_{i \in I}$.*

Then, there exists a correspondence $\tilde{Z}$ from C to $\mathbb{R}^\ell$ such that:

- $\emptyset \neq \tilde{Z}(p) \subseteq Z(p)$ for all $p \in C$;
- $\tilde{Z}$ is upper hemicontinuous on C;
- Let P be a nondegenerate, closed convex cone of vertex 0, and let Z satisfy the assumption of Theorem 1 with $C = \text{co}\,[P \cap S]$ (resp. Theorem 2 with $C = P \cap S$). Then $\tilde{Z}$ satisfies Walras' law on $P \cap S$.
- $\tilde{Z}$ has convex, compact values if Z has convex, compact values.

[4] The first idea to get a global selection is to define $\bar{Z}$ as follows: $\bar{Z}(p) := \cup_{\{i \in I : p \in O_i\}} Z_i(p)$. However, the correspondence $\bar{Z}$ may not be upper hemicontinuous.

Proof Let $C \subseteq \mathbb{R}^\ell$ be compact and let Z be a correspondence satisfying the continuous inclusion property for the family $(Z_i)_{i\in I}$. Since the family $(O_i)_{i\in I}$ is an open covering of the compact set C, there exists a finite set $J \subseteq I$ such that $C = \cup_{i\in J} O_i$. Moreover, there exists a closed refinement $(C_i)_{i\in J}$ of the covering $(O_i)_{i\in J}$, that is, for all $i \in J$, $C_i \subseteq O_i$, C_i is closed and $C = \cup_{i\in J} C_i$. See Lemma 1 of Michael (1953).

We define the correspondences $\tilde{Z}_i$ $(i \in J)$ and $\tilde{Z}$, from C to $\mathbb{R}^\ell$, by

$$\tilde{Z}_i(p) := Z_i(p) \text{ if } p \in C_i, \text{ and } \tilde{Z}_i(p) = \emptyset \text{ otherwise,}$$
$$\tilde{Z}(p) := \cup_{i\in J}\tilde{Z}_i(p) \subseteq Z(p) \text{ for } p \in C.$$

- Clearly, $\tilde{Z}(p) \subseteq Z(p)$ for all $p \in C$. We now prove that $\tilde{Z}$ has nonempty values. Indeed, let $p \in C = \cup_{i\in J} C_i$. Then $p \in C_i \subseteq O_i$ for some $i \in J$. Hence, $\tilde{Z}_i(p) := Z_i(p) \neq \emptyset$ since $p \in O_i$.
- We now prove that $\tilde{Z}$ is upper hemicontinuous on C. Indeed, each correspondence $\tilde{Z}_i$ is clearly upper hemicontinuous on C since Z_i is upper hemicontinuous on $O_i \supseteq C_i$ and C_i is closed. Thus, the finite union of such correspondences $\cup_{i\in J}\tilde{Z}_i(p)$ is clearly upper hemicontinuous on C.
- Assume that Z satisfies Walras' law for the family $(Z_i)_{i\in I}$, that is

  $$\text{(W) } \forall i \in I,\ \forall p \in P \cap S \cap O_i,\ \exists z_i \in Z_i(p),\ p \cdot z_i \le 0.$$

  We now show that $\tilde{Z}$ satisfies (the standard) Walras' law. Indeed, let $p \in P \cap S \subseteq \cup_{i\in J} C_i$. Then $p \in C_i \subseteq O_i$ for some $i \in J$. Hence $\tilde{Z}_i(p) := Z_i(p) \neq \emptyset$. From (W), there exists $z_i \in Z_i(p)$, $p \cdot z_i \le 0$ and

  $$z_i \in Z_i(p) = \tilde{Z}_i(p) \subseteq \cup_{i\in J}\tilde{Z}_i(p) := \tilde{Z}(p).$$

  This proves that $\tilde{Z}$ satisfies also Walras' law.
- Assume that Z has convex, compact values. We need to modify the correspondence $\tilde{Z}$ and define the correspondence $\bar{Z}$, from C to $\mathbb{R}^\ell$ by

  $$\bar{Z}(p) := \text{co cl } \cup_{i\in J} \tilde{Z}_i(p) = \text{co cl}\,\tilde{Z}(p).$$

We now show that $\bar{Z}$ satisfies the conclusion of the lemma. First, for all $p \in C$, $\bar{Z}(p) := \text{co cl } \cup_{i\in J} \tilde{Z}_i(p) \subseteq Z(p)$ since Z has closed, convex values. Second, $\bar{Z}(p)$ is clearly convex and it is compact since it is the convex hull of the compact set $\text{cl } \cup_{i\in J} \tilde{Z}_i(p)$; note that the latter set is compact since it is contained in $Z(p)$ which is compact.

Third, $\bar{Z}$ is upper hemicontinuous on C since $\text{cl}\,\tilde{Z}$ is upper hemicontinuous (as the closure of $\tilde{Z}$ which has been proved to be upper hemicontinuous previously) and $\bar{Z} := \text{co cl}\,\tilde{Z}$ is the convex hull of the upper hemicontinuous correspondence $\text{cl}\,\tilde{Z}$, hence is also upper hemicontinuous.

Finally, $\bar{Z} = \text{co cl}\,\tilde{Z}$ satisfies Walras' law since $\tilde{Z}$ satisfies Walras' law and $\tilde{Z}(p) \subseteq \bar{Z}(p)$ for all p. □

Step 2: Proof of Theorem 1 under strong Walras' law

In view of Lemma 1, we can assume that Z is upper hemicontinuous with nonempty, convex, compact values. The proof below makes the additional assumption that Z satisfies the *strong* Walras' law, that is

$$\forall p \in P \cap S,\ \forall z \in Z(p),\ p \cdot z \leq 0,$$

with the above inequality satisfied for all $z \in Z(p)$ (instead of some $z \in Z(p)$). We let $C := \mathrm{co}\,[P \cap S]$. Then C is clearly convex and compact. It is also nonempty since $P \neq \{0\}$. Since Z is upper hemicontinuous with compact values and C is compact, the set $\cup_{p \in C} Z(p)$ is compact, hence is contained in some nonempty, convex, compact subset K of $\mathbb{R}^{\ell}$. We consider the correspondence F from $C \times K$ to itself defined by

$$F(p, z) = \{\bar{p} \in C \mid \bar{p} \cdot z \geq q \cdot z,\ \forall q \in C\} \times Z(p).$$

One easily checks that F is an upper hemicontinuous correspondence, from $C \times K$ to itself, with nonempty, convex, compact values. Thus, from Kakutani's theorem, the correspondence F has a fixed point (p^*, z^*) in $C \times K$, that is

$$\text{for all } q \in C,\ p^* \cdot z^* \geq q \cdot z^* \text{ and } z^* \in Z(p^*).$$

We now show that $p^* \cdot z^* \leq 0$. Indeed, if $p^* = 0$, this is clearly true. Assume now that $p^* \neq 0$, we claim that $p^*/\|p^*\| \in P \cap S$; indeed, first $p^*/\|p^*\| \in S$ and second, $p^* \in C = \mathrm{co}\,[P \cap S] \subseteq P$ (since P is convex), hence $p^*/\|p^*\| \in P$ (since P is a cone). Consequently, from the *strong* Walras' law, $(p^*/\|p^*\|) \cdot z^* \leq 0$ since $p^*/\|p^*\| \in P \cap S$ and $z^* \in Z(p^*)$. Thus, $p^* \cdot z^* \leq 0$.

Consequently, from the above fixed-point condition, we deduce that $q \cdot z^* \leq p^* \cdot z^* \leq 0$ for every $q \in C$. Clearly the inequality still holds for every $q \in P$; indeed, for $q \in P \setminus \{0\}$, one has $q/\|q\| \in P \cap S \subseteq C$; thus, $(q/\|q\|) \cdot z^* \leq 0$. Hence $q \cdot z^* \leq 0$ for all $q \in P$. Thus, $z^* \in P^0$, which ends the proof Theorem 2.

Step 3: Proof of Theorem 1 in the general case

We now assume that Z satisfies Walras' law (as in Theorem 1) and we approximate the correspondence Z by a sequence of correspondences Z_n satisfying the strong version of Walras' law. Applying Step 2, and going to the limit when $n \to \infty$, will allow us to conclude the proof of Theorem 1.[5]

We define the correspondences Z_n $(n \geq 2)$, from $\mathrm{co}\,[P \cap S]$ to $\mathbb{R}^{\ell}$, by

$$Z_n(p) := \begin{cases} Z(p) & \text{if } \|p\| < 1 - 1/n, \\ \mathrm{co}\left[Z(p) \cup \left\{z \in Z(p/\|p\|) \,:\, p \cdot z \leq 0\right\}\right] & \text{if } \|p\| = 1 - 1/n, \\ \left\{z \in Z(p/\|p\|) \,:\, p \cdot z \leq 0\right\} & \text{if } 1 - 1/n < \|p\| \leq 1. \end{cases}$$

[5] Note that this step is not needed in the proof of Theorem 2, which only needs to consider Step 2 with Z satisfying the Strong Walras' law.

Then, one checks that Z_n is upper hemicontinuous with nonempty convex compact values and satisfies the strong Walras' law. Moreover, there exists some convex compact set K such that $Z_n(p) \subseteq K$ for all p and all n, since the correspondence Z is bounded on the compact set $\mathrm{co}\,[P \cap S]$.

Thus, from Step 2, there exists $(p_n, z_n) \in \mathrm{co}\,[P \cap S] \times K$ such that $z_n \in Z_n(p_n) \cap P^0 \neq \emptyset$. Without any loss of generality, we can assume that $(p_n, z_n) \to (p^*, z^*)$ for some $(p^*, z^*) \in \mathrm{co}\,[P \cap S] \times K$.

Clearly, one has $z^* \in P^0$. We end the proof by showing that $z^* \in Z(p^*)$. Indeed, assume first that $\|p^*\| = 1$, then for n large enough, $p_n \neq 0$ and $z_n \in Z_n(p_n) \subseteq \mathrm{co}\left[Z(p_n) \cup [Z(p_n/\|p_n\|)\right]$; thus, at the limit, when $n \to \infty$, $z^* \in Z(p^*)$. Assume now that $\|p^*\| < 1$. Then, for n large enough, $\|p_n\| < 1 - 1/n$. Thus, $z_n \in Z_n(p_n) = Z(p_n)$. At the limit when $n \to \infty$, one gets $z^* \in Z(p^*)$. □

3.2 Proof of Theorem 2

Step 1 In order to use a fixed-point theorem on the set $P \cap S$, we first prove that $P \cap S$ is a continuous retraction of its convex hull $\mathrm{co}\,[P \cap S]$. The proof below will also prove that $P \cap S$ is homeomorphic to a nonempty, convex, compact set.

Lemma 2 *Let $P \subseteq \mathbb{R}^\ell$ be a closed, convex, cone with vertex* 0, *which is not a vector space. Then there exists a continuous mapping $r : \mathrm{co}\,[P \cap S] \to P \cap S$ such that $r(p) = p$ for all $p \in P \cap S$.*

We prepare the proof with two claims.

Claim 1 *There exists $e \in P \cap -P^0$, $\|e\| = 1$.*

Proof We first prove that $\mathrm{ri}\,P \cap -P^0 \neq \emptyset$, where $\mathrm{ri}\,P$ denotes the relative interior of the convex set P. Suppose on the contrary that $\mathrm{ri}\,P \cap -P^0 = \emptyset$, then we can separate the two nonempty convex sets $\mathrm{ri}\,P$ and $-P^0$, that is:

$$\exists u \neq 0, \alpha := \sup\{u \cdot q \mid q \in \mathrm{ri}\,P\} \leq \beta := \inf\{u \cdot q' \mid q' \in -P^0\}.$$

Since $\mathrm{ri}\,P$ is a cone, we deduce that $\alpha = 0$. Thus, $u \cdot q \leq \alpha = 0$ for all $q \in \mathrm{ri}\,P$, hence also for all $q \in \mathrm{cl}\,\mathrm{ri}\,(P) = P$ since $\mathrm{cl}\,\mathrm{ri}\,(P) = \mathrm{cl}\,P$ by Rockafellar (1970) and $\mathrm{cl}\,P = P$ since P is closed. Thus, $u \in P^0$.

Similarly, one has $\beta = 0$ since P^0 is also a cone. Thus, $u \in P^{00} = P$ from the bipolar theorem (see Rockafellar 1970) since P is a closed convex cone. We have thus proved that $u \in P \cap P^0$. But $P \cap P^0 = \{0\}$, hence $u = 0$, a contradiction with $u \neq 0$.

Thus, we can choose $e \in \mathrm{ri}\,P \cap -P^0 \neq \emptyset$ and it suffices to show that $e \neq 0$ (in which case $e/\|e\| \in \mathrm{ri}\,P \cap -P^0$ which is a cone). Suppose on the contrary that $e = 0$, then $0 \in \mathrm{ri}\,P$. We now show that P is a vector space. Indeed, since $0 \in \mathrm{ri}\,P$, one has $\varepsilon B \cap < P > \subseteq P$ for some $\varepsilon > 0$. Here, $< P >$ denotes the vector space spanned by P (which is also the affine space spanned by P since $0 \in P$). Thus, for every $p \in P \setminus \{0\}$ one has $-\varepsilon p/\|p\| \in \varepsilon B \cap < P > \subseteq P$. Hence $-p \in P$ since P is a cone. Thus P is a vector space, which contradicts the assumption made in the lemma. □

Let e be as in Claim 1 and let $\pi : \mathbb{R}^\ell \to e^\perp$ be the orthogonal projection mapping from $\mathbb{R}^\ell$ to $e^\perp$, that is π is defined by $\pi(x) = x - e\,(x \cdot e)$ for all x. We denote by $\pi|_{P\cap S}$ the restriction of π to $P \cap S$.

Claim 2 *The mapping* $\pi|_{P\cap S} : P \cap S \to \pi(P \cap S)$ *is a homeomorphism and* $\pi(P \cap S) = \pi(\mathrm{co}\,[P \cap S])$, *thus is nonempty, convex, and compact.*

Proof The proof is a consequence of the following claims.

- $\Delta := \pi(P \cap S)$ *is nonempty, convex, and compact.* Indeed, Δ nonempty since the cone P is non-degenerate (recalling that P is not a vector space by assumption). The set Δ is clearly compact since $P \cap S$ is compact and π is continuous. We now show that Δ is convex. Let $\delta_i \in \Delta$ $(i = 1, 2)$, then $\delta_i = \pi(x_i)$ for some $x_i \in P \cap S$ $(i = 1, 2)$. Then, for every $\lambda \in [0, 1]$, and every $t \geq 0$ one has

$$\begin{aligned}\lambda\delta_1 + (1-\lambda)\delta_2 &= \lambda\pi(x_1) + (1-\lambda)\pi(x_2) = \pi(\lambda x_1 + (1-\lambda)x_2)\\ &= \pi(\lambda x_1 + (1-\lambda)x_2 + te).\end{aligned}$$

  However, $\lambda x_1 + (1-\lambda)x_2 + te \in P$ since $x_1 \in P, x_2 \in P$, $e \in P$ (from Claim 1) and P is a convex cone. We now show that we can choose $t \geq 0$ so that $\lambda x_1 + (1-\lambda)x_2 + te \in S$, that is, satisfying for $x := \lambda x_1 + (1-\lambda)x_2$:

$$1 = \|x + te\|^2 = \|x\|^2 + 2tx \cdot e + t^2,$$

  which is possible since the second degree equation in t has always a nonegative solution. Consequently, $\lambda\delta_1 + (1-\lambda)\delta_2 \in \pi(P \cap S) = \Delta$. Thus Δ is convex.
- $\pi(P \cap S) = \pi(\mathrm{co}\,[P \cap S])$. We clearly have $\pi(P \cap S) \subseteq \pi(\mathrm{co}\,[P \cap S]) = \mathrm{co}\,\pi(P \cap S)$ since π is linear. But $\mathrm{co}\,\pi(P \cap S) \subseteq \pi(P \cap S)$, since we have proved previously that $\pi(P \cap S)$ is convex. Thus, $\pi(P \cap S) = \pi(\mathrm{co}\,[P \cap S])$.
- $\pi|_{P\cap S} : P \cap S \to \mathbb{R}^\ell$ *is one-to-one.* Indeed, let x_1, x_2 in $P \cap S$ such that $\pi(x_1) = \pi(x_2)$. Then, $0 = \pi(x_1) - \pi(x_2) = \pi(x_1 - x_2)$, which implies that $x_1 - x_2 = te$ for some $t \in \mathbb{R}$. Thus $1 = \|x_1\|^2 = \|x_2 + te\|^2 = 1 + 2tx_2 \cdot e + t^2$. Hence, $t = 0$ or $t = -2x_2 \cdot e$. In the latter case, we get $x_1 \cdot e = (x_2 + te) \cdot e = -x_2 \cdot e$. Recalling that $-e \in P^0$ and x_1, x_2 belong to P, we deduce that $t = -2x_2 \cdot e = 2x_1 \cdot e \in (-\mathbb{R}_+) \cap \mathbb{R}_+ = \{0\}$. Consequently, $x_1 = x_2$. This proves that the mapping $\pi|_{P\cap S}$ is one-to-one.
- $\varphi := \pi|_{P\cap S} : P \cap S \to \pi(P \cap S)$ *is a homeomorphism.* Clearly, the mapping φ is continuous (since π is continuous), onto, and one-to-one from above. From the compactness of $P \cap S$, one deduces that the inverse φ^{-1} of φ is continuous. Hence, φ is a homeomorphism..

□

Proof of Lemma 2 We define $r := (\pi|_{P\cap S})^{-1} \circ \pi|_{\mathrm{co}[P\cap S]}$, that is

$$r : \mathrm{co}\,[P \cap S] \xrightarrow{\pi|_{\mathrm{co}[P\cap S]}} \pi(\mathrm{co}\,[P \cap S]) = \pi(P \cap S) \xrightarrow{(\pi|_{P\cap S})^{-1}} P \cap S.$$

Then r is well defined since, by Claim 2, $\pi(\mathrm{co}\,[P \cap S]) = \pi(P \cap S)$ and the mapping $\pi|_{P\cap S} : P \cap S \to \pi(P \cap S)$ is a homeomorphism. Thus r is clearly continuous and, for all $p \in P \cap S$, $r(p) = p$ □

Step 2: Proof of Theorem 2. In view of Lemma 1, we can assume that Z is upper hemicontinuous, from $P \cap S$ to $\mathbb{R}^\ell$, with nonempty, convex, compact values. Let $r : \mathrm{co}\,[P \cap S] \to P \cap S$ be the continuous retraction defined by Lemma 2. We define the correspondence $\hat{Z}$, from $\mathrm{co}\,[P \cap S]$ to $\mathbb{R}^\ell$ by

$$\hat{Z}(p) := \left\{ z \in Z(r(p)) : r(p) \cdot z \leq 0 \right\}.$$

Since Z satisfies Walras' law, the correspondence $\hat{Z}$ is nonempty valued and satisfies the strong Walras' law. Moreover, $\hat{Z}$ is clearly upper hemicontinuous with compact convex values. Thus, from Theorem 1[6] we deduce the existence of $\bar{p} \in \mathrm{co}\,[P \cap S]$ such that $\hat{Z}(\bar{p}) \cap P^0 \neq \emptyset$. Taking $p^* := r(\bar{p}) \in P \cap S$, we have $\emptyset \neq \hat{Z}(\bar{p}) \cap P^0 \subseteq Z(p^*) \cap P^0$. □

4 Concluding remarks

The fundamental result by Debreu (1956), Gale (1955), and Nikaido (1956), had a revival of interest after the fundamental work by Dasgupta and Maskin (1986), Reny (1999), and Reny (2016) on the existence of equilibria in games with discontinuous payoffs. He and Yannelis (2017) provide an extensive study of such existence results together with a generalization of the Gale–Nikaido–Debreu's lemma for a class of correspondences satisfying the Continuous Inclusion Property, and we have borrowed this notion in this paper. Finally, Maskin and Roberts (2008) and Tian (2016) provide also extensions of the Gale–Nikaido–Debreu's lemma but for classes of continuous and discontinuous correspondences that are not directly comparable to the one considered here.

In infinite dimensional spaces, Aliprantis and Brown (1983), Podczeck (1997), Yannelis (1985) have stressed the importance of considering the (weak) version of Walras' law we consider in our paper:

$$\forall p \in P \cap S, \;\; \exists z \in Z(p), \; p \cdot z \leq 0,$$

instead of the standard and stronger version considered by Gale (1955), Nikaido (1956), and Debreu (1956):

[6] In fact, at this stage, we only need Theorem 1 with strong Walras' law, as proved in Step 2 of the proof of Theorem 1.

$$\forall p \in P \cap S,\ \ \forall z \in Z(p),\ p \cdot z \leq 0.$$

We have seen in the proof of Theorem 1, that considering the weak version of Walras' law required a specific argument provided in Step 3. This is not the case for Theorem 2, and the following remark shows that the two formulations of Theorem 2, with the weak and the strong version of Walras' law are equivalent, at least with a finite dimensional commodity space.

Remark 1 In finite dimensional spaces, the formulations of Theorem 2, with the weak and the strong version of Walras' law are equivalent. Indeed, if Z satisfies the weak Walras' law, we can define the correspondence $\tilde{Z}$, by $\tilde{Z}(p) := \{z \in Z(p) : p \cdot z \leq 0\}$ and one checks that $\tilde{Z}$ satisfies the strong Walras' law. Moreover, whenever Z is upper hemicontinuity with nonempty convex compact values, $\tilde{Z}$ satisfies the same property, using the joint continuity of the mapping $(z, p) \to p \cdot z$. Note however that this latter property may not hold in infinite dimensional spaces, hence the importance of considering the weak version of Walras' law. □

The proof of Theorem 1 relies on Kakutani's theorem and the following remark recalls (the known result) that, in fact, Kakutani's theorem is also a direct consequence of Theorem 1.

Remark 2 One easily deduces Kakutani's theorem from Theorem 1 as follows. Let F be an upper hemicontinuous correspondence, from B to B, with nonempty, convex, compact values. Let $P := \mathbb{R}^\ell$ and let Z be the correspondence defined by $Z(p) = F(p) - \{p\}$. If $p \in S$, then for all $z \in Z(p)$, $y := z + p \in F(p) \subseteq B$. Thus, $p \cdot z = p \cdot y - 1$ and $p \cdot y \leq \|p\|\|y\| \leq 1$. Consequently, $p \cdot z \leq 0$, which proves that Z satisfies (the strong) Walras' law. Hence, Z satisfies the assumptions of Theorem 1. Thus, there exists $p^* \in B$ such that $\emptyset \neq Z(p^*) \cap P^0 = Z(p^*) \cap \{0\}$ since $P^0 = (\mathbb{R}^\ell)^0 = \{0\}$. Hence, $0 \in Z(p^*)$ and $p^* \in F(p^*)$. □

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