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RESEARCH ARTICLE

# A semi-uniform-price auction for multiple objects

**Peyman Khezr[1] · Flavio M. Menezes[1]**

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**Abstract**
This paper proposes a semi-uniform payment rule for selling multiple homogeneous objects. Under the proposed auction, all bidders pay a uniform price equal to the highest losing bid, except the bidder with the highest losing bid who, under some circumstances, pays the second highest losing bid. We show that bidders in this auction face an incentive, on the margin, to increase their bids vis-a-vis their bids in a uniform-price auction. This incentive is sufficient to eliminate the zero revenue equilibrium that has been identified in the multiple-object, uniform-price auction literature.

**Keywords** Multi-object · Semi-uniform · Uniform-price · Auctions

**JEL Classification:** D44 · D82 · L10

## 1 Introduction

Uniform-price auctions are widely used in the allocation of multiple objects such as treasury bills, emissions permits, and wholesale electricity. Under a uniform-price auction, all successful bidders pay the same price. While our understanding of equilibrium behavior in uniform-price auctions has evolved considerably since Vickrey's original work, there are still important concerns regarding the existence of undesirable equilibria in these auctions.

For instance, Back and Zender (1993) and Menezes (1995) show that there exist equilibria of uniform-price auctions where the auction price does not exceed the seller's valuation. In the special case where the seller's value is equal to zero, this means that the seller's expected revenue is equal to zero. When bidders seek more than one object, they can impact the uniform price in the auction by bidding less than their value for some

✉ Peyman Khezr
p.khezr@uq.edu.au

Flavio M. Menezes
f.menezes@uq.edu.au

[1] School of Economics, University of Queensland, Brisbane 4072, Australia

of the units. This demand reduction effect, which results in a zero-price equilibrium in extreme cases, is a well-known disadvantage of uniform-price auctions. [1] However, governments often favor uniform-price auctions in solving allocation problems as these auctions have the clear advantage of yielding identical prices in cases where price discrimination is not desirable.

Khezr and Menezes (2017) provided a new characterization of equilibrium bidding behavior in a multi-unit uniform-price auction when bidders demand up to two objects. Under such an equilibrium, bidders with values below a certain threshold bid zero for the second unit, which can result in zero revenue for the seller. Their analysis corrected and completed that of Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998). Despite over 3 decades of research on multi-unit outcomes, there still remains considerable interest on the implementation of uniform-price outcomes (Kagal and Levin 2009).

In this paper, we provide an alternative payment rule that, in most instances, retains the advantage of the uniform-price auction, but breaks down the zero-price equilibrium. In the next section, we provide an example that spells out the intuition for such result. Section 3 then sets up the model, characterizes equilibrium behavior in the semi-uniform auction, and shows that under certain conditions; this auction format dominates the standard uniform-price auction in terms of the seller's expected revenue and efficiency. The conclusion is drawn in Sect. 4.

## 2 An illustrative example

Suppose that a seller has $Q = 4$ identical units of a good for sale to $N = 4$ buyers, who each demands two units. Bidders have privates values for each of the units, which are drawn independently and with equal probability from the set $\{1, 2, \ldots, 10\}$. First, suppose that the objects are sold through a sealed-bid uniform-price auction where each bidder submits two bids, one for each unit, the bidders with the highest four bids are allocated the objects, and all winners pay a uniform price equal to the fifth highest bid.

In this auction, it is a weakly dominant strategy for bidders to submit a bid equal to their value for their first unit. The reason for this is analogous to why bidders bid their valuation in a single object Vickrey auction. By bidding their value for the first unit, they will secure an object if their value is one of the highest four values. Therefore, lowering or increasing their bid for the first unit does not change the probability of winning. However, there are incentives to reduce the second bid below valuations. As Engelbrecht-Wiggans and Kahn (1998) and Khezr and Menezes (2017) show, there is an equilibrium where buyers with values below a certain threshold, bid zero for their second unit.

Suppose that the realized values are as shown in Table 1, where we denote the threshold value, in the context of this example, by $v^*$. Bidders' values for their second units are denoted by $\lambda_i$, $i = 1, \ldots, 4$, and we assume that $\lambda_i \leq v^*$ for every $i$. In this case, there is an equilibrium where all bidders bid zero for their second unit, and

[1] Back and Zender (1993).

**Table 1** An example

| | Bidder 1 | Bidder 2 | Bidder 3 | Bidder 4 |
|---|---|---|---|---|
| $(v_1, v_2)$ | $(5, \lambda_1)$ | $(10, \lambda_2)$ | $(1, \lambda_3)$ | $(9, \lambda_4)$ |
| $(b_1, b_2)$ | (5, 0) | (10, 0) | (1, 0) | (9, 0) |
| Payoffs | 5 | 10 | 1 | 9 |

the auction price is also equal to zero, as illustrated in Table 1. The existence of this type of equilibrium is shown in Engelbrecht-Wiggans and Kahn (1998) and Khezr and Menezes (2017).

*The semi-uniform payment rule*

Now, consider the following alternative payment rule. The bidders with the four highest bids still win. However, the amount which they pay will depend upon whether or not they have also placed the fifth highest bid. A bidder who has placed one of the four highest bids, but not the fifth highest bid, will win one object and pays the fifth highest bid. A bidder with one of the four highest bids, who has also placed the fifth highest bid, will pay the sixth highest bid on their first unit.

In other words, under this semi-uniform payment rule, all successful bidders pay the highest losing bid except the bidder who placed it. That particular bidder pays the second highest losing bid, which cannot be her own bid when she had one of the four highest bids and have secured an object. This is because bidders demand at most two units. Therefore, if a bidder wins one unit, and she also has the highest losing bid, then the second highest losing bid cannot be hers.[2]

We can readily check that this semi-uniform payment rule eliminates the zero-price equilibrium. For instance, in our example, Bidder 2 now has an incentive to bid above zero for her second unit. If all other bidders bid zero for the second unit, then Bidder 2 can increase the likelihood of displacing the bidder with the fourth highest bid by raising her bid for the second unit, and in doing so, she does not increase the price it may pay for her first unit. Given that values are private information, at least in some cases where the fourth highest bid is low enough, Bidder 2 could win the second unit and increase her overall payoff.

In contrast, Bidder 2 faces a risk of raising her bid for a second unit in a standard uniform-price auction, as the price she pays for the first unit could rise, while she may still not win a second item. However, in a semi-uniform case, her bid cannot increase the price on the first unit without winning a second unit.

In the next section, we formally show that the intuition illustrated in the example above holds true, and that a semi-uniform payment rule can potentially eliminate the zero-price equilibrium of uniform-price auctions.

[2] In general, with multiple unit demand, this payment rule can be generalized to the highest losing bid among all bidders except the bidder with the marginal bid.

## 3 Equilibrium of the semi-uniform-price auction

Suppose that there are $Q \geq 2$ identical objects available for sale to $N \geq 2$ potential buyers, and each buyer demands two objects. Demand exceeds supply as $Q < 2N$. Bidder $i$'s valuation for the two units are private information and given by a vector $\mathbf{v^i} = (v_1^i, v_2^i)$.

The vectors $\mathbf{v^i} = (v_1^i, v_2^i)$ are distributed independently and identically according to the distribution function $F(.,.)$ on the set $V = \{\mathbf{v} \in [0, \bar{v}]^2 : v_1^i \geq v_2^i\}$, with positive density $f < \infty$. That is, we assume that bidders' valuations exhibit diminishing marginal values. Each bidder submits two bids and the bidders with the highest $Q$ bids win. Under a uniform-price payment rule, winners pay the $Q+1$ highest bid. In contrast, under the semi-uniform payment rule, winners pay the $Q+1$ highest bid, except the bidder who placed that bid. This bidder would pay the highest losing bid which is not placed by her; namely, she would pay the $Q+2$ highest bid if she had a bid among the $Q$ highest bids. Otherwise, she would not be allocated an object, and the outcome of the auction would be the same as in a standard uniform-price auction.

Suppose that each bidder $i$ submits a vector of bids $\mathbf{b^i} = (b_1^i, b_2^i)$ with $b_1^i \geq b_2^i$. It is straightforward to check that bidders will not bid more than their values for each unit.[3] Let vector $\mathbf{c}^{-i} = (c_1^{-i}, c_2^{-i})$ denote the $Q-1$ and the $Q$ highest bids faced by bidder $i$. These bids determine how many units, if any, bidder $i$ wins. Assume further that $\mathbf{c^{-i}}$ is distributed according to a distribution function $H(.)$ with density $h$.

Consider any candidate symmetric equilibrium $\boldsymbol{\beta} = (\beta_1(\cdot), \beta_2(\cdot))$, where $\beta_1(\cdot)$ and $\beta_2(\cdot)$ are increasing in bidders' valuations for their first and second units, respectively. Assume that every bidder follows $\boldsymbol{\beta}$ except for bidder $i$.

If bidder $i$ bids $\mathbf{b^i} = (b_1^i, b_2^i)$, then her expected payoff under the uniform payment rule is equal to:

$$\Pi^u(\mathbf{v}, \mathbf{b^i}) = \int_{c_1^{-i} < b_2^i} (v_1^i + v_2^i - 2c_1^{-i}) h(\mathbf{c^{-i}}) d\mathbf{c^{-i}} + \int_{c_2^{-i} < b_1^i, c_1^{-i} > b_2^i} (v_1^i - \max\{b_2^i, c_2^{-i}\}) h(\mathbf{c^{-i}}) d\mathbf{c^{-i}}. \tag{1}$$

For the semi-uniform payment rule, the expected payoff is equal to:

$$\Pi^{su}(\mathbf{v}, \mathbf{b^i}) = \int_{c_1^{-i} < b_2^i} (v_1^i + v_2^i - 2c_1^{-i}) h(\mathbf{c^{-i}}) d\mathbf{c^{-i}} + \int_{c_2^{-i} < b_1^i, c_1^{-i} > b_2^i} (v_1^i - c_2^{-i}) h(\mathbf{c^{-i}}) d\mathbf{c^{-i}}. \tag{2}$$

The first integral is the same under both payment rules, and it refers to the case where bidder $i$'s lowest bid (and, therefore her highest bid as well) is larger than the $Q-1$ highest bid among her competitors, $c_1^{-i}$. In this case, bidder $i$ is guaranteed to win two

[3] See Krishna (2002) pages 190–192.

units and pays $c_1^{-i}$, which, in this instance, is equal to the $Q + 1$ highest bid among all bidders including bidder $i$.

The second integral represents the case in which bidder $i$ only wins one unit. This happens under both payment rules when bidder $i$'s highest bid is higher than $c_2^{-i}$, and her lowest bid is lower than $c_1^{-i}$. However, under a uniform payment rule, bidder $i$ pays in this instance the maximum of her second bid and $c_2^{-i}$, whereas under the semi-uniform payment rule, her payment is equal to $c_2^{-i}$.

We denote by $H_1^t$ and $H_2^t$ the marginal distributions of random variables $C_1^{-i}$ and $C_2^{-i}$, respectively, for mechanism $t$. Also denote $h_1^t$ and $h_2^t$ as their densities.[4] We can express the probability of the various possible events in terms of $H_1$ and $H_2$ as follows:

$$\begin{cases} H_1(b_2) & Prob[\text{winning two units}] \\ H_2(b_1) & Prob[\text{winning at least one unit}] \\ H_2(b_1) - H_1(b_2) & Prob[\text{winning exactly one unit}]. \end{cases} \tag{3}$$

Using (3), we can rewrite $i$'s expected payoff for the semi-uniform case, when her bid vector is $\mathbf{b^i}$ and her values given by vector $\mathbf{v}$, as follows:

$$\begin{aligned} \Pi^{su}(\mathbf{v}, \mathbf{b^i}) = H_1(b_2)(v_1^i + v_2^i) - 2\int_0^{b_2} c_1^{-i} h_1(c_1^{-i})\mathrm{d}c_1^{-i} \\ + (H_2(b_1) - H_1(b_2))v_1^i - \int_{b_2}^{b_1} c_2^{-i} h_2(c_2^{-i})\mathrm{d}c_2^{-i} \\ - \int_0^{b_2}\int_{b_2}^{\bar{v}} c_2^{-i} h_1(c_1^{-i}) h_2(c_2^{-i})\mathrm{d}c_1^{-i}\mathrm{d}c_2^{-i}. \end{aligned} \tag{4}$$

The first and second terms of (4) represent $i$s expected value of winning two objects and paying $c_1^{-i}$ for each of the objects. The third term represents $i$'s expected value of winning exactly one object, and the fourth and fifth terms represent the expected payment of bidder $i$ when she wins exactly one unit and pays $c_2^{-i}$. Although the payment is always $c_2^{-i}$ when she wins one unit, there are two possible events. The first event corresponds to the case where her second bid is lower than the two competing bids (fourth term), and the other event corresponds to the case where her second bid is between the two competing bids (fifth term). We can rewrite the terms in (4) as follows:

$$\begin{aligned} \Pi^{su}(\mathbf{v}, \mathbf{b^i}) = H_1(b_2)(v_1^i + v_2^i) - 2\int_0^{b_2} c_1^{-i} h_1(c_1^{-i})\mathrm{d}c_1^{-i} \\ + (H_2(b_1) - H_1(b_2))v_1^i - \int_0^{b_2}\left(\int_{b_2}^{\bar{v}} h_1(c_1^{-i})\mathrm{d}c_1^{-i}\right) c_2^{-i} h_2(c_2^{-i})\mathrm{d}c_2^{-i} \end{aligned}$$

[4] For simplicity, we have omitted the superscript $t$ in what follows. However, we note that these distributions could be different for each of the two mechanisms.

$$= H_1(b_2)(v_1^i + v_2^i) - 2\int_0^{b_2} c_1^{-i} h_1(c_1^{-i}) \mathrm{d}c_1^{-i}$$
$$+ (H_2(b_1) - H_1(b_2))v_1^i - \int_0^{b_2} \big(1 - H_1(b_2)\big) c_2^{-i} h_2(c_2^{-i}) \mathrm{d}c_2^{-i}. \quad (5)$$

Differentiating (5) with respect to $b_1$ and $b_2$ yields the following:

$$\frac{\partial \Pi^{su}}{\partial b_1} = h_2(b_1)v_1^i - b_1 h_2(b_1) \quad (6)$$
$$\frac{\partial \Pi^{su}}{\partial b_2} = h_1(b_2)(v_1^i + v_2^i) - 2b_2 h_1(b_2) - h_1(b_2)v_1^i + h_1(b_2)\mathbb{E}[c_2^{-i} | c_2^{-i} < b_2]$$
$$+ (1 - H_1(b_2))b_2 h_2(b_2). \quad (7)$$

By setting (6) and (7) to zero, we have the following:

$$b_1^i = v_1^i \quad (8)$$

and

$$b_2^i = \frac{v_2^i}{2 - \Theta} + \frac{\tilde{c}_2^{-i}}{2 - \Theta}, \quad (9)$$

where $\Theta = (1 - H_1(b_2))\frac{h_2(b_2)}{h_1(b_2)}$, and $\tilde{c}_2^{-i} = \mathbb{E}[c_2^{-i} | c_2^{-i} < b_2]$.

It is straightforward to check that $0 \leq \Theta \leq 1$ given that $H_1(.)$ stochastically dominates $H_2(.)$. We can now characterize the symmetric equilibrium bidding behavior, which is described in the following proposition.

**Proposition 1** *There is a symmetric bidding function for the semi-uniform payment rule in which each bidder's bid for the first unit is equal to their value for the first unit and their bid for the second unit is given by the following:*

$$b_2^i = \frac{v_2^i}{2 - \Theta} + \frac{\tilde{c}_2^{-i}}{2 - \Theta}$$

*with* $0 \leq \Theta \leq 1$.

The bidding behavior in a standard uniform-price auction in a similar set-up is studied by Noussair (1995), Engelbrecht-Wiggans and Kahn (1998), and Khezr and Menezes (2017). They focus on a symmetric undominated set of equilibria and characterize the bidding functions. In what follows, we compare the revenue generated under the two payments rules in the respective undominated symmetric bidding strategy equilibria. To accomplish this, we first rewrite the expected payoff from the uniform-price auction as follows:

$$\Pi^u(\mathbf{v}, \mathbf{b^i}) = H_1(b_2)(v_1^i + v_2^i) - 2\int_0^{b_2} c_1^{-i} h_1(c_1^{-i})\mathrm{d}c_1^{-i} + (H_2(b_1) - H_1(b_2))v_1^i$$
$$- (H_2(b_2) - H_1(b_2))b_2 - \int_{b_2}^{b_1} c_2^{-i} h_2(c_2^{-i})\mathrm{d}c_2^{-i}. \tag{10}$$

The first-order condition is given by the following:

$$\frac{\partial \Pi^u}{\partial b_2} = v_2^i h_1(b_2) - b_2 h_1(b_2) - H_2(b_2) + H_1(b_2) = 0, \tag{11}$$

which yields the following equilibrium bidding strategy for the second unit (whereas bidders bid their valuations for the first unit):

$$b_2^u(v_2^i) = v_2^i - \frac{H_2(b_2^u) - H_1(b_2^u)}{h_1(b_2^u)}. \tag{12}$$

As Engelbrecht-Wiggans and Kahn (1998) and Khezr and Menezes (2017) show that the equilibrium bidding is either given by the above equation or it is equal to zero. Therefore, the bid for the second unit is either increasing in the value for the second unit or equal to zero. Engelbrecht-Wiggans and Kahn (1998) and Khezr and Menezes (2017) further characterize the conditions under which every bidder would bid zero for the second unit, yielding zero revenue for the seller. As it is clear from the analysis above, and as summarized in the following Corollary, there is no zero-price equilibrium under a semi-uniform auction.

**Corollary 1** *Unlike the uniform-price auction, the semi-uniform auction does not result in a zero-price equilibrium.*

Next, as in the aforementioned literature, we focus on the relationship between the number of bidders and objects for sale. First, we assume that the number of bidders is at least as large as the number of units. As Engelbrecht-Wiggans and Kahn (1998), and Khezr and Menezes (2017) show, in this case, there exists a threshold value $v^*$, such that bidders with a value lower than $v^*$ bid zero for the second unit. Hereafter, we refer to $v^*$ as this threshold value.

**Definition 1** A multi-unit auction is efficient as long as the items are allocated to those bidders who have the highest values for them.

The above definition suggests that as long the items are allocated to those who place the highest value for them, the auction is efficient. In other words, in an efficient auction, we would not observe a case where a bidder with a lower value receives an object while there is a bidder with a higher value who still has an unsatisfied demand. In the next definition, we clarify the dominance in terms of efficiency.

**Definition 2** A multi-unit auction is more efficient in a second-best allocation, as long as there is a higher chance that a bidder with a higher value receives an item.

Given the above two definitions, the next Proposition shows the conditions under which the semi-uniform-price auction dominates the uniform-price auction in terms of revenue and efficiency.

**Proposition 2** *If the number of bidders is at least as large as the number of units, and $v^* = \bar{v}$, then the semi-uniform-price auction dominates the uniform-price auction in terms of both the expected revenue and efficiency.*

Proposition 2 show when the number of units is relatively smaller compared to the number of bidders, the semi-uniform payment rule performs better. In this case, it is more likely that the zero-price equilibrium with uniform payment rule, where bidders bid zero for the second unit, will eventuate.

Note that as suggested by (Engelbrecht-Wiggans and Kahn 1998) in their Corollary 4.3, the condition $v^* = \bar{v}$ would consider a case where all bidders bid zero for the second unit. This situation happens when the hazard rate function of values for each bidder is sufficiently low.

## 4 Conclusion

This paper shows that an alternative payment rule, where the bidder with the marginal bid can potentially pay a different price from other successful bidders, breaks down the zero-price equilibrium of the uniform-price auction. More precisely, we provide sufficient conditions under which the semi-uniform-price auction dominates the uniform-price auction in expected revenue and efficiency.

## 5 Appendix

***Proof of Proposition 1*** First, rewrite Eq. (7) as follows:

$$\frac{\partial \Pi^{su}}{\partial b_2} = h_1(b_2)\big(v_2^i - 2b_2 + \tilde{c}_2^i\big) + b_2 h_1(b_2)\Theta. \tag{13}$$

For simplicity, we omit the term $i$ in what follows. Differentiating with respect to $b_2$ gives the second-order condition as follows:

$$\frac{\partial^2 \Pi^{su}}{\partial b_2^2} = h_1(b_2)\big(-2 + \tilde{c}_2'\big) + h_1'(b_2)\big(v_2 - 2b + \tilde{c}_2\big) + h_1(b_2)\Theta + b_2 h_1'(b_2)\Theta + b_2 h_1(b_2)\Theta'. \tag{14}$$

After some manipulations, we have the following:

$$\frac{\partial^2 \Pi^{su}}{\partial b_2^2} = h_1(b_2)\big(-2 + \tilde{c}_2' + \Theta + b_2\Theta'\big) + h_1'(b_2)\big(v_2 - 2b + \tilde{c}_2 + b_2\Theta\big). \tag{15}$$

To ensure that the bidding function for the second unit is a local maxima, we need to show that the above expression is negative in the neighborhood of the optimal bid given by the FOC. The first part of (15) is strictly negative, because $\tilde{c}_2' < 0$, $\Theta \leq 1$, and $\Theta' < 0$. Note that it is straightforward to check that $\Theta' < 0$ given that $H_1$ stochastically dominates $H_2$. The second part of (15) is equal to zero in the neighborhood of $b_2^*$. It follows that $b_2^*$ is a local maxima. □

***Proof of Proposition 2*** Denote by $p^*$ the equilibrium clearing price in the uniform-price auction. According to Engelbrecht-Wiggans and Kahn (1998) and Khezr and Menezes (2017), when the number of bidders is at least as large as the number of units, that is, $N \geq Q$, all bidders bid zero for the second unit in the uniform-price auction, irrespective of their values. Denote by $v_{1(Q+1)}$ the $Q+1$th highest value for the first unit. Since $N \geq Q$ and it is a weakly dominant strategy for bidders to bid their value for the firs unit, then in the uniform-price auction, $p^*$ is either equal to zero or equal to the $v_{1(Q+1)}$. In fact, in the case where the number of bidders is exactly equal to the number of units, the $Q$th highest bid is equal to the lowest value for the first unit among the bidders. Therefore, the clearing price, which is equal to the $Q+1$th highest bid, becomes zero: all bidders bid zero for the second unit, so the aggregation of bids results in $Q$ positive bids which are the values for the first units and $Q$ zero bids, which are bids for the second unit. Further note that when $N > Q$, then the aggregation of bids results in $N$ positive bids where the $Q+1$th highest bid among them sets the clearing price.

In the semi-uniform-price auction, there are two possibilities. Define $\tilde{v}_2$ as the highest realized value for the second unit among bidders. First, if $b_2(\tilde{v})$ is larger than $v_{1(Q+1)}$, then the semi-uniform-price auction results in a higher revenue for sure. To see this, imagine the worst case scenario where only $b_2(\tilde{v}) > v_{1(Q+1)}$ and all other values for the second units are such that $b_2(v) < v_{1(Q+1)}$. In this case, the auction clearing price is greater than $v_{1(Q+1)}$ and the bidder who has $\tilde{v}_2$ pays $v_{1(Q+1)}$. Therefore, the overall auction revenue is strictly higher than the one for the uniform-price auction. Second, is when $b_2(\tilde{v})$ is smaller than $v_{1(Q+1)}$. In this case, the two auctions become equivalent. Thus, the semi-uniform-price auction has a higher expected payoff than the uniform-price auction.

The proof for efficiency follows a similar argument. A mechanism is more efficient as long as it allocates the objects to bidders with higher values. Of course, here, we are analyzing a second-best case and neither of the two mechanisms are fully efficient, that is, always allocate the objects to the buyers with the highest values.

In the first case identified above, the semi-uniform auction is more efficient, because there is a higher chance that it allocates objects to bidders with higher values. This is because in the uniform-price auction, all bidders bid zero for the second unit, and if, for instance, there exist a bidder, such that her value for the second unit is larger than at least the value of some bidders for the first unit, she will never receive a second object. However, given that $b_2 > 0$ for the same bidder in the semi-uniform-price auction, there is a positive probability that this bidder receives a second unit. In the second case, where $N > Q$, both auctions allocate the object in the same way. Thus, we can conclude that the semi-uniform-price auction would allocate the objects more efficiently compared to the uniform-price auction. □

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