Economic Theory Bulletin (2019) 7:245–258
https://doi.org/10.1007/s40505-018-0157-4

RESEARCH ARTICLE

Equilibria in games with weak payoff externalities

Takuya Iimura[1] · Toshimasa Maruta[2] · Takahiro Watanabe[1]

Received: 19 December 2017 / Accepted: 13 October 2018 / Published online: 20 October 2018

Abstract
Ania (J Econ Behav Organ 65:472–488, 2008) shows that in the class of symmetric games with weak payoff externalities, symmetric Nash equilibria are equivalent to symmetric evolutionary equilibria (Schaffer in J Econ Behav Organ 12:29–45, 1989). We introduce a notion of a game with *partial* weak payoff externalities. We show that the class of games with partial weak payoff externalities includes most of previously known classes of games in which the equivalence prevails. We also establish a number of pure strategy Nash equilibrium existence results for a game with weak payoff externalities, and for a class of games that includes games with partial weak payoff externalities. The results include, in particular, the existence of pure strategy Nash equilibrium in some *finite* games.

Keywords Existence of equilibrium · Evolutionary equilibrium · Weak payoff externalities · Weakly unilaterally competitive games · Weakly competitive games · Potential games

JEL Classification C72 (Noncooperative games) · C73 (Evolutionary games)

The authors thank valuable comments from an anonymous referee and an associate editor. An earlier version of this paper was presented at UECE Lisbon Meetings in Game Theory and Applications 2016. This work is supported by JSPS Grant-in-Aid for Scientific Research (C) (KAKENHI) 25380233, 16K0355301, and 17K03631.

✉ Takuya Iimura
t.iimura@tmu.ac.jp

Toshimasa Maruta
maruta.toshimasa@nihon-u.ac.jp

Takahiro Watanabe
contact_nabe10@nabenavi.net

1 Faculty of Economics and Business Administration, Tokyo Metropolitan University, Tokyo 192-0397, Japan

2 College of Economics, Nihon University, Tokyo 101-8360, Japan

1 Introduction

The notion of a game with weak payoff externalities (henceforth, WPE) was introduced by Ania (2008) as an n-person symmetric game in which "the effect of any unilateral deviation on the deviator's payoff is always greater than the effect on the opponents' payoffs" (Ania 2008, p. 478). She showed that in a WPE game or in a constant-sum game, a symmetric Nash equilibrium (SNE) is equivalent to a symmetric evolutionary equilibrium (SEE, Schaffer 1989). Hehenkamp et al. (2010) subsequently showed that either WPE or the "weak competitiveness" at symmetric profiles is sufficient for the equivalence. Recently, Iimura and Watanabe (2016) pointed out that the equivalence holds in any symmetric game that is weakly unilaterally competitive (Kats and Thisse 1992). In these games, a symmetric profile is a Nash equilibrium if and only if it is stable under the evolutionary pressure driven by the relative payoff maximization.

In this paper, we offer an extensive analysis of the class of games in which the Nash-Evolutionary equilibrium equivalence holds. First, by proposing a single condition, we unify most of equilibrium equivalence results obtained thus far. A symmetric game is a game with *partial* weak payoff externalities if in any unilateral deviation in which the deviator's payoff increases (decreases, respectively), the payoff increase (decrease, respectively) of the deviator is greater than that of any other player (see Sect. 3 for the precise definition). We show that the class of games with partial WPE strictly extends many of previously known "competitive" games in which the equilibrium equivalence prevails. We interpret this result as a further extension of the proposition that in a competitive game, own-payoff maximization is equivalent to relative payoff maximization.

Second, we investigate the existence of pure strategy SNE in partial WPE games. We start by showing that in this class of games, a strategy constitutes an SNE if and only if it is a maximal element of a binary relation of strategies defined by profitable deviations at symmetric profiles. It turns out that WPE games (to be precise, a class of games that slightly extends WPE games in the sense of Ania) and partial WPE games call for different treatments. We show that in a WPE game, the binary relation is transitive. This result immediately leads to existence of SNE in a WPE game in which a transitive relation of strategies is guaranteed to have a maximal element, e.g., a finite WPE game.[1] For games with partial WPE, in contrast, we adopt a different line of reasoning. For an n-person symmetric (more generally, quasi-symmetric) game, we begin by associating it with a two-person symmetric game, which we call the two-person reduction.[2] It has the critical property that a strategy forms an SNE in the two-person reduction if and only if the strategy does so in the original n-person game. Moreover, it turns out that the two-person reduction of an n-person partial WPE game belongs to a special class of two-person games, which we have called *pairwise solvable* (PS) games (Iimura et al. 2016). Consequently, we are able to establish equilibrium existence in n-person partial WPE games by deriving new existence results for two-person PS games.

[1] A game is finite if there are only finitely many strategies. We always assume that the number of players is finite.

[2] Moulin (1986, pp. 115–116) is an early example of such a treatment.

To be specific, we establish two types of existence results for two-person PS games. First, we focus on games in which the strategy set is partially ordered. We show that a two-person PS game has an SNE whenever the strategy set has the least upper bound property, the upper contour set at any symmetric profile is a closed interval, and some of which are bounded. It follows that if the game is finite then it has an SNE whenever the strategy set is a lattice and the upper contour set at any symmetric profile is an interval. This result extends one in Iimura and Watanabe (2016), which showed, in an attempt to generalize a result by Duersch et al. (2012), the existence of pure strategy SNE in an n-person symmetric weakly unilaterally competitive finite game that satisfies a quasiconcavity condition. The results for finite games merit a special attention since there are few results in the literature that ensure the existence of pure strategy Nash equilibrium in finite games. Second, invoking a generalization of Knaster–Kuratowski–Mazurkiewicz lemma by Fan (1961), we derive an SNE existence result for a two-person PS game in which the strategy set is a subset of a topological vector space. Note that each existence result for two-person PS games immediately implies corresponding result for n-person partial WPE games through PS two-person reduction.

The rest of the paper is organized as follows. In Sect. 2, we introduce notations and definitions. In Sect. 3, we define a game with partial WPE, and show the equivalence of SNE and SEE in such a game. In Sect. 4, we establish equilibrium existence results for n-person WPE games and two-person PS games, from the latter of which corresponding results for n-person partial WPE games follow. In Sect. 5, we suggest an interpretation of the equivalence result, and discuss the nature of evolutionary equilibria in partial WPE games in a dynamic context.

2 Preliminaries

Notations and equilibrium concepts

An n-person strategic game is an n-tuple of pairs (S_i, u_i), $i = 1, \ldots, n$, where S_i is player i's strategy set and $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ his payoff function. It is *symmetric* if $S_i = S_j$ for all $i, j \in \{1, \ldots, n\}$, and for any permutation π on $\{1, \ldots, n\}$

$$u_{\pi(i)}(s_1, \ldots, s_n) = u_i(s_{\pi(1)}, \ldots, s_{\pi(n)}) \quad \forall i \in \{1, \ldots, n\}$$

for any $s = (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$. Such a symmetric n-person game is concisely described by a pair (S^n, u), where S^n is the n-product of $S = S_1$ and $u = u_1$. We assume that S has at least two elements. We call the game finite if S is finite.

For $i = 1, \ldots, n$, let π_i be the permutation on $\{1, \ldots, n\}$ such that $\pi_i(1) = i$, $\pi_i(i) = 1$, and $\pi_i(j) = j$ for every $j \notin \{1, i\}$. With a slight abuse of notation, let $\pi_i(s) = (s_{\pi_i(1)}, \ldots, s_{\pi_i(n)})$ for any $s = (s_1, \ldots, s_n) \in S^n$. Then $u_i(s) = u(\pi_i(s))$. Denote by x^k the k-repetition of x, i.e., $x^k = x, \ldots, x$ (k times). In $s = (x^{i-1}, y, x^{n-i}) \in S^n$, player i chooses y and all the others x. Applying π_i for this s, we have $u_i(s) = u(y, x^{n-1})$, and $u_j(s) = u(x^{n-1}, y)$ for every $j \neq i$ by symmetry.

A strategy profile $s \in S^n$ is *symmetric* if $s = (x^n)$ for some $x \in S$. A strategy profile (x^n) is a *symmetric Nash equilibrium* (SNE) if

$$u(x, x^{n-1}) \geq u(y, x^{n-1}) \quad \forall y \in S.$$

A strategy profile (x^n) is a *weak symmetric evolutionary equilibrium* (Schaffer 1989) if

$$u(x^{n-1}, y) \geq u(y, x^{n-1}) \quad \forall y \in S.$$

For simplicity, we call a weak symmetric evolutionary equilibrium a *symmetric evolutionary equilibrium* (SEE). The definition of an SEE says that if some player changes his strategy from x to y then his payoff never exceeds the payoff of the opponents.

Weak payoff externalities

Ania (2008) calls a symmetric game (S^n, u) a game with weak payoff externalities if for any $x, y \in S$ such that $x \neq y$ and for any $\xi \in S^{n-1}$

$$|u(y, \xi) - u(x, \xi)| > |u(\pi_i(y, \xi)) - u(\pi_i(x, \xi))| \quad \forall i \neq 1.$$

Note that if there are distinct $x, y \in S$ such that $u(x, \xi) = u(y, \xi)$, i.e., a payoff tie, then this condition is violated. Thus, a game has WPE in Ania's sense if and only if for any $x, y \in S$ such that $x \neq y$ and for any $\xi \in S^{n-1}$

$$u(y, \xi) \neq u(x, \xi) \quad \text{and} \quad |u(y, \xi) - u(x, \xi)| > |u(\pi_i(y, \xi)) - u(\pi_i(x, \xi))| \quad \forall i \neq 1.$$

Let us modify the definition in the following way. We say that a game (S^n, u), possibly with payoff ties, has weak payoff externalities if

(WPE$_0$) for any $x, y \in S$ and $\xi \in S^{n-1}$

$$\left.\begin{array}{l} u(y, \xi) - u(x, \xi) \neq 0 \Rightarrow |u(y, \xi) - u(x, \xi)| > |u(\pi_i(y, \xi)) - u(\pi_i(x, \xi))| \\ u(y, \xi) - u(x, \xi) = 0 \Rightarrow u(\pi_i(y, \xi)) - u(\pi_i(x, \xi)) = 0 \\ \quad \forall i \neq 1. \end{array}\right\}$$

We call (WPE$_0$) the *weak-or-zero payoff externality condition*.[3] Suppose that (x^n) is an SNE and $y \in S$ is an alternative best response to (x^{n-1}). If it were the case that $u(y, x^{n-1}) > u(x^{n-1}, y)$, then (x^n) would not be an SEE. The second condition in (WPE$_0$) excludes this possibility. Note that any game with weak payoff externalities in the sense of Ania must satisfy (WPE$_0$).

Figure 1 shows two examples of games satisfying (WPE$_0$). Clearly, the game G_1 has a dominant strategy SNE (z, z) (in fact, any *two-strategy* game with WPE has a

[3] One might also call this condition weak payoff externalities *with TDI (transference of decisionmaker indifference*; Marx and Swinkels 1997).

	x	y	z
x	0,0	1,2	1,3
y	2,1	3,3	3,3
z	3,1	3,3	3,3

(a) 2-person 3-strategy game G_1

x:

	x	y	z
x	23,23,23	17,30,17	29,11,29
y	30,17,17	24,24,14	16,1,22
z	11,29,29	1,16,22	7,7,18

y:

	x	y	z
x	17,17,30	14,24,24	22,1,16
y	24,14,24	20,20,20	15,4,15
z	1,22,16	4,15,15	8,8,9

z:

	x	y	z
x	29,29,11	22,16,1	18,7,7
y	16,22,1	15,15,4	9,8,8
z	7,18,7	8,9,8	5,5,5

(b) 3-person 3-strategy game G_2

Fig. 1 Examples of games satisfying (WPE$_0$)

dominant strategy SNE; see Lemma 4.1 below). However, this is not always the case. The three-person game G_2 has an SNE that consists of a non-dominant strategy y.

3 Equivalence of SNE and SEE in games with 'partial' WPE

Consider the following condition that generalizes (WPE$_0$).

Definition 3.1 A game (S^n, u) has *partial weak payoff externalities*[4] if

(pWPE) for any $x, y \in S$ and $\xi \in \{x, y\}^{n-1}$

$$\left.\begin{array}{l} u(y,\xi) - u(x,\xi) > 0 \implies u(y,\xi) - u(x,\xi) > u(\pi_i(y,\xi)) - u(\pi_i(x,\xi)) \\ u(y,\xi) - u(x,\xi) < 0 \implies u(y,\xi) - u(x,\xi) < u(\pi_i(y,\xi)) - u(\pi_i(x,\xi)) \\ u(y,\xi) - u(x,\xi) = 0 \implies u(\pi_i(y,\xi)) - u(\pi_i(x,\xi)) = 0 \\ \quad \forall i \neq 1. \end{array}\right\}$$

Note that (pWPE) requires (WPE$_0$) to hold only for any other player whose payoff increases (resp. decreases) when the deviator's payoff increases (resp. decreases), and not in the whole game but in its *two-strategy subgames* $(\{x, y\}^n, u)$, where $x, y \in S$, $x \neq y$, and u is restricted to $\{x, y\}^n$ (note that $\xi \in \{x, y\}^{n-1}$). Note also that the three conditions in (pWPE) can be combined into a single bi-conditional:

$$u(y,\xi) - u(x,\xi) > 0 \iff u(y,\xi) - u(x,\xi) > u(\pi_i(y,\xi)) - u(\pi_i(x,\xi)). \quad (1)$$

Therefore, if $\xi = (x^{n-1})$, in particular, then $u(y, x^{n-1}) - u(x^n) > 0$ if and only if $u(y, x^{n-1}) > u(x^{n-1}, y)$, i.e.,

$$u(y, x^{n-1}) \leq u(x^n) \iff u(y, x^{n-1}) \leq u(x^{n-1}, y).$$

4 The partial WPE is always with TDI, allowing payoff ties.

Fig. 2 A game satisfying (pWPE) but not (WPE_0) nor (WUC)

	x	y	z
x	$2,2$	$0,3$	$1,-1$
y	$3,0$	$1,1$	$1,-1$
z	$-1,1$	$-1,1$	$0,0$

Varying $y \in S$, this says that (x^n) is an SNE if and only if it is an SEE. We thus have:

Proposition 3.2 *For any game with partial WPE, an SNE is equivalent to an SEE.*

In what follows, we show how games with partial WPE relate to weakly unilaterally competitive games and weak competitive games. A game (S^n, u) is said to be *weakly unilaterally competitive* (Kats and Thisse 1992)[5] if

(WUC) for any $x, y \in S$ and $\xi \in S^{n-1}$

$$\left.\begin{aligned} u(y,\xi) > u(x,\xi) &\implies u(\pi_i(y,\xi)) \le u(\pi_i(x,\xi)) \\ u(y,\xi) = u(x,\xi) &\implies u(\pi_i(y,\xi)) = u(\pi_i(x,\xi)) \end{aligned}\right\} \forall i \neq 1.$$

It is straightforward to see that (WUC) implies (pWPE) (but not (WPE_0)). Hence, the class of games with partial WPE includes not only games satisfying (WPE_0), but also games satisfying (WUC).[6] These inclusions are strict. For example, the game of Fig. 2 satisfies (pWPE) but not (WPE_0) because $u(y,x) - u(x,x) \neq 0$ and $|u(y,x) - u(x,x)| < |u(x,y) - u(x,x)|$; also not (WUC) because $u(x,x) > u(z,x)$ and $u(x,x) > u(x,z)$.

A game (S^n, u) is said to be *weakly competitive* (Hehenkamp et al. 2010) if

(WC) for any $x, y \in S$ and $\xi \in S^{n-1}$

$$\left.\begin{aligned} u(y,\xi) > u(x,\xi) &\implies u(\pi_i(y,\xi)) \le u(\pi_i(x,\xi)) \\ u(y,\xi) \le u(x,\xi) &\implies u(\pi_i(y,\xi)) \ge u(\pi_i(x,\xi)) \end{aligned}\right\} \exists i \neq 1.$$

Letting $\xi = (x^{n-1})$, we find that (WC) implies the equivalence of SNE and SEE: for any $y \in S$, $u(y, x^{n-1}) \le u(x^n)$ if and only if $u(y, x^{n-1}) \le u(x^{n-1}, y)$. Also, it can be shown that (WUC) implies (WC), and (WC) is equivalent to (WUC) in two-person games. However, (WC) does not imply (pWPE).[7] Hence a game satisfying (WC) need not be a game with partial WPE. See Fig. 3 for the relationship among the classes of games.

[5] The original definition of weakly unilaterally competitive game by Kats and Thisse (1992) accommodates asymmetric games.

[6] In particular, the class includes two-person symmetric zero-sum games, since any such game satisfies (WUC).

[7] Iimura and Watanabe (2016, Fig. 7) contains a game satisfying (WC) with no SNE that fails to satisfy (pWPE).

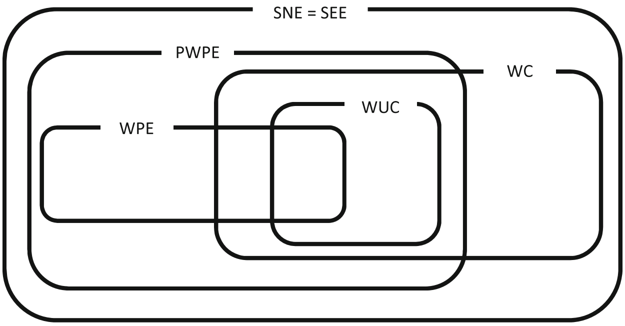

Fig. 3 Relationship among the classes of games

4 Existence of SNE

4.1 Preliminary considerations

In this section, we show several equilibrium existence results. We start with a result that forms the basis of the analysis.

Lemma 4.1 *Let $G = (S^n, u)$ be a game with partial WPE. Then for all distinct $x, y \in S$,*

$$u(y, \xi) > u(x, \xi) \iff u(y, \xi') > u(x, \xi') \quad \forall \xi, \xi' \in \{x, y\}^{n-1}. \tag{D}$$

Proof By symmetry, we only need to show (D) for $\xi, \xi' \in \{x, y\}^{n-1}$ such that $\xi = (x^p, y^q)$ and $\xi' = (x^{p-1}, y^{q+1})$, where $q = 0, \ldots, n-2$ with $p = n-1-q$ (note that $p \geq 1$). Let $\eta = (x^{p-1}, y^q)$, which is void if $n = 2$. It follows from (1) that

$$\begin{aligned} u(y, x, \eta) > u(x, x, \eta) &\iff u(y, x, \eta) - u(x, x, \eta) > u(x, y, \eta) - u(x, x, \eta), \\ u(x, y, \eta) < u(y, y, \eta) &\iff u(x, y, \eta) - u(y, y, \eta) < u(y, x, \eta) - u(y, y, \eta). \end{aligned}$$

The right-hand sides of these equivalences are equivalent. Thus $u(y, x, \eta) > u(x, x, \eta) \iff u(y, y, \eta) > u(x, y, \eta)$. By symmetry $u(y, x, \eta) > u(x, x, \eta) \iff u(y, \eta, y) > u(x, \eta, y)$, which is equivalent to

$$u(y, x^p, y^q) > u(x, x^p, y^q) \iff u(y, x^{p-1}, y^{q+1}) > u(x, x^{p-1}, y^{q+1}).$$

□

Consider a subgame of an n-person game in which strategies are restricted in $\{x, y\} \subseteq S$, $x \neq y$. The condition (D) states that in any such subgame, either y strictly dominates x, x strictly dominates y, or they are equivalent. Thus any *two-strategy* game with partial WPE has a dominant strategy SNE, in particular.

Lemma 4.1 allows us to characterize an SNE in terms of binary relation on the strategy set S. To do this and for later use, let us review relevant definitions. A binary relation $\succ$ on a set X is *asymmetric* if $x \succ y$ implies not $y \succ x$ for every $x, y \in X$. For any $Y \subseteq X$, an element $x \in Y$ *is* $\succ$-maximal *on* Y if there is no $y \in Y$ such that $y \succ x$. A binary relation $\succ$ on X is *transitive* if $x \succ y$ and $y \succ z$ imply $x \succ z$ for all $x, y, z \in X$; *acyclic* if every finite subset $Y \subseteq X$ has a $\succ$-maximal element. A transitive relation $\succ$ is acyclic.

Now, define a binary relation $\succ_u$ on the strategy set S as follows. For $x, y \in S$,

$$x \succ_u y \iff u(x, y^{n-1}) > u(y, y^{n-1}). \tag{2}$$

This relation is asymmetric by (D). Clearly, a symmetric profile (x^n) is an SNE if and only if $x \in S$ is $\succ_u$-maximal on S. Let

$$U(y) = \{z \in S \mid u(z, y^{n-1}) \geq u(y, y^{n-1})\},$$

which we call the *upper contour set* of y at $(y^n) \in S^n$. For games satisfying (D), we have $x \in U(y)$ or $y \in U(x)$ for any $x, y \in S$, since $u(x, y^{n-1}) < u(y, y^{n-1}) \iff u(x, x^{n-1}) < u(y, x^{n-1})$; symmetric profile (x^n) is an SNE if and only if $x \in \bigcap_{y \in S} U(y)$, since $u(x, y^{n-1}) \geq u(y, y^{n-1}) \iff u(x, x^{n-1}) \geq u(y, x^{n-1})$. Let us summarize these observations.

Lemma 4.2 *Let $G = (S^n, u)$ be a game with partial WPE. Then $\succ_u$ is asymmetric, and (x^n) is an SNE if and only if $x \in S$ is $\succ_u$-maximal on S. For every $x, y \in S$, $x \in U(y)$ or $y \in U(x)$, and (x^n) is an SNE if and only if $x \in \bigcap_{y \in S} U(y)$.*

4.2 Games with weak payoff externalities

In this subsection, we show that a game with WPE has an SNE under a reasonable condition. In particular, any *finite* game with WPE must have an SNE. Consider a WPE game possibly with payoff ties, satisfying (WPE_0).

Theorem 4.3 *Let $G = (S^n, u)$ be a game with WPE. Then $\succ_u$ is transitive.*

Proof We begin with proving claims. For an arbitrary $(n-2)$-profile[8] $\zeta \in S^{n-2}$, set $\rho(x, y \mid \zeta) = u(x, y, \zeta) + u(y, x, \zeta)$ for every $x, y \in S$. Note that $\rho(x, y \mid \zeta) = \rho(y, x \mid \zeta)$.

Claim 1. $u(x, z, \zeta) > u(y, z, \zeta) \iff \rho(x, z \mid \zeta) > \rho(y, z \mid \zeta)$.

This follows from the conditions in (WPE_0).

Claim 2. If $u(x, y, \zeta) > u(y, y, \zeta) > u(z, y, \zeta)$, then there exists $w \in \{x, z\}$ such that $u(x, w, \zeta) > u(y, w, \zeta) > u(z, w, \zeta)$.

Recall (D) in Lemma 4.1. Then it follows from (WPE_0) and $u(x, y, \zeta) > u(y, y, \zeta)$ that $u(x, x, \zeta) > u(y, x, \zeta)$. Similarly, $u(y, z, \zeta) > u(z, z, \zeta)$. By Claim 1, $\rho(x, y \mid \zeta) > \rho(y, y \mid \zeta) > \rho(y, z \mid \zeta)$. But if it were the case that $u(z, x, \zeta) > u(y, x, \zeta)$ and $u(y, z, \zeta) > u(x, z, \zeta)$, then Claim 1 would imply that $\rho(y, z \mid \zeta) >$

[8] If $n = 2$ then set ζ to be the empty string.

$\rho(z, x \mid \zeta) > \rho(x, y \mid \zeta)$. Therefore either $u(y, x, \zeta) > u(z, x, \zeta)$ or $u(x, z, \zeta) > u(y, z, \zeta)$.

Now let $x \succ_u y$ and $y \succ_u z$. Then by (D), $u(x, y, y^{n-2}) > u(y, y, y^{n-2}) > u(z, y, y^{n-2})$. By Claim 2, $u(x, w, y^{n-2}) > u(y, w, y^{n-2}) > u(z, w, y^{n-2})$, where $w \in \{x, z\}$. By symmetry of the payoff function, $u(x, y, w, y^{n-3}) > u(y, y, w, y^{n-3}) > u(z, y, w, y^{n-3})$. Applying Claim 2 successively, we arrive at $u(x, \xi) > u(y, \xi) > u(z, \xi)$, where $\xi \in \{x, z\}^{n-1}$. By (D), $x \succ_u z$. □

The next lemma is standard.

Lemma 4.4 *Let $\succ$ be a binary relation on a compact topological space X. If $\succ$ is acyclic and the set $\{y \in X \mid x \succ y\}$ is open for every $x \in X$, then X has a $\succ$-maximal element.*

Proof See Walker (1977). □

Proposition 4.5 *Let $G = (S^n, u)$ be a game with WPE. If S is a compact topological space and the upper contour set $U(x)$ is closed for every $x \in S$, then there is an SNE. In particular, if S is finite then there is an SNE.*

Proof Let $\succ_u$ be as defined by (2). By Theorem 4.3, this relation is transitive, and hence acyclic. Observe that the set $\{y \mid x \succ_u y\} = \{y \mid u(x, y^{n-1}) > u(y, y^{n-1})\}$ is the complement of $U(x) = \{y \mid u(x, x^{n-1}) \leq u(y, x^{n-1})\}$ by (D), so $\{y \mid x \succ_u y\}$ is open for every $x \in S$. Hence by Lemma 4.4, there exists a $\succ_u$-maximal element $x \in S$. Then (x^n) is an SNE by Lemma 4.2. □

If the payoff function is upper semicontinuous in own strategy at any symmetric profile of the others, then the upper contour sets $U(x)$ are closed. Neither the convexity of S or $U(x)$, nor the continuity in strategies of the others, is required.[9] Note that a strategy that forms an SNE in a game with WPE need not be a dominant strategy. See Fig. 1b.

A technical remark is in order. In the proof of Theorem 4.3, the function $\rho(x, y \mid \zeta)$ works as a potential function. More precisely, given any $\zeta \in S^{n-2}$, consider the two-person symmetric game on S defined by $u(x, y, \zeta)$. The function $\rho(x, y \mid \zeta)$ is an *ordinal potential function* (Monderer and Shapley 1996) of this game. In particular, a two-person symmetric game satisfying (WPE_0) is an ordinal potential game.

4.3 Games with partial weak payoff externalities

In a partial WPE games, the binary relation $\succ_u$ need not be transitive, nor even acyclic (e.g., "Rock–Paper–Scissors").[10] Hence we take a different approach. Let us call

[9] In a game $G = (S^n, u)$ satisfying (WPE_0), assume that u is *continuous* in own strategy. Then one can prove that it is continuous in any other's strategy.

[10] One can show that in a partial WPE game the relation is acyclic if the strategy set is *totally* ordered. It follows that such a game has an SNE if it is finite, or it does with some additional topological conditions as in Proposition 4.5. If the strategy set is only *partially* ordered, the relation need not be acyclic. We shall consider such games in Sect. 4.3.1.

an n-person strategic game with a common strategy set S *quasi-symmetric* (Reny 1999, p. 1040) if $u_1(x, y, \ldots, y) = u_2(y, x, y, \ldots, y) = \cdots = u_n(y, \ldots, y, x)$ for all $x, y \in S$. Any symmetric game is quasi-symmetric, and the quasi-symmetry is equivalent to symmetry in two-person games. By letting $u = u_1$, we also denote by (S^n, u) an n-person quasi-symmetric game.

Definition 4.6 *Two-person reduction* of n-person quasi-symmetric game $G = (S^n, u)$ is a two-person symmetric game $G_\tau = (S^2, u_\tau)$ such that

$$u_\tau(x, y) = u(x, y^{n-1}) \quad \forall x, y \in S.$$

By construction, G and G_τ have identical strategy set and identical upper contour sets at symmetric profiles, and (x^n) is an SNE in G if and only if (x, x) is an SNE in G_τ. What does G_τ of an n-person partial WPE game G look like? The following definition is given in Iimura et al. (2016).

Definition 4.7 A two-person symmetric game (S^2, u) is *pairwise solvable* (PS) if

(PS) for any distinct $x, y \in S$

$$u(x, y) > u(y, y) \iff u(x, x) > u(y, x).$$

See Iimura et al. (2016) for the analysis of this class of games. Now, if $G = (S^n, u)$ is a partial WPE game, then by (D) in Lemma 4.1 $u(x, y^{n-1}) > u(y, y^{n-1}) \iff u(x, x^{n-1}) > u(y, x^{n-1})$, i.e., $u_\tau(x, y) > u_\tau(y, y) \iff u_\tau(x, x) > u_\tau(y, x)$: G_τ is PS. Note that, since (PS) equals (D) with $n = 2$, any PS game has all the properties stated in Lemma 4.2. In what follows, we offer SNE existence results for PS games, from which the corresponding results follow for n-person partial WPE games.[11]

4.3.1 Partially ordered strategies

A binary relation $\leq$ on a set L is a *partial order* if it is reflexive, anti-symmetric ($x \leq y$ and $y \leq x$ imply $x = y$), and transitive. A set L is *partially ordered* if it has a partial order. For a subset J of a partially ordered set L, $x \in L$ is a *lower bound* (resp. *upper bound*) if $x \leq y$ (resp. $y \leq x$) for all $y \in J$. J is *bounded from below* (resp. *from above*) if it has a lower bound (resp. an upper bound); *bounded* if it has both. Whenever they exist, the greatest lower bound and the least upper bound are denoted by $\bigwedge J$ and $\bigvee J$, respectively. If $\bigwedge J \in J$ (resp. $\bigvee J \in J$) then it is the *minimum* (resp. *maximum*) of J. A subset I of L is a *closed interval* if $I = \uparrow a = \{x \in L \mid a \leq x\}$ or $I = \downarrow b = \{x \in L \mid x \leq b\}$, or $I = [a, b] = \{x \in L \mid a \leq x \leq b\}$ for some $a, b \in L$. In addition, we regard L as a closed interval. While $\uparrow a$ and $\downarrow b$ are never empty, $[a, b]$ is nonempty if and only if $a \leq b$. A nonempty closed interval $I \neq L$ has the minimum or the maximum. If $I = [a, b]$ with $a \leq b$ then it has both. Note that $\uparrow a$ and $\downarrow b$ may well have both.

[11] Our existence results apply to any n-person game that has a pairwise solvable two-person reduction. Hence they may apply to a WC game if its two-person reduction is pairwise solvable.

A partially ordered set L is said to have the *least upper bound property* (*lubp* for short) if any nonempty subset J has the least upper bound whenever J is bounded from above; or equivalently, if J has both $\bigwedge J$ and $\bigvee J$ whenever J is bounded. L is a *lattice* if it has $\bigwedge\{a, b\}$ and $\bigvee\{a, b\}$ for all $a, b \in L$. L is a *complete lattice* if it has $\bigwedge J$ and $\bigvee J$ for any $J \subseteq L$. Any complete lattice has the minimum and the maximum, and hence is bounded. Any finite lattice is complete.

Theorem 4.8 *Let $\mathscr{I} = \{I_\lambda \mid \lambda \in \Lambda\}$ be a nonempty set of closed intervals in a partially ordered set L such that for all $\lambda, \lambda' \in \Lambda$, $I_\lambda \cap I_{\lambda'} \neq \varnothing$. Then $\bigcap_{\lambda\in\Lambda} I_\lambda \neq \varnothing$ if L has lubp and the minimum of I_λ and the maximum of $I_{\lambda'}$ exist for some $\lambda, \lambda' \in \Lambda$. In particular, the intersection is nonempty if L is a complete lattice.*

Proof If $I_\lambda \in \mathscr{I}$ has a minimum, denote it by $b(\lambda)$. If it has a maximum, $t(\lambda)$. By assumption, neither $\Lambda(b) = \{\lambda \mid b(\lambda) \text{ exists}\}$ nor $\Lambda(t) = \{\lambda \mid t(\lambda) \text{ exists}\}$ is empty. For every $\lambda \in \Lambda(b)$ and $\kappa \in \Lambda(t)$, $b(\lambda) \leq t(\kappa)$, since $I_\lambda \cap I_\kappa \neq \varnothing$ and $\leq$ is transitive. Fixing λ, this is true for any $\kappa \in \Lambda(t)$. Therefore $b(\lambda)$ is a lower bound of $\{t(\kappa) \mid \kappa \in \Lambda(t)\}$. By lubp, $b(\lambda) \leq t^* = \bigwedge\{t(\kappa) \mid \kappa \in \Lambda(t)\}$. Since this is true for any $\lambda \in \Lambda(b)$, t^* is an upper bound of $\{b(\lambda) \mid \lambda \in \Lambda(b)\}$. By lubp, $b^* = \bigvee\{b(\lambda) \mid \lambda \in \Lambda(b)\} \leq t^*$. Thus $[b^*, t^*]$ is nonempty. Let $a \in [b^*, t^*]$ and consider $I_\lambda \in \mathscr{I}$. If $I_\lambda = L$, $a \in I_\lambda$. If not, either $b(\lambda)$ or $t(\lambda)$ exists. If $b(\lambda)$ exists, then $b(\lambda) \leq b^* \leq a$. If there is no $t(\lambda)$, then $a \in I_\lambda = \uparrow b(\lambda)$. If $t(\lambda)$ exists, then $a \in I_\lambda = [b(\lambda), t(\lambda)]$ since $a \leq t^* \leq t(\lambda)$. Similarly, $a \in I_\lambda$ if $t(\lambda)$ exists. Therefore $[b^*, t^*] \subseteq I_\lambda$ for any $I_\lambda \in \mathscr{I}$. Finally, note that any complete lattice is bounded, and possesses lubp. □

Proposition 4.9 *Let S be a partially ordered set and $G = (S^2, u)$ be a PS game in which the upper contour set $U(x)$ is a closed interval for every $x \in S$. Then G has an SNE if S has lubp and the minimum of $U(x)$ and the maximum of $U(y)$ exist for some $x, y \in S$. In particular, G has an SNE if S is a finite lattice, or more generally, a complete lattice.*

Proof Let $\mathscr{I} = \{U(x) \mid x \in S\}$. By assumption, it is a collection of closed intervals. By Lemma 4.2, $U(x) \cap U(y) \neq \varnothing$ for all $x, y \in S$, and (x, x) is an SNE if and only if $x \in \bigcap_{y\in S} U(y)$. Now the results follow from Theorem 4.8. □

The crucial assumption in Proposition 4.9 is that the upper contour set is a closed interval. This is satisfied, for example, if S is a finite lattice and u is *order-closed-quasiconcave* in own strategy in the sense of Cigola and Licalzi (1997, p. 29): letting $f_x(y) = u(y, x)$, $f_x(z) \geq \min\{f_x(y), f_x(y')\}$ for every $z \in [y \wedge y', y \vee y']$ for all $y, y' \in S$, for every $x \in S$. Note that in Proposition 4.9, S need not be a lattice.

In passing, we note that in the proof of Theorem 4.8, we showed that $[b^*, t^*] \subseteq \bigcap_{\lambda\in\Lambda} I_\lambda$. One can verify the converse inclusion as well. It follows then that the set of strategies that appear in some SNE in the game of Proposition 4.9 forms a closed interval.

4.3.2 The KKM-type argument

By Lemma 4.2, a PS game $G = (S^2, u)$ has an SNE if and only if $\bigcap_{x\in S} U(x)$ is nonempty. This suggests the following application of Fan's generalization of Knaster–

Kuratowski–Mazurkiewicz lemma (Fan 1961). For any subset Y of a vector space X, let $\text{co}\, Y$ denote the convex hull of Y.[12] Following Aliprantis and Border (2006, p. 577), we call a correspondence $\psi : Y \twoheadrightarrow X$ a *KKM correspondence* if $\text{co}\{x_1, \ldots, x_K\} \subset \bigcup_{i=1}^{K} \psi(x_i)$ for every finite subset $\{x_1, \ldots, x_K\}$ of Y.

Proposition 4.10 *Let $G = (S^2, u)$ be a PS game in which S is a compact subset of a topological vector space. Then G has an SNE if $U(x)$, as a correspondence $U : S \twoheadrightarrow S$, is a closed-valued KKM correspondence.*

Proof By Fan (1961, Lemma 1), the assumptions imply that $\bigcap_{x \in S} U(x)$ is nonempty.[13] Hence an SNE exists by Lemma 4.2. □

The question is under what condition $U(x)$ becomes a KKM correspondence. Inspired by Aliprantis and Border (2006, Lemma 17.47, p. 579), we have the following result.

Proposition 4.11 *Let $G = (S^2, u)$ be a PS game in which S is a compact subset of a topological vector space. Then G has an SNE if $U(x)$ is closed and $x \notin \text{co}\, P(x)$ for each $x \in S$, where $P(x) = \{y \in S \mid u(y, x) > u(x, x)\}$.*

Proof By Proposition 4.10, it suffices to show that if U is not a KKM correspondence, then there is $x \in S$ such that $x \in \text{co}\, P(x)$. Assume that there are $x_i \in S$, $\alpha_i \geq 0$, and $\sum_i \alpha_i = 1$, $i = 1, \ldots, K$, such that $x = \sum_i \alpha_i x_i \in \text{co}\{x_1, \ldots, x_K\}$ but $x \notin \bigcup_{i=1}^{K} U(x_i)$. Then it follows that $x \notin U(x_i)$, or $u(x_i, x_i) > u(x, x_i)$. By (PS), $u(x_i, x) > u(x, x)$, or $x_i \in P(x)$. Since this is true for all i, $x = \sum_i \alpha_i x_i \in \text{co}\, P(x)$. □

In fact, it follows from Aliprantis and Border (2006, Lemma 17.47, p. 579) that the condition $x \notin \text{co}\, P(x)$ is not only sufficient but also necessary to make $U(x)$ a KKM correspondence.[14] It is satisfied if $P(x) = \text{co}\, P(x)$, as in a game with the payoff function that is quasiconcave in own strategy at symmetric profiles. Notice, however, that neither $U(x)$ nor $P(x)$ is required to be convex in Proposition 4.11. It is straightforward to verify that if $U : S \twoheadrightarrow S$ is a KKM correspondence, then S is convex. It is clear that $U(x)$ is closed whenever the payoff function is upper semicontinuous in own strategy. No continuity is required in the opponent's strategy.

For results in this subsection (in particular, Propositions 4.9 and 4.11), assumptions are concerned with either the strategy set or the upper contour sets at symmetric profiles. As noted earlier, an n-person quasi-symmetric game and its two-person reduction have identical strategy set and identical upper contour sets at symmetric profiles, and the two-person reduction of an n-person partial WPE game is a PS game. Therefore, we now have corresponding existence results for n-person partial WPE games.

Corollary 4.12 (to Proposition 4.9) *Let S be a partially ordered set and $G = (S^n, u)$ be a partial WPE game in which the upper contour set $U(x)$ is a closed interval for*

[12] Recall that $x \in \text{co}\, Y$ if and only if there are $x_i \in Y$, $\alpha_i \geq 0$, and $\sum_i \alpha_i = 1$, $i = 1, \ldots, K$, such that $x = \sum_i \alpha_i x_i$.

[13] The topological vector space need not be Hausdorff. On this, see, for example, Yuan (1998, p. 6).

[14] We can see this as well by reversing the argument in the proof of Proposition 4.11.

every $x \in S$. Then G has an SNE if S has lubp and the minimum of $U(x)$ and the maximum of $U(y)$ exist for some $x, y \in S$. In particular, G has an SNE if S is a finite lattice, or more generally, a complete lattice.

Corollary 4.13 (to Proposition 4.11) *Let $G = (S^n, u)$ be a partial WPE game in which S is a compact subset of a topological vector space. Then G has an SNE if $U(x)$ is closed and $x \notin \text{co}\, P(x)$ for each $x \in S$, where $P(x) = \{y \in S \mid u(y, x) > u(x, x)\}$.*

5 Concluding remarks

The notion of weakly unilaterally competitive game (Kats and Thisse 1992) is an extension of the notion of a competitive game, which is exemplified in a two-person zero-sum game. Recall from Sect. 3 that these games need not have weak payoff externalities, but they must exhibit a *partial* form of weak payoff externalities. This observation leads us to regard the notion of a game with partial weak payoff externalities as a further extension of the notion of a competitive game. Viewed in this way, and together with the fact that an evolutionary equilibrium is a Nash equilibrium in the relative payoff game (Schaffer 1989), an intuitive understanding suggests itself: in a competitive game, own-payoff maximization is equivalent to relative payoff maximization.

Given an n-person game with partial WPE, let us examine a strategy revision process, which may be described as "beat the others or imitate if beaten". For simplicity, let the game be a three-person finite game with partial WPE. The process starts from (x, x, x), which we assume is not an SEE. Sooner or later, a player finds (and switches to) a strategy y that beats the others in that $u(y, x, x) > u(x, y, x)$. Note that by (pWPE), y beats the others if and only if $u(y, x, x) > u(x, x, x)$. Since the payoff of the deviator is higher than that of the others, the others may have an incentive to imitate. So a second player switches to y to bring about (y, y, x). The imitator realizes a payoff improvement by the imitation, i.e., $u(y, y, x) > u(x, y, x)$ by Lemma 4.1. Also by Lemma 4.1, $u(y, y, y) > u(x, y, y)$, thus the third player still has an incentive to switch to y. In this way, they arrive at a new symmetric profile (y, y, y). Now, does the iteration of this process lead to an SEE, which we know is also an SNE? If the game has WPE, the process does lead to an SEE by Theorem 4.3. For the games with partial WPE, however, the process need not reach an SEE in general.[15]

In the above process each strategy revision involves revising player's payoff improvement, both absolutely and relatively. It need not be the case, however, that the new symmetric profile Pareto-dominates the old. To see this, look at the three-person game in Fig. 1b. From (x, x, x) to (y, x, x), the revising player gains by 7 but the others lose by 6. From (y, x, x) to (y, y, x), the revising player gains by 7 but the others lose by 6 or 3. From (y, y, x) to (y, y, y), the revising player gains by 6 but the others lose by 4. At each step, the gain of the revising player is greater than the loss(es) of the others. This is a consequence of weak payoff externalities. Each player enjoys, however, the payoff increase once, but suffers a loss twice. Consequently, they find themselves impoverished at equilibrium (y, y, y). This argument reveals that in *two-*

15 But see footnote 10.

person games with weak payoff externalities, no symmetric profile Pareto-dominates a symmetric equilibrium.[16] In a game with *partial* weak payoff externalities, by contrast, a symmetric equilibrium may well be Pareto-dominated even in a two-person game, e.g., see Fig. 2.

References

Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis: A Hitchhiker's Guide, 3rd edn. Springer, Berlin (2006)

Ania, A.B.: Evolutionary stability and Nash equilibrium in finite populations, with an application to price competition. J. Econ. Behav. Organ. **65**, 472–488 (2008)

Cigola, M., Licalzi, M.: Ordinally quasiconcave functions. Optimization **40**, 25–39 (1997)

Duersch, P., Oechssler, J., Schipper, B.C.: Pure strategy equilibria in symmetric two-player zero-sum games. Int. J. Game Theory **41**, 553–564 (2012)

Fan, K.: A generalization of Tichonoff's fixed point theorem. Math. Ann. **142**, 305–310 (1961)

Hehenkamp, B., Possajennikov, A., Guse, T.: On the equivalence of Nash and evolutionary equilibrium in finite populations. J. Econ. Behav. Organ. **73**, 254–258 (2010)

Iimura, T., Maruta, T., Watanabe, T.: Two-person pairwise solvable games. NUPRI Working Paper 2016-02, Nihon University (2016)

Iimura, T., Watanabe, T.: Pure strategy equilibrium in finite weakly unilaterally competitive games. Int. J. Game Theory **45**, 719–729 (2016)

Kats, A., Thisse, J.-F.: Unilaterally competitive games. Int. J. Game Theory **21**, 291–299 (1992)

Marx, L.M., Swinkels, J.M.: Order independence for iterated weak dominance. Games Econ. Behav. **18**, 219–245 (1997)

Monderer, D., Shapley, L.S.: Potential games. Games Econ. Behav. **14**, 124–143 (1996)

Moulin, H.: Game Theory for the Social Sciences, 2nd and revised ed., New York University Press, New York (1986)

Reny, P.J.: On the existence of pure and mixed strategy Nash equilibria in discontinuous games. Econometrica **67**, 1029–1056 (1999)

Schaffer, M.E.: Are profit-maximizers the best survivors? A Darwinian model of economic natural selection. J. Econ. Behav. Organ. **12**, 29–45 (1989)

Walker, M.: On the existence of maximal elements. J. Econ. Theory **16**, 470–474 (1977)

Yuan, G.X.-Z.: The Study of Minimax Inequalities and Applications to Economics and Variational Inequalities, Memoirs of the American Mathematical Society No. 625, American Mathematical Society (1998)

[16] Actually, one can show that the symmetric equilibrium is Pareto-efficient: no asymmetric profile Pareto-dominates it either.