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RESEARCH ARTICLE

Weak maximal elements and weak equilibria in ordinal games with applications to exchange economies

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Abstract We study binary relations (preferences) and ordinal games in the case where no continuity-like properties are assumed at all. We introduce generalizations of the maximal element and Nash equilibrium, called, respectively, the *weak maximal element* and *weak equilibrium*, and give existence results when binary relations satisfy only convexity conditions. The weak maximal element (weak equilibrium) is equivalent to the maximal element (Nash equilibrium) if and only if a generalization of continuity is given. Moreover, we obtain the existence of *quasi-Pareto optimal* allocations in exchange economies.

Keywords Binary relations · Preference relations · Weak maximal element · Ordinal games · Weak equilibrium · Exchange economies · Quasi-Pareto optimality

JEL Classification C72 · D51

1 Introduction

As it is well known, in order to obtain the existence of maximal elements of binary relations and Nash equilibria in ordinal games, one needs to assume properties of convexity and continuity. The aim of this paper was to study binary relations and ordinal games where the sets of alternatives (strategies) are convex and compact subsets of Hausdorff topological vector spaces and binary relations satisfy only convexity properties. We introduce new concepts for binary relations and ordinal games, called,

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respectively, the *weak maximal element* and the *weak equilibrium* and prove that the set of weak maximal elements and the set of weak equilibria are non-empty and compact in our framework.

An alternative $\bar{x}$ is a weak maximal element if, for any x and any open neighborhood O of $\bar{x}$, there exists some $z \in O$ such that x is not preferred to z (so, a maximal element is a weak maximal element). We prove that the set of weak maximal elements coincides with the set of maximal elements if and only if a general topological property introduced by Tarafdar (1977) is satisfied. This allows us to clarify the role of topological assumptions on binary relations (that is, the continuity and its generalizations) in maximal element existence results. More precisely, convexity guarantees the existence of maximal elements in an *weak sense* and continuity allows such weak points to be optimal ones.

A strategy profile $\bar{x}$ of an ordinal game is a weak equilibrium if, for any strategy profile x and any open neighborhood O of $\bar{x}$, there exists some $z \in O$ such that (x_i, z_{-i}) is not preferred to z for each player i (so, every Nash equilibrium is a weak equilibrium). We give an example to show that the convexity condition used in several Nash equilibrium existence results (see Shafer and Sonnenschein 1975; Yannelis and Prabhakar 1983; Wu and Shen 1996; Scalzo 2015; He and Yannelis 2016) is not sufficient to obtain the existence of weak equilibria. Hence, we introduce a stronger condition which implies that the set of weak equilibria is non-empty and compact (we show that the condition is not connected with generalizations of continuity used in Nash equilibrium existence results). Moreover, we prove that every weak equilibrium is a Nash equilibrium if and only if the game has the *single deviation property* (Reny 2009). So, convexity guarantees the existence of *weak* Nash equilibria, while the single deviation property allows such weak equilibria to be Nash equilibria. Let us remark that the single deviation property is a very general topological condition and does not guarantee the existence of Nash equilibria under standard convexity assumptions (Reny 2009). If the binary relations of players are represented by payoff functions, we obtain that the set of weak equilibria is included in both the set of Reny equilibria (Bich and Laraki 2012) and the set of quasi-Nash equilibria (Scalzo 2016).

Finally, we apply the results to finite exchange economies and introduce a generalization of Pareto optimal allocations that we call *quasi-Pareto optimal* allocations. An allocation $\bar{x}$ is quasi-Pareto optimal if, for any open neighborhood O of $\bar{x}$, there are no allocations which improve upon each allocation which belongs to O. We obtain that the set of quasi-Pareto optimal allocations is non-empty and compact whether the preferences of consumers are complete and strictly convex. Moreover, when the preferences are also transitive, there exist quasi-Pareto optimal allocations which belong to the topological closure of the set of individually rational allocations.

2 Preliminaries and the weak maximal element

Let X be a non-empty subset of a Hausdorff topological vector space and $\succ$ be an asymmetric (but not necessarily complete or transitive) binary relation on X. Define the correspondence (map) $P : X \rightrightarrows X$ by $P(x) = \{y \in X : y \succ x\}$[1] for all

[1] As usual, $y \succ x$ denotes that $(y, x) \in \succ$.

$x \in X$. So, $P(x)$ is the set of elements which are *better* than x (we identify the binary relation $\succ$ with the correspondence P). An element $x^* \in X$ is said to be *maximal* for $\succ$ if $P(x^*) = \emptyset$; we also say that x^* is a *P-maximal* element. Suppose that X is convex and compact. In order to obtain the existence of P-maximal elements, one needs to assume convexity and continuity properties: we refer to the seminal work by Yannelis and Prabhakar (1983), where, for any $x \in X$, (1) $x \notin \mathrm{co}P(x)$ and (2) $P^{-1}(x) = \{z \in X : x \in P(z)\}$ is open in X.[2] Using the property below (see Tarafdar 1977)[3]

$$P(x) \neq \emptyset \implies \begin{cases} \text{there exists } x' \in X \text{ and an open neighborhood } O_x \text{ of } x \\ \text{such that} x' \in P(z) \quad \text{for all } z \in O_x \cap X, \end{cases} \tag{1}$$

condition (2) can be relaxed, and one obtains the following result (see Wu and Shen 1996, where (1) is called the *local intersection property*):

Lemma 1 *Let X be a convex and compact subset of a Hausdorff topological vector space and* $P : X \rightrightarrows X$. *Assume that* $x \notin \mathrm{co}P(x)$ *for each* $x \in X$ *and property (1) is satisfied. Then, the set of P-maximal elements is non-empty and compact.*

Remark 1 In a recent paper (Scalzo 2015), the existence of P-maximal elements has been obtained for maps which satisfy (i) above and a generalization of (1), that is, there exists a well-behaved map $\xi_x : O_x \rightrightarrows X$ such that $\xi_x(z) \subseteq P(z)$ for all $z \in O_x \cap X$ (*well-behaved* means that the map is upper hemicontinuous with non-empty, convex and compact values). This new property is an extension of a condition introduced by Corson and Lindenstrauss (1966) which characterizes the existence of continuous selections from a map (see page 495, where ξ_x is a single valued and continuous function).

It is easy to find examples of correspondences which satisfy every assumption of Lemma 1 except (1) and maximal elements fail to exist. In these cases, one would know if there are elements that, in some sense, can be looked as *weak* maximal elements. More precisely, it would be interesting to obtain the existence of a non-empty set which includes the maximal elements and coincides with the set of maximal elements when the property (1) is satisfied. A positive answer to the question is given through the definition below:

Let $P : X \rightrightarrows X$ be a correspondence. The set of P-maximal elements is denoted by E_P. Given an element $x \in X$, with $\tau(x)$ we denote the set of open neighborhoods of x relative to X.

Definition 1 $\bar{x} \in X$ is said to be a *weak P-maximal* element if for all $x \in X$ and all $O \in \tau(\bar{x})$, there exists $z \in O$ such that $x \notin P(z)$. The set of weak P-maximal elements is denoted by E_P^W.

It is clear that $E_P \subseteq E_P^W$. The converse does not hold: for instance, consider the function $u : [0, 1] \longrightarrow [0, 1]$ defined by $u(x) = x$ if $x \in [0, 1[$ and $u(1) = 0$, and let

[2] Given $A \subseteq X$, we denote by co$A$ and cl$A$, respectively, the convex hull and the topological closure in X of the set A.

[3] I thank Nicholas Yannelis for the reference of Tarafdar (1977).

$P : [0,1] \rightrightarrows [0,1]$ such that $P(x) = \{z \in [0,1] : u(z) > u(x)\}$. We have $E_P = \emptyset$ and $E_P^W = \{1\}$. The next proposition shows some properties of E_P^W.

Proposition 1 *Assume that X is a convex and compact subset of a Hausdorff topological vector space. The following statements hold:*

1. $E_P = E_P^W$ *if and only if P satisfies (1).*
2. *If $x \notin \text{co}P(x)$ for any $x \in X$, then $E_P^W \neq \emptyset$.*
3. E_P^W *is a closed (and compact) set.*

Proof Let D_P be the subset of X where the property (1) does not hold for the map P, that is the set of all $x \in X$ such that $P(x) \neq \emptyset$ and, for all $x' \in X$ and all $O \in \tau(x)$, there exists some $z \in O$ such that $x' \notin P(z)$. So, one has $E_P^W = E_P \cup D_P$, which proves (1). Concerning (2), if $E_P^W = \emptyset$, we get $E_P = E_P^W$ and P has the property (1). Hence, in light of Lemma 1, we obtain $E_P \neq \emptyset$, which is a contradiction. Finally, if $x \notin E_P^W$, for some x' and an open neighborhood O of x, we get $x' \in P(z)$ for any $z \in O$. But $O \in \tau(z)$ for each $z \in O$; so, $O \cap E_P^W = \emptyset$, which proves (3). □

3 The weak equilibrium in ordinal games

An *ordinal game* is a collection $\Gamma = \langle X_i, P_i \rangle_{i \in N}$ where N is the set of players, that we assume to be finite, and, for any $i \in N$, X_i is a non-empty subset of a Hausdorff topological vector space. The set of strategy profiles is denoted by $X = \prod_{i \in N} X_i$ and, for any $i \in N$, $X_{-i} = \prod_{j \neq i} X_j$; given $x \in X$, we set $x = (x_i, x_{-i})$. We assume that each player i compares strategy profiles by means of an asymmetric binary relation $\succ_i$. So, $P_i : X \rightrightarrows X_i$ is the correspondence so that $z_i \in P_i(x)$ if and only if $(z_i, x_{-i}) \succ_i x$. A *Nash equilibrium* (*equilibrium* in short) of Γ is a strategy profile x^* such that $P_i(x^*) = \emptyset$ for each $i \in N$; the set of equilibria of Γ is denoted by E_Γ. Sufficient conditions for the existence of equilibria have been recently provided by Carmona and Podczeck (2016), He and Yannelis (2016), Prokopovych (2013) and (2016), Reny (2016), Scalzo (2015).

Our purpose was to study ordinal games where some convexity condition is satisfied and no topological assumptions on the correspondences P_i are given at all. We aim to identify a non-empty set of *weak* equilibria. So, we give the following definition:

Definition 2 $\bar{x} \in X$ is said to be a *weak equilibrium* of Γ if for all $x \in X$ and all $O \in \tau(\bar{x})$ there exists $z \in O$ such that $x_i \notin P_i(z)$ for each $i \in N$. The set of weak equilibria of Γ is denoted by E_Γ^W.

We have $E_\Gamma \subseteq E_\Gamma^W$, and $E_\Gamma = E_\Gamma^W$ under the topological property introduced below (see Reny 2009):

Definition 3 Γ has the *single deviation property* if whenever $x \in X$ is not an equilibrium, there exists $x' \in X$ and $O \in \tau(x)$ such that for every $z \in O$ there is a player i for whom $x'_i \in P_i(z)$.

Proposition 2 $E_\Gamma = E_\Gamma^W$ *if and only if Γ has the single deviation property.*

Proof First, assume that $E_\Gamma = E_\Gamma^W$. If x is not an equilibrium, since $x \notin E_\Gamma^W$, we have that Γ has the single deviation property. On the other hand, suppose that Γ has the single deviation property and $x \in E_\Gamma^W \setminus E_\Gamma$. From the definition of weak equilibrium, we obtain that, for any strategy profile x' and any open neighborhood O of x, there exists $z \in O$ such that $x'_i \notin P_i(z)$ for each $i \in N$, which is in contrast with the single deviation property. □

Remark 2 The following condition is used in several equilibrium existence results (see Shafer and Sonnenschein 1975; Yannelis and Prabhakar 1983; Wu and Shen 1996; Scalzo 2015; He and Yannelis 2016):

$$x_i \notin \mathrm{co}P_i(x) \quad \text{for all } x \in X \text{ and all } i \in N. \tag{2}$$

Now, assume that the sets of strategies are convex and compact subsets of Hausdorff topological vector spaces. We note that *the single deviation property and* (2) *are not sufficient conditions for the existence of equilibria*. Indeed, consider the ordinal game Γ with 3 players and the correspondences P_i defined by $P_i(x) = \{y_i \in X_i : u_i(y_i, x_{-i}) > u_i(x)\}$, where the sets X_i and the functions u_i are given by Reny (2009) (pages 4 and 5). The function $u_i(\cdot, x_{-i})$ is quasi-concave for all x_{-i} and all i; so, the correspondences P_i satisfy the property (2).[4] Moreover, Γ has the single deviation property and $E_\Gamma = \emptyset$ (Reny 2009).

Remark 3 Prokopovych (2013) identified a convexity condition which guarantees the existence of equilibria in ordinal games having the single deviation property (see Theorem 2). More precisely, assume that Γ has the single deviation property: for any x which is not an equilibrium, $d(x)$ and O_x are, respectively, the deviations profile and the open neighborhood of x as given by Definition 3. The condition introduced by Prokopovych (which supposes that the game has the single deviation property) is the following:[5]

$$\begin{aligned}&\text{for all } \{x^1, ..., x^k\} \text{ such that } \{x^1, ..., x^k\} \cap E_\Gamma = \emptyset \text{ and for all } z \in \cap_{h=1}^k O_{x^h}\\ &\text{there exists } i \in N \text{ such that } z_i \notin \mathrm{co}\left\{d(x^h)_i : h \in \{1, ..., k\}\right\}.\end{aligned} \tag{3}$$

Remark 4 Consider the setting of ordinal games where the sets of strategies are convex and compact subsets of Hausdorff topological vector spaces. We have that: *property* (2) *is not a sufficient condition for the existence of weak equilibria*. In fact, if $E_\Gamma^W \neq \emptyset$ for any Γ which satisfies property (2), for games that have also the single deviation property, we obtain $E_\Gamma \neq \emptyset$ (see Proposition 2), which contradicts Remark 2.

In light of Remark 4, we need a condition different from (2) in order to obtain the existence of weak equilibria in ordinal games where no other assumptions on the

[4] Let $G = \langle X_i, u_i \rangle_{i \in N}$ be a normal form game and $\Gamma = \langle X_i, P_i \rangle_{i \in N}$ be the ordinal game where $P_i(x) = \{y_i \in X_i : u_i(y_i, x_{-i}) > u_i(x)\}$ for any $x \in X$ and any $i \in N$. It is easy to prove that property (2) holds if and only if $u_i(\cdot, x_{-i})$ is quasi-concave for any $x_{-i} \in X_{-i}$ and any $i \in N$.

[5] For our purpose, it is sufficient to refer to the version of Theorem 2 by Prokopovych (2013) for ordinal games. However, this theorem has been given for games which satisfy a condition more general than the single deviation property (see also Theorem 5 ibidem).

correspondences are given. First, for any $\Gamma = \langle X_i, P_i \rangle_{i \in N}$, let us consider the correspondence $P : X \rightrightarrows X$ defined as below:

$$y \in P(x) \iff \text{there exists} \quad i \in N \quad \text{such that } y_i \in P_i(x) \tag{4}$$

It is easy to see that P satisfies property (1) if and only if Γ has the single deviation property.

Proposition 3 *Given $\Gamma = \langle X_i, P_i \rangle_{i \in N}$, assume that any X_i is a convex and compact subset of a Hausdorff topological vector space. Let P be the correspondence defined by (4) and assume that*

$$x \notin \mathrm{co} P(x) \quad \forall\, x \in X. \tag{5}$$

Then, E_Γ^W is non-empty and compact.

Proof It is sufficient to note that $E_\Gamma = E_P$ and $E_\Gamma^W = E_P^W$. So, Proposition 1 applies and the thesis follows. □

The connections between (5) and properties (2) and (3) are given in the proposition below (let us emphasize that, differently from (3), (5) does not suppose any topological property on Γ).

Proposition 4 *Assume that Γ satisfies property (5). Then, (2) holds, and, when Γ has the single deviation property, also (3) holds.*

Proof First, if (2) is not true, there exists $x \in X$ and $i \in N$ such that $x_i \in \mathrm{co}\{x_i^1, ..., x_i^k\}$ with $x_i^h \in P_i(x)$ and $h = 1, ..., k$. So, $(x_i^h, x_{-i}) \in P(x)$ for $h = 1, ..., k$, and we get $x \in \mathrm{co}\{(x_i^h, x_{-i}) : h = 1, ..., k\} \subseteq \mathrm{co} P(x)$, which is in contrast with (5).

Finally, assume that Γ has the single deviation property. If (3) is not satisfied, for at least one finite set $\{x^1, ..., x^k\}$ of strategy profiles which are not equilibria, taken the corresponding set of deviations $\{d(x^1), ..., d(x^k)\}$ and the open neighborhoods $O_{x^1}, \ldots O_{x^k}$ given by the single deviation property, there exists $z \in \cap_{h=1}^k O_{x^h}$ such that $z_i \in \mathrm{co}\{d(x^1)_i, ..., d(x^k)_i\}$ for all $i \in N$; so, $z \in \mathrm{co}\{d(x^1), ..., d(x^k)\}$. On the other hand, from the single deviation property we have that, for every $h \in \{1, ..., k\}$, there exists $i_h \in N$ such that $d(x^h)_{i_h} \in P_{i_h}(z)$, which implies $d(x^h) \in P(z)$ for any h. So, we have $z \in \mathrm{co} P(z)$, which contradicts (5). □

If Γ does not satisfy the single deviation property, condition (5) does not imply (3), as it is shown by the example below:

Example 1 Let $u_1 : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ be such that $u_1(x_1, x_2) = 1$ if $x_1 > x_2$, $u_1(x_1, x_2) = 0$ if $x_1 < x_2$ and $x_1 = x_2 = 0$, $u_1(x_1, x_2) = 1$ if $x_1 = x_2 > 0$, and let u_2 be defined by $u_2(x_1, x_2) = 1 - u_1(x_1, x_2)$ if $x_1 \neq x_2$ and $u_2(x_1, x_2) = u_1(x_1, x_2)$ otherwise. Consider the ordinal game $\Gamma = \langle X_i, P_i \rangle_{i=1,2}$ where $X_1 = X_2 = [0, 1]$ and $P_i(x) = \{y_i \in X_i : u_i(y_i, x_{-i}) > u_i(x)\}$ for all $x \in X$ and $i = 1, 2$.

If $x \in X$ and $x_1 > x_2$, one has $P_1(x) = \emptyset$ and $P_2(x) = [x_1, 1]$; so, $P(x) = [0, 1] \times [x_1, 1]$ which is a convex set and $x \notin [0, 1] \times [x_1, 1]$. If $x_1 < x_2$, one has

$P_1(x) = [x_2, 1]$ and $P_2(x) = \emptyset$; so, $x \notin P(x) = [x_2, 1] \times [0, 1]$. For $x_1 = x_2 > 0$ we have $P(x) = \emptyset$. Finally, we get

$$P(0,0) = \big(]0,1] \times [0,1]\big) \cup \big([0,1] \times]0,1]\big) = \big([0,1] \times [0,1]\big) \setminus \{(0,0)\}$$

Hence, Γ satisfies (5). On the other hand, it is easy to see that the single deviation property fails to be verified for $x = (0, 0)$. However, we have that $E_\Gamma = \{x : x_1 = x_2 > 0\}$ and $E_\Gamma^W = \{x : x_1 = x_2\}$. So, in light of Proposition 2, we know that Γ does not have the single deviation property. Finally, property (3) is not satisfied.

Remark 5 The game introduced in the example above satisfies property (5) but not the assumptions of the equilibrium existence results given by Scalzo (2015), He and Yannelis (2016), Reny (2009). Indeed, these assumptions imply that the set of equilibria is closed, differently from the game presented in Example 1.

Remark 6 From Propositions 2 and 3, we obtain the following equilibrium existence result: assume that $\Gamma = \langle X_i, P_i \rangle_{i \in N}$ is an ordinal game where, for each player i, X_i is a convex and compact subset of a Hausdorff topological vector space. If Γ has the single deviation property and condition (5) is satisfied, then the set of equilibria of Γ is non-empty and compact. Indeed, (5) implies that E_Γ^W is non empty and compact and the single deviation property provides $E_\Gamma = E_\Gamma^W$. However, in light of Proposition 4, this result is a corollary of Theorem 5 by Prokopovych (2013).

4 The weak equilibrium in strategic form games

Let $G = \langle X_i, u_i \rangle_{i \in N}$ be a strategic form game and $\Gamma(G) = \langle X_i, P_i \rangle_{i \in N}$ be the ordinal game such that $P_i(x) = \{y_i \in X_i : u_i(y_i, x_{-i}) > u_i(x)\}$ for every $x \in X$ and $i \in N$ (with N we denote also the number of players). The set of Nash equilibria of G is $E_{\Gamma(G)}$. With E_G^W we denote the set of weak equilibria of $\Gamma(G)$, that we call *weak equilibria* of G.[6]

In this section, we compare the weak equilibrium in strategic form games with other generalizations of the Nash equilibrium. The following definitions have been introduced by Bich and Laraki (2012) and Scalzo (2016), respectively:

Definition 4 A strategy profile $\bar{x}$ is said to be a *Reny equilibrium* of G if there exists $\bar{w} \in \mathbb{R}^N$ such that $(\bar{x}, \bar{w})$ belongs to the closure of the graph of the function $u = (u_1, ..., u_N)$ and $\liminf_{z \to \bar{x}} u_i(x_i, z_{-i}) \leq \bar{w}_i$ for any $x_i \in X_i$ and any $i \in N$. The set of Reny equilibria is denoted by RE_G.[7]

Definition 5 A strategy profile $\bar{x}$ is said to be a *quasi-Nash equilibrium* of G if

$$\liminf_{z \to \bar{x}} \Phi_G(x, z) \leq 0 \quad \forall\, x \in X,$$

[6] The weak equilibrium in strategic form games was introduced by Bich and Laraki (2012). Differently from what was stated in their paper, weak equilibria do not exist in every quasi-concave game (see the previous Remark 4).

[7] The definition of Reny equilibrium is connected with the definition of *better-reply secure* games given by Reny (1999).

where the function Φ_G is defined on $X \times X$ by

$$\Phi_G(x, z) = \sum_{i \in N} \left[u_i(x_i, z_{-i}) - u_i(z)\right]$$

The set of quasi-Nash equilibria is denoted by Q_G.[8]

Proposition 5 $E_G^W \subseteq RE_G \cap Q_G$.

Proof Assume that $\bar{x} \in E_G^W$. If $x \in X$, for any $O \in \tau(\bar{x})$ there is $z \in O$ such that

$$u_i(x_i, z_{-i}) - u_i(z) \leq 0 \quad \forall\, i \in N \tag{6}$$

This implies that, fixed $i \in N$, $\liminf_{z \to \bar{x}} u_i(x_i, z_{-i}) \leq \bar{w}_i$ for all $x_i \in X_i$, where $\bar{w}_i = \limsup_{z \to \bar{x}} u_i(z)$. So, $\bar{x} \in RE_G$. Moreover, (6) leads to $\liminf_{z \to \bar{x}} \Phi_G(x, z) \leq 0$ for all $x \in X$, that is, $\bar{x} \in Q_G$. □

The inclusion in Proposition 5 can be strict, as the example below shows:

Example 2 Let G be the 1-player game where $X_1 = [0, 1]$, $u_1(0) = 1$, $u_1(x_1) = x_1$ if $x_1 \in]0, 1[$ and $u_1(1) = 0$. One has $E_G = E_G^W = \{0\}$ and $RE_G = Q_G = \{0, 1\}$.

In the previous section, we have showed that property (5) implies (2), and (2) is equivalent to the quasi-concavity of $u_i(\cdot, x_{-i})$ for any x_{-i} and any $i \in N$. Baye et al. (1993) introduced a property called *diagonal transfer quasi-concavity*, that is, for any $\{x^1, ..., x^k\}$ there exists $\{z^1, ..., z^k\}$ such that, if $z = \sum_{j=1}^{s} \lambda_{i_j} z^{i_j}$ with $\lambda_{i_j} > 0$ for any j, we have $\min\left\{\Phi_G(x^{i_j}, z) : j = 1, ..., s\right\} \leq 0$. The connection between (5) and diagonal transfer quasi-concavity is given below.

Proposition 6 *Property (5) implies the diagonal transfer quasi-concavity.*

Proof Assume that $\{x^1, ..., x^k\}$ and $z = \sum_{h=1}^{k} \lambda_h x^h$, with $\lambda_h > 0$ for any h, are such that $\min\left\{\Phi_G(x^h, z) : h = 1, ..., k\right\} > 0$. So, for any h there exists $i_h \in N$ so that $u_{i_h}(x_{i_h}^h, z_{-i_h}) > u_{i_h}(z)$, that is, $x_{i_h}^h \in P_{i_h}(z)$ (P_{i_h} is the correspondence of player i_h in $\Gamma(G)$). Using the arguments of the proof of Proposition 4, we get $z \in \mathrm{co}P(z)$, which contradicts (5). Hence, (5) implies the following property: for any $\{x^1, ..., x^k\}$, if $z = \sum_{j=1}^{s} \lambda_{i_j} x^{i_j}$ with $\lambda_{i_j} > 0$ for any j, we have $\min\left\{\Phi_G(x^{i_j}, z) : j = 1, ..., s\right\} \leq 0$. Now, it is sufficient to set $\{z^1, ..., z^k\} = \{x^1, ..., x^k\}$ and the proposition is proved. □

5 Quasi-Pareto optimality in exchange economies

In this section, we introduce a weakening of Pareto optimality in exchange economies. It is known that Pareto optimal allocations existence results include continuity and convexity assumptions on the preferences of consumers (see, for example, Aliprantis et al. 1990). Now, we focus on economies where the preferences are convex and not

[8] A strategy profile x^* is a Nash equilibrium of G if and only if $\Phi_G(x, x^*) \leq 0$ for all $x \in X$.

necessarily continuous. In this setting, we obtain the existence of *quasi Pareto optimal* allocations, that are allocations $\bar{x}$ such that, for any allocation x, there is at least one consumer that prefers an approximation of $\bar{x}$ to x. Moreover, we get the existence of quasi Pareto optimal allocations which belong to the closure of the set of individually rational allocations.

Assume that $\mathcal{E}$ is an exchange economy with a finite number ℓ of commodities and a finite set I of consumers. The set of bundles, denoted by $\mathbb{K}$, is a convex and compact subset of $\mathbb{R}^{\ell}_{+}$ containing the null vector and with non-empty interior.[9] We suppose that any consumer $i \in I$ is endowed with a complete but not necessarily transitive binary relation $\succsim_i$ defined on $\mathbb{R}^{\ell}_{+}$ ($\succ_i$ denotes the asymmetric part of $\succsim_i$) and no topological assumptions, like continuity ant its generalizations, are assumed at all. Let $e_i \in \mathbb{K}\backslash\{\text{the null vector}\}$ be the endowment of i and $\mathcal{A}(\mathcal{E})$ be the set of *feasible allocations* (in short *allocations*), that is, the set of $x = (x_i)_{i\in I} \in \mathbb{K}^{|I|}$ such that $\sum_{i\in I} x_i \leq \sum_{i\in I} e_i$.[10] It is easy to see that $\mathcal{A}(\mathcal{E})$ is non-empty, convex and compact.

We recall that an allocation x is said to be *Pareto optimal*, if there are no allocations y such that (a) $y_i \succsim_i x_i$ for all $i \in I$ and (b) $y_j \succ_j x_j$ for some $j \in I$; *individually rational*, if $x_i \succsim_i e_i$ for all $i \in I$. We denote by $PO(\mathcal{E})$ the set of Pareto optimal allocation of $\mathcal{E}$ and by $\mathcal{A}_r(\mathcal{E})$ the set of individually rational allocations.

Definition 6 An allocation $\bar{x}$ of $\mathcal{E}$ is said to be *quasi-Pareto optimal* if for every allocation x and every open neighborhood O of $\bar{x}$, there exists an allocation $z \in O$ such that either $z_j \succ_j x_j$ for some $j \in I$ or $z_i \succsim_i x_i$ for any $i \in I$. The set of quasi-Pareto optimal allocations is denoted by $QPO(\mathcal{E})$.

Proposition 7 *Assume that the preferences of consumers are complete and strictly convex,*[11] *that is*

$$y_i \succsim_i x_i \quad \text{and} \quad z_i \succsim_i x_i \quad \text{with} \quad y_i \neq z_i \implies (1-t)y_i + tz_i \succ_i x_i \quad \forall\, t \in]0,1[\tag{7}$$

Then

1. $QPO(\mathcal{E})$ *is non-empty and compact;*
2. $QPO(\mathcal{E}) \cap \mathrm{cl}\mathcal{A}_r(\mathcal{E})$ *is non-empty and compact providing that* $\succsim_i$ *is transitive for any* $i \in I$.

Proof In order to prove (1), consider the binary relation $\succ$ defined on $\mathcal{A}(\mathcal{E})$ by $y \succ x$ *if and only if* (a) and (b) above hold: we have that the quasi-Pareto optimal allocations are nothing but the weak maximal elements of $\succ$. Property (7) implies that $P(x)$ is convex; moreover, $x \notin P(x)$. So, Proposition 1 applies and (1) follows.

Let us prove (2). First, note that $\mathrm{cl}\mathcal{A}_r(\mathcal{E})$ is convex and compact. Hence, in light of (7) and Proposition 1, there exists at least one weak maximal element $\bar{x}$ of $\succ$ restricted to $\mathrm{cl}\mathcal{A}_r(\mathcal{E})$. If $\bar{x}$ is not a quasi-Pareto optimal allocation, there exists an allocation x and an open neighborhood O of $\bar{x}$ such that $x \in P(z)$ for all $z \in O \cap \mathcal{A}(\mathcal{E})$. Since

9 In real-life markets, the quantity of each commodity is limited. So, it is reasonable to assume that the space of bundles is compact.

10 Given two elements x and y of $\mathbb{R}^{\ell}$, the inequality $x \leq y$ means $x_i \leq y_i$ for each $i \in \{1, \ldots, \ell\}$.

11 See Aliprantis et al. (1990).

$O \cap \mathcal{A}_r(\mathcal{E}) \neq \emptyset$, for at least one individually rational allocation z belonging to O we have $x \succ z$, which implies $x \in \mathcal{A}_r(\mathcal{E})$. Because $\bar{x}$ is a weak maximal element of $\succ$ on $\mathrm{cl}\mathcal{A}_r(\mathcal{E})$, there exists $z' \in \mathrm{cl}\mathcal{A}_r(\mathcal{E}) \cap O$ such that $x \notin P(z')$. But $\mathrm{cl}\mathcal{A}_r(\mathcal{E}) \subset \mathcal{A}(\mathcal{E})$, and we get a contradiction. So, (2) is proved. □

6 Conclusion

In this paper, we have studied binary relations and ordinal games when only some convexity assumption was given. In this setting, we have obtained the existence of *weak maximal elements* for binary relation (Proposition 1) and *weak equilibria* for ordinal games (Proposition 3). Any maximal element is a weak maximal element and the converse holds for binary relations that satisfy a condition introduced by Tarafdar (1977) (Proposition 1). Similarly, any Nash equilibrium is a weak equilibrium; Nash equilibria and weak equilibria coincide in ordinal games which have the single deviation property (Reny 2009) (Proposition 2). These results have permitted us to clarify the role of the topological assumptions on binary relations in maximal element and Nash equilibrium existence results. In fact, we have obtained that convexity guarantees the existence of maximal elements and Nash equilibria in a *weak sense* (weak maximal elements and weak equilibria), while generalizations of continuity allow such weak elements to be optimal points. Finally, we have applied the results to exchange economies where the preferences of consumers were assumed to be complete and strictly convex; in this setting, we have obtained the existence of *quasi-Pareto optimal* allocations.

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