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RESEARCH ARTICLE

On the existence of price equilibrium in economies with excess demand functions

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Abstract This paper provides a price equilibrium existence theorem in economies where commodities may be indivisible and excess demand functions may not be continuous. We introduce a weak notion of continuity, called *recursive transfer lower semi-continuity*, which, together with Walras' law, guarantees the existence of price equilibrium in economies with excess demand functions. The condition is also necessary, and thus our results generalize all the existing results on the existence of price equilibrium in economies where excess demand is a function.

Keywords Price equilibrium · Recursive transfer lower semi-continuity · Discontinuity · Excess demand function

JEL Classification C62 · C72 · D44 · D61 · D82

1 Introduction

This paper presents a theorem on the existence of price equilibrium in terms of excess demand function, which fully characterizes the existence of equilibrium price systems in general economies where commodities may be indivisible, aggregate excess demand

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functions may not be continuous, and no forms of convex/lattice structures are imposed so that no fixed-point-related theorems can be applied.

One of the great achievements in economics in the last century is the general equilibrium theory. It aims at studying the behavior of demand, supply, and prices in a whole economy, by considering equilibrium in many markets simultaneously. It is a benchmark model to study market economy and also an abstraction from a real economy. It can be used for either considering equilibrium prices as long-term prices or considering actual prices as deviations from equilibrium.

A price equilibrium is defined as a state where the aggregate demand does not exceed the aggregate supply for all markets. The proof of the existence of general equilibrium is generally considered one of the most important and robust results in microeconomic theory. While there are different ways of establishing the existence of general equilibrium, all the classic proofs use a fixed-point theorem (Arrow and Debreu 1954; Debreu 1982). It includes the "excess demand approach" which solves the problem by showing that there is a price system at which excess demand can be non-positive. The significance of such an approach lies in the fact that supply may not be continuous or even not be necessarily derived from profit-maximizing behavior of price-taking firms, but is determined by prices in completely different ways. It is well known that Walrasian equilibrium precludes the existence of an equilibrium in the presence of increasing returns to scale and assumes price-taking and profit-maximizing behavior. As such, some other alternative pricing rules have been proposed, such as loss-free, average cost, marginal cost, voluntary trading, and quantity-taking pricing rules in the presence of increasing returns to scale or more general types of non-convexities (cf. Beato 1982; Brown and Heal 1983; Brown et al. 1992; Cornet 1988; Bonnisseau 1988a, b, 1990; Kamiya 1988; Vohra 1988).

At the heart of the excess demand approach is a technical result known as the Gale–Nikaido–Debreu lemma (Gale 1955; Nikaido 1956; Debreu 1956). Many existence results in terms of excess demand functions or correspondences have been given. Some use Kakutani's Fixed-Point Theorem, as in Debreu (1974, 1982, 1983), while some others use modifications of the excess demand functions, as in Dierker (1974), McKenzie (1954), and Neuefeind (1980). When preferences and production sets are strictly convex, excess demand from the Walrasian pricing rule is a function rather than a correspondence. In addition, when preferences are strictly monotone, it can be defined only on the open price simplex. Some then use a technique that simplex is exhausted by an increasing sequence of compact subsets so that the Gale–Nikaido–Debreu lemma can be applied. The resulting sequence of price systems then is shown to converge to an equilibrium price system (cf. Hildenbrand 1983; Hildenbrand and Kirman 1988).

However, all the existing theorems impose convex or lattice structures on economic environments and only provide sufficient conditions for the existence of price equilibrium. To apply a fixed-point theorem (say, Brouwer, Kakutani, Tarski's fixed-point theorem, or KKM lemma, etc.), they all need to assume some forms of convexity (or transitivity/monotonicity) and continuity of preferences, in addition to convexity of commodity space. While it may be the convex/lattice structures that easily connect economics to mathematics, in many important situations there are no convex or lattice structures. A typical situation is that commodities are invisible so that the choice spaces are discrete. As a result, convexity assumption excludes the possibility of considering discrete choice problems, and consequently seriously limits the applicability of general equilibrium theory. As such, the intrinsic nature of price equilibrium has not been fully understood yet. Why does or does not a market have an equilibrium? This paper sheds some light in answering the question.

This paper fully characterizes the existence of price equilibrium in economies where commodities may be indivisible and excess demand functions may be discontinuous or do not have any structures except Walras' law. We establish a condition, called *recursive transfer lower semi-continuity*, which guarantees the existence of general equilibrium in such economies. The condition is also necessary, and thus generalizes all the existing results on the existence of equilibrium in economies with aggregate excess demand functions.

The remainder of the paper is organized as follows. In Sect. 2, we present the notion of recursive transfer lower semi-continuity. In Sect. 3, we prove that recursive transfer lower semi-continuity is a necessary and sufficient condition for the existence of equilibrium. Concluding remarks are offered in Sect. 4.

2 Notation and definitions

Consider an economy where there are L commodities and the aggregate excess demand correspondence from a pricing rule that may be Walrasian, loss-free, average cost, marginal cost, voluntary trading, or quantity-taking pricing rule is a single-valued function. The aggregate excess demand correspondence becomes a function when preferences and production possibility sets are both strictly convex for Walrasian pricing rule.

Let Δ be the closed $L-1$ dimensional unit simplex defined by

$$\Delta = \left\{ p \in \Re^L_+ : \sum_{l=1}^{L} p^l = 1 \right\}, \tag{1}$$

and let $\hat{z}(\cdot) : \Delta \to R^L$ denote the (aggregate) excess demand function for the economy. A very important property of excess demand function is Walras' law, which can take one of the following three forms:

(1) strong Walras' law

$$p \cdot \hat{z}(p) = 0 \quad \text{for all } p \in \Delta;$$

(2) weak Walras' law

$$p \cdot \hat{z}(p) \leq 0 \quad \text{for all } p \in \Delta;$$

(3) interior Walras' law

$$p \cdot \hat{z}(p) = 0 \quad \text{for all } p \in \text{int}\, \Delta,$$

where int Δ denotes the set of interior points of Δ.

Definition 2.1 A price vector $p^* \in \Delta$ is said to be a *price equilibrium* iff $\hat{z}(p^*) \leq 0$.

In words, the equilibrium price problem is to find a price vector $p^* \in \Delta$ which clears the markets for all commodities (i.e., the aggregate excess demand $\hat{z}(p) \leq 0$ for the free disposal equilibrium price or $\hat{z}(p) = 0$) under the assumption of Walras' law.

Let X be a topological space. A function $f : X \to R$ is said to be *lower semi-continuous* iff for each point x', we have

$$\liminf_{x \to x'} f(x) \geq f(x'),$$

or equivalently, iff its epigraph epi $f \equiv \{(x, a) \in X \times R : f(x) \leq a\}$ is a closed subset of $X \times R$. An excess demand function $\hat{z}(\cdot) : \Delta \to R^L$ is lower semi-continuous iff $\hat{z}^l(\cdot) : \Delta \to R$ is lower semi-continuous for $l = 1, \ldots, L$.

We say that price system p *upsets* price system q iff q's excess demand is not affordable at price p, i.e., $p \cdot \hat{z}(q) > 0$.

The following weak notion of lower semi-continuity is introduced in Tian (1992a).

Definition 2.2 An excess demand function $\hat{z}(\cdot) : \Delta \to R^L$ is transfer lower semi-continuous iff, whenever price system p upsets price system q for $q, p \in \Delta$, there exists a price system $p' \in \Delta$ and some neighborhood $\mathcal{N}(q)$ of q such that $p' \cdot \hat{z}(\mathcal{V}_q) > 0$. Here $p' \cdot \hat{z}(\mathcal{V}_q) > 0$ means that $p' \cdot \hat{z}(q') > 0$ for all $q' \in \mathcal{N}(q)$.

Remark 2.1 The transfer lower semi-continuity of $\hat{z}(\cdot)$ means that, whenever the aggregate excess demand $\hat{z}(q)$ at price vector q is not affordable at price vector p, there exists some price vector p' such that $\hat{z}(q')$ are also not affordable for all price vectors p', provided q' are sufficiently close to q. In this case, we say p' secures the upsetting relation at q locally. Note that, since $p \geq 0$, this condition is satisfied if $\hat{z}(\cdot)$ is lower semi-continuous by letting $p' = p$.

Tian (1992a) shows that transfer lower semi-continuity, together with weak Walras' law, guarantees the existence of price equilibrium. However, under Walras' law, transfer lower semi-continuity is just a sufficient, but not necessary condition for the existence of price equilibrium. To have a necessary and sufficient condition for the existence of price equilibrium, we define the following concepts.

Definition 2.3 (*Recursive Upset Pricing*) Let $\hat{z}(\cdot) : \Delta \to R^L$ be an excess demand function. We say that a price system $p^0 \in \Delta$ is *recursively upset by* $p \in \Delta$ if there exists a finite set of price systems $\{p^1, p^2, \ldots, p^m\}$ with $p^m = p$ such that $p \cdot \hat{z}(p^{m-1}) > 0$, $p^{m-1} \cdot \hat{z}(p^{m-2}) > 0, \ldots, p^1 \cdot \hat{z}(p^0) > 0$. We say that $p^0 \in \Delta$ is *m-recursively upset by* $p \in \Delta$ if the number of sequence of such upsetting relations is m.

In words, non-equilibrium price system p^0 is recursively upset by p means that there exist finite upsetting price systems $p^1, p^2, \ldots, p^m$ with $p^m = p$ such that p^0's excess demand is not affordable at p^1, p^1's excess demand is not affordable at p^2, and p^{m-1}'s excess demand is not affordable at p^m. Note that, by definition, when p^0 is recursively upset by p, it must be a non-equilibrium price system (since $p^1 \cdot \hat{z}(p^0) > 0$).

For convenience, we say p^0 is *directly upset by p* when $m = 1$, and *indirectly upset by p* when $m > 1$. Recursive upsetting says that non-equilibrium price system p^0 can be directly or indirectly upset by a price system q through sequential upsetting price systems $\{p^1, p^2, \ldots, p^{m-1}\}$ in a recursive way that p^0 is upset by p^1, p^1 is upset by p^2, ..., and p^{m-1} is upset by p.

Definition 2.4 (*Recursive Transfer Lower Semi-Continuity*) An excess demand function $\hat{z}(\cdot) : \Delta \to R^L$ is said to be *recursively transfer lower semi-continuous* on Δ iff, whenever $q \in \Delta$ is not an equilibrium price system, there exists some price vector $p^0 \in \Delta$ (possibly $p^0 = q$) and a neighborhood $\mathcal{V}_q$ such that $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for any p that recursively upsets p^0, i.e., for any sequence of price vectors $\{p^0, p^1, \ldots, p^{m-1}, p\}$, $p \cdot \hat{z}(p^{m-1}) > 0$, $p^{m-1} \cdot \hat{z}(p^{m-2}) > 0, \ldots, p^1 \cdot \hat{z}(p^0) > 0$ for $m \geq 1$ imply that $p \cdot \hat{z}(\mathcal{V}_q) > 0$.

In the definition of recursive transfer lower semi-continuity, q is transferred to p^0 that could be any point in Δ. Roughly speaking, recursive transfer lower semi-continuity of $\hat{z}(\cdot)$ means that, whenever q is not an equilibrium price system, there exists another non-equilibrium price vector p^0 such that all excess demands in some neighborhood of q are not affordable at any price vector p^m, where $\{p^1, p^2, \ldots, p^m\}$ is a finite set of price vectors and the excess demand at p^{k-1} is not affordable at p^k, $k = 0, 1, \ldots, m$. This implies that, if an excess demand function $\hat{z}(\cdot) : \Delta \to R^L$ is not recursively transfer lower semi-continuous, then there is some non-equilibrium price system q such that for every other price system p^0 and every neighborhood of q, excess demand of some price system q' in the neighborhood becomes affordable at price system p that recursively upsets p^0. We can similarly define m-recursive diagonal transfer continuity.

Definition 2.5 (*m-Recursive Transfer Lower Semi-Continuity*) An excess demand function $\hat{z}(\cdot) : \Delta \to R^L$ is said to be *m-recursively transfer lower semi-continuous* on Δ iff, whenever $q \in \Delta$ is not an equilibrium price system, there exists $p^0 \in \Delta$ and a neighborhood $\mathcal{V}_q$ such that $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for any $p \in \Delta$ that m-recursively upsets p^0.

Remark 2.2 By recursive transfer lower semi-continuity, when $p \cdot \hat{z}(p^{m-1}) > 0$, $p^{m-1} \cdot \hat{z}(p^{m-2}) > 0, \ldots, p^1 \cdot \hat{z}(p^0) > 0$, we have not only $p \cdot \hat{z}(\mathcal{V}_q) > 0$, but also $p^{m-1} \cdot \hat{z}(\mathcal{V}_q) > 0, \ldots, p^1 \cdot \hat{z}(\mathcal{V}_q) > 0$ since it is also k-recursively transfer lower semi-continuous for $k = 1, 2 \ldots, m-1$. That means all of the points in $\mathcal{V}_q$ are upset by $\{p^1, \ldots, p^{m-1}, p^m\}$ that directly or indirectly upset p^0. Thus, an excess demand function $\hat{z}(\cdot) : \Delta \to R^L$ is recursively transfer lower semi-continuous on Δ if it is m-recursively transfer lower semi-continuous on Δ for all $m = 1, 2 \ldots$

3 The existence of price equilibrium

Before proceeding to our main result, we describe the main idea why the recursive transfer lower semi-continuity ensures the existence of price equilibria. When an economy fails to have a price equilibrium, every price vector q is upset by some price vector p^0. Then, by recursive transfer lower semi-continuity, there is some open set of candidate solutions containing q, all of which will be upset by some solution p that directly or indirectly upsets p^0. Then there are finite price vectors $\{q^1, q^2, \ldots, q^n\}$ whose neighborhoods cover Δ. Then, all of the points in a neighborhood, say $\mathcal{V}_{q^1}$, will be upset by a corresponding price vector p^1, which means p^1 cannot be a point in $\mathcal{V}_{q^1}$. If it is in some other neighborhood, say, $\mathcal{V}_{q^2}$, then it can be shown that p^2 will upset all points in the union of $\mathcal{V}_{q^1}$ and $\mathcal{V}_{q^2}$ so that p^2 is not in the union. We suppose $p^2 \in \mathcal{V}_{q^3}$, and then we can similarly show that p^3 is not in the union of $\mathcal{V}_{q^1}$, $\mathcal{V}_{q^2}$ and $\mathcal{V}_{q^3}$. Repeating such a process, we can show that price vectors in $\{p^1, \ldots, p^n\}$ will not be in Δ, which is impossible. Thus recursive transfer lower semi-continuity guarantees the existence of a price equilibrium.

Now we state our main result on the existence of price equilibrium in economies that have single-valued excess demand functions.

Theorem 3.1 *Let Δ be the closed standard $L - 1$ dimensional unit simplex. Suppose an excess demand function $\hat{z}(\cdot) : \Delta \to R^L$ satisfies either the strong or the weak form of Walras' law. If $\hat{z}(\cdot)$ is recursively transfer lower semi-continuous on Δ, then there exists an equilibrium price system $p^* \in \Delta$.*

Proof Suppose, by way of contradiction, that there is no price equilibrium. Then, by recursive transfer lower semi-continuity of $\hat{z}(\cdot)$, for each $q \in \Delta$, there exists p^0 and a neighborhood $\mathcal{V}_q$ such that $p \cdot \hat{z}(\mathcal{V}_q) > 0$ whenever $p^0 \in \Delta$ is recursively upset by p, i.e., for any sequence of recursive price systems $\{p^1, \ldots, p^{m-1}, p\}$ with $p \cdot \hat{z}(p^{m-1}) > 0$, $p^{m-1} \cdot \hat{z}(p^{m-2}) > 0$, ..., $p^1 \cdot \hat{z}(p^0) > 0$ for $m \geq 1$, we have $p \cdot \hat{z}(\mathcal{V}_q) > 0$. Since there is no price equilibrium by the contrapositive hypothesis, p^0 is not a price equilibrium and thus, by recursive transfer lower semi-continuity, such a sequence of recursive price systems $\{p^1, \ldots, p^{m-1}, p\}$ exists for some $m \geq 1$.

Since Δ is compact and $\Delta \subseteq \bigcup_{q \in \Delta} \mathcal{V}_q$, there is a finite set $\{q^1, \ldots, q^T\}$ such that $\Delta \subseteq \bigcup_{i=1}^T \mathcal{V}_{q^i}$. For each of such q^i, the corresponding initial price system is denoted by p^{0i} so that $p^i \cdot \hat{z}(\mathcal{V}_{q^i}) > 0$ whenever p^{0i} is recursively upset by p^i.

Since there is no price equilibrium, for each of such p^{0i}, there exists p^i such that $p^i \cdot \hat{z}(p^{0i}) > 0$, and then, by 1-recursive transfer lower semi-continuity, we have $p^i \cdot \hat{z}(\mathcal{V}_{q^i}) > 0$. Now consider the set of price systems $\{p^1, \ldots, p^T\}$. Then, $p^i \notin \mathcal{V}_{q^i}$; otherwise, by $p^i \cdot \hat{z}(\mathcal{V}_{q^i}) > 0$, we will have $p^i \cdot \hat{z}(p^i) > 0$, contradicting to Walras' law. So we must have $p^1 \notin \mathcal{V}_{q^1}$.

Without loss of generality, we suppose $p^1 \in \mathcal{V}_{q^2}$. Since $p^2 \cdot \hat{z}(p^1) > 0$ by noting that $p^1 \in \mathcal{V}_{q^2}$ and $p^1 \cdot \hat{z}(p^{01}) > 0$, then, by 2-recursive transfer lower semi-continuity, we have $p^2 \cdot \hat{z}(\mathcal{V}_{q^1}) > 0$. Also, $q^2 \cdot \hat{z}(\mathcal{V}_{q^2}) > 0$. Thus $p^2 \cdot \hat{z}(\mathcal{V}_{q^1} \cup \mathcal{V}_{q^2}) > 0$, and consequently $p^2 \notin \mathcal{V}_{q^1} \cup \mathcal{V}_{q^2}$.

Again, without loss of generality, we suppose $p^2 \in \mathcal{V}_{q^3}$. Since $p^3 \cdot \hat{z}(p^2) > 0$ by noting that $p^2 \in \mathcal{V}_{p^3}$, $p^2 \cdot \hat{z}(p^1) > 0$, and $p^1 \cdot \hat{z}(p^{01}) > 0$, by 3-recursive transfer lower semi-continuity, we have $p^3 \cdot \hat{z}(\mathcal{V}_{q^1}) > 0$. Also, since $p^3 \cdot \hat{z}(p^2) > 0$ and $p^2 \cdot \hat{z}(p^{02}) > 0$, by 2-recursive transfer lower semi-continuity, we have $p^3 \cdot \hat{z}(\mathcal{V}_{q^2}) > 0$. Thus, we have $p^3 \cdot \hat{z}(\mathcal{V}_{q^1} \cup \mathcal{V}_{q^2} \cup \mathcal{V}_{q^3}) > 0$, and consequently $p^3 \notin \mathcal{V}_{q^1} \cup \mathcal{V}_{q^2} \cup \mathcal{V}_{q^3}$.

With this recursive process going on, we can show that $p^k \notin \mathcal{V}_{q^1} \cup \mathcal{V}_{q^2} \cup \ldots, \cup \mathcal{V}_{q^k}$, i.e., p^k is not in the union of $\mathcal{V}_{q^1}, \mathcal{V}_{q^2}, \ldots, \mathcal{V}_{q^k}$ for $k = 1, 2, \ldots, T$. In particular, for $k = T$, we have $p^L \notin \mathcal{V}_{q^1} \cup \mathcal{V}_{q^2} \ldots \cup \mathcal{V}_{q^T}$ and thus $p^T \notin \Delta \subseteq \mathcal{V}_{q^1} \cup \mathcal{V}_{q^2} \ldots \cup \mathcal{V}_{q^T}$, a contradiction.

Thus, there exists $p^* \in \Delta$ such that $p \cdot \hat{z}(p^*) \leq 0$ for all $p \in \Delta$. Letting $p^1 = (1, 0, \ldots, 0)$, $p^2 = (0, 1, 0, \ldots, 0)$, and $p^L = (0, 0, \ldots, 0, 1)$, we have $\hat{z}^l(p^*) \leq 0$ for $l = 1, \ldots, L$ and thus p^* is a price equilibrium. □

Remark 3.1 Recursive transfer lower semi-continuity is, in fact, also a necessary condition for the existence of price equilibrium. Indeed, suppose p^* is a price equilibrium and $p \cdot \hat{z}(q) > 0$ for $q, p \in \Delta$. Let $p^0 = p^*$ and $\mathcal{V}_q$ be a neighborhood of q. Then the recursive transfer lower semi-continuity holds trivially.

Remark 3.2 Although recursive transfer lower semi-continuity is necessary for the existence of price equilibrium, it may not be sufficient for the existence of price equilibrium when an excess demand function is well defined only on the set of positive price vectors. As such, recursive transfer lower semi-continuity cannot be regarded as being equivalent to the definition of price equilibrium. To see this, consider the following counterexample.

Example 3.1 Consider a two-commodity economy and let $p = (p_1, 1 - p_1)$ denote the price vector where $p_1 \in [0, 1]$. Let $\hat{z}(\cdot)$ be a function defined on int Δ such that

$$\hat{z}(p) = \left(\frac{1}{p_1}, -\frac{1}{1 - p_1}\right).$$

This excess demand function clearly satisfies Walras' law and does not have a competitive equilibrium. However, it is recursively transfer lower semi-continuous on int Δ. To see this, observe that for prices $p = (p_1, 1 - p_1)$ and $q = (q_1, 1 - q_1)$, $p \cdot \hat{z}(q) = \frac{p_1}{q_1} - (1 - p_1)\frac{1}{1-q_1} > 0$ if and only if $p_1 > q_1 > 0$.[1] As such, $\hat{z}(q)$ is upset by p if and only if $p_1 > q_1$ for $q_1 \neq 1$.

Now for any two price vectors $r, q \in$ int Δ with $r \cdot \hat{z}(q) > 0$, choose $\epsilon > 0$ such that $(q_1 - \epsilon, q_1 + \epsilon) \times (q_2 - \epsilon, q_2 + \epsilon) \subset$ int Δ. Let $p^0 = (q_1 + \epsilon, q_2 + \epsilon) \in$ int Δ and $\mathcal{V}_q \subseteq (q_1 - \epsilon, q_1 + \epsilon) \times (q_2 - \epsilon, q_2 + \epsilon)$. Then, for any sequence of price vectors $\{p^0, p^1, p^2, \ldots, p^{m-1}, p\}$ with $p \cdot \hat{z}(p^{m-1}) > 0, \ldots, p^1 \cdot \hat{z}(p^0) > 0$ (as such we must have $p_1^0 < p_1^1 < p_1^2 < \cdots < p_1^{m-1} < p_1$ by the above observation), we have $p \cdot \hat{z}(q') > 0$ for all $q' \in \mathcal{V}_q$, which means the excess demand function is recursively transfer lower semi-continuous on Δ.

Thus, Theorem 3.1 assumes that the excess demand function is well defined for all prices in the closed unit simplex Δ, including zero prices. However, when preferences are strictly monotone, excess demand functions are not well defined on the boundary of Δ. Then, some boundary conditions have been used to show the existence of price equilibrium in the case of an open set of price systems for which excess demand is

[1] Note that, even for any $p_1 < 0$, we still have $p \cdot \hat{z}(q) = \frac{p_1}{q_1} - (1 - p_1)\frac{1}{1-q_1} > 0$ as q_1 goes to 1. Thus the necessity part is not true when $q_1 = 1$.

defined—cf. Neuefeind (1980). In this case, we naturally cannot use Theorem 3.1 to fully characterize the existence of price equilibrium.

Nevertheless, Theorem 3.1 can be extended to the case of any set, especially the positive price open set, of price systems for which excess demand is defined. To do so, we introduce the following version of recursive transfer lower semi-continuity.

Definition 3.1 Let D be a subset of int Δ. An excess demand function $\hat{z}(\cdot)$: int $\Delta \rightarrow R^L$ is said to be *recursively transfer lower semi-continuous* on int Δ with respect to D iff, whenever $q \in$ int Δ is not an equilibrium price system, there exists some price system $p^0 \in D$ (possibly $p^0 = q$) and a neighborhood $\mathcal{V}_q$ such that (1) whenever p^0 is upset by a price system in int $\Delta \backslash D$, it is upset by a price system in D, and (2) $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for any finite subset of price vectors $\{p^1, \ldots, p^m\} \subset D$ with $p^m = p$ and $p \cdot \hat{z}(p^{m-1}) > 0$, $p^{m-1} \cdot \hat{z}(p^{m-2}) > 0$, $p^1 \cdot \hat{z}(p^0) > 0$ for $m \geq 1$.

Condition (1) in the above definition ensures that if q is not an equilibrium price vector, it must not be an equilibrium price vector when int Δ is constrained to be D. Note that, while $\{p^1, \ldots, p^{m-1}, p\}$ are required to be in D, p^0 is not necessarily in D but can be any point in int Δ. Also, when $D = \Delta$, recursive transfer lower semi-continuity on Δ with respect to D reduces to recursive transfer lower semi-continuity on Δ.

We then have the following theorem that fully characterizes the existence of price equilibrium in economies with possibly indivisible commodity spaces and discontinuous excess demand functions.

Theorem 3.2 *Suppose an excess demand function $\hat{z}(\cdot)$: int $\Delta \rightarrow R^L$ satisfies Walras' law: $p \cdot \hat{z}(p) = 0$ for all $p \in$ int Δ. Then there exists a compact subset $D \subseteq$ int Δ such that $\hat{z}(\cdot)$ is recursively transfer lower semi-continuous on int Δ with respect to D if and only if there exists a price equilibrium $p^* \in$ int Δ.*

Proof The proof of necessity is essentially the same as that of sufficiency in Theorem 3.1 and we just outline the proof here. We first show that there exists a price equilibrium p^* on D if it is recursively transfer lower semi-continuous on Δ with respect to D. Suppose, by way of contradiction, that there is no price equilibrium on D. Then, since $\hat{z}$ is recursively transfer lower semi-continuous on Δ with respect to D, for each $q \in D$, there exists $p^0 \in D$ and a neighborhood $\mathcal{V}_q$ such that (1) whenever p^0 is upset by a price system in $\Delta \backslash D$, it is upset by a price system in D and (2) $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for any finite subset of price systems $\{p^1, \ldots, p^m\} \subset D$ with $p^m = p$ and $p \cdot \hat{z}(p^{m-1}) > 0$, $p^{m-1} \cdot \hat{z}(p^{m-2}) > 0, \ldots, p^1 \cdot \hat{z}(p^0) > 0$ for $m \geq 1$. Since there is no price equilibrium by the contrapositive hypothesis, p^0 is not a price equilibrium and thus, by recursive transfer lower semi-continuity on Δ with respect to D, such a sequence of recursive securing price systems $\{p^1, \ldots, p^{m-1}, p\}$ exists for some $m \geq 1$.

Since D is compact and $D \subseteq \bigcup_{q \in \Delta} \mathcal{V}_q$, there is a finite set $\{q^1, \ldots, q^T\} \subseteq D$ such that $D \subseteq \bigcup_{i=1}^T \mathcal{V}_{q^i}$. For each of such q^i, the corresponding initial deviation price system is denoted by p^{0i} so that $p^i \cdot \hat{z}(\mathcal{V}_{q^i}) > 0$ whenever p^{0i} is recursively upset by p^i through any finite subset of securing price systems $\{p^{1i}, \ldots, p^{mi}\} \subset D$ with $p^{mi} = p^i$. Then, by the same argument as in the proof of Theorem 3.1, we will obtain that z^k is not in the union of $\mathcal{V}_{q^1}, \mathcal{V}_{q^2}, \ldots, \mathcal{V}_{q^k}$ for $k = 1, 2, \ldots, T$. For $k = T$, we have $p^T \notin \mathcal{V}_{q^1} \cup \mathcal{V}_{q^2} \ldots \cup \mathcal{V}_{q^T}$ and thus $p^T \notin D \subseteq \bigcup_{i=1}^{T} \mathcal{V}_{q^i}$, which contradicts that p^T is an upsetting price in D. Thus, there exists a price system p^* such that $p \cdot \hat{z}(p^*) \leq 0$ for all $p \in D$.

We now show that $p \cdot \hat{z}(p^*) \leq 0$ for all $p \in \text{int}\,\Delta$. Suppose not. p^* would be upset by a price system in int $\Delta \setminus D$, but it is upset by a price system in D, a contradiction. We want to show that p^* in fact is a price equilibrium on Δ. Note that int Δ is open and D is a compact subset of int Δ. One can always find a sequence of price vector $\{q_n^l\} \subseteq \text{int}\,\Delta \setminus D$ such that $q_n^l \to p^l$ as $n \to \infty$, where $p^l = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the unit vector that has only one argument—the lth argument—with value 1 and others with value 0. Since $p \cdot \hat{z}(q)$ is continuous in p, we have $\hat{z}^l(p^*) \leq 0$ for $l = 1, \ldots, L$ and thus p^* is a price equilibrium on Δ.

Similarly, we can show that recursive transfer lower semi-continuity on int Δ with respect to the existence of a compact D is also a necessary condition for the existence of price equilibrium.

Example 3.2 Consider again Example 3.1. We know that, although the excess demand function is recursively transfer lower semi-continuous on int Δ, there does not exist a price equilibrium. In fact, the necessity part of the above theorem implies the excess demand function is not recursively transfer lower semi-continuous on int Δ with respect to any compact subset D of int Δ. In other words, there does not exist any compact set $D \subset \text{int}\,\Delta$ such that the game is recursively transfer lower semi-continuous on int Δ with respect to D.

We can verify this directly. Indeed, for a given compact set D, let $\bar{r}_1$ be the maximum of r_1 for all $(r_1, r - q_1) \in D$. Choosing $q \in \text{int}\,\Delta \setminus D$ with $q_1 > \bar{r}_1$, we have $p \cdot \hat{z}(q) < 0$ for all $p \in D$. Also, for any p with $p_1 > q_1$, we have $p \cdot \hat{z}(q) > 0$. Now we show that one cannot find any price vector $p^0 \in \text{int}\,\Delta$ and a neighborhood $\mathcal{V}_q$ of q such that (1) whenever p^0 is upset by a price vector in int $\Delta \setminus D$, it is upset by a price vector in D and (2) $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for every price vector $p \in D$ that upsets directly or indirectly p^0. We show this by considering two cases.

Case 1 $p_1^0 \geq \bar{r}_1$. p^0 can be upset by a price vector profile $p' \in \text{int}\,\Delta \setminus D$ with $p_1' > p_1^0$, but it cannot be upset by any price vector in D.

Case 2 $p_1^0 < \bar{r}_1$. For any price vector $p \in D$ that upsets directly or indirectly p^0, we have $p \cdot \hat{z}(\mathcal{V}_q) < 0$, but not $p \cdot \hat{z}(\mathcal{V}_q) > 0$.

Thus, we cannot find any price system $p^0 \in \text{int}\,\Delta \setminus D$ and a neighborhood $\mathcal{V}_q$ of q such that (1) whenever p^0 is upset by a price system in int $\Delta \setminus D$, it is upset by a price system in D and (2) $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for every price system p that recursively upsets p^0. Hence, the excess demand function is not recursively transfer lower semi-continuous on int Δ with respect to D.

Thus, Theorem 3.2 generalizes all the existing results on the existence of price equilibrium in economies with single-valued excess demand functions, such as those in Gale (1955), Nikaido (1956), Debreu (1974, 1982), Neuefeind (1980), Aliprantis and Brown (1983), Tian (1992b), Wong (1977), Momi (2003), and Ruscitti (2011).

In general, the weaker the conditions in an existence theorem, the harder it is to verify whether the conditions are satisfied in a particular game. For this reason, it is useful to provide some sufficient conditions for recursive transfer lower semi-continuity.

Definition 3.2 (*Deviation Transitivity*) An excess demand function $\hat{z}(\cdot) : \Delta \to R^L$ is said to be *deviational transitive* if $p^2 \cdot \hat{z}(p^1) > 0$ and $p^1 \cdot \hat{z}(p^0) > 0$ implies that $p^2 \cdot \hat{z}(p^0) > 0$.

Deviation transitivity means that if excess demand at p^0 is not affordable at p^1 and excess demand at p^1 is not affordable at p^2, then excess demand at p^0 is not affordable at p^2.

We then have the following result.

Proposition 3.1 *Suppose excess demand function $\hat{z}(\cdot) : \Delta \to R^L$ is deviational transitive and satisfies weak Walras' law. If it is also 1-recursively transfer lower semi-continuous, then there exists a price equilibrium.*

Proof We only need to show that, when $\hat{z}(\cdot)$ is deviational transitive, 1-recursive transfer lower semi-continuity implies m-recursive transfer lower semi-continuity for $m \geq 1$. Suppose q is not a price equilibrium. Then, by 1-recursive transfer lower semi-continuity, there exists a price vector $p^0 \in \Delta$ and a neighborhood $\mathcal{V}_q$ of q such that $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for any finite subset of price vectors $\{p^1, \ldots, p^m\} \subset D$ with $p^m = p$ and $p \cdot \hat{z}(p^{m-1}) > 0$, $p^{m-1} \cdot \hat{z}(p^{m-2}) > 0$, $p^1 \cdot \hat{z}(p^0) > 0$ for $m \geq 1$. Then, by deviation transitivity of $\hat{z}(\cdot)$, we have $p^m \cdot \hat{z}(p^0) > 0$. Thus, by 1-recursive transfer lower semi-continuity, we have $p \cdot \hat{z}(\mathcal{V}_q) > 0$. Since m is arbitrary, $\hat{z}(\cdot)$ is recursively transfer lower semi-continuous. □

4 Conclusion

The existing results only provide sufficient conditions for the existence of price equilibrium, and no characterization result has been given for studying the essence of price equilibrium in the literature. This paper fills this gap by providing a necessary and sufficient condition for the existence of price equilibrium. We establish a condition, called *recursive transfer lower semi-continuity*, which fully characterizes the existence of price equilibrium. As such, it strictly generalizes all the existing theorems on the existence of price equilibrium.

The basic transfer method is systematically developed in Tian (1992a, 1993), Tian and Zhou (1992, 1995), Zhou and Tian (1992), and Baye et al. (1993) for studying the maximization of binary relations that may be nontotal or nontransitive and the existence of equilibrium in games that may have discontinuous or nonquasiconcave payoffs. The notion of recursive transfer continuity extends usual transfer continuity from single transfer to allowing recursive (sequential) transfers so that it turns out to be a necessary and sufficient condition for the existence of price equilibrium in economies with excess demand functions.

This recursive transfer method is somewhat similar to extending the weak axiom of revealed preference (WARP) to strong axiom of revealed preference (SARP) to fully characterize individuals' rational behavior in the consumer choice theory. Incorporating recursive transfers into various transfer continuities can also be used to obtain full characterization results for many other solution problems. Indeed, transfer irreflexive lower continuity (TILC) that shares the same feature as recursive transfer continuities has been introduced in Rodríguez-Palmero and Garcíg-Lapresta (2002) for studying the existence of maximal elements for irreflexive binary relations. Tian (2015) also uses this recursive transfer method to fully characterize the existence of equilibrium in games with general strategy spaces and payoffs, which in turn extends many existing results in discontinuous games such as those in Dasgupta and Maskin (1986), Baye et al. (1993), He and Yannelis (2015) and their references therein.

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