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An explicit tuning of the fractional order controller using a novel time delay approximation

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Abstract

In this paper, new simple tuning rules for the fractional order PID controller cascaded with a first order lead compensator are extracted analytically for standard transfer functions of time delay systems. The proposed direct synthesis controller is designed to achieve exact desired frequency domain specifications with guaranteed iso-damping property. Furthermore, the effect of the pure time delay is compensated locally around the gain crossover frequency point ω_c using a direct inversion of the proposed time delay approximation. In addition, the flexibility of the fractional order PID controller is exploited by using specific structures that allow simplifying the derivation of explicit expressions of the five parameters from desired specifications and the plant's model. These special structures can be used either by deleting the proportional action or by adding another integral or derivative action depending on the process transfer function. Tuning formulas are summarized for ten types of standard transfer functions with illustrative examples which demonstrate the implementation simplicity of the proposed tuning method with significant robustness and performance improvement.

Keywords Analytical design · Fractional order PID controller · Bode's ideal loop transfer function · Iso-damping property · Time delay systems

1 Introduction

In recent years, fractional order control has attracted great interest from the research community. In fact, several works have proved that the use of the fractional concept in the control system field offers more improvement in terms of performance and robustness of the closed-loop systems [1, 2]. The success achieved by the fractional order PID controller, which is a generalization of the classical PID, is mainly due to its flexibility where it has two more degrees of freedom that can extend the control region. In addition, different approximations techniques have been proposed to approximate the fractional order operator by a rational transfer functions [3], such as the two commonly used continuous approximations proposed in [4] and [5], which allows an easy analog or digital implementation of the fractional order controller. However, the usage of the $PI^{\lambda}D^{\mu}$ controller faces a real challenge, mainly due to the tuning complexity of its parameters, where it has five instead of three as in the classical case. During the last two decades, a lot of tuning methods have been proposed in order to improve the performance and robustness of the closed-loop system and facilitate the usage of such controller. According to the review paper [1], these tuning methods are divided into three categories: rules-based as in [6] and [7], numerical as in [8–10], and analytical design techniques as in [11, 12]. Analytical design techniques that derivate the controller parameters as an explicit expression function of the plant and desired closed-loop are the best-required techniques from practical applications [13]. The advantages of analytical design compared to numerical design are their simple implementation which gives the possibility to implement it as online indirect adaptive control.

Time delay systems pose a real challenge for controller designers. Indeed, the pure time delay decreases the system phase linearly with respect to the frequency ω which can reduce performance or destabilize the plant in a closed-loop configuration. In addition, this delay causes a limitation in the desired closed-loop system bandwidth [14] which forces the designer to decelerate the system in order to maintain its stability. In general, the main difficulty can be found precisely in the control of dominated dead time systems where the time

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delay is longer than the time constant of the plant [15]. In the literature, many works used the fractional order control for time delay systems as in [16]. Some of these methods considered the iso-damping property in the controller design such as in [8, 17–19] and [20], where authors used a flat phase condition described by the open loop derivative of the argument to be equal to zero at the gain crossover frequency. However, analytical methods are very limited in these variants such as [6] where the tuning rules were extracted from the numerical design technique and other works are proposed in [7] and [21] where the tuning is based on frequency domain specifications without verifying of the iso-damping property. In addition, the research work presented in [22] where the iso-damping property is loosed because of using an open loop function that include a time delay. Using the same delayed Bode's loop transfer function, the work [23] proposed a new design methodology for a special fractional order model with dead time without consideration of the iso-damping property. A robust controller has been proposed in [19] for the first order plus time delay processes, where the frequency domain and the time domain specifications are taken into consideration to achieve an optimal controller. A recent analytical controller, designed with fulfilling five frequency domain specifications simultaneously, including the iso-damping property, has been proposed in [20] which is dedicated to first order normalized plus time delay model where the tuning procedure is a bit complex for a less-experienced controller designer. To the extent knowledge of authors, there has been no research work in the literature that proposed explicit analytical tuning rules for time delay systems with guaranteed iso-damping property.

In the present work, an analytical tuning method of the fractional order controller for time delay systems is proposed. This method consists in one hand, of compensating the system poles by using the flexibility of the fractional $PI^{\lambda}D^{\mu}$ controller for the purpose of setting the closed-loop system equivalent to the Bode's ideal transfer function, and in the other hand to compensate the pure time delay effect by using a first order lead compensator in series with the $PI^{\lambda}D^{\mu}$ controller. The compensation of the pure time delay is carried out, for the first time, using the direct inversion of a novel rational minimum phase approximation around the gain crossover frequency of the projected feedback system. The validity of this approximation has been studied and a validity condition has been derived as a function of the selected system bandwidth.

2 Bode's Ideal loop transfer function as reference model

Consider the fractional order system given by the following transfer function [24]:

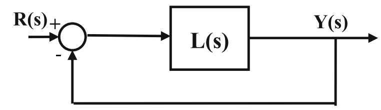

Fig. 1 Bode's ideal loop

$$G_d(s) = \frac{1}{1 + \left(\frac{s}{\omega_c}\right)^m} \tag{1}$$

where m and ω_c are real positive that gives the system's order and the gain crossover frequency.

This system includes several dynamics behaviors depending on its fractional order m, in the case of $1 < m < 2$ its presents an oscillation mode. System (1) is the closed-loop transfer function of the fractional order integrator with a static gain equal to $\omega_c{}^m$ as shown in Fig. 1.

The open loop transfer function L(s) is given as:

$$L(s) = \frac{1}{\left(\frac{s}{\omega_c}\right)^m} \tag{2}$$

Bode called the transfer function L(s), the ideal open loop transfer function [25]. In the Bode diagram, the magnitude of L(s) is a straight line with constant slop of -20mdB/dec, and its phase curve is a horizontal line at $-20m\pi/2$ rad, which indicates that L(s) has important robustness against the static gain variation (called the iso-damping property) where the phase is independent to the gain crossover frequency. This important characteristic motivated the majority of proposed works in the literature, starting with the CRONE controller [5], to use the system (1) as a reference system during the fractional order controller design. This robustness property is the main additional tangible advantage given by using the fractional order control compared to the classical control concept.

3 Problem formulation

Figure 1 shows the conventional control scheme of the SISO system with unity feedback.

$G_p(s)$ and C(s) are the transfer functions of the plant and the fractional order controller, respectively.

The considered plant transfer function is given as:

$$G_p(s) = G_f(s)\, e^{-\theta s} \tag{3}$$

where $G_f(s)$ is the delay-free part of the plant transfer function, θ is time delay of the considered plant.

In order to design a closed-loop system has the iso-damping property, it is necessary to satisfy the following

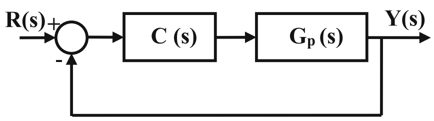

Fig. 2 Unity feedback control system

desired open loop transfer function:

$$G_o(s) = C(s)G_f(s)e^{-\theta s} \approx \frac{1}{(s/\omega_c)^m} \quad (4)$$

Mathematically, an equivalent open loop such as (2) needs a special structure of the controller C(s) that can compensate the effect of the exponential term. Unfortunately, the term of ***exp (+ θs)*** is not realizable which imply that the design of the exact Bode's open loop cannot be achieved for time delay systems.

Model-based tuning methods, which are divided into the internal model (IMC) and the direct synthesis (DS) approaches, are widely approximates the time delay part either by the Padè approximation as in [26]:

$$e^{-\theta s} = \frac{1 - \theta/2\, s}{1 + \theta/2s} \quad (5)$$

or by the first order Taylor series approximation which is given by:

$$e^{-\theta s} = 1 - \theta s \quad (6)$$

These two approximations has been used in [27] and [28] where the IMC controller has not satisfied a flat phase due to no minimum phase part which gives an unstable controller. This problem has been avoided in [27] and [28], where the IMC technique has been used and only the minimum phase part has been taken in controller design (Fig. 2).

The commonly used Padé approximation (5) gives an exact magnitude for all frequency ranges but not for the phase where the argument is an arctangent function of ω which approximates the exact phase -θω around 0. Indeed, there is no rational approximation that can give the exact phase of -θω for all the frequency ranges.

The problem investigated in this paper consists of the direct conception of the desired open loop of (4) for general stable minimum phase plant given by (3).

4 Proposed controller design:

The condition (4) can be partially satisfied as follows:

The proposed controller is composed of two cascaded parts as:

$$C(s) = C_m(s)\, C_n(s) \quad (7)$$

The open loop transfer function is designed as:

$$\underbrace{(C_m(s)G_f(s))}_{\approx \frac{1}{(s/\omega_c)^m}}\ \underbrace{(C_n(s)e^{-\theta s})}_{\approx 1} \approx \frac{1}{(s/\omega_c)^m} \quad (8)$$

The first part $C_m(s)$ is the fractional order controller that satisfies the Bode's ideal open loop for the delay-free system G_f (s) and the second part $C_n(s)$ is a rational minimum phase transfer function that compensates the time delay effect around the specific point $s = j\omega_c$.

The proposed controller is designed using two steps:

Step1: Design q rational minimal phase approximation of the pure time delay $f(s) \approx e^{-\theta s}$ then, the compensator Cn(s) is obtained as:

$$C_n(s) = \frac{1}{f(s)} \quad (9)$$

in section (4.1) the details of this proposition is presented with an illustrative simulation.

Step 2: Design the controller $C_m(s)$ for the delay-free system as follows:

Form equation (8), the controller $C_m(s)$ is derived as:

$$C_m(s) = \frac{1}{(s/\omega_c)^m}\frac{1}{G_f(s)} \quad (10)$$

For a stable plant, this controller has a structure similar to the fractional PID controller defined by [29]:

$$C_m(s) = k_c + \frac{T_i}{s^{\lambda}} + T_d s^{\mu} \quad (11)$$

In Sect. (4.2), a general study of different cases for the linear system $G_f(s)$ will be presented with extraction of analytical tuning formulas of the five parameters (K_c, T_i, T_d, λ and μ) that satisfy equivalence between $C_m(s)G_f(s)$ and the fractional order integral $1/(s/\omega_c)^m$.

4.1 Design of the time delay compensator $C_n(s)$

4.1.1 Main result:

Theorem 1. *Consider the following transfer function:*

$$f(s) = g\left(\frac{1+as}{1+bs}\right) \quad (12)$$

With $b > a$

This transfer function is a lag compensator which gives a negative phase in a specific frequency band (inferior to -45° in [1/b 1/a]). Indeed, the pure time delay system $e^{-\theta s}$can be approximated around the specific point $s = j\omega_c$by the rational minimal phase transfer (12).

where the three parameters a, b and g are given by:

$$\begin{cases} \text{a is chosen as} \quad 1/a >> \omega_c \\ b = \frac{1}{\omega_c}\tan\left\{\tan^{-1}(a\omega_c) + \theta\,\omega_c\right\} \\ g = \frac{\sqrt{1+\tan^2\{\tan^{-1}(a\omega_c)+\theta\,\omega_c\}}}{\sqrt{1+(a\omega_c)^2}} \end{cases} \tag{13}$$

In order to obtain the same phase slope, the specific frequency $\omega_c$must be chosen with the following condition:

$$\theta\,\omega_c \leq 0.5 \tag{14}$$

Proof. The transfer function of Eq. (12) is a lag compensator, then it gives a negative phase inferior of -π/4 between [1/b 1/a]. For this reason, the parameter a is a positive parameter chosen as 1/b<<1/a where the parameter b is generally obtained around ω_c, so the parameter a must satisfy the condition.

The equivalence between the pure time delay e-θs and its approximated function around $s = j\omega_c$ can be obtained by satisfying the following three conditions:

$$\begin{cases} \left|f(j\omega_c)\right| = \left|e^{-j\omega_c\theta}\right| = 1 & \text{(i)} \\ \angle f(j\omega)|_{\omega=\omega_c} = \angle e^{-j\omega\theta}\big|_{\omega=\omega_c} = -\theta\omega_c & \text{(ii)} \\ \frac{d}{d\omega}\angle f(j\omega)\big|_{\omega=\omega_c} = \frac{d}{d\omega}\angle e^{-j\omega\theta}\big|_{\omega=\omega_c} = -\theta & \text{(iii)} \end{cases} \tag{15}$$

Analytical resolution of condition (i) and (ii) allow to obtain the two parameters b and g, the third condition (iii) is considered as a constraint for choosing specific frequency ω_c.

Let two functions of ω $A_1(\omega)$ and $A_2(\omega)$, which are argument of (5) and (12), respectively, given as:

$$\begin{cases} A_1(\omega) = \arg(e^{-\theta\,\omega\,j}) = -\,\theta\,\omega \\ A_2(\omega) = arg(f(\omega j)) = \tan^{-1}(a\,\omega) - \tan^{-1}(b\,\omega) \end{cases} \tag{16}$$

The parameter b is chosen such as the following condition is satisfied (ii):

$$A_1(\omega)\,|_{\omega=\omega_c} = A_2(\omega)\,|_{\omega=\omega_c} \tag{17}$$

So:

$$\tan^{-1}(a\omega_c) - \tan^{-1}(b\omega_c) = -\theta\omega_c \tag{18}$$

$$\Rightarrow \quad b = \frac{1}{\omega_c}\tan\left\{\tan^{-1}(a\omega_c) + \theta\,\omega_c\right\} \tag{19}$$

Then, the parameter g is calculated for first condition (i):

$$|f(j\omega)||_{\omega=\omega_c} = 1 \qquad \Leftrightarrow g\left(\frac{\sqrt{1+(a\omega_c)^2}}{\sqrt{1+(b\omega_c)^2}}\right) = 1 \tag{20}$$

Which allow to derivate the parameter g as:

$$g = \left(\frac{\sqrt{1+(b\omega_c)^2}}{\sqrt{1+(a\omega_c)^2}}\right) \tag{21}$$

The condition (iii) must be verified for two reasons:

- The lead compensator gives a maximum negative phase equal to -π/2 which imposes a first limitation in as $\theta\,\omega_c \leq 1.57$ where the slop in the phase curve at the limitation point 1.57 is equal to 0.
- The objective of taking the same slop in the phase between the pure time delay and its approximation.

Let $A_1'(\omega)$ and $A_2'(\omega)$ derivatives of tow functions of (16) with respect to ω.

$$\begin{cases} A_1'(\omega) = -\,\theta \\ A_2'(\omega) = \frac{a}{1+(a\omega)^2} - \frac{b}{1+(b\omega)^2} \end{cases} \tag{22}$$

$$A_2'(\omega_c) = \frac{a}{1+(a\omega_c)^2} - \frac{b}{1+(b\omega_c)^2} = -\frac{b}{1+(b\omega_c)^2} \tag{23}$$

$$\begin{cases} b = \frac{1}{\omega_c}\tan\left\{\tan^{-1}(a\omega_c) + \theta\,\omega_c\right\} \\ a\omega_c << 1 \end{cases} \Rightarrow b \approx \frac{1}{\omega_c}\tan\{a\,\omega_c + \theta\,\omega_c\} \tag{24}$$

$$a\,\omega_c << \theta\,\omega_c \Rightarrow b \approx \frac{1}{\omega_c}\tan(\theta\omega_c) \tag{25}$$

The function tan(x) has a linear characteristic for a small value of x.

With $\theta\omega_c \leq 0.5 \Rightarrow b \approx \theta$, we have: tan(x) = x for x < 0.5 as it shown in Fig. 3

Finally:

$$\begin{cases} A_2'(\omega_c) = -\frac{\theta}{1+(\theta\omega_c)^2} \\ \theta\omega_c \leq 0.5 \end{cases} \Rightarrow A_2'(\omega_c) = -\theta = A_1'(\omega_c) \tag{26}$$

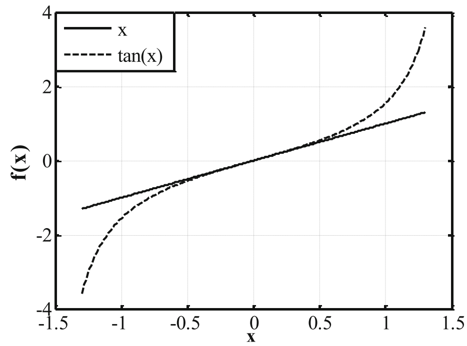

Fig. 3 Curve of the tangent function

4.1.2 Illustrative example

Let the pure time delay transfer function given by:

$$f_1(s) = e^{-50s} \tag{27}$$

The rational first order Padé approximation of $f_1(s)$ is given by:

$$f_2(s) = \frac{1-25s}{1+25s} \tag{28}$$

The minimal phase part of this rational function is given by:

$$f_3(s) = \frac{1}{1+25s} \tag{29}$$

The proposed approximation around $s = 0.008j$ ($\omega_c = 0.008$) is given by:

$$f_4(s) = 1.0861\left(\frac{1+0.1s}{1+52.8609s}\right) \tag{30}$$

Figure 4 shows the Bode diagrams of the four functions (27)-(30). It is clear that the Padè approximation gives an exact value of magnitude in all frequency range in addition it is the most large approximation around zero point for argument, but the proposed one gives a more exact approximation around its specific point $\omega_c = 0.008$.

In order to show the effect of condition (14), the specific point ω_c is changed from 0.002 to 0.018 which equivalent to $0.1 \leq \theta\,\omega_c \leq 0.9$. The phase of approximated function is shown in Fig. 5. It is clear that for ω_c increase the slop of argument (f(s)) decrease and the condition (iii) is looses.

4.2 Tuning of the fractional order controller $C_m(s)$ for delay-free system

In this section, a simple analytical tuning method for fractional order PID controller is proposed. The controller $C_m(s)$

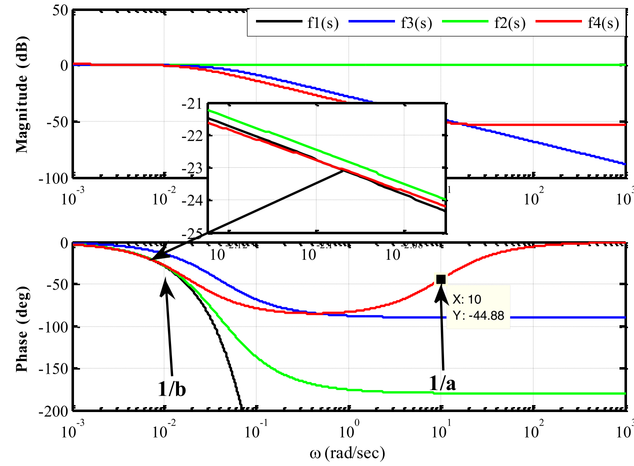

Fig. 4 Bode diagrams of the four functions (27)-(30)

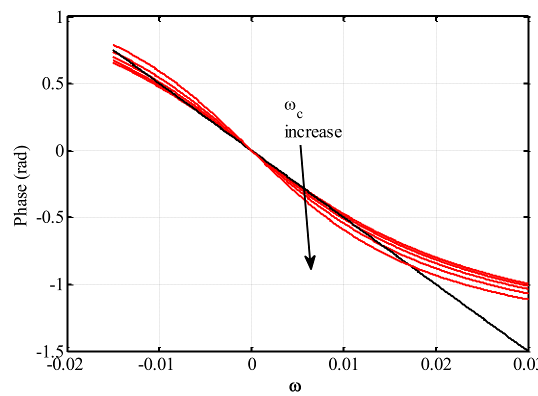

Fig. 5 Argument of $f_4(s)$ for different values of ω_c (red lines)

is derived from Eq. (10) in order to taking the open loop equivalent to the Bode's transfer function. Different standard transfer functions are considered in the present study.

4.2.1 Generalized first order system

The general form transfer function of the fractional first order system is known as following:

$$G_f(s) = \frac{k}{1+T_s^{\alpha}} \tag{31}$$

$$0 < \alpha < 2$$

where k and T are, respectively, the static gain and the time constant of the considered system. This system is delay-free part of the One Non-Integer Order plus Time Delay system (NIOPTD-I) proposed in [30]. In addition, the model of Eq. (31) included the conventional first order system is case of $\alpha = 1$.

Using the fractional order $PI^{\lambda}D^{\mu}$ controller of Eq. (11), the open loop transfer function is given as:

$$G_o(s) = \left(k_c + \frac{T_i}{s^{\lambda}} + T_d s^{\mu}\right)\frac{k}{(1+Ts^{\alpha})} = T_i\left(\frac{1+\frac{k_c}{T_i}s^{\lambda}+\frac{T_d}{T_i}s^{\mu+\lambda}}{s^{\lambda}}\right)\frac{k}{(1+Ts^{\alpha})} \quad (32)$$

in order to take the loop transfer function $G_o(s)$ equal to the fractional order integrator (8), the controller (11) is designed such that it compensate the process denominator by its numerator as following:

- The proportional action is set to zero (kc = 0) than the open loop will be:

$$G_o(s) = T_i\left(\frac{1+\frac{T_d}{T_i}s^{\mu+\lambda}}{s^{\lambda}}\right)\frac{k}{(1+Ts^{\alpha})} \quad (33)$$

From Eq. (33), the function $G_o(s)$ can be equal to the desired function (8) if we choose the controller's parameters as follows:

$$\begin{aligned} &\lambda = m \\ &T_i k = \omega_c^m \quad \Rightarrow \quad T_i = \omega_c^m/k \\ &T_d/T_i = T \quad \Rightarrow \quad T_d = T\,T_i \\ &\mu+\lambda = \alpha \quad \Rightarrow \quad \mu = \alpha - m \end{aligned} \quad (34)$$

Finally, for any generalized first order system, the FOPID controller can be tuned by Eq. (34) to satisfy equivalence between the open loop transfer function and the Bode's ideal transfer function (8). Hence, for a required dynamics specification given in terms of unity gain crossover frequency ω_c and phase margin φ_m the five parameters of FOPID are given by:

$$\begin{cases} k_c = 0 \\ T_i = \omega_c^m/k \\ T_d = T_2\omega_c^m/k \\ \lambda = m \\ \mu = \alpha - m \end{cases} \quad (35)$$

The last formulas is valid for the integer order system by taking $\alpha = 1$ in (35).

4.2.2 Generalized first order system plus fractional integrator

Consider the fractional order system defined by the following transfer function:

$$G_f(s) = \frac{k}{s^{\alpha}\left(1+Ts^{\beta}\right)} \quad (36)$$

where α, β are positive real gives the orders of the plant.

Using the fractional order PID controller of Eq. (11), the open loop transfer function is given as:

$$G_o(s) = T_i\left(\frac{1+\frac{k_c}{T_i}s^{\lambda}+\frac{T_d}{T_i}s^{\mu+\lambda}}{s^{\lambda}}\right)\frac{k}{s^{\alpha}\left(1+Ts^{\beta}\right)} \quad (37)$$

In order to take the function (37) to be equivalent to the desired function of Eq. (8), the following considerations are taking:

$$\begin{aligned} &k_c = 0 \\ &\lambda+\alpha = m \quad \Rightarrow \quad \lambda = m-\alpha \\ &T_i k = \omega_c^m \quad \Rightarrow \quad T_i = \omega_c^m/k \\ &T_d/T_i = T \quad \Rightarrow \quad T_d = T\,T_i \\ &\mu+\lambda = \beta \quad \Rightarrow \quad \mu = \beta+\alpha-m \end{aligned} \quad (38)$$

So, the tuning formulas of the fractional $PI^{\lambda}D^{\mu}$ controller are given by:

$$\begin{cases} k_c = 0 \\ T_i = \omega_c^m/k \\ T_d = T_2\omega_c^m/k \\ \lambda = m-\alpha \\ \mu = \beta+\alpha-m \end{cases} \quad (39)$$

The last formulas is valid for the integer first order plus integrator system by taking $\alpha = 1$ and $\beta = 1$ in (39).

4.2.3 Generalized second order system

Consider the fractional order system defined by the following transfer function:

$$G_f(s) = \frac{k}{a_1 s^{\beta} + a_2 s^{\alpha} + 1} \quad (40)$$

wherea_1 and a_2 are real coefficients parameters and k is the static gain of the plant.

This last is delay-free part of the Two Non-Integer Order plus Time Delay system (NIOPTD-II) proposed in [30]. Note that this model included the ordinary second order system in the special case where $\alpha = 1$ and $\beta = 2$. In this case the system parameters are given as

$$\begin{aligned} &\omega_n = 1/\sqrt{a_1} \\ &\xi = 0.5\,a_2/\sqrt{a_1} \end{aligned} \quad (41)$$

where ω_n and ζ are the parameters of the ordinary second order system.

Using the fractional order PID controller of Eq. (11), the open loop transfer function is given as:

$$G_o(s) = T_i\left(\frac{1+\frac{k_c}{T_i}s^{\lambda}+\frac{T_d}{T_i}s^{\mu+\lambda}}{s^{\lambda}}\right)\frac{k}{\left(a_1 s^{\beta}+a_2 s^{\alpha}+1\right)} \tag{42}$$

In order to set the loop transfer function equivalent to desired one of Eq. (8), the numerator of Eq. (42) must be equal to its denominator except the term of s^{α}. This can be satisfied by taking the following conditions:

$$\begin{aligned} &\lambda = m \\ &T_i k = \omega_c^m \quad \Rightarrow \quad T_i = \omega_c^m/k \\ &T_d/T_i = a_1 \quad \Rightarrow \quad T_d = a_1 T_i \\ &\mu+\lambda = \beta \quad \Rightarrow \quad \mu = \beta - m \end{aligned} \tag{43}$$

It is clear that it still one term which is the terms of s^{α} that must be satisfying the condition:

$$k_c \frac{k}{\omega_c} s^{\lambda} \approx a_2 s^{\alpha} \tag{44}$$

This condition is realized by using a special structure of controller as follows:

$$C_m(s) = \frac{k_c}{s^{\lambda-\alpha}} + \frac{T_i}{s^{\lambda}} + T_d s^{\mu} \tag{45}$$

By using this controller, the open loop transfer function will be as follows:

$$G_o(s) = T_i\left(\frac{1+\frac{k_c}{T_i}s^{\alpha}+\frac{T_d}{T_i}s^{\mu+\lambda}}{s^{\lambda}}\right)\frac{k}{\left(a_1 s^{\beta}+a_2 s^{\alpha}+1\right)} \tag{46}$$

This function can be set equivalent to the desired function (8) by taking the parameters values of Eq. (43) with choosing the proportional action $k_c = T_i\, a_2$.

The tuning formulas of FOC defined by Eq. (45) are summarized by:

$$\begin{cases} k_c = a_2\omega_c^m/k \\ T_i = \omega_c^m/k \\ T_d = a_1\omega_c^m/k \\ \lambda = m \\ \mu = \beta - m \end{cases} \tag{47}$$

4.2.4 Generalized second order plus integrator system

Consider the fractional order system defined by:

$$G_f(s) = \frac{k}{\left(a_1 s^{\beta}+a_2 s^{\alpha}+1\right) s^{\gamma}} \tag{48}$$

where a_1 and a_2 are real coefficients parameters and (α, β and γ) are positive real orders.

Using the same structure of FO controller of Eq. (45), the loop transfer function is given as:

$$G_o(s) = T_i\left(\frac{\frac{k_c}{T_i}s^{\alpha}+1+\frac{T_d}{T_i}s^{\mu+\lambda}}{s^{\lambda}}\right)\frac{k}{\left(a_1 s^{\beta}+a_2 s^{\alpha}+1\right) s^{\gamma}} \tag{49}$$

Similar to the previous subsection, the controller's parameters are given by:

$$\begin{cases} k_c = a_2\omega_c^m/k \\ T_i = \omega_c^m/k \\ T_d = a_1\omega_c^m/k \\ \lambda = m - \gamma \\ \mu = \beta + \gamma - m \end{cases} \tag{50}$$

4.2.5 Fractional order system with two times constant

Consider the fractional order system defined by the following transfer function:

$$G_f(s) = \frac{k}{(1+T_1 s^{\alpha})\left(1+T_2 s^{\beta}\right)} \tag{51}$$

where T_1 and T_2 are the time constants and (α, β) are positive real orders.

In this case, a special structure of FO-PID controller is used as:

$$C_m(s) = \frac{k_c}{s^{\lambda-\alpha}} + \frac{T_b}{s^{\lambda-\beta}} + \frac{T_i}{s^{\lambda}} + T_d s^{\mu} \tag{52}$$

By using this controller, the open loop transfer function will be as following:

$$G_o(s) = T_i\left(\frac{1+\frac{k_c}{T_i}s^{\alpha}+\frac{T_b}{T_i}s^{\beta}+\frac{T_d}{T_i}s^{\mu+\lambda}}{s^{\lambda}}\right) \frac{k}{\left(T_1 T_2 s^{\alpha+\beta}+T_1 s^{\alpha}+T_2 s^{\beta}+1\right)} \tag{53}$$

Table 1 Explicit expressions of controller's parameters for delay-free systems

$G_f(s)$	$C_m(s)$	k_c	T_i	T_d	λ	μ	T_b
$\frac{k}{(1+Ts)}$	$k_c+\frac{T_i}{s^\lambda}+T_d s^\mu$	0	$\frac{\omega_c^m}{k}$	$T\frac{\omega_c^m}{k}$	m	1-m	
$\frac{k}{s(1+Ts)}$	$k_c+\frac{T_i}{s^\lambda}+T_d s^\mu$	0	$\frac{\omega_c^m}{k}$	$T\frac{\omega_c^m}{k}$	m-1	2-m	
$\frac{k}{\left(\frac{s}{\omega_n}\right)^2+\frac{2\xi}{\omega_n}s+1}$	$\frac{k_c}{s^{\lambda-1}}+\frac{T_i}{s^\lambda}+T_d s^\mu$	$\frac{2\xi\omega_c^m}{k\omega_n}$	$\frac{\omega_c^m}{k}$	$\frac{\omega_c^m}{k\omega_n^2}$	m	2-m	
$\frac{k}{\left(\left(\frac{s}{\omega_n}\right)^2+\frac{2\xi}{\omega_n}s+1\right)s}$	$\frac{k_c}{s^{\lambda-1}}+\frac{T_i}{s^\lambda}+T_d s^\mu$	$\frac{2\xi\omega_c^m}{k\omega_n}$	$\frac{\omega_c^m}{k}$	$\frac{\omega_c^m}{k\omega_n^2}$	m-1	3-m	
$\frac{k}{(1+T_1s)(1+T_2s)}$	$\frac{k_c}{s^{\lambda-1}}+\frac{T_i}{s^\lambda}+T_d s^\mu$	$\frac{(T_1+T_2)\omega_c^m}{k}$	$\frac{\omega_c^m}{k}$	$\frac{T_1T_2\omega_c^m}{k}$	m	2-m	
$\frac{k}{1+Ts^\alpha}$	$k_c+\frac{T_i}{s^\lambda}+T_d s^\mu$	0	$\frac{\omega_c^m}{k}$	$T\frac{\omega_c^m}{k}$	m	α-m	
$\frac{k}{s^\alpha(1+Ts^\beta)}$	$k_c+\frac{T_i}{s^\lambda}+T_d s^\mu$	0	$\frac{\omega_c^m}{k}$	$T\frac{\omega_c^m}{k}$	m-α	β + α- m	
$\frac{k}{a_1s^\beta+a_2s^\alpha+1}$	$\frac{k_c}{s^{\lambda-\alpha}}+\frac{T_i}{s^\lambda}+T_d s^\mu$	$\frac{a_2\omega_c^m}{k}$	$\frac{\omega_c^m}{k}$	$a_1\frac{\omega_c^m}{k}$	m	β-m	
$\frac{k}{(a_1s^\beta+a_2s^\alpha+1)\ s^\gamma}$	$\frac{k_c}{s^{\lambda-\alpha}}+\frac{T_i}{s^\lambda}+T_d s^\mu$	$\frac{a_2\omega_c^m}{k}$	$\frac{\omega_c^m}{k}$	$a_1\frac{\omega_c^m}{k}$	m-γ	β + γ- m	
$\frac{k}{(1+T_1s^\alpha)(1+T_2s^\beta)}$	$\frac{k_c}{s^{\lambda-\alpha}}+\frac{T_b}{s^{\lambda-\beta}}+\frac{T_i}{s^\lambda}+T_d s^\mu$	$T_1\frac{\omega_c^m}{k}$	$\frac{\omega_c^m}{k}$	$T_1T_2\frac{\omega_c^m}{k}$	m	β + α- m	$T_1\frac{\omega_c^m}{k}$

This function can be set equivalent to the desired function (8) by taking the following controller parameters:

$$\begin{cases} k_c = T_1\omega_c^m/k \\ T_i = \omega_c^m/k \\ T_d = T_1T_2\omega_c^m/k \\ T_b = T_2\omega_c^m/k \\ \lambda = m \\ \mu = \alpha+\beta-m \end{cases} \tag{54}$$

Table 1 summarizes the results of this work where tow case of the plant are considered (fractional and integer order). As it shown in subSects. 4.2, all analytical controllers presented for delay-free systems can be cascaded with the inverse of proper filter f(s) of Eq. (12) for controlling the same system with dead time θ. For example, if the first order plus dead time system (FOPDT) is considered, the analytical controller is obtained as:

$$C(s) = \left(\frac{T_i}{s^\lambda}+T_d s^\mu\right)\frac{1}{f(s)} \tag{55}$$

where T_i,T_d, λ and μ are given from Table 1 and f(s) is computed from Eqs. (12) and (13).

5 Simulation results

5.1 Example 1: Dominated dead time system

Consider the system given by the following transfer function

$$G_p(s) = \frac{1}{1+0.05s}e^{-s} \tag{56}$$

This system has dominated dead time ($\theta >> T$) which take its control more difficult compared with lag dominated system ($T >> \theta$) [15]. Usually, this type of system is decelerating in closed-loop configuration. In the work presented in [31], authors are proposed the following controller:

$$C_{ph}(s) = \left(0.41+\frac{0.24}{s}\right)\left(\frac{1+0.1439\ s^{0.666}}{s^{0.666}}\right) \tag{57}$$

This controller was satisfied the following dynamics specifications:

$$\begin{cases} \omega_c = 0.521\ \text{rad/sec} \\ \phi_m = 44.8^\circ \end{cases} \tag{58}$$

For a fairly comparison, the proposed controller is designed for the same specification and its transfer function

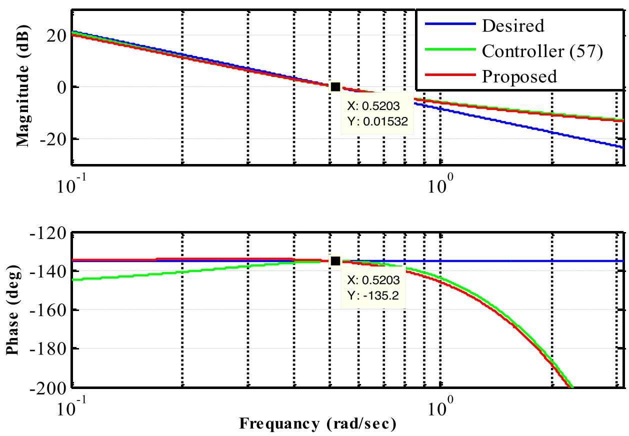

Fig. 6 Bode diagrams of open loops with the two controllers of example 1

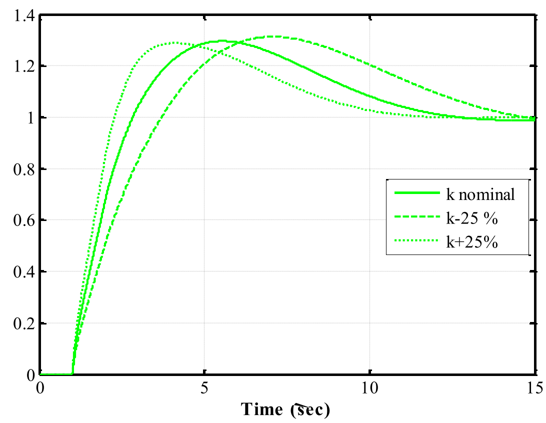

Fig. 8 Step responses of plant (56) with controller (57) for different static gains k

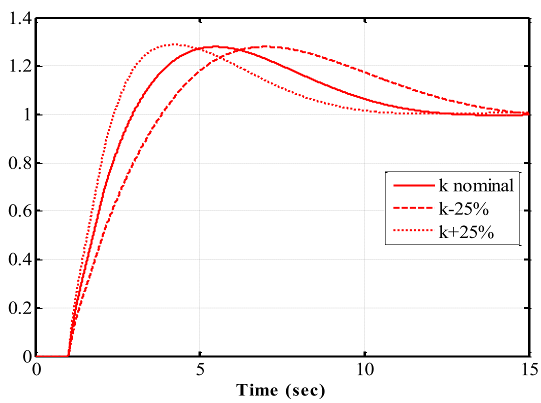

Fig. 7 Step responses of plant (56) with proposed controller for different static gains k

is given by:

$$C(s) = 0.8544\left(\frac{1+1.169\,s}{1+0.05\,s}\right)\left(\frac{0.3755}{s^{1.5}} + 0.0188\ s^{-0.5}\right) \quad (59)$$

Figure 6 shows the Bode diagram of open loop system obtained by two controllers (57) and (59). It is clear that the two controllers are satisfies the desired dynamic performances with a flat phase around the gain crossover frequency ω_c.

Figures 7 and 8 show, respectively, the step response of the two feedback systems by the controllers (57) and (59), when the plant's static gain changed about $\pm$ 25% from its nominal value. It is clear that the iso-damping property is assured by the two feedback controllers.

Table 2 shows the different characteristics of the closed-loop system with the tow controllers compared to the desired specifications where the third column gives the phase open loop derivative at ω_c, F_{ISEm} and F_{ISEph} gives, respectively, the integral square error ISE between desired and achieved values of the magnitude and phase. t_{ISE} is the time ISE between desired and achieved responses. Σov is the sum of variations in overshoot in case of static gain change.

From Table 2 it is clear that the proposed controller gives more equivalent characteristics to the desired one than the controller (57).

5.2 Example 2: Second order plus dead time system

Let the second order system given by the following transfer function:

$$G_p(s) = \frac{0.0015}{s^2 + 0.1547\,s + 0.004357}e^{-8\,s} \quad (60)$$

The required specifications are given by:
$\omega_c = 0.01$ rad/sec and $\varphi_m = 70°$

The controller obtained using the proposed design method is given by:

$$C(s) = 0.9964\left(\frac{1+8.5205\,s}{1+0.5\,s}\right)\left(\frac{0.3706}{s^{0.222}} + \frac{0.0104}{s^{1.222}} + 2.3959\ s^{0.78}\right) \quad (61)$$

This last is compared with a fractional $PI^{\lambda}D^{\mu}$ designed for the same specification using an optimization technique in [17], its transfer function is given as:

$$C_{FOPID}(s) = 1.1030 + \frac{0.0141}{s^{1.1741}} + 18.1322s^{1.2893} \quad (62)$$

For a fair comparison, another analytical controller (63) is designed for the same specification using method proposed in [28] where the internal model control is used to extract an analytical formulas for conventional controller. Not that this

Table 2 Frequency and time domain characteristics of example 1

	ω_c	φ_m	$Darg/d\omega\|_{\omega c}$	F_{ISEm}	F_{ISEph}	t_{ISE}	Σov
Desired	0.521	44.8	0	0	**0**	**0**	**0**
Proposed controller	0.521	44.8	−0.013	1.160	**7.8684**	**0.0027**	**7.487 10^{-5}**
Controller (57)	0.521	44.8	**−0.00013**	0.954	25.379	0.0031	34.868 10^{-5}

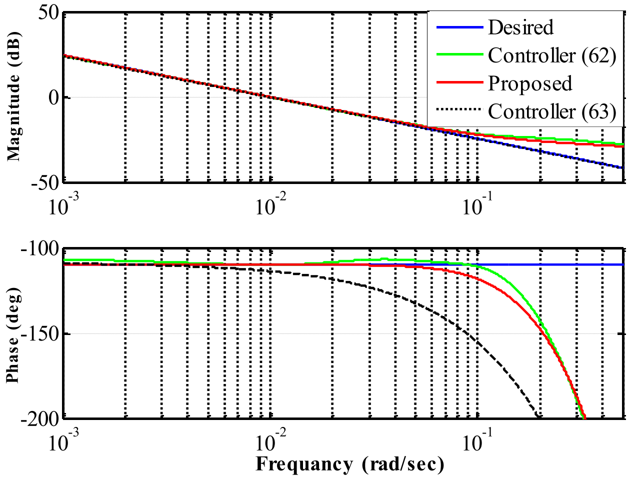

Fig. 9 Bode diagrams of open loops with the three controllers for example 2

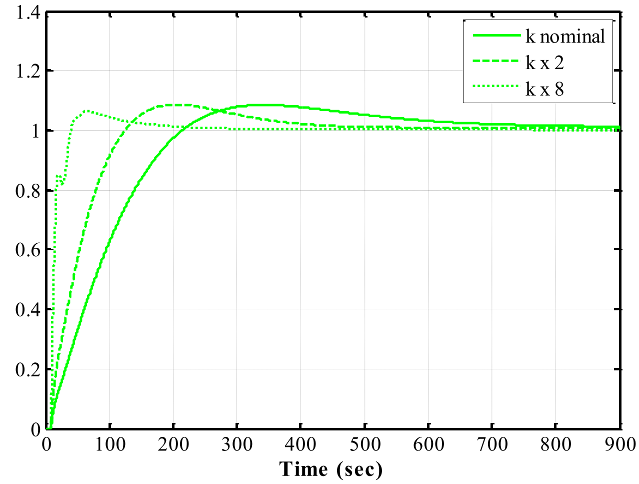

Fig. 11 Step responses of plant (60) with controller (62) for different static gains k

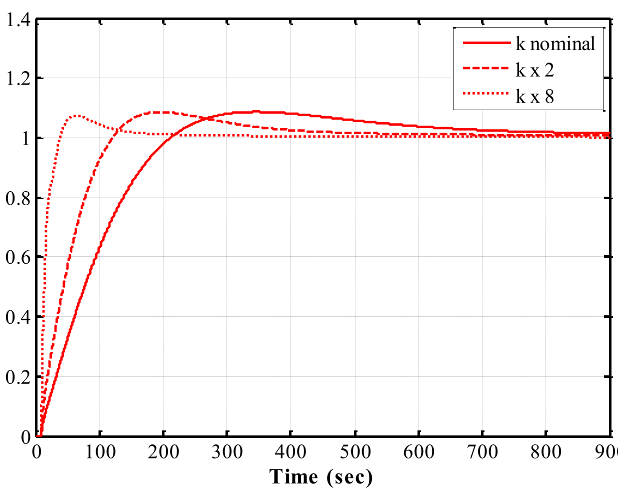

Fig. 10 Step responses of plant (60) with proposed controller for different static gains k

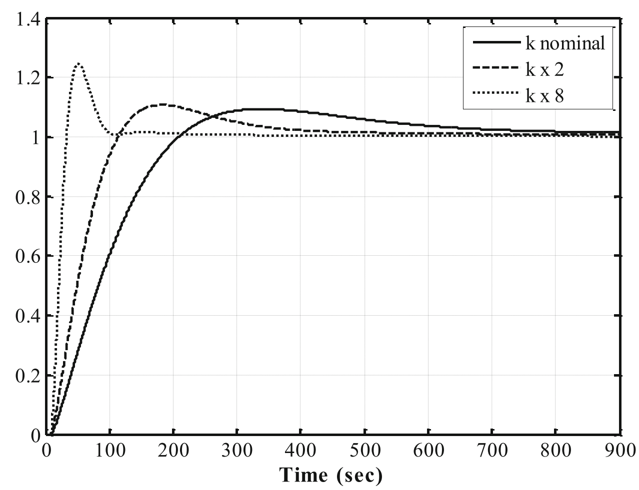

Fig. 12 Step responses of plant (60) with controller (63) for different static gains k

technique use the well-known Padè and Taylor approximations for time delay system.

$$C_{IMC}(s) = \left(\frac{25.7833}{1 + 69.56s^{0.22}}\right)\left(1 + \frac{1}{35.5061\ s} + 6.4641\ s\right) \quad (63)$$

Figure 9 shows the Bode diagram of the open loop system obtained by three controllers (61)-(63). It is clear that two controllers (61) and (62) satisfy the desired dynamic performances with a large flat phase around the gain crossover frequency ω_c. The step responses of the closed-loop system for a large change in static gain with three used controllers are shown in Figs. 10, 11 and 12.

Table 3 shows the different characteristics of the closed-loop system with the three controllers compared to the desired specifications.

Table 3 Frequency and time domain characteristics of example 2

	ω_c	φ_m	$\mathrm{Darg}/\mathrm{d}\omega\vert_{\omega c}$	Fisem	F_{ISEph}	t_{ISE}	$\Sigma\Delta ov$
Desired	0.01	70	0	0	0	0	0
Proposed controller	**0.01**	**70**	$\mathbf{-1.6\ 10^{-5}}$	**0.0302**	**0.0698**	$\mathbf{6.195\ 10^{-6}}$	$\mathbf{1.6758\ 10^{-4}}$
Controller of (62)	0.0099	69.9	$2.9\ 10^{-5}$	0.0478	3.767	$3.02\ 10^{-5}$	$3.76\ 10^{-4}$
Controller of (63)	0.0097	66.3	$3.1\ 10^{-3}$	0.1351	60.2610	$2.52\ 10^{-4}$	$231\ 10^{-4}$

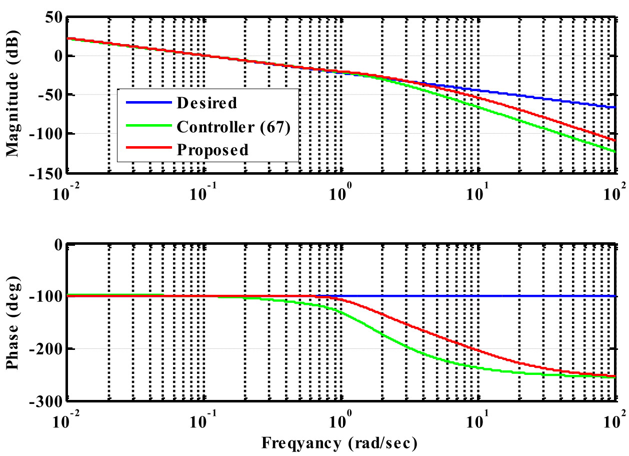

Fig. 13 Bode diagrams of open loops with different controllers for example 3

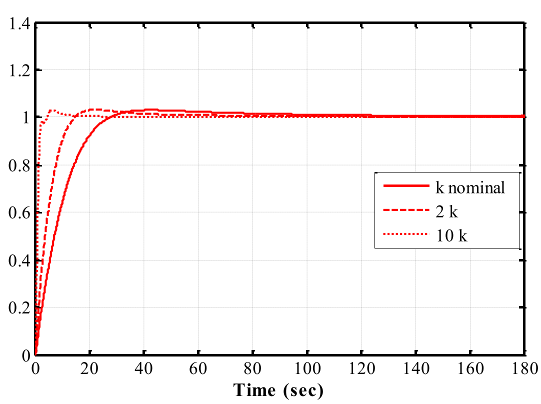

Fig. 14 Step responses of plant (64) with the proposed controller for different static gains k

From Table 3 it is clear that the proposed controller gives more equivalent characteristics to the desired one than the tow controllers (62) and (63).

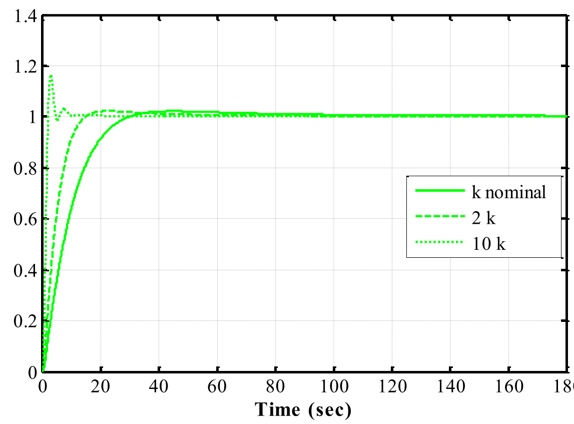

Fig. 15 Step responses of plant (64) with controller (67) for different static gains k

5.3 Example 3: Two non-integer order plus time delay system (NIOPTD-II)

Consider the following high order system:

$$G_p(s) = \frac{1}{(1+s)^4} \tag{64}$$

The transfer function $G_p(s)$ is approximated by the following non-integer plus dead time model [30]:

$$G_p(s) = \frac{0.22287\ e^{-0.5532s}}{s^{2.2251} + 0.8631s^{1.0389} + 0.22394} \tag{65}$$

The required specifications are:
$\omega_c = 0.1$ rad/sec and $\varphi_m = 80°$
The controller obtained using the proposed design method is given by:

$$C(s) = 0.9964\left(\frac{1+0.6541\,s}{1+0.1\,s}\right)\left(\frac{0.2999}{s^{0.0722}} + \frac{0.0778}{s^{1.111}} + 0.3474\ s^{1.114}\right) \tag{66}$$

Table 4 Frequency and time domain characteristics of example 3

	ω_c	φ_m	$\mathrm{Darg}/\mathrm{d}\omega\vert_{\omega c}$	Fisem	F_{ISEph}	t_{ISE}	$\Sigma\Delta ov$
Desired	0.1	80	0	0	0	0	0
Proposed controller	0.099	**80.09**	**14.5 10^{-5}**	**0.1712**	**1.0087**	**2.12 10^{-6}**	**2.06 10^{-6}**
controller of (67)	0.099	80.10	−130.8 10^{-5}	0.186	76.9984	46.32 10^{-6}	0.0209

In the work presented in [22], for the same required specifications, the following controller was proposed:

$$C_F(s) = \frac{0.3252}{s^{0.0369}}\left(1 + \frac{0.2594}{s^{1.0389}} + 1.1585\, s^{1.1862}\right) \quad (67)$$

Figure 13 shows the Bode diagram of open loop system obtained by the controllers (66) and (67). It is clear that the proposed controller gives a flat phase in a frequency band larger than that obtained by controller (67). Figures 14 and 15 show the step responses of the closed-loop system with static gain variation for the two used controllers. It is noted that the proposed controller was guaranteed the iso-damping property for a large variation in plant's static gain.

Table 4 shows the different characteristics of the closed-loop system with the tow controllers compared to the desired specifications. It is clear that the proposed controller gives more equivalent characteristics to the desired one than the controller (67).

6 Conclusion

In this work, a new simple method is proposed for tuning the parameters of fractional PID controllers. The main advantage of the proposed technique is that the eight parameters of the proposed controller, represented by the $PI^{\lambda}D^{\mu}$ with the proposed additional filter, are given with explicit formulas function of the plant and the desired closed-loop parameters. In addition, the tuning approach has been applied for different classes of linear time delay systems and tested with different examples where the simulation results show that the proposed controller has the well-known iso-damping robustness property. The analytical formulas obtained in this work allow extending the proposed technique to indirect adaptive control in order to design a closed-loop system insensitive to the plant's parameters change. The different simulation examples show that the proposed tuning has the more precise results than others analytical methods.

Two new results are obtained in this work, where the first one is that the user of the fractional controller is not restricted by the general $PI^{\lambda}D^{\mu}$ form proposed in [29]. Instead, any form can be chosen depending on the process transfer function as in [32] where the designer can choose any controller structure that simplify its analytical tuning with the proposed method. The second one is that although most existing direct synthesis-based methods use a reference closed-loop model that involve the same time delay of the plant [21] [22] [33], the present work proved that a delay-free model can be used as reference desired model to produce an analytical controller where the equivalence between the closed-loop system and desired model can be only assured in the limited frequency band. As a main result, any stable time delay plant can be controlled with the proposed controller to achieve the desired gain crossover frequency and phase margin specifications. This feedback system can maintain the iso-damping property with a limitation in the closed-loop system bandwidth where the gain crossover frequency cannot exceed the value $(0.5/\theta)$.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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