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Traveling wave solution and Painleve' analysis of generalized fisher equation and diffusive Lotka–Volterra model

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Abstract
In this paper, we have obtained the traveling wave solution for generalized Fisher equation and Lotka–Volterra (L-V) model with diffusion using hyperbolic function method. The Painleve' analysis has been used to check both of the system's integrability. Obtained solutions have also been plotted to represent their spatio-temporal dependence. The three dimensional plot shows a monotonic profile of the solutions.

Keywords Traveling wave solutions · Hyperbolic function method · Fisher equation · Lotka–Volterra equation · Painleve' analysis

1 Introduction

Nonlinear waves play significant roles in different phenomena related to fluid mechanics [1], plasma physics [2], biology [3], chemistry [4] etc. Because of the wide applications, nonlinear wave theory has made a lot of progress. It has got a new boom with the advancements of computation as well as the theory of dynamical systems. The reaction–diffusion (R-D) models are studied extensively in the context of biological and chemical systems. The interaction of reaction and diffusion together becomes the cause of formation of traveling waves. Traveling wave solutions in R-D systems has been found in neurology, chemistry, epidemiology etc [5].

The exact solution of nonlinear partial differential equations, if available, facilitates the verification of numerical solvers and aids in the stability analysis of solutions. It can also provide much physical information and more insight into the physical aspects of the nonlinear physical problem. During the past decades, much effort has been spent on the subject of obtaining the exact analytical solutions to the nonlinear evolution PDEs. Fisher's equation is one of the most useful reaction diffusion systems [6]. It is utilized to display the propagation of genes [7], logistic population growth [3], flame propagation [8], nuclear reactor theory [9] etc. There are different strategies to solve this equation, some of them are inverse scattering method [10], Hirota's bilinear methods [11], homogeneous balance method [12], tanh-function method [13], exp-function method [14], hyperbolic function method [15] etc. In recent years, the direct search for exact solutions of PDEs becomes more and more attractive partly due to the availability of computer symbolic systems like Maple or Mathematica, which allows us to perform complicated tedious algebraic calculations on computer. In particular one of the most effective direct methods to construct exact solutions of PDEs is the hyperbolic function method. Lin and Ruan [16] have concerned with the traveling wave solutions of delayed reaction–diffusion systems. By using Schauder's fixed point theorem they have shown that the existence of traveling wave solutions may be reduced to the existence of generalized upper and lower solutions. Fan [17] obtained the analytic solution to the generalized Fisher's equation with higher degree of nonlinearity. Kudryashov [18] have presented a new approach to look for exact solutions of nonlinear ordinary differential equations. He has used a simple nonlinear equation with general solution in order to express special solution of nonlinear differential equation of higher order. Zhou et al. [19] have studied the generalized Fisher equation analytically by using three tools of integration, like: improved tan $\left(\frac{\phi(\xi)}{2}\right)$-

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expansion method, the generalized Kudryashov method and the extended G'/G-expansion method. Using these methods, they have derived the bright-like, dark-like and singular-like solitary wave solutions. Kyrychko and Blyuss [20] studied traveling wave solutions to the generalized Fisher's equation with fourth ordered derivative. Numerically, they have studied the behavior of travelling waves when the long-range diffusion coefficient becomes larger. Also, they have found that starting with some values, solutions of the model lose monotonicity and become oscillatory.

Among the models on mathematical biology, Lotka–Volterra model is one of the most famous models. It has been used in lot of biological models related to predator-prey competition models, cooperative models, diffusive models etc. Many investigators have used Lotka–Volterra related equations for ecological modeling and simulations, in an effort to understand the most basic features of a spatially distributed interaction [21–24]. Dunbar [25], established the existence of traveling wave solutions for two reaction diffusion systems based on the Lotka–Volterra model for predator and prey interactions. Ma and Guo [26], investigated the dynamics of a class of diffusive Lotka–Volterra equations with time delay subject to the homogeneous Dirichlet boundary condition in a bounded domain.

In this paper, exact solutions have been obtained to diffusive Lotka–Volterra equations by using hyperbolic function method. Bai [27] proposed hyperbolic function method to get exact solutions for nonlinear partial differential equations. In this paper, we apply this method to solve Fisher's equation and diffusive Lotka–Volterra equations for predator-prey models.

The general form of a partial differential equation is $f(v, v_x, v_t, v_{xx}, v_{xt}, v_{tt}, \ldots) = 0$, where f is a function. To get the traveling wave solution we introduce a new variable ξ such that $\xi = k(x - \lambda t + c)$ and $v(x,t) = v(\xi)$, where k, λ are constants and c is an arbitrary constant. With this substitution the general form of a partial differential equation transforms to an ordinary differential equation as:

$$f(v, v^{'}, v^{''}, \ldots) = 0, \tag{1}$$

where $v^{'} = \frac{dv}{d\xi}$. Integrating Eq. (1) we shall get the solution and the solution $v(x,t) = v(\xi)$ is written in the form,

$$v(\xi) = \sum_{i=1}^{n} \sinh^{i-1}\omega(a_i \cosh\omega + b_i \sinh\omega) + a_0, \tag{2}$$

with

$$\frac{d\omega}{d\xi} = \sinh\omega, \tag{3}$$

where, $a_i, b_i, i = 1, 2, \ldots, n$ and a_0 are constant. Balancing the highest-order nonlinear term and the highest-order partial derivative term in the given Eq. (1), we shall get the value of the parameter 'n'. After this, we substitute Eq. (2) into the Eq. (1) and to replace ξ by ω we use Eq. (3). Equating the coefficients of different terms of the form $\sinh^k\omega \cosh^l\omega$ equals to zero and solving we shall get the value of the constants k, λ, a_i and b_i, $\quad i = 1, 2, \ldots$

Integrability plays a crucial role in the study of nonlinear differential equations. The integrability of a differential equation provides a lot of interesting and vital properties of that equation: its behavior near movable singularity, existence of infinitely many conserved quantities, its Bäcklund transformations etc [28,29]. Here an important thing is how one can check whether a nonlinear differential equation is integrable or not without finding its general solution. Painlevé test [29] is an powerful tool to check this. An ordinary differential equation (ODE) is said to have the Painlevé property if its solution does not have any movable singularity other than pole. ie. the solution of that differential equation can be expressed in Laurent series near movable singularity (if there be any). Ablowitz et al. [30] conjectured that every exact reduction in case of integrable nonlinear partial differential equation (NPDE), to ODE gives rise to an ODE having the Painlevé property. Contrapositively, we can say that if we can find an ODE by exact reduction from a NPDE such that the ODE does not has the Painlevé property then we conclude that this PDE is not integrable.

The paper is organized into several sections. In Sect. 2, we have considered the generalized Fisher's equation and sinh function method has been used to get the traveling wave solution. Diffusive Lotka–Volterra equation for predator-prey model has been considered in Sect. 3. Here also, using sinh function method, the solutions of the diffusive Lotka–Volterra predator-prey model have been obtained. We perform the Painleve' test to check the integrability for both the models (Fisher's equation and Lotka–Volterra equation) in Sect. 4. Finally, in Sect. 5, the paper has been completed with the discussion of the obtained results.

2 Fisher's equation

Fisher's equation belongs to the class of reaction–diffusion equations: in fact, it is one of the simplest semilinear reaction–diffusion equations, the one which has the inhomogeneous term

$$g(u, x, t) = ru(1 - u)$$

which can exhibit traveling wave solutions that switch between equilibrium states given by $g(u) = 0$. Such equations occur, e.g., in ecology, physiology, combustion, crystallization, plasma physics, and in general phase transition problems [18]. Fisher proposed this equation in his 1937 paper. It is used to study the wave of advance of advantageous populations in the context of population dynamics to describe the spatial spread of an advantageous species and explored its traveling wave solutions [31]. For every wave speed $c \geq 2\sqrt{rD}$, ($c \geq 2$ in dimensionless form) it admits traveling wave solutions of the form $v(x,t) = v(x \pm ct) \equiv v(z)$, where v is increasing and $\lim_{z\to-\infty} v(z) = 0$, $\lim_{z\to\infty} v(z) = 1$, that is, the solution switches from the equilibrium state $u = 0$ to the equilibrium state $u = 1$.

The generalized fisher equation is given by

$$u_t = Du_{xx} + pu(1-u^r)(q+u^r), \tag{4}$$

where D is the diffusion coefficient. If $q = 0$, $r = 1$, Eq. (4) reduces to Huxley equation and for $r = 1$, $q = -\theta_1$, Eq. (4) reduces to Fitzhugh–Nagumo equation [32]. Here we have assumed the case for $q \neq 0$ and $r = \frac{1}{2} \in (0, 1)$. Let $\xi = k(x - \lambda t + c)$ and $u(x,t) = \phi(\xi)$, here k and λ are constants that need to be determined and c is an arbitrary constant. Thus, the Eq. (4) becomes

$$Dk^2\phi_{\xi\xi} + k\lambda\phi_\xi + p\phi\left[q + (1-q)\phi^r - \phi^{2r}\right] = 0. \tag{5}$$

Now $\phi(\xi)$ is expressed in terms of hyperbolic functions as

$$\phi(\xi) = \sum_{i=1}^{n} \sinh^{i-1}\omega(a_i\cosh\omega + b_i\sinh\omega) + a_0. \tag{6}$$

Since there are two non-linear terms, so, $n = 2$. Thus,

$$\phi(\xi) = a_1\cosh\omega + b_1\sinh\omega + a_2\sinh\omega\cosh\omega + b_2\sinh^2\omega + a_0, \tag{7}$$

$$\frac{d\omega}{d\xi} = \sinh\omega. \tag{8}$$

Differentiating Eq. (7) with respect to ξ and using Eq. (8) we get-

$$\begin{aligned}\phi_\xi &= a_1\sinh^2\omega + b_1\sinh\omega\cosh\omega + 2a_2\sinh^3\omega + a_2\sinh\omega \\ &\quad + 2b_2\sinh^2\omega\cosh\omega,\end{aligned} \tag{9}$$

$$\begin{aligned}\phi_{\xi\xi} &= 2a_1\sinh^2\omega\cosh\omega + 2b_1\sinh^3\omega + b_1\sinh\omega \\ &\quad + 6a_2\sinh^3\omega\cosh\omega + a_2\sinh\omega\cosh\omega \\ &\quad + 6b_2\sinh^4\omega + 4b_2\sinh^2\omega.\end{aligned} \tag{10}$$

From Eq. (5), using Eqs. (7), (8), (9) and (10), then simplifying and equating the coefficients of $\sinh^l\omega\cosh^m\omega$ to zero, we have a set of solutions given in "Appendix-A" and solving them we get $D = D$, $a_0 = \frac{1}{2}$, $a_1 = \pm\frac{1}{2}$, $a_2 = 0$, $b_1 = 0$, $b_2 = 0$, $k = k$, $p = 8Dk^2$, $q = 1$, $\lambda = \pm 2Dk(2q+1)$. From Eq. (8), we get

$$\sinh\omega = -\mathrm{cosech}\xi, \tag{11}$$

$$\cosh\omega = -\coth\xi, \tag{12}$$

where the integration constant is taken zero. Thus, from Eq. (7) we get

$$u = \phi(\xi) = a_1\cosh\omega + a_0, \tag{13}$$

where

$$\xi = k(x - \lambda t + c). \tag{14}$$

Thus, the solution which satisfies the boundary conditions, $u(-\infty) = 0$, $u(\infty) = 1$, is

$$u = \frac{1}{2}(\coth\xi + 1), \tag{15}$$

where

$$\xi = k(x \mp 6Dkt + c). \tag{16}$$

3 Application to diffusive Lotka–Volterra equation for predator-prey model

One of the dominant themes in both ecology and mathematical ecology is the dynamic relationship between predators and their prey due to its universal existence and importance in population dynamics. From a biological perspective, individual organisms are distributed in space and typically interacting with the physical environment and other organisms in their spatial neighborhood [33]. The prey-predator system exhibits such a phenomenon: predator pursuing prey while prey escaping the predator [34]. In the same manner, there is a tendency that the predators would get closer to the preys and the chase velocity of predators may be proportional to the dispersive velocity of preys [35]. This is often done in terms of diffusion which also models ecological processes such as searching for food, escaping high infection risks, etc. [36]. For examples, individuals tend to diffuse in the direction of lower density of population, where there are more resources; individuals may move along the gradient of infectious individuals to avoid higher infections [37]. In our work, we would include diffusion processes in both prey and predator. The model is as follows:

$$u_t = D_1 \frac{\partial^2 u}{\partial x^2} + Au - Buw,$$
$$w_t = D_2 \frac{\partial^2 w}{\partial x^2} - Cw + Euw. \tag{17}$$

Here we use the transformation $U = \frac{Eu}{C}, W = \frac{Bw}{A}, t' = At, x' = \frac{x}{\sqrt{\frac{D_2}{A}}}, D = \frac{D_1}{D_2}, \rho = \frac{C}{A}$. The system of Eqs. (17) transform to:

$$U_t = DU_{xx} + U(1 - W), \tag{18}$$

$$W_t = W_{xx} + \rho W(U - 1). \tag{19}$$

For traveling wave solution, we take, $U = \phi(\xi), W = \psi(\xi)$ where $\xi = k(x - \lambda t + c)$ with the boundary conditions $U(-\infty) = 0, U(+\infty) = 1, W(-\infty) = 0, W(+\infty) = 1$. Here k, λ are constants that will be determined and $'c'$ is an arbitrary constant.

Now $U(x,t) = \phi(\xi), W(x,t) = \psi(\xi)$, then from Eqs. (18) and (19) we get,

$$Dk^2\phi_{\xi\xi} + k\lambda\phi_\xi + \phi(1 - \psi) = 0, \tag{20}$$

$$k^2\psi_{\xi\xi} + k\lambda\psi_\xi + \rho\psi(\phi - 1) = 0, \tag{21}$$

where,

$$\phi(\xi) = \sum_{i=1}^{n} \sinh^{i-1} w(b_i \sinh w + a_i \cosh w) + a_0, \tag{22}$$
$$\psi(\xi) = \sum_{j=1}^{m} \sinh^{j-1} w(B_j \sinh w + A_j \cosh w) + A_0, \tag{23}$$
$$\frac{dw}{d\xi} = \sinh w. \tag{24}$$

By equating the highest nonlinear terms and the highest order partial derivative term in Eqs. (18) and (19) we get $n = 2 = m$. Therefore,

$$\phi(\xi) = a_1 \cosh w + b_1 \sinh w + a_2 \sinh w \cosh w + b_2 \sinh^2 w + a_0,$$
$$\psi(\xi) = A_1 \cosh w + B_1 \sinh w + A_2 \sinh w \cosh w + B_2 \sinh^2 w + A_0. \tag{25}$$

Now,

$$\phi_\xi = b_1 \sinh w \cosh w + a_1 \sinh^2 w + 2b_2 \sinh^2 w \cosh w + 2a_2 \sinh^3 w + a_2 \sinh w,$$
$$\phi_{\xi\xi} = 6b_2 \sinh^4 w + 2b_1 \sinh^3 w + 6a_2 \sinh^3 w \cosh w + 4b_2 \sinh^2 w + 2a_1 \sinh^2 w \cosh w + b_1 \sinh w + a_2 \sinh w \cosh w,$$
$$\psi_\xi = B_1 \sinh w \cosh w + A_1 \sinh^2 w + 2B_2 \sinh^2 w \cosh w + 2A_2 \sinh^3 w + A_2 \sinh w,$$
$$\psi_{\xi\xi} = 6B_2 \sinh^4 w + 2B_1 \sinh^3 w + 6A_2 \sinh^3 w \cosh w + 4B_2 \sinh^2 w + 2A_1 \sinh^2 w \cosh w + B_1 \sinh w + A_2 \sinh w \cosh w. \tag{26}$$

Substituting Eqs. (25) and (26) into Eqs. (20) and (21) and simplify the expression by using Mathematica. Equating the coefficients of $\sinh^p w \cosh^q w$ to zero, we obtain a set of equations given in "Appendix-B" and solving those system of equation we get the solutions as: $a_0 = A_0 = \frac{1}{2}, a_1 = A_1 = -\frac{1}{2}, a_2 = b_1 = B_1 = 0, \rho = -1, D = 1, b_2 = \frac{3}{2}, B_2 = 1, b_1 = B_1, a_2 = A_2, \lambda = \frac{5}{6k}a_1, k = \frac{1}{2}$.

Using Eqs. (11) and (12), the solutions of Eqs. (20) and (21), which satisfies the boundary conditions, are

$$U = \phi(\xi) = \frac{3}{2}\text{cosech}^2\xi + \frac{1}{2}\coth\xi + \frac{1}{2}, \tag{27}$$

$$W = \psi(\xi) = \text{cosech}^2\xi + \frac{1}{2}\coth\xi + \frac{1}{2}, \tag{28}$$

where

$$\xi = \frac{1}{2}\left(x - \frac{5}{6}t + c\right). \tag{29}$$

4 Painleve' analysis

4.1 Painleve' analysis on Fisher's equation

To check whether Eq. (4) possesses the Painlevé property, we perform Painlevé ODE test on the reduced ordinary differential Eq. (5). Collecting the leading terms from Eq. (5), we get

$$Dk^2\phi_{\xi\xi} - p\phi^{2r+1} \approx 0. \tag{30}$$

Let us put $\phi = a_0\xi^{-m}$, where m is a natural number, in Eq. (30) and get

$$a_0 = \left[\frac{(1+r)Dk^2}{pr^2}\right]^{\frac{1}{2r}}, \quad m = \frac{1}{r}. \tag{31}$$

We have to choose r in such a way that $\frac{1}{r}$ belongs to the set of all naturals.

Therefore, the first term in the Laurent series expansion of the solution of Eq. (5) is

$$\left[\frac{(1+r)Dk^2}{pr^2}\right]^{\frac{1}{2r}}(\xi-\xi_0)^{-\frac{1}{r}}. \tag{32}$$

To find the Fuch's indices [30,38] we put

$$\phi \approx \left[\frac{(1+r)Dk^2}{pr^2}\right]^{\frac{1}{2r}}(\xi-\xi_0)^{-\frac{1}{r}}+a_j(\xi-\xi_0)^{j-\frac{1}{r}}, \tag{33}$$

in Eq. (30) and collect the coefficients of a_j and make them equal to zero. Thus we get

$$j=-1, 2\left(1+\frac{1}{r}\right), \tag{34}$$

as the Fuch's indices. We shall discuss the case when $\frac{1}{r}=1$. Then the Fuch's indices are $j_1=-1$, $j_2=4$. To pass the Painleve$^{'}$ test the coefficient a_4 of the Laurent series expansion of the solution of Eq. (5) must be arbitrary. To check this we put the expression

$$\begin{aligned}\phi &\approx \left[\frac{(1+r)Dk^2}{pr^2}\right]^{\frac{1}{2r}}(\xi-\xi_0)^{-\frac{1}{r}}+a_1+a_2(\xi-\xi_0)\\ &\quad+a_3(\xi-\xi_0)^2+a_4(\xi-\xi_0)^3\\ \text{i.e.}\quad \phi &\approx \left(\frac{2Dk^2}{p}\right)^{\frac{1}{2}}(\xi-\xi_0)^{-1}+a_1+a_2(\xi-\xi_0)\\ &\quad+a_3(\xi-\xi_0)^2+a_4(\xi-\xi_0)^3,\end{aligned} \tag{35}$$

in Eq. (5) and equate the coefficients of the same powers of $\xi-\xi_0$ to zero. We obtain that the coefficient a_4 can not be chosen arbitrary. Hence, Eqs. (4), (5) fail to pass the Painleve$'$ test for the case $r=1$. We have also checked it for $r=\frac{1}{2}$, in these case $j=-1, 6$. So in order to pass the Painlevé test the coefficient a_6 in the corresponding Laurent series must be arbitrary. But we found that a_6 can not be chosen arbitrary. So, the Eq. (4) fails to possess the Painlevé poperty in this case also. However, for the other values of r, the integrability of Eq. (4) may be checked. Here, we have omitted those cases.

4.2 Painleve$'$ analysis on diffusive Lotka–Volterra equation

To examine the Painlevé property of Eqs. (20), (21) we follow the procedure used by Kudryashov and Zakharchenko [39]. We reduce the system of Eqs. (20) and (21) into a single ordinary differential equation as

$$\begin{aligned}&Dk^4\phi^2\phi_\xi\phi_{\xi\xi\xi}-Dk^4\phi^3\phi_{\xi\xi\xi\xi}-(Dk^4-3k^3\lambda)\phi^2\phi_\xi\phi_{\xi\xi}\\ &\quad-2k^3\lambda\phi\phi_\xi^3+(Dk^4-k^3\lambda-Dk^3\lambda)\phi^3\phi_{\xi\xi\xi}+(Dk^3\lambda\\ &\quad-k^2\lambda^2+Dk^2\rho)\phi^3\phi_{\xi\xi}+k^2\lambda^2\phi^2\phi_\xi^2+\rho\phi^5\\ &\quad-Dk^2\rho\phi^4\phi_{\xi\xi}-k\lambda\phi^4\phi_\xi-\phi^4+k\lambda\phi^3\phi_\xi=0\end{aligned} \tag{36}$$

Collecting the leading terms from Eq. (36) we have

$$\begin{aligned}&Dk^4\phi^2\phi_\xi\phi_{\xi\xi\xi}-Dk^4\phi^3\phi_{\xi\xi\xi\xi}-Dk^2\rho\phi^4\phi_{\xi\xi}=0\\ &\quad\Rightarrow k^2\phi^2\phi_\xi\phi_{\xi\xi\xi}-k^2\phi^3\phi_{\xi\xi\xi\xi}-\rho\phi^4\phi_{\xi\xi}=0.\end{aligned} \tag{37}$$

By putting $\phi=\frac{a_0}{\xi^p}$ [30,38] in Eq. (37) we get

$$k^2a_0^4\xi^{-4p-4}-k^2a_0^4\xi^{-4p-4}-\rho a_0^5\xi^{-5p-2}=0. \tag{38}$$

Since $a_0\neq 0$, then from Eq. (38), we take $\rho=0$ and $p=2$. Therefore, the first term in the Laurent series expansion of the solution of Eq. (36) is

$$\frac{a_0}{(\xi-\xi_0)^2}. \tag{39}$$

Now we are going to find the Fuchs indices [30,38] and so we put

$$\phi\approx\frac{a_0}{(\xi-\xi_0)^2}+a_j\xi^{j-2}, \tag{40}$$

in Eq. (37). Then equating the coefficients of a_j to zero we get the Fuchs indices as

$$\begin{aligned}j_1&=3+\frac{1}{2}\sqrt{\frac{1}{3}(2+\alpha+\beta)}\\ &\quad-\frac{1}{2}\sqrt{\frac{4}{3}-\frac{\alpha}{3}-\frac{\beta}{3}-\frac{48}{\sqrt{\frac{1}{3}(2+\alpha+\beta)}}},\\ j_2&=3+\frac{1}{2}\sqrt{\frac{1}{3}(2+\alpha+\beta)}\\ &\quad-\frac{1}{2}\sqrt{\frac{4}{3}-\frac{\alpha}{3}-\frac{\beta}{3}-\frac{48}{\sqrt{\frac{1}{3}(2+\alpha+\beta)}}},\\ j_3&=3-\frac{1}{2}\sqrt{\frac{1}{3}(2+\alpha+\beta)}\\ &\quad-\frac{1}{2}\sqrt{\frac{4}{3}-\frac{\alpha}{3}-\frac{\beta}{3}+\frac{48}{\sqrt{\frac{1}{3}(2+\alpha+\beta)}}},\\ j_4&=3-\frac{1}{2}\sqrt{\frac{1}{3}(2+\alpha+\beta)}\\ &\quad+\frac{1}{2}\sqrt{\frac{4}{3}-\frac{\alpha}{3}-\frac{\beta}{3}+\frac{48}{\sqrt{\frac{1}{3}(2+\alpha+\beta)}}},\end{aligned} \tag{41}$$

where $\alpha = \frac{1729}{\sqrt[3]{12959+72i\sqrt{964662}}}, \beta = \sqrt[3]{12959+72i\sqrt{964662}}$.

To pass the Painleve′ test all the Fuchs indices must be an integer. But here all the Fuch's indices are imaginary. Thus we can say that Eq. (36) and hence the system of ordinary differential Eqs. (20) and (21) do not pass the Painleve′ test. Therefore, the system of Eqs. (20) and (21) fails to satisfy the Painleve′ property. So, Eq. (17) does not satisfy Painleve′ test.

5 Results and discussions

The generalized Fisher equation models a system which is subjected to reaction and diffusion simultaneously. The Lotka–Volterra diffusion model and their extensions have been applied to understand the spread of population, spread of diseases etc. Our results may help to identify the key parameters which govern the dynamics of the system. Hyperbolic function method has been used to obtain exact solutions for generalized Fisher's equation and diffusive Lotka–Volterra equations. The obtained solutions are made to satisfy appropriate boundary conditions. Painleve' test has been carried out to check their integrability. Finally, the obtained analytical solutions have been plotted. Equation (15) represents the solution for the generalized Fisher's equation and Eqs. (27) and (28) correspond to the solution for Lotka–Volterra equation.

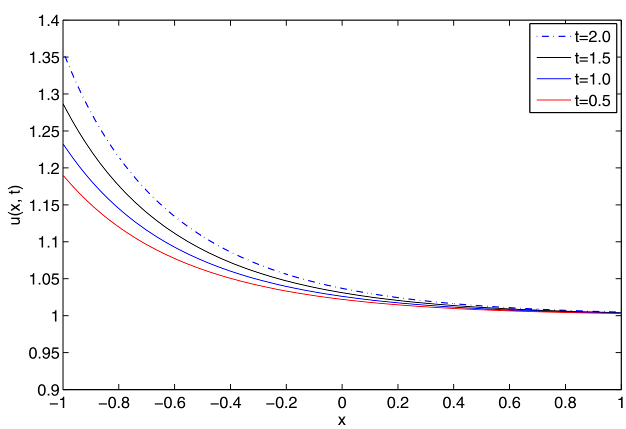

Fig. 1 For different value of t, u versus x have been plotted for the solution (15)

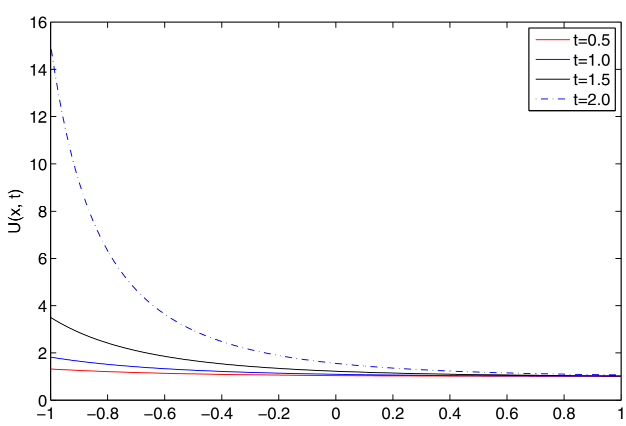

Fig. 2 For different value of t, U versus x have been plotted

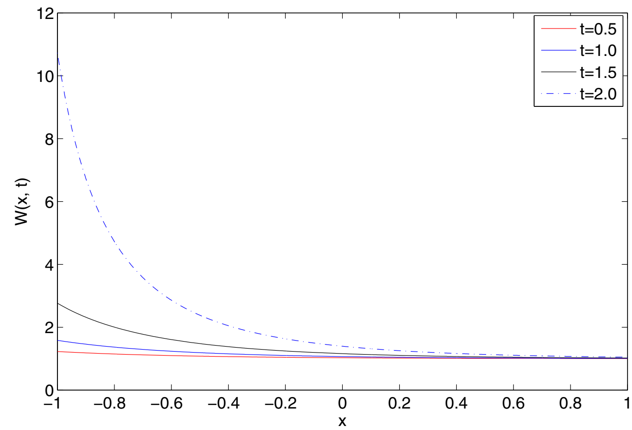

Fig. 3 For different value of t, W versus x have been plotted

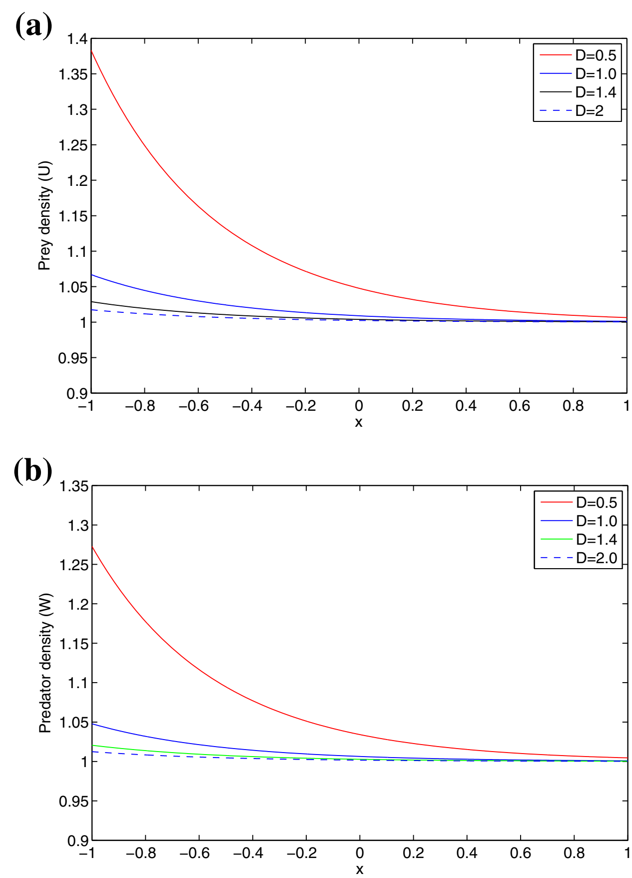

Fig. 4 For different values of D, U and W are plotted in **a** and **b** respectively

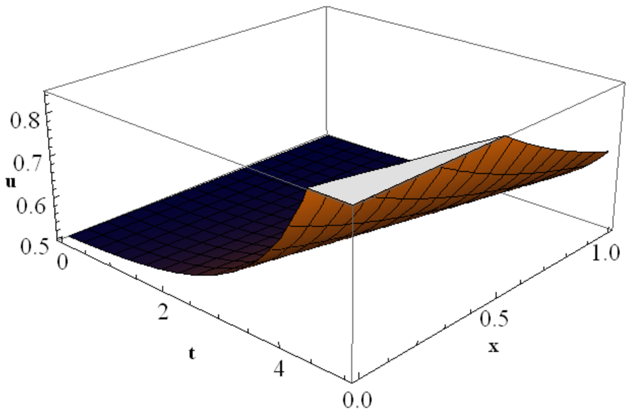

Fig. 5 Plot of the solution (13) with respect to the change of time and 'x'

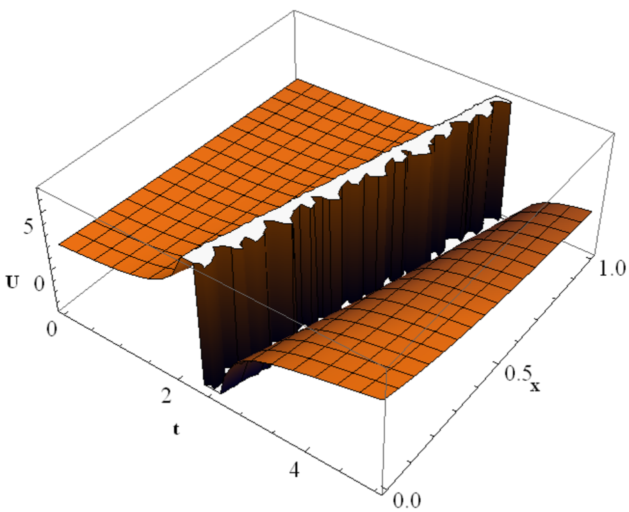

Fig. 6 A three dimensional diagram has been plotted with respect to the change of time and 'x' for the solution U, i.e., diagram for U has been plotted

In Fig. 1, for different values of the time t, the solution Eq. (15) has been plotted with the variable x. In this figure we can see that for different times, initially we have different u, but as x increases form negative to positive all the u values coincide at $u = 1$. Also, as t increases $u(x, t)$ increases.

In Fig. 2, for different values of the time t, the solution Eq. (27) has been plotted with the variable x. In this figure we can see that for different time, initially we have different U, but as x increases form negative to positive all the U values coincide at $U = 1$. Here, for fixed x, $U(x, t)$ increases as t increases. Similar result can be found in Fig. 3 for the solution Eq. (28).

Also, increasing $D = \frac{D_1}{D_2}$ with x, it is found that the number of preys and the predators decreases (Fig. 4).

In addition to these, three dimensional diagram for the solutions Eqs. (13), (27) and (28) have been plotted in Figs. 5, 6 and 7 respectively, with respect to the change of 'x' and time (t). Here the figures show the solutions to be monotonic.

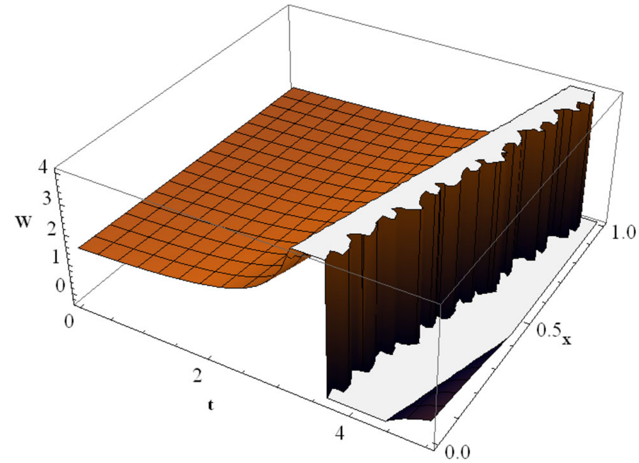

Fig. 7 A three dimensional diagram has been plotted with respect to the change of time and 'x' for the solution W

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Appendix-A

$$3a_2^2 b_2 p + b_2^3 p = 0, \tag{A.1}$$

$$3a_2^2 b_1 p + 6a_1 a_2 b_2 p + 3b_1 b_2^2 p = 0, \tag{A.2}$$

$$\begin{aligned} &a_2^2 p - 3a_0 a_2^2 p - 6a_1 a_2 b_1 p - 3a_1^2 b_2 p + b_2^2 p \\ &\quad - 3a_0 b_2^2 p - 3a_2^2 b_2 p - 3b_1^2 b_2 p \\ &\quad - a_2^2 pq - b_2^2 pq + 6b_2 Dk^2 = 0, \end{aligned} \tag{A.3}$$

$$\begin{aligned} &2a_1 a_2 p - 6a_0 a_1 a_2 p - 3a_1^2 b_1 p - 3a_2^2 b_1 p - b_1^3 p \\ &\quad - 6a_1 a_2 b_2 p + 2b_1 b_2 p \\ &\quad - 6a_0 b_1 b_2 p - 2a_1 a_2 pq - 2b_1 b_2 pq + 2b_1 Dk^2 \\ &\quad + 2a_2 k\lambda = 0, \end{aligned} \tag{A.4}$$

$$\begin{aligned} &a_1^2 p - 3a_0 a_1^2 p + a_2^2 p - 3a_0 a_2^2 p - 6a_1 a_2 b_1 p + b_1^2 p \\ &\quad + 3a_0 b_1^2 p + 2a_0 b_2 p - 3a_0^2 b_2 p \\ &\quad - 3a_1^2 b_2 p - a_1^2 pq - a_2^2 pq - b_1^2 pq + b_2 pq \\ &\quad - 2a_0 b_2 pq + 4b_2 Dk^2 + a_1 k\lambda = 0, \end{aligned} \tag{A.5}$$

$$\begin{aligned} &2a_1 a_2 p - 6a_0 a_1 a_2 p + 2a_0 b_1 p + 3a_0^2 b_1 p - 3a_1^2 b_1 p \\ &\quad - 2a_1 a_2 pq + b_1 pq \\ &\quad - 2a_0 b_1 pq + b_1 Dk^2 + a_2 k\lambda = 0, \end{aligned} \tag{A.6}$$

$$2a_0 a_1 p - 3a_0^2 a_1 p - a_1^3 p + a_1 pq - 2a_0 a_1 pq = 0, \tag{A.7}$$

$$2a_0a_2p - 3a_0^2a_2p - 3a_1^2a_2p + 2a_1b_1p - 6a_0a_1b_1p + a_2pq - 2a_0a_2pq - 2a_1b_1pq + a_2Dk^2 + b_1k\lambda = 0, \quad (A.8)$$

$$-a_1^3p - 3a_1a_2^2 + 2a_2b_1p - 6a_0a_2b_1p - 3a_1b_1^2p + 2a_1b_2p - 6a_0a_1b_2p - 2a_2b_1pq - 2a_1b_2pq + 2a_1Dk^2 + 2b_2k\lambda = 0, \quad (A.9)$$

$$-3a_1^2a_2p - a_2^3p - 3a_2b_1^2p + 2a_2b_2p - 6a_0a_2b_2p - 6a_1b_1b_2p - 2a_2b_2pq + 6a_2Dk^2 = 0, \quad (A.10)$$

$$3a_1a_2^2p + 6a_2b_1b_2p + 3a_1b_2^2p = 0, \quad (A.11)$$

$$a_2^3p + 3a_2b_2^2p = 0, \quad (A.12)$$

$$a_0^2p - a_0^3p + a_1^2p - 3a_0a_1^2p + a_0pq - a_0^2pq - a_1^2pq = 0. \quad (A.13)$$

Appendix-B

$$6b_2Dk^2 - b_2B_2 - a_2A_2 = 0, \quad (B.1)$$

$$2b_1Dk^2 + 2a_2k\lambda - b_1B_2 - a_1A_2 - b_2B_1 - a_2A_1 = 0, \quad (B.2)$$

$$4b_2Dk^2 + a_1k\lambda - b_1B_1 - a_1A_1 + b_2 - b_2A_0 - a_2A_2 - a_0B_2 = 0, \quad (B.3)$$

$$b_1Dk^2 + a_2k\lambda + b_1 - b_1A_0 - a_1A_2 - a_2A_1 - a_0B_1 = 0, \quad (B.4)$$

$$a_2Dk^2 + b_1k\lambda - b_1A_1 - a_1B_1 + a_2 - a_2A_0 - a_0A_2 = 0, \quad (B.5)$$

$$2a_1Dk^2 + 2b_2k\lambda - b_1A_2 - a_1B_2 - b_2A_1 - a_2B_1 = 0, \quad (B.6)$$

$$6a_2Dk^2 - b_2A_2 - a_2B_2 = 0, \quad (B.7)$$

$$a_1 - a_1A_0 - a_0A_1 = 0, \quad (B.8)$$

$$a_0 - a_0A_0 - a_1A_1 = 0, \quad (B.9)$$

$$6B_2k^2 + b_2B_2\rho + a_2A_2\rho = 0, \quad (B.10)$$

$$2B_1k^2 + 2A_2k\lambda + b_2B_1\rho + b_1B_2\rho + a_2A_1\rho + a_1A_2\rho = 0, \quad (B.11)$$

$$4B_2k^2 + A_1k\lambda + \rho b_1B_1 + a_0B_2\rho - B_2\rho + A_0b_2\rho + a_1A_1\rho + a_2A_2\rho = 0, \quad (B.12)$$

$$B_1k^2 + A_2k\lambda + a_0B_1\rho - B_1\rho + A_0b_1\rho + a_2A_1\rho + a_1A_2\rho = 0, \quad (B.13)$$

$$A_2k^2 + B_1k\lambda + a_1B_1\rho + b_1A_1\rho + a_0A_2\rho - A_2\rho + A_0a_2\rho = 0, \quad (B.14)$$

$$2A_1k^2 + 2B_2k\lambda + a_2B_1\rho + b_2A_1\rho + a_1B_2\rho + b_1A_2\rho = 0, \quad (B.15)$$

$$6A_2k^2 + a_2B_2\rho + b_2A_2\rho = 0, \quad (B.16)$$

$$a_0A_1\rho - A_1\rho + A_0a_1\rho = 0, \quad (B.17)$$

$$a_1A_1 + a_0A_0 - A_0 = 0. \quad (B.18)$$

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