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Synchronization on the adaptive sliding mode controller for fractional order complex chaotic systems with uncertainty and disturbances

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Abstract
The present paper purports to examine and analyse the concept of non identical complex chaotic systems of fractional order with external bounded disturbances and uncertainties. Hybrid projective synchronization has been achieved between fractional order complex Lu-system (drive system) and complex T-system (slave system). The adaptive sliding mode control technique has been used to design control law through suitable sliding surface and estimate the uncertainties and external disturbances in order to establish the stability of controlled system by using allied theorems. Also we have compared our results with prior published literature results to determine the supremacy of considered methodology. Computer simulations outcomes have established the efficacy and adeptness of the prospective scheme.

Keywords Fractional order complex chaotic system · Hybrid projective synchronization · Adaptive sliding mode control technique

Mathematics Subject Classification 37D45 · 37E99 · 37F99 · 37N10

1 Introduction

Chaos can be described as unconditional distraction, randomness or unpredictability. Chaotic dynamics [1] has become very interesting and attractive area for researchers. Chaotic dynamical systems are unstable and uncertain. Henri Poincare discovered a well known intrinsic property of the chaotic systems, the sensitive dependence on the initial conditions i.e. two neighbouring points in state space get isolated very quickly as they emerge in time. In general, chaos being the intrinsic property of non-linear systems has numerous applications such as in viscoelasticity [2], dielectric polarization, electromagnetic waves [3], diffusion, signal processing, mathematical biology and in many more disciplines. Different techniques are used to investigate the chaotic behaviour few of them are by plotting phase portraits, poincare section, bifurcation diagram or by finding Lyapunov exponents. The most reliable and widely used among the above technique is Lyapunov exponent spectrum. If the largest Lyapunov exponent is positive, we say that the system is chaotic and if more than one Lyapunov exponents are positive, then the system is said to be hyperchaotic.

It was the pioneering work of Pecora and Caroll who gave the concept of synchronization to control and utilize the chaos in the porper way. Synchronization means the trajectories of the coupled systems evolve with time to a usual pattern. Various techniques have been developed by researchers in this direction during last two decades. Numerous synchronization schemes have been proposed such as lag synchronization [4], complete synchronization [5], phase and anti-phase synchronization [6], anti-synchronization [7], hybrid synchronization [8], projective synchronization [9], hybrid funchtion projective synchronization [10], generalised synchronization [11], multi-switching synchronization [12] etc. To achieve synchronization different techniques have been designed some of them are adaptive feedback control,optimal control,linear and nonlinear feedback synchronization [13], active control [14], sliding mode control [15], adaptive sliding mode technique [16], time delay feedback approach [17],

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tracking control [18], backstepping design method [18] and so on.

In recent years, a lot of pioneering work has been done in the field of fractional calculus [19]. It was first suggested by Leibnitz and L'Hospital in 1675 and they gave the theory of integrals and derivatives of random order which combines the concept of integer order differentiation and n-fold integration. These studies describes the significant work in the real life systems and have a lot of multidisciplinary applications. As compared to integer order network the fractional order system add a degree of freedom by employing fractional derivative. Many types of fractional order chaotic and hyperchaotic systems have been introduced by researchers like Lorenz system [20], Chen system [21], Rossler system [21], Lu-system [22], Lui-system [23], Chua system [24] etc to explain the various physical processes. hlIn order to increase the complexity , researchers also introduced numerous fractional order complex chaotic systems like complex Lorenz system [25], T-sytem [26], Lu-system [27], Chen system [25] etc.

Various efforts have been made to synchronize identical and non-identical systems with different techniques. In this manuscript we have synchronized the two different fractional order complex chaotic systems by taking unknown bounded external disturbances and uncertainties. We have considered fractional order complex chaotic Lu-system as drive system and complex chaotic T-system as response system with external bounded disturbances and bounded uncertainties which has not been discussed in any literature to the best of our knowledge.The uncertainties and disturbances have a great impact on chaotic systems, dynamics and synchronization action and reduce the act of actual systems.Therefore to examine the synchronization of chaotic systems with various kind of disturbances and uncertainties, researchers have introduced different types of synchronization schemes [28,29]. Generally sliding mode control technique [30] is an efficient approach concerning the uncertainty and disturbances. In our paper we have used adaptive sliding mode control scheme [31] to synchronize the considered systems. To decline their effect we have chosen suitable sliding surface and estimated the disturbances and uncertainties through adapting control rule.

In [27], author synchronized fractional order complex Lu-system and complex T-system by active control method. As we have taken uncertainties and disturbances into consideration, despite of that our methodology shows better results when compared with the previous work [27]. Numerical simulations have been done to validate and visualize our results in the form of plots and demonstrates that our results are in excellent agreement with the theoretical results.

2 Preliminaries

The fractional order systems is continuation to the integer order calculus. As compared to integer order network the fractional order system add a degree of freedom by employing fractional derivative. Also, fractional order derivatives show better results when modelling real life processes as compared to integer order derivatives.The fractional order derivative can be defined in various forms [19], such as Riemann–Lioville's derivative, Grünwald Letnikov's derivative, Caputo's derivative etc.

The Riemann Liouville's derivative is defined as

$$ {}_{t_0}D_t^{\alpha} f(t) = \frac{d^n}{dt^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau\right], \quad t > t_0 $$

where α is fractional derivative, $n-1 < \alpha < n, n \in \mathbb{N}$, $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}$ is the Gamma function.

The Caputo's derivative is defined as

$$ {}_{t_0}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \quad t > t_0 $$

The Grünwald Letnikov's derivative is defined as

$$ {}_{t_0}D_t^{\alpha} f(t) \mid_t = kh = lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{j=0}^{\lfloor \frac{t-c}{h}\rfloor}\omega_j(\alpha) f(kh-jh) $$

where h shows the sample time. $\lfloor . \rfloor$ is the floor function and the coefficients

$$ \omega_j^{\alpha} = \frac{(-1)^j\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}, \quad j = 0, 1, 2, \ldots, k. $$

Since the Caputo's fractional derivative of a constant is zero,in this paper we choose Caputo's definition.

3 System description

3.1 Master system

Considering the fractional order complex Lu-system [27] given by

$$ \begin{aligned} D^{\alpha}u_1' &= a_1(u_2' - u_1') \\ D^{\alpha}u_2' &= a_2u_2' - u_1'u_3' \\ D^{\alpha}u_3' &= \frac{1}{2}(\overline{u_2'}u_2' + u_1'\overline{u_2'}) - a_3u_3' \end{aligned} \qquad (1) $$

where $u' = [u_1', u_2', u_3']^T$ is the state variable vector, $u_1' = u_1 + iu_2$, $u_2' = u_3 + iu_4$ are complex variables, $u_3' = u_5$ is the real variable and a_1, a_2, a_3 are real constant parameters.

Separating the real and imaginary parts, we obtain the system (1) as

$$\begin{aligned} D^\alpha u_1 &= a_1(u_3 - u_1)\\ D^\alpha u_2 &= a_1(u_4 - u_2)\\ D^\alpha u_3 &= a_2u_3 - u_1u_5\\ D^\alpha u_4 &= a_2u_4 - u_2u_5\\ D^\alpha u_5 &= u_1u_3 - u_2u_4 - a_3u_5 \end{aligned} \tag{2}$$

For the values of parameters as $a_1 = 42, a_2 = 22, a_3 = 5$, initial conditions as $u(0) = [1, 2, 3, 4, 5]^T$ and $\alpha = 0.95$, the system is chaotic.

3.2 Slave system

The fractional order complex T-system [26] is

$$\begin{aligned} D^\alpha v_1' &= b_1(v_2' - u_1')\\ D^\alpha v_2' &= (b_2 - b_1)u_1' - b_1v_1'v_3'\\ D^\alpha v_3' &= \frac{1}{2}(\overline{v_1'}v_2' + v_1'\overline{v_2'}) - b_3v_3' \end{aligned} \tag{3}$$

where $y' = [v_1', v_2', v_3']^T$ is the state variable vector of the system,$v_1' = v_1 + iv_2$, $v_2' = v_3 + iv_4$ are complex variables, $v_3' = v_5$ is the real variable and $b_1, b_2, b_3$are real constant parameters.

Separating real and imaginary parts, we have

$$\begin{aligned} D^q v_1 &= b_1(v_3 - v_1)\\ D^q v_2 &= b_1(v_4 - v_2)\\ D^q v_3 &= (b_2 - b_1)v_1 - b_1v_1v_5\\ D^q v_4 &= (b_2 - b_1)v_2 - b_1v_2v_5\\ D^q v_5 &= v_1v_3 + v_2v_4 - b_3v_5 \end{aligned} \tag{4}$$

For the values of parameters as $b_1 = 2.1, b_2 = 30, b_3 = 0.6$, initial conditions as $v(0) = [8, 7, 5, 6, 10]^T$ and $\alpha = 0.95$, the system is chaotic.

4 Synchronization scheme

The fractional order complex chaotic system (2) is taken as drive system. The fractional order complex chaotic T-system (4) with uncertainty and disturbance is taken as response system given by

$$\begin{aligned} D^q v_1 &= b_1(v_3 - v_1) + \Delta g_1(v_1, v_2, v_3, v_4, v_5)+\omega_1(t)+\Theta_1\\ D^q v_2 &= b_1(v_4 - v_2)+\Delta g_2(v_1, v_2, v_3, v_4, v_5)+\omega_2(t)+\Theta_2\\ D^q v_3 &= (b_2 - b_1)v_1 - b_1v_1v_5 + \Delta g_3(v_1, v_2, v_3, v_4, v_5)\\ &\quad + \omega_3(t) + \Theta_3\\ D^q v_4 &= (b_2 - b_1)v_2 - b_1v_2v_5 + \Delta g_4(v_1, v_2, v_3, v_4, v_5)\\ &\quad + \omega_4(t) + \Theta_4\\ D^q v_5 &= v_1v_3 + v_2v_4 - b_3v_5 + \Delta g_5(v_1, v_2, v_3, v_4, v_5)\\ &\quad + \omega_5(t) + \Theta_5 \end{aligned} \tag{5}$$

$\Delta g_i(v_1, v_2, v_3, v_4, v_5)$ are bounded uncertainties, $\omega_i(t)$ are bounded disturbances and Θ_i are appropriate control inputs of the response system for $i = 1, 2, 3$ which will be designed later.
Here we assume that $|\ \Delta g_i\ | \leqslant \varpi_i$ and $|\ \omega_i(t)\ | \leqslant \upsilon_i$, where ϖ_i and υ_i are positive constants.Also $\hat{\varpi}_i$ and $\hat{\upsilon}_i$ represents the estimated values of ϖ_i and υ_i respectively.

Now,the error state is defined as

$$e_i = v_i - \sigma_iu_i, i = 1, 2, 3, 4, 5. \tag{6}$$

where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ are scaling factors.

Definition The drive system (2) and response system (5) are said to be in hybrid projective synchronization, if there exists suitable controller $\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_m)$, such that

$$\begin{aligned} &\lim_{t\to\infty} \parallel e(t) \parallel\\ &\quad = \lim_{t\to\infty} \parallel v_i(t) - \sigma u_i(t) \parallel = 0, i = 1, 2, 3, 4, 5. \end{aligned} \tag{7}$$

The synchronization error is asymptotically stable between the state variables of drive system (3) and state variables of response system (2). The error dynamics is obtained as

$$\begin{aligned} D^\alpha e_1 &= b_1(e_3 - e_1) + (b_1\sigma_3 - a1\sigma_1)u_3 + \sigma_1(a1 - b1)u_1\\ &\quad + \Delta g_1 + \omega_1(t) + \Theta_1\\ D^\alpha e_2 &= b_1(e_4 - e_2) + (b_1\sigma_4 - a1\sigma_2)u_4 + \sigma_2(a_1 - b_1)u_2\\ &\quad + \Delta g_2 + \omega_2(t) + \Theta_2\\ D^\alpha e_3 &= (b_2 - b_1)e_1 - b_1v_1v_5 - \sigma_1(b_2 - b_1)u_1 - \sigma_3a_2u_3\\ &\quad + \sigma_3u_1u_5 + \Delta g_3 + \omega_3(t) + \Theta_3\\ D^\alpha e_4 &= (b_2 - b_1)e_2 - b_1v_1v_5 - \sigma_2(b_2 - b_1)u_2 - \sigma_4a_2u_4\\ &\quad + \sigma_4u_2u_5 + \Delta g_4 + \omega_4(t) + \Theta_4\\ D^\alpha e_5 &= v_1v_3 + v_2v_4 - b_3e_5 + b_3\sigma_5u_5 - \sigma_5(u_1u_3 + u_2u_4)\\ &\quad - \sigma_5a_3u_5 + \Delta g_5 + \omega_5(t) + \Theta_5 \end{aligned} \tag{8}$$

In order to minimize the error,we choose the suitable sliding surface which is as follows:

$$s_i(t) = D^{\alpha-1}e_i(t) + \phi_i \int_0^t e_i(\xi)d\xi \tag{9}$$

To accomplish the error dynamic system (8) at chosen sliding surface (13), it is necessary that it should satisfy the following condition

$s_i(t)=0, \dot{s_i}(t)=0, \quad i=1,2,3,4,5.$

The derivative of (9), yields the following equation

$$\dot{s_i}(t)=D^{\alpha}e_i(t)+\phi_i e_i(t), \quad i=1,2,3,4,5. \tag{10}$$

Then, by considering the necessary condition $\dot{s_i}(t)=0$, we obtain

$$D^{\alpha}e_i(t)=-\phi_i e_i(t), \quad i=1,2,3,4,5. \tag{11}$$

Hence, the system (9) is asymptotically stable by using Matignon theorem [32]. Therefore, the control laws by using (11), (8) and SMC theory are obtained as follows

$$\begin{aligned}
\Theta_1 &= -b_1(e_3-e_1)-(b_1\sigma_3-a_1\sigma_1)u_3-\sigma_1(a_1-b_1)u_1\\
&\quad -\phi_1 e_1-(\hat{\varpi}_1+\hat{\upsilon}_1+\gamma_1)signs_1\\
\Theta_2 &= -b_1(e_4-e_2)-(b1\sigma_4-a_1\sigma_2)u_4-\sigma_2(a_1-b_1)u_2\\
&\quad -\phi_2 e_2-(\hat{\varpi}_2+\hat{\upsilon}_2+\gamma_2)signs_2\\
\Theta_3 &= -(b_2-b_1)e_1+b_1v_1v_5+\sigma_1(b_2-b_1)u_1+\sigma_3a_2u_3\\
&\quad -\sigma_3u_1u_5-\phi_3e_3-(\hat{\varpi}_3+\hat{\upsilon}_3+\gamma_3)signs_3\\
\Theta_4 &= -(b_2-b_1)e_2+b_1v_2v_5+\sigma_2(b_2-b_1)u_2+\sigma_4a_2u_4\\
&\quad -\sigma_4u_2u_5-\phi_4e_4-(\hat{\varpi}_4+\hat{\upsilon}_4+\gamma_4)signs_4\\
\Theta_5 &= -v_1v_3-v_2v_4+b_3e_5-b_3\sigma_5u_5+\sigma_5(u_1u_3+u_2u_4)\\
&\quad +\sigma_5a_3u_5-\phi_5e_5-(\hat{\varpi}_5+\hat{\upsilon}_5+\gamma_5)signs_5
\end{aligned} \tag{12}$$

where $sign(.)$ denotes the signum function and γ_i are positive constant parameters. The adaptive parameter update laws are

$$\begin{aligned}
\dot{\hat{\varpi}}_i &= m_i \mid s_i \mid, i=1,2,3,4,5.\\
\dot{\hat{\upsilon}}_i &= n_i \mid s_i \mid, i=1,2,3,4,5.
\end{aligned} \tag{13}$$

where m_i and n_i are positive constants and γ_i are gain constants of the controllers for $i=1,2,3,4,5$.

Theorem 4.1 *The fractional order complex chaotic system (2) and the slave system (5) with uncertain dynamics are globally and asymptotically stable and synchronized with adaptive sliding mode control laws (12) and parameter update laws (13).*

Proof To discuss the stability of the fractional order chaotic systems, we have used Lyapunov's direct method [33, Ch-5]. Here our main focus is to take a positive definite function V and would show the derivative of V negative definite which would imply that our error converges asymptotically to zero. □

$$V=V_1+V_2+V_3+V_4+V_5 \tag{14}$$

where

$$\begin{aligned}
V_1 &= \frac{1}{2}s_1^2+\frac{1}{m_1}(\hat{\varpi}_1-\varpi_1)^2+\frac{1}{n_1}(\hat{\upsilon}_1-\upsilon_1)^2\\
V_2 &= \frac{1}{2}s_2^2+\frac{1}{m_2}(\hat{\varpi}_2-\varpi_2)^2+\frac{1}{n_2}(\hat{\upsilon}_2-\upsilon_2)^2\\
V_3 &= \frac{1}{2}s_3^2+\frac{1}{m_3}(\hat{\varpi}_3-\varphi_3)^2+\frac{1}{n_3}(\hat{\upsilon}_3-\upsilon_3)^2\\
V_4 &= \frac{1}{2}s_4^2+\frac{1}{m_4}(\hat{\varpi}_4-\varpi_4)^2+\frac{1}{n_4}(\hat{\upsilon}_4-\upsilon_4)^2\\
V_5 &= \frac{1}{2}s_5^2+\frac{1}{m_5}(\hat{\varpi}_5-\varpi_5)^2+\frac{1}{n_5}(\hat{\upsilon}_5-\upsilon_5)^2
\end{aligned} \tag{15}$$

The dynamics of Lyapunov function is

$$\begin{aligned}
\dot{V}_1 &= s_1\dot{s_1}+\frac{1}{m_1}(\hat{\varpi}_1-\varpi_1)\dot{\hat{\varpi}}_1+\frac{1}{n_1}(\hat{\upsilon}_1-\upsilon_1)\dot{\hat{\upsilon}}_1\\
\dot{V}_2 &= s_2\dot{s_2}+\frac{1}{m_2}(\hat{\varpi}_2-\varpi_2)\dot{\hat{\varpi}}_2+\frac{1}{n_2}(\hat{\upsilon}_2-\upsilon_2)\dot{\hat{\upsilon}}_2\\
\dot{V}_3 &= s_3\dot{s_3}+\frac{1}{m_3}(\hat{\varpi}_3-\varpi_3)\dot{\hat{\varpi}}_3+\frac{1}{n_3}(\hat{\upsilon}_3-\upsilon_3)\dot{\hat{\upsilon}}_3\\
\dot{V}_4 &= s_4\dot{s_4}+\frac{1}{m_4}(\hat{\varpi}_4-\varpi_4)\dot{\hat{\varpi}}_4+\frac{1}{n_4}(\hat{\upsilon}_4-\upsilon_4)\dot{\hat{\upsilon}}_4\\
\dot{V}_5 &= s_5\dot{s_5}+\frac{1}{m_5}(\hat{\varpi}_5-\varpi_5)\dot{\hat{\varpi}}_5+\frac{1}{n_5}(\hat{\upsilon}_5-\upsilon_5)\dot{\hat{\upsilon}}_5
\end{aligned} \tag{16}$$

Substituting the values of $s_i's$, we obtain

$$\begin{aligned}
\dot{V}_1 &= s_1(D^{\alpha}e_1+\phi_1e_1)+\frac{1}{m_1}(\hat{\varpi}_1-\varpi_1)\dot{\hat{\varpi}}_1\\
&\quad +\frac{1}{n_1}(\hat{\upsilon}_1-\upsilon_1)\dot{\hat{\upsilon}}_1\\
\dot{V}_2 &= s_2(D^{\alpha}e_2+\phi_2e_2)+\frac{1}{m_2}(\hat{\varpi}_2-\varpi_2)\dot{\hat{\varpi}}_2\\
&\quad +\frac{1}{n_2}(\hat{\upsilon}_2-\upsilon_2)\dot{\hat{\upsilon}}_2\\
\dot{V}_3 &= s_3(D^{\alpha}e_3+\phi_3e_3)+\frac{1}{m_3}(\hat{\varpi}_3-\varpi_3)\dot{\hat{\varpi}}_3\\
&\quad +\frac{1}{n_3}(\hat{\upsilon}_3-\upsilon_3)\dot{\hat{\upsilon}}_3\\
\dot{V}_4 &= s_4(D^{\alpha}e_4+\phi_4e_4)+\frac{1}{m_4}(\hat{\varpi}_4-\varpi_4)\dot{\hat{\varpi}}_4\\
&\quad +\frac{1}{n_4}(\hat{\upsilon}_4-\upsilon_4)\dot{\hat{\upsilon}}_4\\
\dot{V}_5 &= s_5(D^{\alpha}e_5+\phi_5e_5)+\frac{1}{m_5}(\hat{\varpi}_5-\varpi_5)\dot{\hat{\varpi}}_5\\
&\quad +\frac{1}{n_5}(\hat{\upsilon}_5-\upsilon_5)\dot{\hat{\upsilon}}_5
\end{aligned} \tag{17}$$

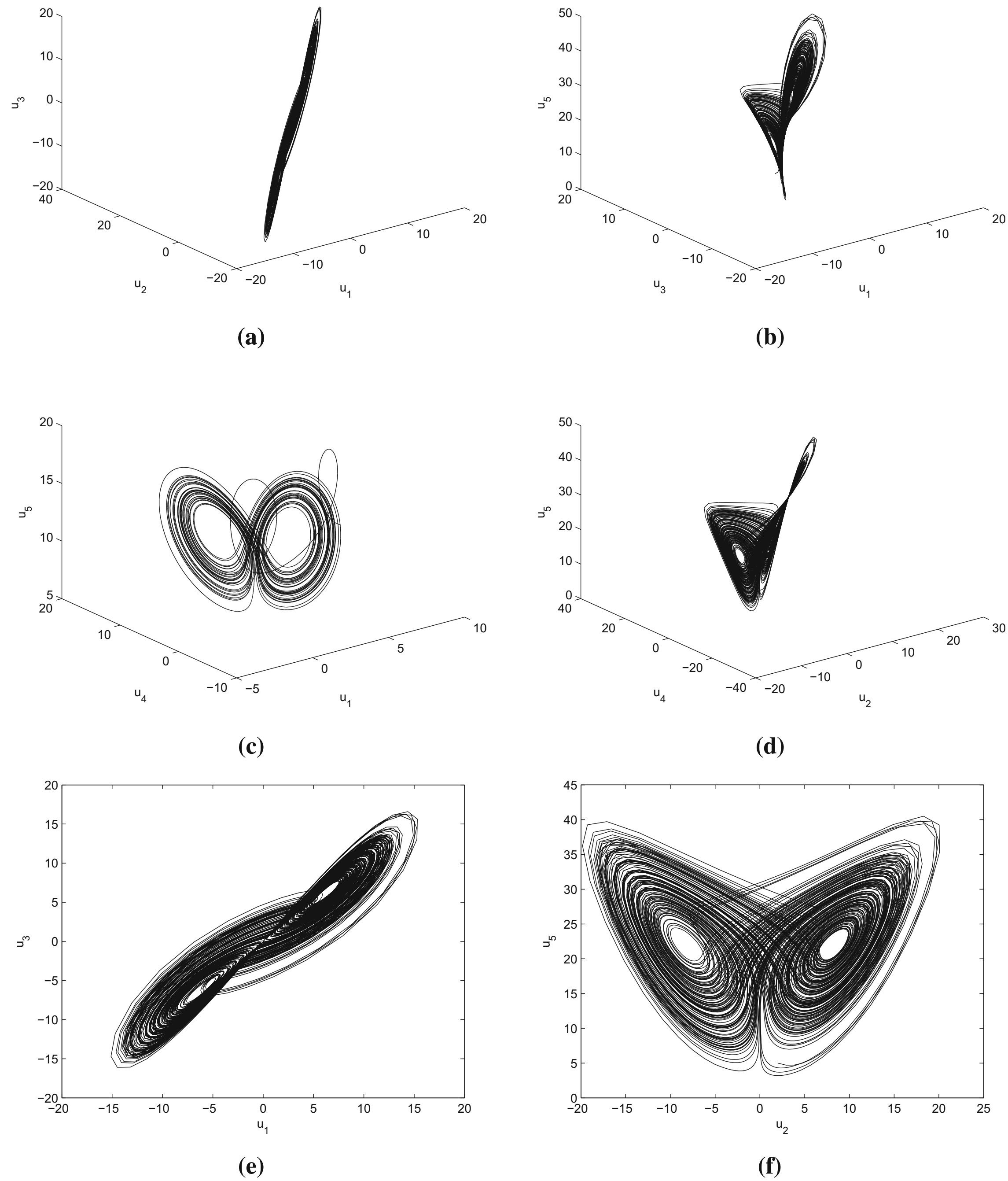


Fig. 1 Phase portraits of fractional order complex Lu-system for fractional order $\alpha = 0.95$ **a** $u_1 - u_2 - u_3$ **b** $u_1 - u_3 - u_5$ **c** $u_1 - u_4 - u_5$ **d** $u_2 - u_4 - u_5$ **e** $u_1 - u_3$ **f** $u_2 - u_5$

By substituting the values of $D^q e_i$, $\dot{\hat{\varpi}}_i$ and $\dot{\hat{\upsilon}}_i$ in (17) , we obtain

$$\begin{aligned}
\dot{V}_i &= s_1[\Delta g_i + \omega_i - (\hat{\varpi}_i + \hat{\upsilon}_i + \gamma_i) signs_i] \\
&\quad + (\hat{\varpi}_i - \varpi_i) \mid s_i \mid + (\hat{\upsilon}_i - \upsilon_i) \mid s_i \mid \\
&\leqslant (\mid \Delta g_i \mid + \mid \omega_i \mid) \mid s_i \mid - (\hat{\varpi}_i + \hat{\upsilon}_i + \gamma_i) \mid signs_i \mid \\
&\quad + (\hat{\varpi}_i - \varpi_i) \mid s_i \mid + (\hat{\upsilon}_i - \upsilon_i) \mid s_i \mid \\
&< (\varpi_i + \upsilon_i) \mid s_i \mid - (\hat{\varpi}_i + \hat{\upsilon}_i + \gamma_i) \mid signs_i \mid \\
&\quad + (\hat{\varpi}_i - \varpi_i) \mid s_i \mid + (\hat{\upsilon}_i - \upsilon_i) \mid s_i \mid \\
&= -P_i \mid s_i \mid
\end{aligned}$$

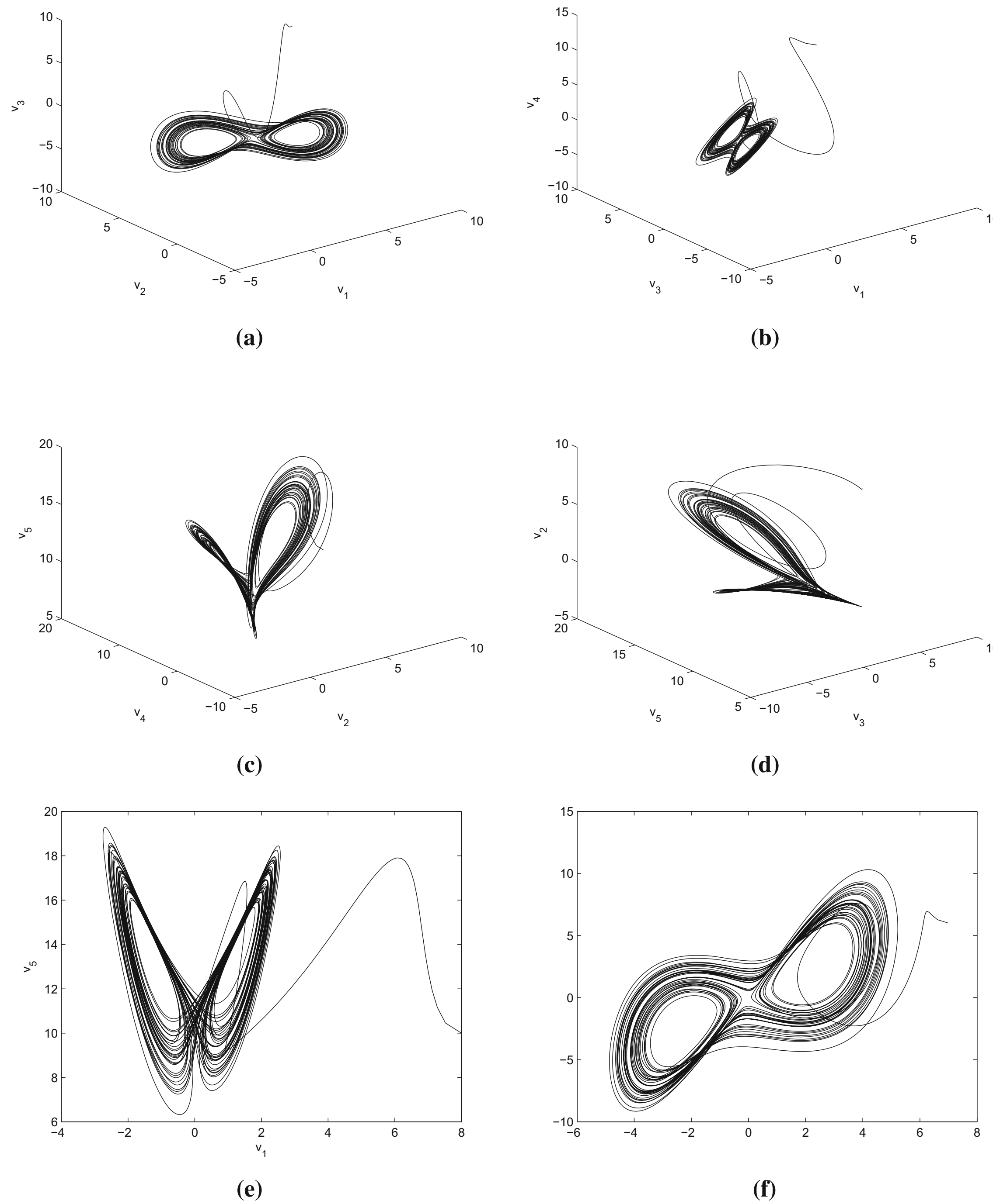


Fig. 2 Phase portraits of fractional order complex T-system for fractional order $\alpha = 0.95$ **a** $v_1 - v_2 - v_3$ **b** $v_1 - v_3 - v_4$ **c** $v_2 - v_4 - v_5$ **d** $v_3 - v_5 - v_2$ **e** $v_1 - v_5$ **f** $v_2 - v_4$

Finally, we get

$$\begin{aligned} \dot{V} &= \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 + \dot{V}_5 \\ &< -(P_1 \mid s_1 \mid + P_2 \mid s_2 \mid + P_3 \mid s_3 \mid + P_4 \mid s_4 \mid + P_5 \mid s_5 \mid) \end{aligned} \tag{18}$$

Thus there exist a non negative real number P such that $(P_1 \mid s_1 \mid + P_2 \mid s_2 \mid + P_3 \mid s_3 \mid + P_4 \mid s_4 \mid + P_5 \mid s_5 \mid) > P$, then (20) becomes

$$\begin{aligned} \dot{V} &< -P\sqrt{s_1^2 + s_2^2 + s_3^2 + s_4^2 + s_5^2} \\ &< 0 \end{aligned} \tag{19}$$

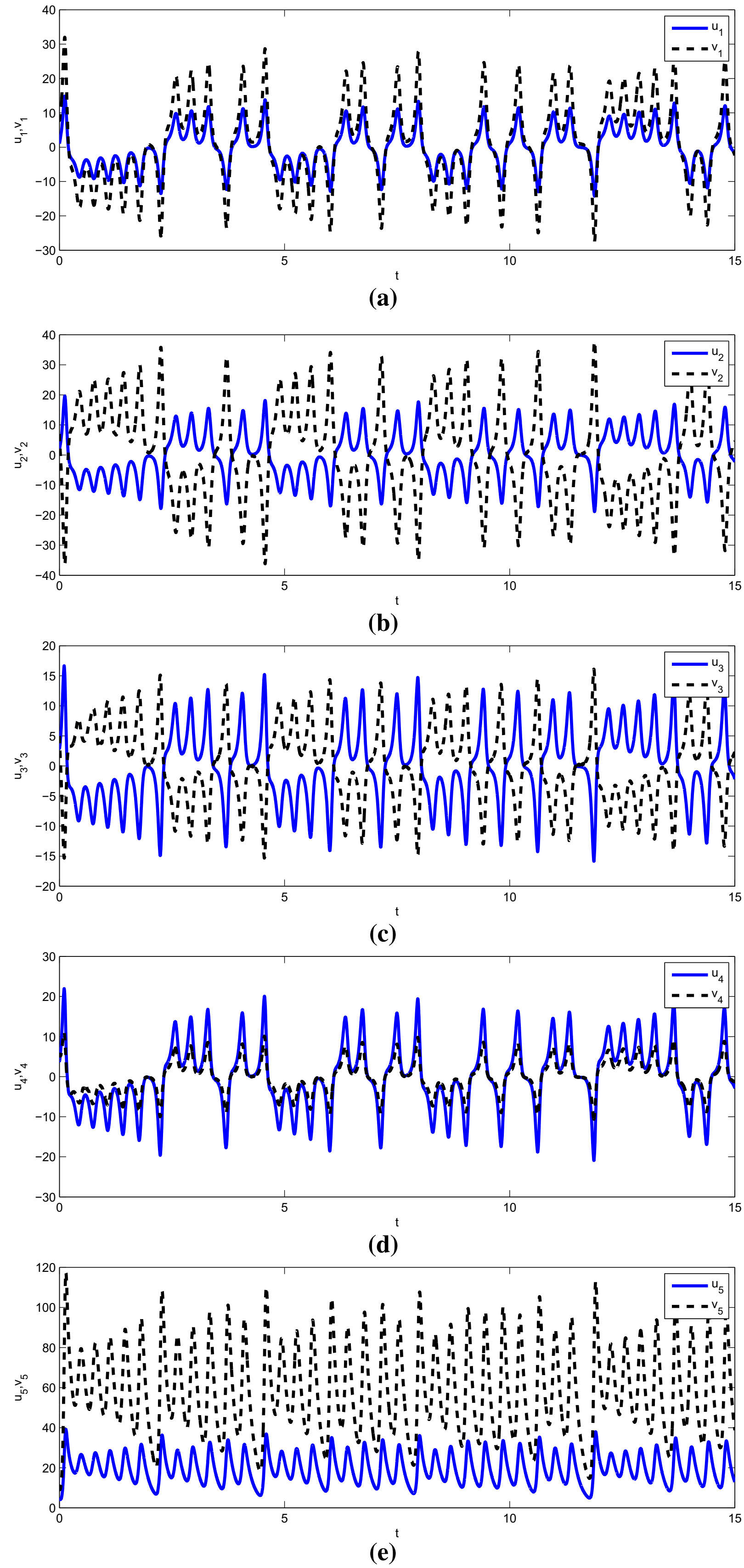


Fig. 3 Synchronized state trajectories of mater system and controlled slave system **a** $u_1 - v_1$ **b** $u_2 - v_2$ **c** $u_3 - v_3$ **d** $u_4 - v_4$ **e** $u_5 - v_5$

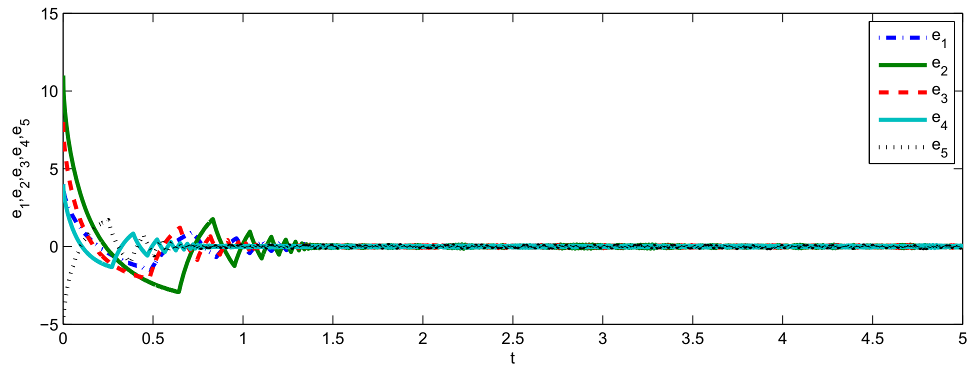


Fig. 4 Error converges to zero at t = 1.5 s (approx.)

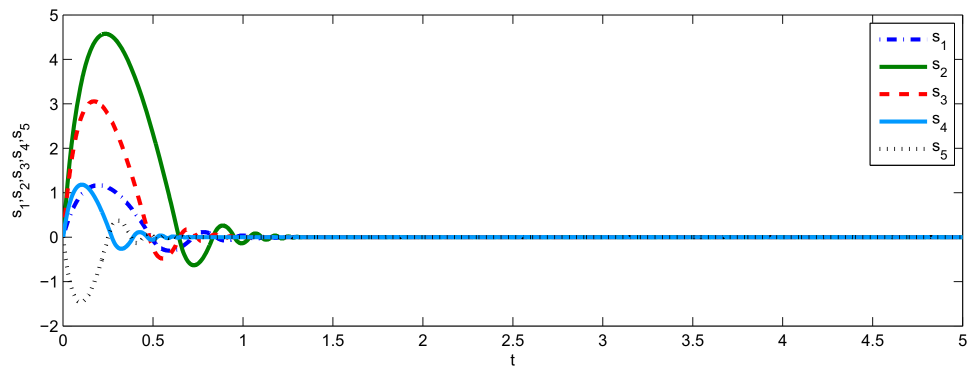
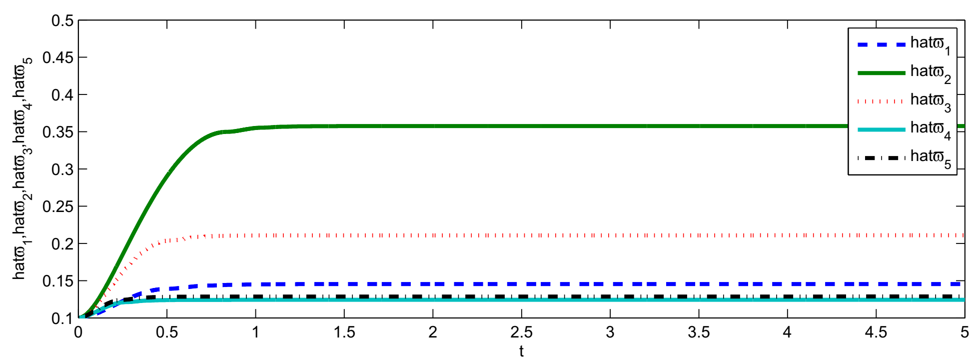
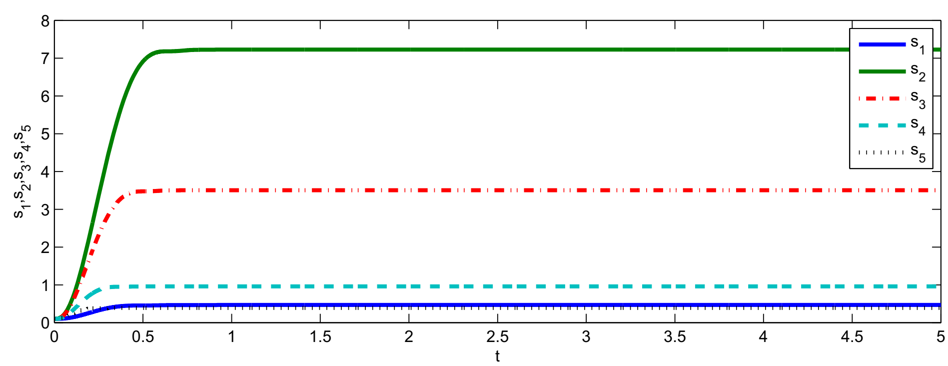


Fig. 5 Sliding surface converges to zero at t = 1.3 s (approx.)



(a)



(b)

Fig. 6 Estimated values of **a** uncertainties bounds, **b** disturbances bounds

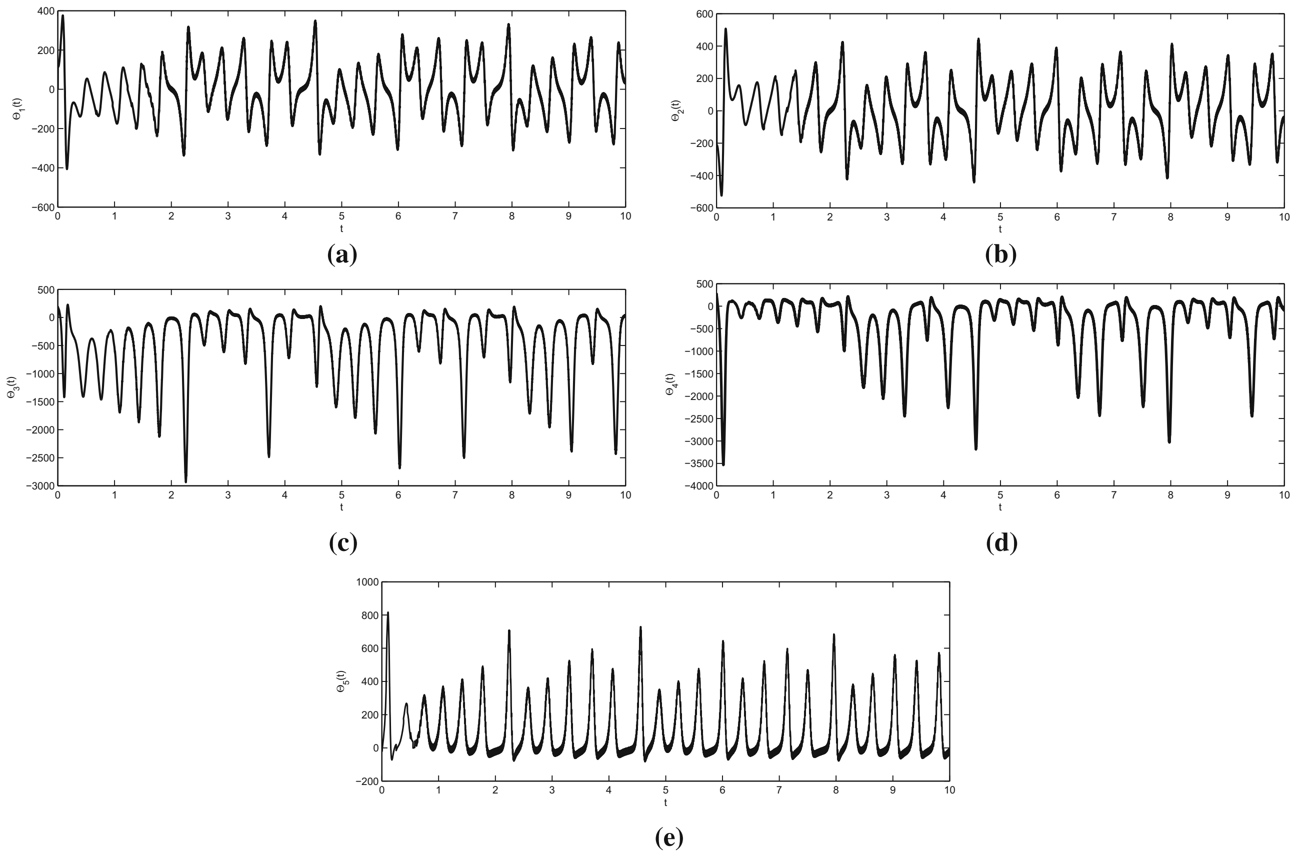


Fig. 7 Time response of controllers used to synchronization

Hence, By Lyapunov stability theory $\| \; s_i \; \| \rightarrow 0$ as $t \rightarrow \infty$.Thus the error dynamical system (10) asymptotically converges to $s_i = 0$. Therefore the trajectories of state variables of projection of master system and chaotic slave system are asymptotically and globally adjusted to desired set of points with control laws (12) and adaptive laws (13).

5 Numerical simulations

Simulations have been performed (using Matlab) to validate and visualize the effectiveness of the proposed scheme for the synchronization between master system and chaotic slave system.In simulations we have taken fractional order $\alpha = 0.95$ with step size 0.001. The parameter of master system are taken as $a_1 = 42, a_2 = 22, a_3 = 5$, and of slave system as $b_1 = 2.1, b_2 = 30, b_3 = 0.6$.Initial conditions for drive system and response system are [1,2,3,4,5], [8,7,5,6,10] respectively. Figures 1 and 2 show the Phase Portraits of respective drive and response systems.We have considered the bounded uncertainties as $\Delta g_1 = cos\pi v_2, \Delta g_2 = cos\pi v_2, \Delta g_3 = 0.1cos\frac{2}{3}\pi(v_1 + v_5), g_4 = 0.1sin\frac{2}{3}\pi(v_1 + v_5)$ and $g_3 = 0.5cos2\pi v_2$, bounded disturbances as $\omega_1(t) = cos\pi t, \omega_2(t) = sin\pi t, \omega_3(t) = 0.5cos\frac{3}{2}\pi t, \omega_4(t) = 0.5sin\frac{3}{2}\pi t$, and $\omega_5 = sign(cos\pi t)$ initial condition for estimating the parameters as $\hat{\varpi}(0) = (0.1, 0.1, 0.1)$, $\hat{\upsilon}(0) = (0.1, 0.1, 0.1)$ and designed control parameters as $m_1 = m_2 = m_3 = m_4 = m_5 = 0.1$, $n_1 = n_2 = n_3 = n_4 = n_5 = 0.5$, $\gamma_1 = 8.5, \gamma_2 = 11, \gamma_3 = 13.5, \gamma_4 = 14.5$ and $\gamma_5 = 18.5$ and $\phi_1 = 4.5, \phi_2 = 6, \phi_3 = 7.3, \phi_4 = 8.5$ and $\phi_5 = 9$. Figure 3 exhibits the trajectories of drive system and controlled response system behaving alike and also Fig. 4 shows that the synchronization error becomes zero as time increasing. The scaling factors are taken as $\sigma_1 = 2, \sigma_2 = -2, \sigma_3 = -1, \sigma_4 = 0.5, \sigma_5 = 3$. By choosing different scaling factors, we can synchronize the given system upto desired level. Figure 5 shows that the chosen sliding surface converges to s = zero and hence stable. Figure 6 shows the estimated values for bounds of uncertainties and disturbances. Figure 7 shows the time response of controllers used for synchronization in present scheme and Figs. 8 and 9 shows the ratios of controllers with the corresponding uncontrolled slave system and master system signals respectively.

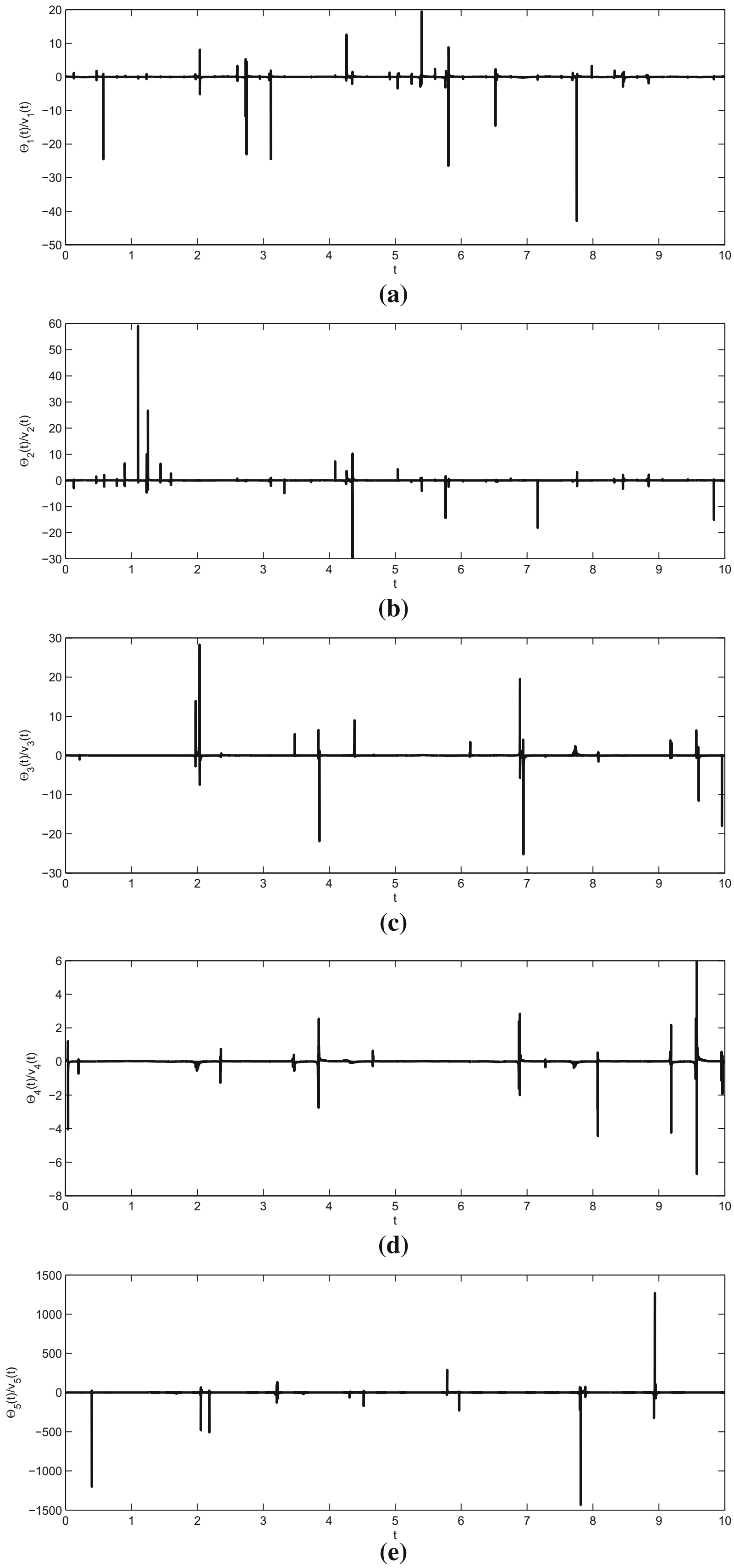
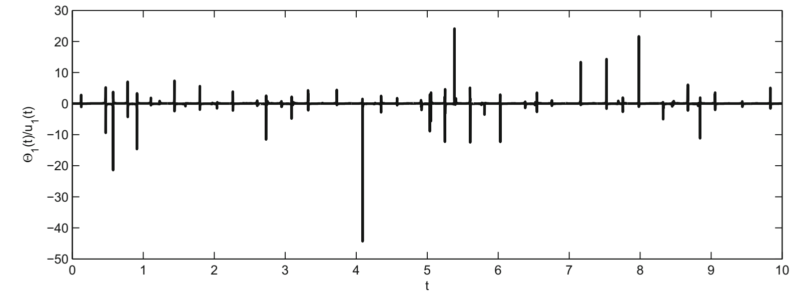
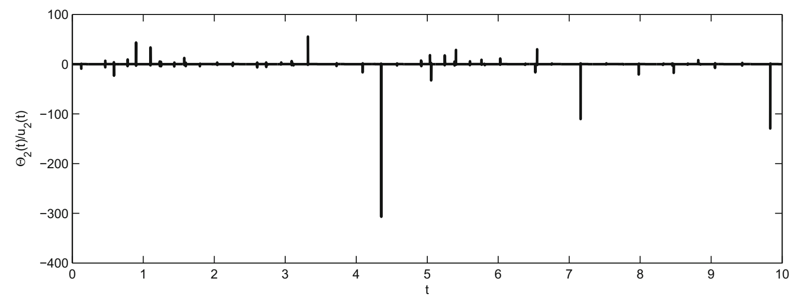


Fig. 8 Ratios of the controllers and the uncontrolled slave system signal **a** Θ_1/v_1 **b** Θ_2/u_2 **c** Θ_3/u_3 **d** Θ_4/u_4 **e** Θ_5/u_5

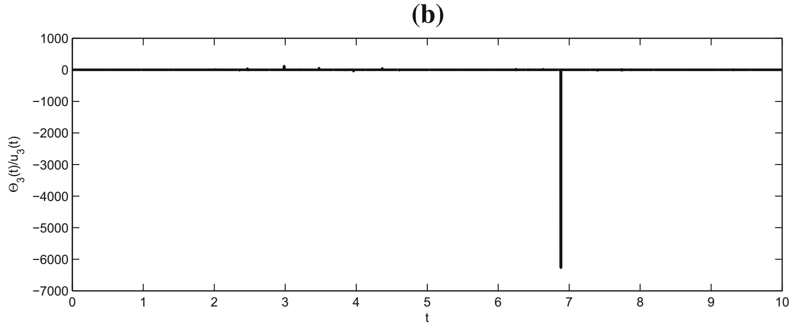
Fig. 9 Ratios of the controllers and the master system signal **a** Θ_1/u_1 **b** Θ_2/u_2 **c** Θ_3/u_3 **d** Θ_4/u_4 **e** Θ_5/u_5



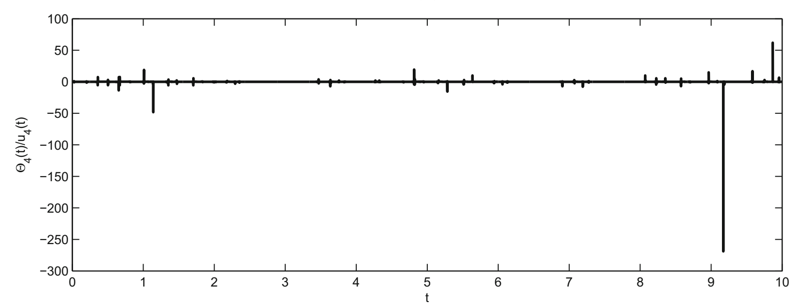
(a)



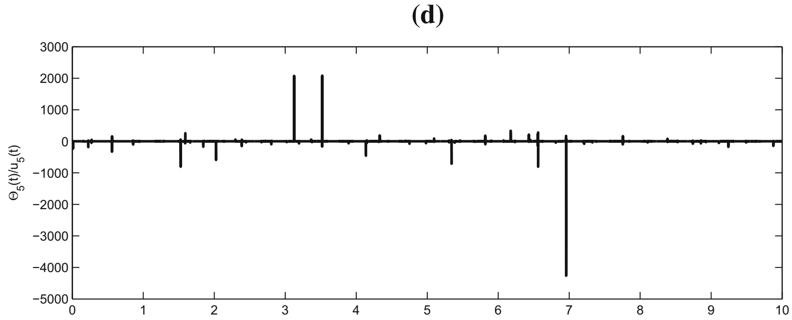
(b)



(c)



(d)



(e)

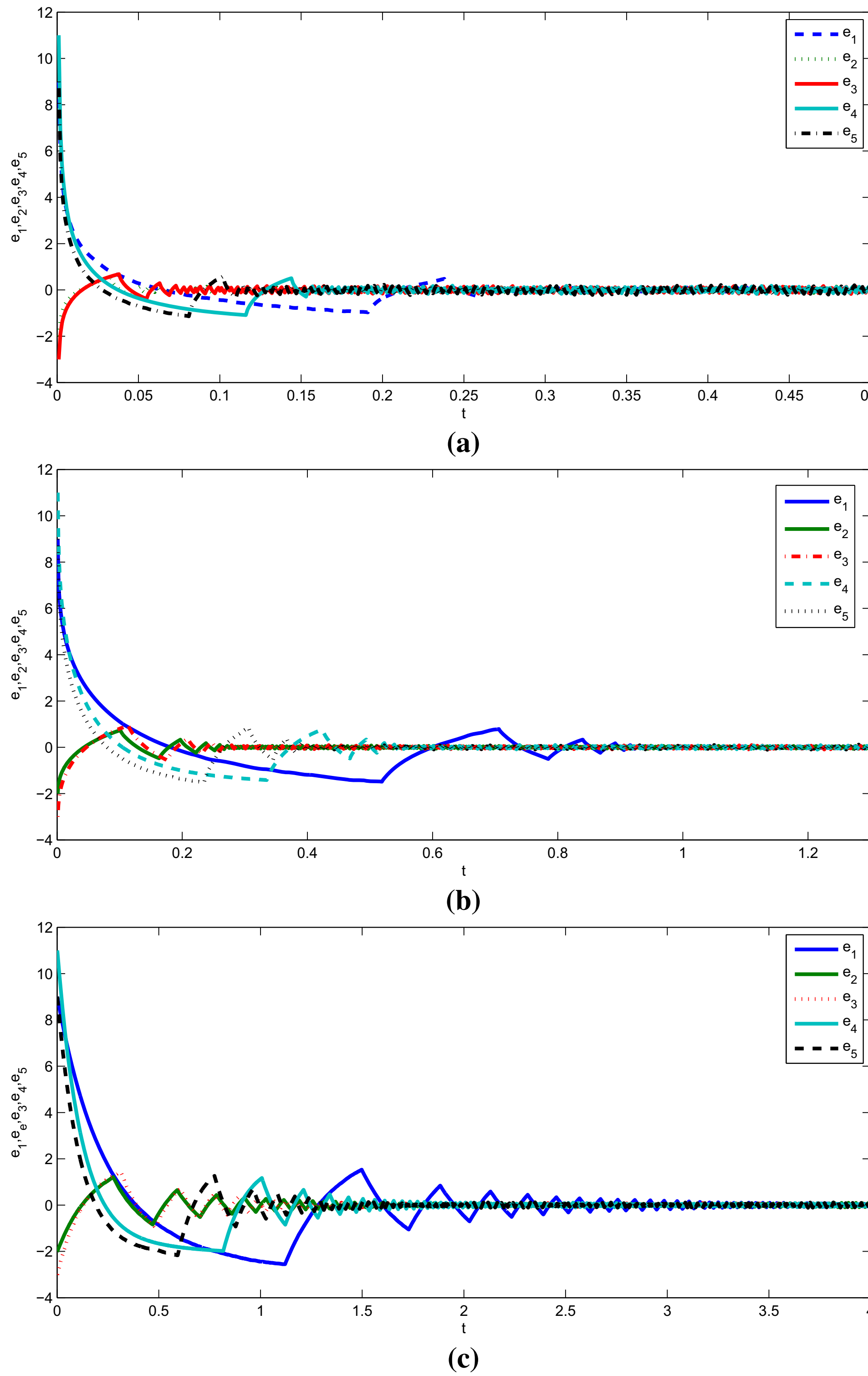


Fig. 10 Synchronization error for complex Lu-system and T-system at **a** $\alpha = 0.7$, **b** $\alpha = 0.85$, and **c** $\alpha = 1$

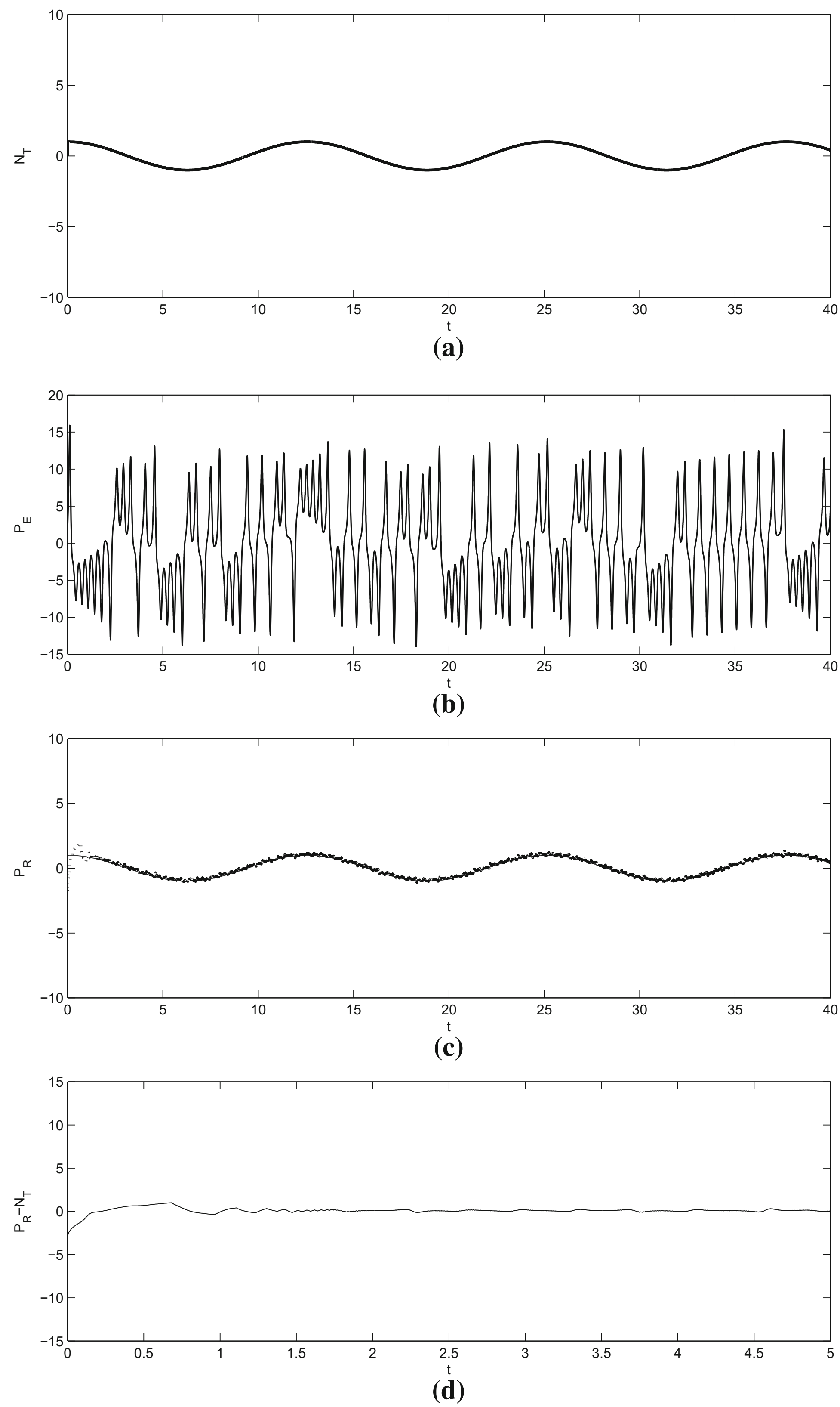


Fig. 11 Trajectories for secure communication based on additive encryption masking scheme **a** information signal $N_T = \cos(0.5t)$ **b** encrypted signal $P_E = N_T + u_1$ **c** decrypted signal $P_R = P_E - v_1/\sigma_1$ **d** error between decrypted signal and original signal $P_R - N_T$

5.1 Comparison of given synchronization with previous published literature

In [27], author studies active control technique to synchronize fractional order complex Lu-system and T-system. For $\alpha = 0.7, \alpha = 0.85$, and $\alpha = 1$ the synchronization is achieved at $t = 4\,\mathrm{s}(approx.), t = 5\,\mathrm{s}(approx.)$ and $t = 6\,\mathrm{s}(approx.)$ respectively whereas in present scheme we achieve synchronization at $t = 0.3\,\mathrm{s}(approx.), t = 0.9\,\mathrm{s}(approx.)$ and $t = 3.5\,\mathrm{s}(approx.)$ respectively given in Fig. 10 which is much lesser than the synchronization time of [27]. Therefore, our results are far better than the results obtained by the previous author.

5.2 Applications in secure communications

As chaos synchronization has great application in secure communication. The main reason behind the process is that, in order to transmit the original containing some secret message, we add this message into a chaotic signal which is transmitted to a prescribed receiver that would recover the original message from the chaotic signal.
In order to demonstrate our scheme, we explain it by taking a simple additive encryption masking scheme which is given below.

Here we choose periodic function $N_T = \cos(0.5t)$as an information signal and the chaotic carrier as u_1. The encrypted information is $P_E = N_T + u_1$. To recover the original information, hybrid projective synchronization between master system and slave system can be attained by controller Θ_1 by using the above methodology. So, the recover signal is $P_R = P_E - v_1/\sigma_1$. The results are shown in Fig. 11 .

6 Conclusion

In this paper, an adaptive sliding mode technique has been consigned.Hybrid Projective synchronization has been used to synchronize different fractional order complex chaotic systems. We have chosen the suitable sliding surface and designed parameters by update laws to achieve desired synchronization and to decline the consequence of external uncertainties and disturbances and chattering problem.Since the synchronization of fractional order complex chaotic system in the presence of uncertainties and disturbances has not been examined in the prior literature, we have interrogated, and synchronized the considered fractional order complex Lu-system and complex chaotic T-systems in the presence of uncertainties and disturbances. Also we have compared our results with previous published literature results which have established that our scheme gives better synchronization time than the used technique in prior literature.Although we have taken complex system with uncertainties and disturbances but still our synchronization results are better. Also,this scheme will perform significant role to enhance security in communication. Computational methods evaluate the efficiency of the considered scheme.

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