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Hesitant fuzzy computational algorithm for multiobjective optimization problems

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Abstract
Often decision maker wants to optimize various conflicting objectives of real life optimization problem simultaneously, classical techniques do not give suitable solution to the optimization problem due to partial information about values of parameters. In many computational methods, fuzzy set or intuitionistic fuzzy set and its other version were used. But, it's very interesting to note that the experts opinion regarding values of parameters of optimization problem are different, therefore, the single membership degree does not deal optimization problem properly than collection of degrees of values of parameters. In this situation, hesitant fuzzy set plays important role instead of fuzzy set or intuitionistic fuzzy set. In this article, we introduce hesitant membership function for each conflicting objective to present uncertainty and imprecision of multiobjective linear programming (MOLP) problems. A computational algorithm based on hesitant fuzzy set is constructed for the solution of MOLP and it is numerically verified by example and results of proposed algorithm are compared with existing methods.

Keywords Hesitant fuzzy sets · Linear programming problem · Objective function · Multiple membership functions

1 Introduction

Multiobjective linear programming problem is one of the famous optimization problem which occurs daily in real life. In formulation of a MOLP problem, various factors of the real world should be reflected in the description of parameters of the optimization problem, and possible values of parameters may be assigned by the experts. Due to this reason, the possible values of these parameters are imprecise or ambiguous. In this case, it may be more appropriate to interpret the experts understanding about parameters as fuzzy numerical data than crisp data. Thus, multiobjective linear programming problem involving fuzzy parameters would be viewed as a more realistic version than the conventional one. Zadeh [1] introduced fuzzy set for dealing problem with uncertainty and imprecision, and which occurs naturally in management science, industrial, and forecasting. Various kinds of multiobjective linear programming models have been proposed to deal with different decision-making situations that involve fuzzy values in objective function parameters, constraints parameters, or goals.

In several optimization problems, it has been observed that a small violation in given constraints or conditions may lead to more efficient solution to the problem. Such situations appear in frequent way in real life modeling. Many times it is not practical to fix accurate parameters as many of these are obtained through approximation or through some kind of human observation. For example in a production optimization problem, it is not necessary that all the produced are of good quality and are completely sellable on a fixed price. There is possibility that some of the products may be defective and are not sellable on the fixed price. Further prices of raw material as well as market price of finished product may vary depending on its surplus/deficiency in the market due to some uncontrollable situations. Thus it is evident that prices and/or productions are not purely deterministic but in general these are imprecise or nondeterministic and thus such problems of optimization are to be dealt with help of some non classical methods. Modeling of most of real life problems involving optimization process turns out to be multiobjective programming problem in a natural way. Such multiobjective programming problems may in general comprise of conflicting objectives. For example, if we consider a problem of agricultural production planning, the optimal

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model should have the objectives of maximizing the profit and minimizing the inputs and cost of cultivation. Thus these objectives are conflicting in nature and hence solution of such problems are in general compromise solutions which satisfy each objective function to a degree of satisfaction and a concept of belonging and non belonging arises in such situations.

It was Zimmermann [2] who first used the fuzzy set introduced by Zadeh [1] for solving the fuzzy multiobjective mathematical programming problem. Optimization in fuzzy environment was further studied and was applied in various areas by many researchers such as Tanaka and Asai [3], Luhandjula [4], Sakawa and Yano [5], etc. A brief review of studies of various research workers on optimization under uncertainty can be found in work of Sahinidis [6]. Recently, Bharati and Singh [7] have presented a computational algorithm based on deviation degree of two trapezoidal fuzzy numbers. In view of growing use of fuzzy set in modeling of problems under situations when information available is imprecise, vague or uncertain, various extension of fuzzy sets immersed. In such extensions. Further, it is observed that parameters of real life problem involves imprecision and hesitation. Such problem cannot be dealt with fuzzy set theory properly.

Atanassov [8] introduced the intuitionistic fuzzy sets as a powerful extension of fuzzy set, in his studies emphasized that in view of handling imprecision, vagueness or uncertainty in information both the degree of belonging and degree of non belonging should be considered as two independent properties as these are not complement of each other. This concept of membership and non membership was considered by Angelov [9] in optimization problem and gave intuitionistic fuzzy approach to solve optimization problems. Later, Bharati and Singh [10–17] studied intuitionistic fuzzy set and used it in various real optimization problems, Jana and Roy [18] studied the multiobjective intuitionistic fuzzy linear programming problem and applied it to transportation problem. Further, Garai and Ray [19] have studied generalized intuitionistic fuzzy set and presented an optimization method for MOLP problem. Luo and Yu [20] applied the inclusion degree of intuitionistic fuzzy set to multi criteria decision making problem. Further many workers such as Mahapatra et al. [21], Nachammai and Thangaraj [22] etc. have also studied linear programming problem under intuitionistic fuzzy environment. Dubey and Mehra [23] studied linear programming problem in intuitionistic fuzzy environment using intuitionistic fuzzy number and interval uncertainty in fuzzy numbers. Further, it is very interesting to note that experts opinions about possible values of the parameters of real life problem are conflicting. In such situation fuzzy or intuitionistic fuzzy cannot present the real solution to the problem. Torra and Narukawa [24] presented the concept of hesitant fuzzy set in which an element is characterized by a collection of membership degrees.

Torra [25] introduced the concept of hesitant fuzzy set which is an effective extension of Zadeh fuzzy set. Hesitant fuzzy set is a very useful tool in situations where there are some difficulties in determining the membership of an element to a set caused by a doubt between a few different values. Hesitant fuzzy set whose membership degree presented by a collection of possible values, are a new useful tool to express non-statistical imprecision and human hesitation of real life problems more accurately. The present study gives a computational algorithm for the solution of multiobjective linear programming problem under hesitant fuzzy circumstances. Here, we define hesitant fuzzy membership functions for each conflicting objectives. Present paper is organized as follows: Sect. 2, we develop the computational algorithm by introducing some new terms concerning hesitant fuzzy set, Sect.. 3 contains computational algorithms based on hesitant fuzzy set and the algorithm has been verified by using production planning problem in Sect. 4 and the result obtained has been placed in Sect. 5 followed by references.

2 Preliminaries

Definition 1 Multiobjective optimization problem is occurred in optimization of multiple, conflicting, non-commensurable objective functions subject to certain conditions. In general, a multiobjective optimization problem with p objectives, q constraints and n decision variables, is follows as:

$$\begin{aligned} &\text{Maximize} f = \{f_1, f_2, \ldots, f_p\} \\ &\text{Such that } g_j(x) \le 0 \; i = 1, 2, \ldots, q \\ &x_j \ge 0, \;\; i = 1, 2, \ldots, n. \end{aligned} \tag{1}$$

Definition 2 (Torra [25], Torra and Narukawa [24]) Let X be a fixed set, a hesitant fuzzy set on X is in terms of a function that when applied to X returns a subset of $[0, 1]$. Further, Xia and Xu [26] expressed it mathematically by: $A = \{(x, h_A(x))|x \in X\}$, where h_A is set of some values in $[0, 1]$, is called the possible membership degree of the element $x \in X$.

Definition 3 Let h_1 and h_2 be two hesitant fuzzy sets. Then union and intersection of h_1 and h_2 are defined as:

(i) $h_1 \cap h_2 = \bigcup_{\lambda_1 \in h_1, \lambda_2 \in h_2} min\{\lambda_1, \lambda_2\}$

(ii) $h_1 \bigcup h_2 = \bigcup_{\lambda_1 \in h_1, \lambda_2 \in h_2} max\{\lambda_1, \lambda_2\}$

Definition 4 Multiobjective linear programming problem (1) can be converted into uncertain multiobjective linear programming problem which is represented below:

$$\begin{aligned}&\text{Find } x\\&\text{Such that } Z_k(x) \gtrsim g_k,\ k = 1, 2, \ldots, p\\&g_j(x) \le 0,\ j = 1, 2, \ldots, q\\&x \ge 0.\end{aligned} \tag{2}$$

Definition 5 Let X be the feasible space of the problem (2), then x^0 is said to be complete hesitant fuzzy optimal solution for problem (2) if there exist $x^0 \in X$ such that $h_k(x^0) \ge h_k(x), k = 1, 2, \ldots, p$, for all $x \in X$. However, in general such complete hesitant fuzzy optimal solutions that simultaneously maximize all of the multiobjective functions are conflicting in nature. Thus instead of a complete hesitant fuzzy optimal solution a solution concept, called pareto hesitant fuzzy optimality was introduced in multiobjective programming problems.

Definition 6 Let X be the feasible space of the problem (2), then $x^0 \in X$ is said to be pareto optimal solution for (2) if there does not exist another $x \in X$ such that $h_k(x^0) \ge h_k(x)$ for all $k = 1, 2, \ldots, p$, $h_j(x^0) > h_j(x)$ for at least one $j \in \{1, 2, \ldots, p\}$.

Definition 7 Fuzzy optimization method [2] for the uncertain multiobjective linear programming problem (2) is described below:

$$\begin{aligned}&\text{Maximize } \alpha\\&\text{Such that } \mu_k(x) \ge \alpha,\ \text{for all } k\\&g_j(x) \le 0,\ j = 1, 2, \ldots, q\\&x \ge 0.\end{aligned} \tag{3}$$

Definition 8 Angelov [9] Intuitionistic fuzzy optimization method for the uncertain multiobjective linear programming problem problem (2) is stated below:

$$\begin{aligned}&\text{Maximize } (\alpha - \beta)\\&\text{subject to } \alpha \le \mu_k(x)\ k = 1, 2, \ldots, p + q\\&\beta \ge \nu_k(x)\ k = 1, 2, \ldots, p + q\\&\alpha + \beta \le 1\\&\alpha \ge \beta, \beta \ge 0\\&x \in X.\end{aligned} \tag{4}$$

3 Development of computational algorithm based on hesitant fuzzy set

In present paper, for development of proposed method first we introduce union and intersection of hesitant fuzzy sets. These are defined in following manner: Let $h_1 = (\delta_1, \delta_2, \ldots, \delta_n)$ and $h_2 = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ where $\delta_1 \le \delta_2 \le \cdots \le \delta_n; \gamma_1 \le \gamma_2 \le \cdots \le \gamma_n; 0 \le \delta_i \le 1; 0 \le \gamma_i \le 1, i = 1, 2, \ldots, n$ be two hesitant fuzzy sets. Then union and intersection of h_1 and h_2 are defined as:

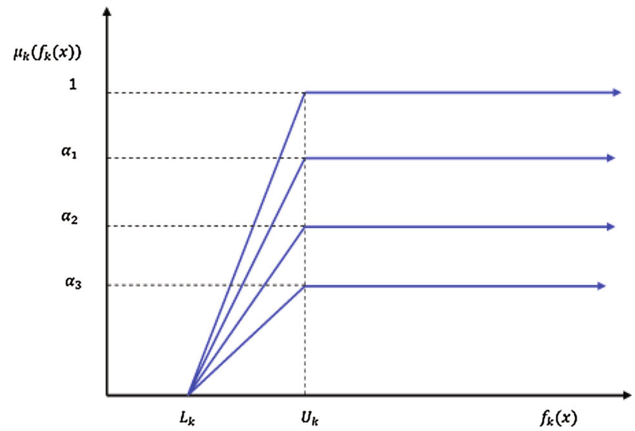

Fig. 1 Graphical representation of hesitant fuzzy membership of objective functions

(i) $h_1 \cap h_2 = \bigcup\{\delta_1 \wedge \lambda_1, \delta_2 \wedge \lambda_2, \ldots, \delta_n \wedge \lambda_n\}$
(ii) $h_1 \bigcup h_2 = \bigcup\{\delta_1 \vee \lambda_1, \delta_2 \vee \lambda_2, \ldots, \delta_n \vee \lambda_n\}$.

Problem (1) is reconsidered as:

$$\begin{aligned}&\text{Find } x\\&\text{Such that } Z_k(x) \gtrsim g_k,\ k = 1, 2, \ldots, p\\&g_j(x) \le 0,\ j = 1, 2, \ldots, q\\&x \ge 0.\end{aligned} \tag{5}$$

where g_k, for all x, denote the goals and all objective functions are assumed to be maximized, and $\gtrsim$ is hesitant fuzzy inequality which is represented by Fig. 1. Here objective are considered as hesitant fuzzy constraints. To establish possible membership functions of various objective functions we could first obtain the Table of positive solutions (PIS). Using PIS we obtain the lower and upper bound of each objective function moreover we define possible membership functions for each objective function. Zimmermann [2] first used the Max–min operator given by Bellman and Zadeh [27] to solve multiobjective linear programming problem. Further, it was extended by Angelov [9]. Here, we develop a method of MOLP problem as:

$$F \cap C = \bigcup\{\delta_1 \wedge \lambda_1, \delta_2 \wedge \lambda_2, \ldots, \delta_n \wedge \lambda_n\} \tag{6}$$

where, F is hesitant fuzzy objective and C denotes hesitant fuzzy constraints. Further, the hesitant fuzzy decision set (HFDS) denoted as $\tilde{D}$:

$$D = F \cap C = \{(x, \bigcup\{\delta_1 \wedge \lambda_1, \delta_2 \wedge \lambda_2, \ldots, \delta_n \wedge \lambda_n\}(x) | x \in X\}$$
$$= \{(x, \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n}) | x \in X\}$$
$$\alpha_1 = \delta_1 \wedge \lambda_1 = \min\{\delta_1, \lambda_1\} \quad (7)$$
$$\alpha_2 = \delta_2 \wedge \lambda_2 = \min\{\delta_2, \lambda_2\}$$
$$\ldots$$
$$\ldots$$
$$\alpha_n = \delta_n \wedge \lambda_n = \min\{\delta_n, \lambda_n\}$$

where, $\mu_D(x)$ denotes the degree of acceptance of hesitant fuzzy decision solution of hesitant fuzzy decision. The above problem (1) can be transformed into the following system:

$$\begin{aligned}
&\text{Maximize } \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} \\
&\text{Such that } \mu_k^{E_1}(f_k(x)) \geq \alpha_1, \text{ for all } k \\
&\mu_k^{E_2}(f_k(x)) \geq \alpha_2, \text{ for all } k \\
&\ldots \\
&\ldots \\
&\mu_k^{E_n}(f_k(x)) \geq \alpha_n, \text{ for all } k \\
&0 \leq \alpha_1, \alpha_2, \ldots, \alpha_n \leq 1, \\
&g_j(x) \leq 0, \ j = 1, 2, \ldots, q \\
&x \geq 0.
\end{aligned} \quad (8)$$

where,

$\mu_k^{E_1}(f_k(x))$ membership degrees are given by 1^{st} expert,
$\mu_k^{E_2}(f_k(x))$ membership degrees are given by 2^{nd} expert,
...
...
$\mu_k^{E_n}(f_k(x))$ membership degrees are given by n^{th} expert.

4 Computational algorithm

Step 1 Taking the first objective function from set of k objectives of the problem and solve it as a single objective subject to the given constraints. Find value of objective functions and decision variables.
Step 2 From values of these decision variables compute values of remaining $(k-1)$ objectives.
Step 3 Repeat the step 1 and step 2 for remaining $(k-1)$ objective functions.
Step 4 Tabulate values of objective functions thus obtained from step 1, step 2 and step 3 to form a Table 1 known as PIS.
Step 5 From step 4, obtain the lower bounds and upper bounds for each objective functions, where f_k^* and $f_k^{'}$ are the maximum, minimum values respectively.

Table 1 Positive ideal solution

Maximum	f_1	f_2	f_3	...	f_k	X
Maximum f_1	f_1^*			...	$f_k(X_1)$	X_1
Maximum f_2		f_2^*		...	$f_k(X_2)$	X_2
Maximum f_3			f_3^*	...	$f_k(X_3)$	X_3
...				...	...	...
Maximum f_k				...	f_k^*	X_k
Minimum	f_1'	f_2'	f_3'	...	f_k'	

Step 6 Here, we denote and define upper and lower bounds by $U_K^{\mu} = max(Z_K(X_r))$ and $L_K^{\mu} = min(Z_K(X_r))$, $1 \leq r \leq p$ respectively for each uncertain and imprecise objective functions of MOLP problems.
Step 7 In this step, we present uncertain and imprecise objectives of MOLP by using the following linear hesitant membership functions $\mu_k^{E^1}(f_k(x))$:

$$\mu_k^{E^1}(f_k(x)) = \begin{cases} 0, \ if \ f_k(x) \leq L_K^{\mu} \\ \alpha_1 \frac{f_k(x) - L_K^{\mu}}{U_K^{\mu} - L_K^{\mu}}, \ if \ L_K^{\mu} \leq f_k(x) \leq U_K^{\mu} \\ 1, if \ f_k(x) \geq L_K^{\mu} \end{cases} \quad (9)$$

$$\mu_k^{E^2}(f_k(x)) = \begin{cases} 0, \ if \ f_k(x) \leq L_K^{\mu} \\ \alpha_2 \frac{f_k(x) - L_K^{\mu}}{U_K^{\mu} - L_K^{\mu}}, \ if \ L_K^{\mu} \leq f_k(x) \leq U_K^{\mu} \\ 1, if \ f_k(x) \geq L_K^{\mu} \end{cases} \quad (10)$$

$$\ldots$$
$$\ldots$$

$$\mu_k^{E^n}(f_k(x)) = \begin{cases} 0, \ if \ f_k(x) \leq L_K^{\mu} \\ \alpha_n \frac{f_k(x) - L_K^{\mu}}{U_K^{\mu} - L_K^{\mu}}, \ if \ L_K^{\mu} \leq f_k(x) \leq U_K^{\mu} \\ 1, if \ f_k(x) \geq L_K^{\mu} \end{cases} \quad (11)$$

where, $0 \leq \alpha_1, \alpha_2, \ldots, \alpha_n \leq 1$.
Step 8 Now the hesitant fuzzy optimization method for MOLP problem (1) with linear membership functions gives a equivalent linear programming problem as:

$$\begin{aligned}
&\text{Maximize } \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{3} \\
&\text{Such that } \mu_k^{E_1}(x) \geq \alpha_1, \text{ for all } k \\
&\mu_k^{E_2}(x) \geq \alpha_2, \text{ for all } k \\
&\ldots \\
&\ldots \\
&\mu_k^{E_n}(x) \geq \alpha_n, \text{ for all } k \\
&0 \leq \alpha_1, \alpha_2, \ldots, \alpha_n \leq 1, \\
&g_j(x) \leq 0, \ j = 1, 2, \ldots, q \\
&x \geq 0.
\end{aligned} \quad (12)$$

Table 2 Physical parameter values

Machine type	Machine hours	Unit price	Products		
			x_1	x_2	x_3
Milling	1400	0.75	12	17	0
Lather	1000	0.60	3	9	8
Grinder	1750	0.35	10	13	15
Jig saw	1325	0.50	6	0	16
Drill press	900	1.15	0	12	7
Band saw	1075	0.65	9.5	9.5	4
Total capacity cost	4658.75				

Step 9 The above linear programming problem (12) can be easily solved by the above simplex method.

5 Numerical verification of the proposed computational algorithm

5.1 Production planning problem

Consider a park of six machine types whose capacities are to be devoted to production of three products. A current capacity portfolio is available, measured in machine hours per weak for each machine capacity unit priced according to machine type. The necessary data in Table 2 is summarized.

Let x_1, x_2, x_3 denote three products, then the complete mathematical formulation of the above mentioned problem as a multiobjective linear programming problem is given as:

$$\begin{aligned}
&\text{Maximize } f_1(x) = 50x_1 + 100x_2 + 17.5x_3 \text{ (Profit)}\\
&\text{Maximize } f_2(x) = 92x_1 + 75x_2 + 50x_3 \text{ (Quality)}\\
&\text{Maximize } f_3(x) = 25x_1 + 100x_2 + 75x_3\\
&\quad \text{(Worker satisfaction)}\\
&\text{Subject to the constraints}\\
&12x_1 + 17x_2 \le 1400\\
&3x_1 + 9x_2 + 8x_3 \le 1000\\
&10x_1 + 13x_2 + 15x_3 \le 1750\\
&6x_1 + 16x_3 \le 1325\\
&x_1, x_2, x_3 \ge 0.
\end{aligned} \tag{13}$$

Stepwise numerical verification of the proposed algorithm are presented below:

Step 1 In this step, we reduce multiobjective linear programming problem into single linear programming problem which is given below:

Table 3 Positive ideal solution

	f_1	f_2	f_3	X
Maximum f_1	8041	10,020.33	9319.25	X_1
Maximum f_2	5452.63	10,950.59	5903.00	X_2
Maximum f_3	7983.60	10,056.99	9355.90	X_3

$$\begin{aligned}
&\text{Maximize } f_1(x) = 50x_1 + 100x_2 + 17.5x_3\\
&\text{Subject to the constraints}\\
&12x_1 + 17x_2 \le 1400\\
&3x_1 + 9x_2 + 8x_3 \le 100\\
&10x_1 + 13x_2 + 15x_3 \le 1750\\
&6x_1 + 16x_3 \le 1325\\
&12x_2 + 7x_3 \le 900\\
&9.5x_1 + 9.5x_2 + 4x_3 \le 1075\\
&x_1, x_2, x_3 \ge 0.
\end{aligned} \tag{14}$$

Solving single objective linear programming problem (14), we get the following optimum solutions: $x_1 = 44.93, x_2 = 50.63, x_3 = 41.77, (f_1)_1 = 8041.14$.

Step 2 With these decision variables, computed values of other remaining objective functions are: $(f_2)_1 = 10{,}020.33$, $(f_3)_1 = 9319.25$.

Step 3 Step 1 and Step 2 are repeated for other objective functions f_2, f_3.

Step 4 The Positive Ideal Solution obtained are placed in Table 3.

Step 5

Step 6 In this step, we calculate lower and upper bounds for each objective functions:

$$\begin{aligned}
L_1^{\mu} &= 5452.63,\ U_1^{\mu} = 8041.14\\
L_2^{\mu} &= 10{,}020.33,\ U_2^{\mu} = 10{,}950.59\\
L_3^{\mu} &= 9355.90,\ U_3^{\mu} = 5903.00
\end{aligned}$$

Step 7 Use following linear membership function $\mu_k^{E^1}(f_k(x))$ for each objective functions:

$$\mu^{E^1}(50x_1 + 100x_2 + 17.5x_3) = \begin{cases} 0, \text{ if } 50x_1 + 100x_2 + 17.5x_3 \le 5452.63\\ 0.96\frac{50x_1+100x_2+17.5x_3-5452.63}{8041.14-5452.63}, \text{ if } 5452.63\\ \quad \le 50x_1 + 100x_2 + 17.5x_3 \le 8041.14\\ 1, \text{ if } 50x_1 + 100x_2 + 17.5x_3 \ge 5452.63 \end{cases} \tag{15}$$

$$\mu^{E^2}(50x_1+100x_2+17.5x_3) = \begin{cases} 0, \text{ if } 50x_1+100x_2+17.5x_3 \le 5452.63 \\ 0.98\frac{50x_1+100x_2+17.5x_3-5452.63}{8041.14-5452.63}, \text{ if } 5452.63 \\ \quad \le 50x_1+100x_2+17.5x_3 \le 8041.14 \\ 1, \text{if } 50x_1+100x_2+17.5x_3 \ge 5452.63 \end{cases} \tag{16}$$

$$\mu^{E^3}(50x_1+100x_2+17.5x_3) = \begin{cases} 0, \text{ if } 50x_1+100x_2+17.5x_3 \le 5452.63 \\ \frac{50x_1+100x_2+17.5x_3-5452.63}{8041.14-5452.63}, \text{ if } 5452.63 \\ \quad \le 50x_1+100x_2+17.5x_3 \le 8041.14 \\ 1, \text{if } 50x_1+100x_2+17.5x_3 \ge 5452.63 \end{cases} \tag{17}$$

$$\mu^{E^1}(92x_1+75x_2+50x_3) = \begin{cases} 0, \text{ if } 92x_1+75x_2+50x_3 \le 10{,}020.33 \\ 0.96\frac{92x_1+75x_2+50x_3-10{,}020.33}{10{,}950.59-10{,}020.33}, \text{ if } 10{,}020.33 \\ \quad \le 92x_1+75x_2+50x_3 \le 10{,}950.59 \\ 1, \text{if } 92x_1+75x_2+50x_3 \ge 10{,}020.33 \end{cases} \tag{18}$$

$$\mu^{E^2}(92x_1+75x_2+50x_3) = \begin{cases} 0, \text{ if } 92x_1+75x_2+50x_3 \le 10{,}020.33 \\ 0.98\frac{92x_1+75x_2+50x_3-10{,}020.33}{10{,}950.59-10{,}020.33}, \text{ if } 10{,}020.33 \\ \quad \le 92x_1+75x_2+50x_3 \le 10{,}950.59 \\ 1, \text{if } 92x_1+75x_2+50x_3 \ge 10{,}020.33 \end{cases} \tag{19}$$

$$\mu^{E^3}(92x_1+75x_2+50x_3) = \begin{cases} 0, \text{ if } 92x_1+75x_2+50x_3 \le 10{,}020.33 \\ \frac{92x_1+75x_2+50x_3-10{,}020.33}{10{,}950.59-10{,}020.33}, \text{ if } 10{,}020.33 \\ \quad \le 92x_1+75x_2+50x_3 \le 10{,}950.59 \\ 1, \text{if } 92x_1+75x_2+50x_3 \ge 10{,}020.33 \end{cases} \tag{20}$$

$$\mu^{E^1}(25x_1+100x_2+75x_3) = \begin{cases} 0, \text{ if } 25x_1+100x_2+75x_3 \le 5903.00 \\ 0.96\frac{25x_1+100x_2+75x_3-5903.00}{9355.90-5903.00}, \text{ if } 5903.00 \\ \quad \le 25x_1+100x_2+75x_3 \le 9355.90 \\ 1, \text{if } 25x_1+100x_2+75x_3 \ge 5903.00 \end{cases} \tag{21}$$

$$\mu^{E^2}(25x_1+100x_2+75x_3) = \begin{cases} 0, \text{ if } 25x_1+100x_2+75x_3 \le 5903.00 \\ 0.98\frac{25x_1+100x_2+75x_3-5903.00}{9355.90-5903.00}, \text{ if } 5903.00 \\ \quad \le 25x_1+100x_2+75x_3 \le 9355.90 \\ 1, \text{if } 25x_1+100x_2+75x_3 \ge 5903.00 \end{cases} \tag{22}$$

$$\mu^{E^3}(25x_1+100x_2+75x_3) = \begin{cases} 0, \text{ if } 25x_1+100x_2+75x_3 \le 5903.00 \\ \frac{25x_1+100x_2+75x_3-5903.00}{9355.90-5903.00}, \text{ if } 5903.00 \\ \quad \le 25x_1+100x_2+75x_3 \le 9355.90 \\ 1, \text{if } 25x_1+100x_2+75x_3 \ge 5903.00 \end{cases} \tag{23}$$

Table 4 Solution based on proposed algorithm

x_1	x_2	x_3	α_1	α_2	α_3
70.7681	20.677	51.5679	0.3916	0.3997	0.4079

Step 8 Linear programming problem in hesitant fuzzy sense is presented below:

$$\begin{aligned}
&\text{Maximize } \frac{\alpha_1+\alpha_2+\alpha_3}{3} \\
&\text{Such that } 0.96\frac{50x_1+100x_2+17.5x_3-5452.63}{8041.14-5452.63} \ge \alpha_1 \\
&0.96\frac{92x_1+75x_2+50x_3-10{,}020.33}{10{,}950.59-10{,}020.33} \ge \alpha_1 \\
&0.96\frac{25x_1+100x_2+75x_3-5903.00}{9355.90-5903.00} \ge \alpha_1 \\
&0.98\frac{50x_1+100x_2+17.5x_3-5452.63}{8041.14-5452.63} \ge \alpha_2 \\
&0.98\frac{92x_1+75x_2+50x_3-10{,}020.33}{10{,}950.59-10{,}020.33} \ge \alpha_2 \\
&0.98\frac{25x_1+100x_2+75x_3-5903.00}{9355.90-5903.00} \ge \alpha_2 \\
&\frac{50x_1+100x_2+17.5x_3-5452.63}{8041.14-5452.63} \ge \alpha_3 \\
&\frac{92x_1+75x_2+50x_3-10{,}020.33}{10{,}950.59-10{,}020.33} \ge \alpha_3 \\
&\frac{25x_1+100x_2+75x_3-5903.00}{9355.90-5903.00} \ge \alpha_3 \\
&0 \le \alpha_1 \le 1 \\
&0 \le \alpha_2 \le 1 \\
&0 \le \alpha_3 \le 1 \\
&12x_1+17x_2 \le 1400 \\
&3x_1+9x_2+8x_3 \le 1000 \\
&10x_1+13x_2+15x_3 \le 1750 \\
&6x_1+16x_3 \le 1325 \\
&x_1, x_2, x_3 \ge 0.
\end{aligned} \tag{24}$$

Step 9 Applying the proposed algorithm, the solutions of the mentioned MOLP is listed in Table 4.

6 Conclusions

In recent decades, many extensions and generalizations of fuzzy optimization techniques have been proposed in literature for the solution of multiobjective programming problems for example, interval-valued fuzzy optimization [28], intuitionistic fuzzy optimization [9], generalized intuitionistic fuzzy optimization [19], etc. In present article a new con-

Table 5 Optimal solutions obtained by various methods

Decision variables objective	Fuzzy optimization technique	Intuitionistic fuzzy optimization technique	Hesitant fuzzy optimization technique
x_1	65.2571	58.4833	70.7681
x_2	26.9187	34.5907	20.6770
x_3	49.8324	47.6992	51.5679
f_1	6826.7920	7217.9710	6508.5432
f_2	10,514.1757	10,359.7261	10,639.8352
f_3	8060.6952	8498.5925	7704.4950

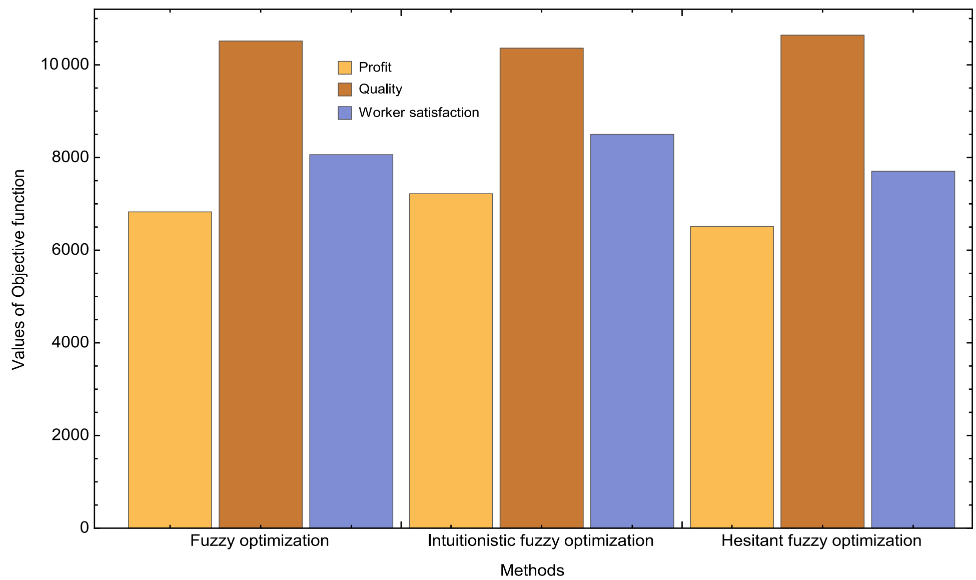

Fig. 2 Comparison of results with existing methods

cept of optimality is presented and in view of comparing the hesitant fuzzy optimization with fuzzy optimization, and intuitionistic fuzzy optimization, we also obtained the solution of the undertaken numerical problem by fuzzy optimization method given by Zimmermann [2] and took the best result obtained for comparison with present study. We considered the best solution obtained by the developed algorithm and are placed in Table 5 for comparison with each other and also to compare with the results obtained by fuzzy optimization method.

The objective of the present study is to give the effective algorithm for hesitant fuzzy optimization method for getting optimal solutions to a multiobjective linear programming problem. The merit of the method lies with fact that it gives a set of solutions based on various experts levels. The decision makers may choose a suitable optimal solution according to the demand of the actual situation. Further, the comparisons of results obtained for the undertaken problem clearly show the superiority with respect to quality of products of hesitant fuzzy optimization over fuzzy and intuitionistic fuzzy optimizations which are presented in Fig. 2. Present work can be further applied in nonlinear multiobjective programming problems, multiobjective transportation problems, multiobjective assignment problems, multiobjective fractional problems, game theory, etc.

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