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Implementation of numerical approximations in studying vibration of functionally graded beams

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Abstract

In this investigation, a brief review on three efficient computational techniques viz. Finite Element Method, Differential Quadrature Method and Rayleigh–Ritz Method along with their mathematical formulation to study free vibration of thin Functionally Graded (FG) beams subject to various classical boundary supports have been presented. The deformation of FG beam is based on the framework of classical beam theory. Three different FG beam constituents assumed in this study are Al/Al_2O_3, Al/ZrO_2 and SUS304/Si_3N_4, in which the first component is meant for the metal constituent and the second for ceramic constituent respectively. The material properties of FG beam are assumed to vary continuously along thickness direction in a power-law form. The objective is to outline exemplary works carried out by various researchers on the concerned problem and also to find the effect of volume fraction of FG constituents on natural frequencies. The natural frequencies of different FG beams under four sets of classical edge supports have been evaluated along with two-dimensional mode shapes after finding the convergence with reference to concerned numerical methods and validation with available literature.

Keywords Vibration · Functionally graded beam · FEM · DQM · RRM

1 Introduction

Functionally Graded (FG) materials are emerging advanced composites in recent decade for their thermal resistance properties, which was first discovered by a group of material scientists in Japan to withstand a huge temperature fluctuation across a very thin cross-section in a space-plane project. Major components of FG composites are metal and ceramic materials, in which the constituent properties vary spatially along thickness direction in a specific mathematical pattern. As their microstructure has not yet been revealed, the mechanics and governing equations related to homogeneous case are assumed to be true for elastic FG composites. Studying dynamics of functionally graded beam is one of the interesting problems in current era and literature related to such problems using various methods have been briefly mentioned herein.

1.1 Development of FEM

One of the recent trends on solving the titled problems is based on Finite Element Method (FEM) and its research development has been reviewed first. The method is first developed in 1956 for the analysis of aircraft structural problems. Thereafter, the finite element technique has drawn enough attention within a decade for solving a wide variety of problems in applied science and engineering and has been developed over the years [1]. As regards, Öz [2] have calculated natural frequencies of an Euler-Bernoulli beam-mass system using this approach. Two C^0 assumed finite element formulations of Reddy's higher-order theory were used by Nayak et al. [3] to obtain the natural frequencies of composite and sandwich plates. Chakraborty et al. [4] proposed a new beam finite element based on the first-order shear deformation theory to study the thermoelastic behavior of functionally graded beam structures. Ribeiro [5] has applied the shooting, Newton and p-version, heirarchical finite element methods to investigate geometrically nonlinear periodic vibrations of elastic and isotropic, beams and plates. Şimşek

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[6] has examined vibration response of a simply-supported FG beam to a moving mass by using Euler–Bernoulli, Timoshenko and the third-order shear deformation beam theories. Alshorbagy et al. [7] have used finite element method to detect the free vibration characteristics of a functionally graded beam. Shahba et al. [8] investigated free vibration and stability analysis of axially functionally graded Timoshenko tapered beams using classical and non-classical boundary conditions through finite element approach. The bending and flexural vibration behavior of sandwich FG plates have been provided by Natarajan and Manickam [9] using QUAD-8 shear flexible element developed on higher order structural theory. Vo et al. [10] have developed a finite element model for vibration and buckling of FG sandwich beams based on refined shear deformation theory. Free vibration and stability of axially functionally graded tapered Euler–Bernoulli beams have been investigated using finite element method by Shahba and Rajasekaran [11]. Vo et al. [12] have presented static and vibration analysis of FG beams using refined shear deformation theory by using finite element formulation. A novel Timoshenko beam element based on the framework of strain gradient elasticity theory is presented in [13] for the analysis of the static bending, free vibration and buckling behaviors of Timoshenko microbeams. Very recently, Hui et al. [14] have given a family of beam higher-orders finite elements based on a hierarchical one dimensional unified formulation for a free vibration analysis of three-dimensional sandwich structures. To name a few out of recent findings, one may easily find finite element solutions of structural members in [15,16] and also literature available therein.

1.2 Development of DQM

On the other hand, differential quadrature method (DQM) is an efficient numerical procedure, which is also taken into account in present study. The DQ method, equivalent to the conventional integral quadrature method, approximates the derivative of a function at any location by a linear combination of the functional values within a closed domain. The key procedure the DQM lies in the determination of the weighting coefficients with respect to specific order derivatives [17]. Initially, the foundation of DQM is proposed mainly by Bellman and Casti [18] and implemented in various classes of partial differential equations by reducing them to ordinary differential equations and then to finite dimensional systems. A precise idea on ways to develop DQM in various forms, numerical solution of different classes of linear and nonlinear partial differential equations, splines and efficiency of this method can be observed in [19–21]. Following the investigation of Bellman et al. [19], Quan and Chang [22,23] have provided new insights in solving distributed system equations by the method of differential quadrature along with results of a series of numerical experiments. The analysis of laminated composite structures has been performed by Bert and Malik [24] using this method. After a close observation to previous studies [19,22,23] on DQM, Shu and Du [25] have developed Generalized Differential Quadrature (GDQ) method for implementing clamped and simply supported boundary conditions for the free vibration analysis of beams and plates. In the similar fashion, a few major changes have also been incorporated in DQM in recent decades by combining with other numerical methods. As such, a mixed Ritz-DQ method has been used in free and forced vibration of functionally graded beams and isotropic rectangular plates in [26–28]. In a recent work, Yas et al. [29] have investigated free vibration of Euler–Bernoulli FG beams resting on elastic foundation by means of GDQ. A general formulation of the quadrature element method is presented by Jin and Wang [30] to estimate vibration of FG beams.

1.3 Development of RRM

In addition, it is also worth to organise the investigations performed by means of Rayleigh–Ritz method. Rayleigh's classical book 'Theory of Sound' was first published in 1877. In this book, lots of examples can be found on evaluating fundamental natural frequencies of free vibration of continuum systems by assuming the mode shape and setting the maximum values of potential and kinetic energy in a cycle of motion equal to each other. This procedure is well known as Rayleigh's Method. In 1908, Ritz laid out his famous method for determining frequencies and mode shapes, choosing multiple admissible displacement functions, and minimizing a functional involving both potential and kinetic energies. Subsequently, this technique is referred to as Rayleigh–Ritz method and has taken major attention among researchers till date. These fact on the origin of Rayleigh and Ritz methods are clearly addressed in Leissa [31]. By assuming a modification, Bhat [32] have considered the characteristic orthogonal polynomials in Rayleigh–Ritz method to estimate transverse vibration response of rotating cantilever beam with a tip mass. The natural frequencies of rectangular plates using characteristic orthogonal polynomials have also been computed in [33,34] using Rayleigh–Ritz method. Singh and Chakraverty [35–37] have solved transverse vibration of elliptic and circular plates using orthogonal polynomials in Rayleigh–Ritz method satisfying different boundary conditions viz. completely-free, simply-supported and clamped respectively. Rayleigh–Ritz method is used by Abrate [38] to find vibration of some non-uniform rods and beams with one end completely fixed. Ding [39] has developed a fast converging series consisting of a set of static beam functions to study vibration characteristics of thin rectangular plates. Aydogdu and Taskin [40] have investigated free vibration of a simply supported FG beam within the framework of Euler-Bernoulli beam theory, parabolic shear deformation theory and exponential shear deformation theory. Analytical solution is proposed by Ece et al. [41] to study vibration of isotropic beam with variable cross-section. Sina et al. [42] have given an analytical solution for free vibration of functionally graded beams. Refined plate theories have been implemented by Carrera et al. [43] in finding accurate free vibration analysis of anisotropic, simply supported plates. The flapwise and chordwise bending vibration analysis of rotating pre-twisted Timoshenko beam are examined in [44] by the use of Rayleigh–Ritz method. Free vibration of the baffled circular plates with radial side cracks and in contact with water on one side is studied by Si et al. [45] based on Rayleigh–Ritz method. Moreover, Chakraverty and Pradhan [46] have given an excellent monograph on vibration of functionally graded beams and plates with various geometries (rectangular, elliptic and triangular) along with the effect of complicating environments.

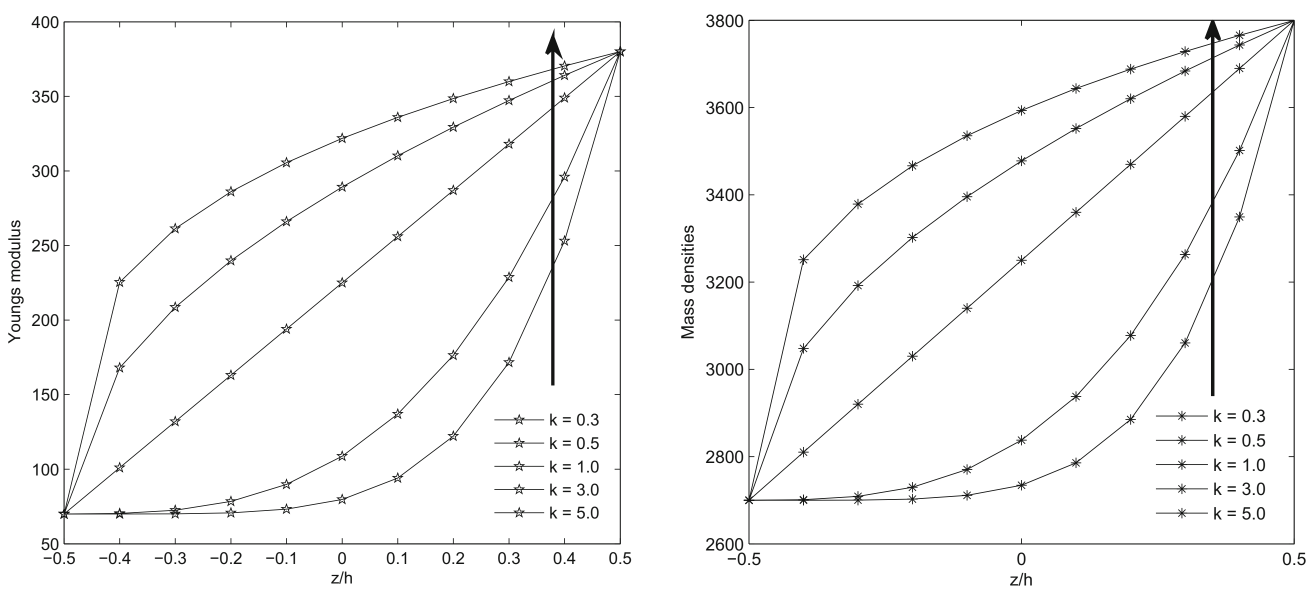
In the above discussion, a brief idea on implementing three efficient numerical techniques in handling structural problems has been presented. To the best of authors' knowledge, no investigation has covered on implementing efficient numerical solutions in studying vibration of Euler–Bernoulli FG beams (estimating six natural frequencies of three FG beams consisting of various constituents and comparison of their numerical approach) along with their recent developments. As such, present study is associated with numerical approach of finite element, differential quadrature and Rayleigh–Ritz methods towards free vibration of Euler–Bernoulli functionally graded beam. The conventional procedures followed by these methods have been discussed in detail. The material properties of FG constituents vary continuously along thickness direction in power-law form. The main objective of this investigation is to address varieties of related significant works and to analyze the effect of volume fractions of constituents on natural frequencies. New results for natural frequencies along with two-dimensional (2-D) mode shapes are incorporated after checking test of convergence and validation with previous literature.

2 Functionally graded beam

A straight FG beam of length L, width b and thickness h, having rectangular cross-section with cartesian coordinate system is considered here. In addition, the material properties of FG beam vary along the thickness direction according to power-law form as shown in Fig. 1. The power-law variation used in Sina et al. [42] is considered

$$\mathbf{P}(z) = (\mathbf{P}_c - \mathbf{P}_m)\left(\frac{z}{h} + \frac{1}{2}\right)^{\mathrm{k}} + \mathbf{P}_m \tag{1}$$

where $\mathbf{P}_c$ and $\mathbf{P}_m$ denote the values of the material properties of the ceramic and metal constituents of the FG beam respectively. k (power-law exponent) is a non-negative variable parameter. According to this distribution, the bottom surface ($z = -h/2$) of FG beam is pure metal, whereas the top surface ($z = h/2$) is pure ceramic and for different values of k one can obtain different volume fractions of material beam as mentioned in Aydogdu and Taskin [40]. For our present formulations, the material properties viz. Young's



(a) Variation of Young's modulus **(b)** Variation of mass densities

Fig. 1 Power-law variation of material properties of Al/Al_2O_3 functionally graded beam modulus (E) and mass density (ρ) are considered to vary along thickness direction except Poisson's ratio (ν) remaining as constant. In Fig. 1, FG constituents own the properties of Al (metal) and Al_2O_3 (ceramic) [42]: $E_m = 70$ GPa, $\rho_m = 2700$ kg/m^3, $E_c = 380$ GPa and $\rho_c = 3800$ kg/m^3.

Here, classical beam theory is assumed to define the deformation of thin functionally graded beam as referred by Şimşek and Kocaturk [6], Alshorbagy et al. [7], Aydogdu and Taskin [40], Sina et al. [42] and Şimşek [47]

$$\begin{aligned} u_x(x,z) &= -z\frac{\partial w}{\partial x} \\ u_z(x,z) &= w(x,t). \end{aligned} \tag{2}$$

Now, Eq. (3) represent the kinematic relations with respect to displacement field of Eq. (2)

$$\begin{aligned} \varepsilon_{xx} &= -z\frac{\partial^2 w}{\partial x^2} \\ \gamma_{xz} &= 0 \end{aligned} \tag{3}$$

where ε_{xx} and γ_{xz} are the normal and shear strains respectively. By assuming the material constituents of FG beam to obey the generalized Hooke's law, the state of stresses in the beam can be written as

$$\begin{aligned} \sigma_{xx} &= Q_{11}\varepsilon_{xx} \\ \tau_{xz} &= Q_{55}\gamma_{xz}. \end{aligned} \tag{4}$$

where σ_{xx} and τ_{xz} in Eq. (4) are the normal and the shear stresses respectively and Q_{ij} are the transformed stiffness constants in the beam co-ordinate system and are defined as [48]

$$Q_{11} = \frac{E(z)}{1-\nu^2}, \quad Q_{55} = \frac{E(z)}{2(1+\nu)}.$$

3 Numerical formulations

First of all, the classical boundary conditions for transverse displacement (w) at the specified beam end can be introduced as provided by Öz [2]

$$\begin{aligned} &\text{Clamped (C):}\ \ w = 0;\ \ \frac{\partial w}{\partial x} = 0 \\ &\text{Simply supported (S):}\ \ w = 0;\ \ \frac{\partial^2 w}{\partial x^2} = 0 \\ &\text{Free (F):}\ \ \frac{\partial^2 w}{\partial x^2} = 0;\ \ \frac{\partial^3 w}{\partial x^3} = 0. \end{aligned} \tag{5}$$

Further, numerical procedures of three well-known numerical techniques viz. FEM, GDQ and Rayleigh–Ritz method have been addressed to obtain the generalized eigenvalue problem for free vibration of FG beam as given below.

3.1 Finite element method

In general, the transverse displacement and rotation (slope) describe the deformed shape of the beam and these components at each end of the beam are treated as the unknown degrees of freedom. As there are four nodal displacements at the beam ends, the cubic displacement model as mentioned in Rao [1] can be defined as

$$w(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \tag{6}$$

where a_1, a_2, a_3 and a_4 are the unknown coefficients and can be found by using the edge conditions. The displacement (w) and rotation (θ) at $x = 0$ and L can be substituted in Eq. (6) and yield

$$\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{pmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \tag{7}$$

Solving Eq. (7), the shape function $[N]$ takes the form

$$[N] = [N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x)] \tag{8}$$

where

$$\begin{aligned} N_1(x) &= \frac{2x^3 - 3Lx^2 + L^3}{L^3} \\ N_2(x) &= \frac{x^3 - 2Lx^2 + L^2x}{L^3} \\ N_3(x) &= \frac{2x^3 - 3Lx^2}{L^3} \\ N_4(x) &= \frac{x^3 - Lx^2}{L^3} \end{aligned}$$

which helps to find the kinetic energy and also the connectivity matrix $[B]$ for elastic strain energy.

$$\begin{aligned} [B] &= [B_1(x) \quad B_2(x) \quad B_3(x) \quad B_4(x)] \\ &= \left[\frac{d^2N_1}{dx^2} \quad \frac{d^2N_2}{dx^2} \quad \frac{d^2N_3}{dx^2} \quad \frac{d^2N_4}{dx^2}\right] \end{aligned} \tag{9}$$

From Eq. (8) and (9), one may find respective element inertia and stiffness matrices as

$$[M_e] = \frac{\rho(z)AL}{420}\begin{pmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{pmatrix} \tag{10}$$

and

$$[K_e] = \frac{E(z)I}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix} \tag{11}$$

The formulation for local stiffness and mass matrix are clearly given in Öz [2] in case of isotropic beams. Prior to the fact that Young's modulus in stiffness matrix and mass density in mass matrix are dependent on the thickness, there occurs a slight modification in the expression of corresponding matrices. If discretization of total length of the FG beam is considered, discretized element inertia and stiffness matrices will be combined to obtain the global inertia and stiffness matrix respectively. The equation of motion for free vibration of FG beam can be obtained from

$$[M]\left\{w''\right\} + [K]\{w\} = \{0\} \tag{12}$$

where $[K]$ and $[M]$ are the global stiffness and mass matrices respectively and $\{w\}$ is the system displacement vector. Then substituting the harmonic displacement in the form $w(x,t) = W(x)\exp(i\omega t)$ with $i = \sqrt{-1}$ and ω as natural frequency and W as amplitude of displacement, we can write Eq. (38) as

$$\left[[K] - \Omega^2[M]\right]\{W\} = \{0\} \tag{13}$$

For non-trivial solution of Eq. (13), it is assumed that the determinant of coefficient matrix must be zero and it provides

$$\det([K] - \Omega^2[M]) = 0 \tag{14}$$

and is referred to as the generalized eigenvalue problem. Consequently, natural frequencies are to be solved by incorporating various sets of classical boundary conditions.

3.2 Differential quadrature method

Analog to the governing equation for vibration of isotropic beams given by Shu and Du [25], free vibration of Euler–Bernoulli functionally graded beam can be governed by

$$\left\{\int_A z^2 E(z)\mathrm{d}A\right\}\frac{\partial^4 w}{\partial x^4} + \left\{\int_A \rho(z)\mathrm{d}A\right\}\frac{\partial^2 w}{\partial t^2} = 0. \tag{15}$$

The material properties $E(z)$ and $\rho(z)$ are the Young's moduli and mass densities of FG material constituents, which are assumed to vary along thickness direction in power-law form (as stated in Eq. 1). By considering the non-dimensional parameters are $-\frac{1}{2} \le \xi = x/L \le \frac{1}{2}$ and $-\frac{1}{2} \le \bar{z} = z/h \le \frac{1}{2}$, Eq. (15) yields

$$\frac{\widetilde{E}I}{L^4}\frac{\partial^4 w}{\partial \xi^4} + \widetilde{\rho}A\frac{\partial^2 w}{\partial t^2} = 0 \tag{16}$$

Here, $\widetilde{E} = \frac{E_m}{1-\nu^2}\left\{1 + 12(E_r - 1)\left(\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4(k+1)}\right)\right\}$ and $\widetilde{\rho} = \rho_m\left\{1 + \left(\frac{\rho_r - 1}{k+1}\right)\right\}$. Introducing the harmonic type displacement as $w(\xi,t) = W(\xi)\exp(i\omega t)$, where $W(x)$ are the amplitude in displacement component and ω is the natural frequency. Next Eq. (16) transforms into

$$\overline{E}\frac{d^4 W}{d\xi^4} - \Omega^2\overline{\rho}W = 0 \tag{17}$$

where $\overline{E} = \frac{1}{1-\nu^2}\left\{1 + 12(E_r - 1)\left(\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4(k+1)}\right)\right\}$; $\overline{\rho} = \left\{1 + \left(\frac{\rho_r - 1}{k+1}\right)\right\}$ and $\Omega^2 = \frac{\omega^2 L^4 \rho_m A}{E_m I}$.

In present discussion, the power-law variations of E and ρ are to be controlled by the components E_r and ρ_r respectively. Moreover, k also plays major role in evaluating both these material properties. In addition while assuming GDQ procedure, Bellman et al. [19] have assumed a sufficiently smooth function $f(x)$ over the interval $[a,b]$, so that its first derivative $f_x^{(1)}(x)$ at any grid point over $[a,b]$ can be approximated by the following approximation

$$f_x^{(1)}(x_i) \cong \sum_{j=1}^{N} c_{ij}^{(1)} f(x_j), \quad i = 1, 2, \ldots, N \tag{18}$$

with the coefficient matrix $(c_{ij}^{(1)})$ can be determined in various fashions. $f_x^{(1)}(x_i)$ finds the first order derivative of $f(x)$ with respect to x at x_i. Necessarily, the key procedure in this method is to compute the weighting coefficients $c_{ij}^{(1)}$. By demanding Eq. (18) to be exact for all polynomials of degree less than or equal to $N-1$. Different approaches towards finding the weighting coefficients are mentioned below:

1. In first approach by Bellman et al. [19], the test functions $g_k(x) = x^{k-1}$; $k = 1, 2, \ldots, N$, which gives a set of linear algebraic equations

$$\sum_{j=1}^{N} c_{ij}^{(1)} x_j^k = k x_i^{k-1}, \quad i = 1, 2, \ldots, N;$$
$$k = 0, 1, \ldots, N-1.$$

   As the matrix is of Vandermonde form, this system of equations has a unique solution. But unfortunately, the concerned matrix becomes ill-conditioned and its inversion is difficult when N is very large.
2. On the other hand, the second approach by Bellman et al. [19] defines the test function as $g_k(x) =$

$\frac{L_N(x)}{(x-x_k)L_N^{(1)}(x_k)}$; $k = 1, 2, \ldots, N$, where $L_N(x)$ is the Nth order Legendre polynomial and $L_N^{(1)}(x)$ is the first order derivative of $L_N(x)$. In this approach, the necessary condition is that the coordinates of grid points should be the roots of an Nth order Legendre polynomial.

3. To overcome such ambiguities of DQM, the generalized differential quadrature (GDQ) approach has been developed by Shu and Du [25] for the determination of weighting coefficients.

3.2.1 Weighting coefficients of first order derivative [25]

In view of the above, Shu and Du [25] has taken the benefit of two approaches of Bellman et al. [19] and approach of Quang and Cheng [22,23] to derive the test functions in GDQ and kept no restrictions in deciding the grid points over the domain. For generality, GDQ chooses the base polynomials (or test functions) $g_k(x)$ to be the Lagrange interpolating polynomial

$$g_k(x) = \frac{M(x)}{(x - x_k)M^{(1)}(x_k)} \tag{19}$$

where $M(x) = \prod_{j=1}^{N}(x - x_j)$; $M^{(1)}(x) = \prod_{j=1, j\neq k}^{N}(x_k - x_j)$ with $x_1, x_2, \ldots, x_N$ are the coordinates of the grid points and may be chosen arbitrarily. For simplicity, it is considered that

$$M(x) = N(x, x_k)(x - x_k), \;\; k = 1, 2, \ldots, N$$

with $N(x_i, x_j) = M^{(1)}(x_i)\delta_{ij}$, where δ_{ij} is the Kronecker operator. With these assumptions, Eq. (19) converts to

$$g_k(x) = \frac{N(x, x_k)}{M^{(1)}(x_k)} \tag{20}$$

By substituting Eq. (20) into Eq. (18), we obtain

$$c_{ij}^{(1)} = \frac{N^{(1)}(x_i, x_j)}{M^{(1)}(x_j)} \tag{21}$$

We can easily find $M^{(1)}(x_j)$ by using its concerned expression. To evaluate $N^{(1)}(x_i, x_j)$, let us differentiate $M(x)$ successively with respect to x and we obtain the following recurrence formulation

$$M^{(m)}(x) = N^{(m)}(x, x_k)(x - x_k) + mN^{(m-1)}(x, x_k) \tag{22}$$

where $k = 1, 2, \ldots, N$; $m = 1, 2, \ldots, N - 1$; $M^{(m)}(x)$ and $N^{(m)}(x, x_k)$ are the m^{th} order derivatives of $M(x)$ and $N(x, x_k)$ respectively. The expression of $N^{(1)}(x_i, x_j)$ can be obtained from Eq. (22) as $N^{(1)}(x_i, x_j) = \begin{cases} \frac{M^{(1)}(x_i)}{x_i - x_j}; & i \neq j \\ \frac{M^{(2)}(x_i)}{2}; & i = j \end{cases}$.
Substituting this expression in Eq. (21), we get

$$c_{ij}^{(1)} = \begin{cases} \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}; & i \neq j \\ \frac{M^{(2)}(x_i)}{2M^{(1)}(x_i)}; & i = j \end{cases} \tag{23}$$

Equation (23) is a simple expression for the computation of $c_{ij}^{(1)}$ without the restriction of choosing grid points x_i. Rather than evaluating $M^{(2)}(x_i)$, it is worth to mention that one set of base polynomials can be derived uniquely by linear combination of another set of base polynomials in a vector space. Moreover, $c_{ij}^{(1)}$ satisfies the relation; $\sum_{j=1}^{N} c_{ij}^{(1)} = 0$ which may be obtained by the base polynomials x^{k-1} when $k = 1$. Here, $c_{ii}^{(1)}$ can easily be determined from $c_{ij}^{(1)}$, $i \neq j$.

3.2.2 Weighting coefficients of second and higher order derivatives [25]

The second and higher order derivatives of the smooth function $f(x)$ may be written with the linear constrained relationships as follows:

$$f_x^{(m)}(x_i) \cong \sum_{j=1}^{N} c_{ij}^{(m)} f(x_j), \quad i = 1, 2, \ldots, N \tag{24}$$

Then, the $(m - 1)$th order derivatives can be expressed as

$$f_x^{(m-1)}(x_i) \cong \sum_{j=1}^{N} c_{ij}^{(m-1)} f(x_j), \quad i = 1, 2, \ldots, N \tag{25}$$

Now let us substitute Eq. (20) into Eqs. (24) and (25) and using Eqs. (22) and (23), a recurrence relation may be written as below:

$$c_{ij}^{(m)} = \begin{cases} m\left(c_{ij}^{(1)} c_{ii}^{(m-1)} - \frac{c_{ij}^{(m-1)}}{x_i - x_j}\right); & i \neq j \\ \frac{M^{(m+1)}(x_i)}{(m+1)M^{(1)}(x_i)}; & i = j \end{cases}$$
$$\text{for } i, \; j = 1, 2, \ldots, N; m = 2, 3, \ldots, N - 1 \tag{26}$$

In $N-$dimensional vector space, the system of equations for $c_{ij}^{(m)}$ derived from Lagrange interpolating polynomials should also be equivalent to that derived from the base polynomials x^{k-1}, $k = 1, 2, \ldots, N$. As discussed for the weighting coefficients of first order derivatives, the weighting coefficients of higher order derivatives also follow the equation $\sum_{j=1}^{N} c_{ij}^{(m)} = 0$ obtained from the base polynomials x^{k-1} when $k = 1$, i.e. $c_{ii}^{(m)} = \sum_{j=1, i\neq j}^{N} c_{ij}^{(m)}$.

3.2.3 Discretization of governing equation

The non-homogeneous grid points in case of DQM are to be considered as Chebyshev–Gauss–Lobatto points in axial direction [25]. The governing equation (Eq. 17) for free vibration of FG beam can be transformed into the following expression by substituting the weighting coefficients of required derivatives,

$$\overline{E}\sum_{j=1}^{N} c_{ij}^{(4)} W(\xi_j) - \Omega^2 \overline{\rho} W(\xi_i) = 0; \quad i = 1, 2, \ldots, N \quad (27)$$

Moreover, the classical boundary supports given in Eq. (5) may be defined as

$$\text{Clamped (C):} \;\; w_\zeta = 0; \;\; \sum_{j=1}^{N} c_{\zeta,j}^{(1)} W(\xi_j) = 0$$

$$\text{Simply supported (S):} \;\; w_\zeta = 0; \;\; \sum_{j=1}^{N} c_{\zeta,j}^{(2)} W(\xi_j) = 0$$

$$\text{Free (F):} \;\; \sum_{j=1}^{N} c_{\zeta,j}^{(2)} W(\xi_j) = 0; \;\; \sum_{j=1}^{N} c_{\zeta,j}^{(3)} W(\xi_j) = 0 \quad (28)$$

with $\zeta = 1$ or N (the edges of FG beam). Now, the discretized governing equation of Eq. (27) may be modified to by using the method of modification of involved weighting coefficient matrices. We have given a simple illustration of modifying weighting coefficient matrix for clamped edge at $\xi = 0$ as mentioned below:

$$\begin{Bmatrix} w_1' \\ w_2' \\ \vdots \\ w_{N-1}' \\ w_N' \end{Bmatrix} = \begin{pmatrix} c_{1,1}^{(1)} & c_{1,2}^{(1)} & \ldots & c_{1,N-1}^{(1)} & c_{1,N}^{(1)} \\ c_{2,1}^{(1)} & c_{2,2}^{(1)} & \ldots & c_{2,N-1}^{(1)} & c_{2,N}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{N-1,1}^{(1)} & c_{N-1,2}^{(1)} & \ldots & c_{N-1,N-1}^{(1)} & c_{N-1,N}^{(1)} \\ c_{N,1}^{(1)} & c_{N,2}^{(1)} & \ldots & c_{N,N-1}^{(1)} & c_{N,N}^{(1)} \end{pmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{Bmatrix} = [C^{(1)}]\{w_j\} \quad (29)$$

The modification of weighting coefficient matrix involved in Eq. (29) occurs by considering $w_1 = 0$ and $w_1' = 0$ at $\xi = 0$ and takes the form

$$\begin{Bmatrix} w_1' \\ w_2' \\ \vdots \\ w_{N-1}' \\ w_N' \end{Bmatrix} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & c_{2,2}^{(1)} & \ldots & c_{2,N-1}^{(1)} & c_{2,N}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_{N-1,2}^{(1)} & \ldots & c_{N-1,N-1}^{(1)} & c_{N-1,N}^{(1)} \\ 0 & c_{N,2}^{(1)} & \ldots & c_{N,N-1}^{(1)} & c_{N,N}^{(1)} \end{pmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{Bmatrix} = [\widetilde{C}^{(1)}]\{w_j\} \quad (30)$$

In Eq. (30), $[\widetilde{C}^{(1)}]$ is the modified weighting coefficient matrix for FG beam with clamped edge at $\xi = 0$. With the usual DQ analog, the higher (Nth) order weighting coefficient matrix can be developed from $[\widetilde{C}^{(N)}] = [\widetilde{C}^{(N-1)}][C^{(1)}]$ by incorporating different sets of edge conditions. Further, the modified matrices are to be substituted in Eq. (27) to get the natural frequencies and mode shapes for free vibration of functionally graded beam.

3.3 Rayleigh–Ritz method

Finally, Rayleigh–Ritz method has also been implemented in present investigation. Using the constitutive relation of Eq. (4), the strain energy S and the kinetic energy T of the beam at any instant in cartesian co-ordinates may be written as

$$S = \frac{1}{2}\int_0^L \int_A (\sigma_{xx}\varepsilon_{xx} + \tau_{xz}\gamma_{xz})\, \mathrm{d}A\, \mathrm{d}x \quad (31)$$

$$T = \frac{1}{2}\int_0^L \int_A \rho(z) \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}A\, \mathrm{d}x \quad (32)$$

where A and ρ are the area of cross-section and the mass density of the beam respectively. In Euler–Bernoulli beam theory, Eqs. (31) and (32) become

$$S = \frac{1}{2}\int_0^L D_z w_{,xx}^2\, \mathrm{d}x \quad (33)$$

$$T = \frac{1}{2}\int_0^L I_z \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \quad (34)$$

The stiffness and inertial coefficients appearing in Eqs. (33) and (34) are defined as

$$D_z = \int_{-h/2}^{h/2} Q_{11} z^2\, \mathrm{d}z$$
$$I_z = \int_{-h/2}^{h/2} \rho(z)\, \mathrm{d}z \quad (35)$$

Assuming harmonic type transverse deflection $w(x,t) = W(x)\sin\omega t$ with $W(x)$ and ω as respective amplitude and natural frequency of the beam. Incorporating harmonic displacement and the non-dimensionalized length parameter $0 \le \xi = x/L \le 1$ in Eqs. (33) and (34) yield the maximum strain energy (S_{max}) and the maximum kinetic energy (T_{max}) as

$$S_{max} = \frac{\overline{D}_z E_m}{2L^4} \int_0^1 D_z W_{,xx}^2 \,\mathrm{d}\xi \tag{36}$$

$$T_{max} = \frac{\omega^2}{2} \int_0^1 I_z W^2 \,\mathrm{d}\xi \tag{37}$$

Next the amplitude of vibration are expanded in terms of algebraic polynomial functions by the following series

$$W = \sum_{i=1}^{n} c_i \phi_i$$

where c_i are the unknown constant coefficients to be determined and ϕ_i are the admissible functions, which must satisfy the essential boundary conditions and can be represented as [49]

$$\phi_i = f x^{i-1}, \quad i = 1, 2, \ldots, n.$$

Here, n is the number of polynomials involved in the admissible functions and $f = x^p (x-1)^q$ where $p, q = 0, 1$ or 2 may be expressed as per different sets of classical boundary conditions (BCs). The parameter $p = 0$, 1 or 2 according as the side $x = 0$ is free (F), simply supported (S) or clamped (C). Similar interpretation can be given to the parameter q corresponding to the sides $x = 1$. Furthermore, Rayleigh Quotient (ω^2) can be obtained by equating S_{max} and T_{max}. Taking partial derivative of the Rayleigh Quotient with respect to the constant coefficients involved in the admissible functions as follows

$$\frac{\partial \omega^2}{\partial c_i} = 0; \quad i = 1, 2, \ldots, n$$

which results in the governing equation for the free vibration of FG beam in the form of generalized eigenvalue problem as mentioned below

$$\left(\mathbf{K} - \Omega^2 \mathbf{M}\right)\{\Delta\} = 0 \tag{38}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the stiffness and inertia matrices respectively and $\{\Delta\}$ is the column vector of unknown coefficients. The eigenvalues (Ω) for the above eigenvalue problem (Eq. 38) are non-dimensional frequencies for the concerned vibration problem. The non-dimensional frequency (Ω) evaluated in all above mentioned methods takes the expression $\Omega = \omega L^2 \sqrt{\frac{\rho_m A}{E_m I}}$. In subsequent sections, present study involves the evaluation of these frequencies after the test of convergence and validation with existing results.

4 Convergence and validation studies

This section involves the test of convergence of natural frequencies in Tables 1, 2, 3, 4, 5 and 6 along with comparison with previously obtained results. The significant facts associated with convergence and validation studies may be summarized as given below:

- The convergence of six lowest natural frequencies of isotropic (k is taken as nullity) and FG beams are carried out in Tables 1, 2, 3, 4, 5 and 6 by means of the above discussed numerical techniques with reference to their corresponding parameters. The methods of finite element and differential quadrature assume such parameter to be the number of discretized elements (discretization of domain), whereas Rayleigh–Ritz method will certainly take the number of polynomials involved in transverse displacement respectively.
- The FEM is implemented in Tables 1 and 4, whereas DQM is considered in Tables 2 and 5 and RRM in Tables 3 and 6 respectively. Moreover, discretization of element domain in FEM and DQM are the non-homogeneous grids based on Gauss–Chebyshev–Lobatto points as assumed by Shu and Du [25].
- Assuming the isotropic beam ($E_r = \rho_r = 1$), first six non-dimensional frequencies are evaluated in Tables 1, 2 and 3 using the formulation $\left(\Omega = \omega L^2 \sqrt{\frac{\rho_m A}{E_m I}}\right)$ and are validated with Öz [2], Shu and Du [25], Abrate [38] and Ece et al. [41]. It can be observed that present results are in excellent agreement with existing literature.
- In a similar fashion, first six eigenfrequencies of FG beam with the formulation $\Omega^2 \left(= \omega L^2 \sqrt{\frac{\rho_m A}{E_m I}}\right)$ have been computed in Tables 4, 5 and 6 with E_r =0.5, 2.0 and k = 0, 0.1, 2.0 and constant mass density and compared with Şimşek and Kocaturk [6] and Alshorbagy et al. [7]. One can easily see a good agreement also in these computations.
- The effect of slenderness (length-to-thickness) ratio is redundant in case of Euler–Bernoulli beam as it doesn't even occur in the formulation. As such, three lowest natural frequencies of S-S FG beam mentioned in [6,7] have been validated in Tables 4, 5 and 6 corresponding to the largest slenderness ratio ($L/h = 100$ or very thin FG beam).
- It is interesting to note that the results found using all three numerical techniques are nearly same, but the convergence is faster in RRM with desired accuracies.

Table 1 Convergence and comparison of six lowest natural frequencies of isotropic beam ($E_r = \rho_r = 1$) using finite element method

BCs	No. of elements	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
C-C	6	22.4335	62.2158	130.4372	211.6678	348.4148	497.9560
	8	22.3892	61.9058	122.7981	201.4761	330.3318	466.7648
	10	22.3792	61.7640	121.5043	201.8458	310.4589	421.6543
	15	22.3743	61.6890	121.0110	200.3150	299.9984	420.8040
	18	22.3738	61.6803	120.9536	200.0738	299.2426	418.7980
	20	22.3736	61.6776	120.9358	199.9981	299.0020	418.1718
	Öz [2]	22.3733	61.6729	120.9039	199.8616	298.5627	–
	Shu and Du [25]	22.3733	61.6728	120.9021	199.9365	299.3886	–
	Abrate [38]	22.3732854	61.672823	120.903392	–	–	–
	Ece et al. [41]	22.37327	61.67281	120.90338	199.85945	298.55552	–
C-S	6	15.4363	50.3112	109.2249	186.8891	314.4239	468.6745
	8	15.4231	50.0853	105.3144	180.5077	294.2674	420.5870
	10	15.4200	50.0110	104.6211	179.8376	279.4785	391.8562
	15	15.4185	49.9730	104.3149	178.5895	273.1206	388.5222
	18	15.4184	49.9686	104.2790	178.4199	272.5481	386.9595
	20	15.4183	49.9673	104.2679	178.3668	272.3667	386.4633
	Öz [2]	15.2752	45.5766	79.3377	133.4684	217.8558	–
	Shu and Du [25]	15.4182	49.9648	104.2471	178.4642	273.1126	–
	Abrate [38]	15.4182	49.9649	104.248	178.270	272.032	385.533
S-S	6	9.8743	39.6323	92.1283	160.4390	283.3474	432.8413
	8	9.8709	39.5323	89.4581	159.2565	266.6035	375.8417
	10	9.8701	39.4990	89.0481	159.0056	252.4690	357.7353
	15	9.8697	39.4820	88.8663	158.1318	247.5481	357.6446
	18	9.8696	39.4801	88.8450	158.0160	247.1226	356.4209
	20	9.8696	39.4795	88.8384	157.9798	246.9882	356.0326
	Öz [2]	9.8695	39.4784	88.8267	157.9147	246.7413	–
	Shu and Du [25]	9.8696	39.4784	88.8249	158.0619	248.4716	–
	Ece et al. [41]	9.86960	39.47841	88.82643	157.91367	246.74011	–
C-F	6	3.5161	22.0780	62.1149	129.7833	210.4847	348.1611
	8	3.5160	22.0459	61.8978	122.6933	201.2669	329.6807
	10	3.5160	22.0388	61.7763	121.4640	201.7579	310.2725
	15	3.5160	22.0352	61.7113	121.0028	200.2997	299.9689
	18	3.5160	22.0348	61.7037	120.9490	200.0668	299.2290
	20	3.5160	22.0347	61.7014	120.9323	199.9936	298.9933
	Ece et al. [41]	3.51602	22.03449	61.69721	120.90191	199.85953	–

- It is also evident that natural frequencies in case of FG beam are increasing with increase in k for $E_r < 1$ and follow descending pattern with increase in k while considering $E_r > 1$.

5 Numerical results

In view of the test of convergence and validation, it is worth evaluating the first six non-dimensional frequencies of FG beams having different constituents. The three different FG beam constituents (Al/Al_2O_3, Al/ZrO_2 and $SUS304/Si_3N_4$) have been considered in Tables 7, 8, 9, 10, 11 and 12 to find their corresponding results and their material properties are reported as follows.

Al/Al_2O_3: $E_m = 70$ GPa, $\rho_m = 2702$ kg/m^3, $E_c = 380$ GPa, $\rho_c = 3960$ kg/m^3 and $\nu_m = \nu_c = 0.3$.

Al/ZrO_2: $E_m = 70$ GPa, $\rho_m = 2700$ kg/m^3, $E_c = 200$ GPa, $\rho_c = 5700$ kg/m^3 and $\nu_m = \nu_c = 0.3$.

$SUS304/Si_3N_4$: $E_m = 208$ GPa, $\rho_m = 8166$ kg/m^3, $E_c = 322$ GPa, $\rho_c = 2370$ kg/m^3 and $\nu_m = \nu_c = 0.3$.

Table 2 Convergence and comparison of six lowest natural frequencies of isotropic beam ($E_r = \rho_r = 1$) using differential quadrature method

BCs	No. of elements	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
C-C	10	22.3723	61.5963	123.5385	236.0434	255.7144	293.5479
	15	22.3733	61.6728	120.9021	199.9365	299.3886	412.3276
	18	22.3733	61.6728	120.9034	199.8600	298.5299	416.7246
	19	22.3733	61.6728	120.9034	199.8597	298.5597	416.8916
	20	22.3733	61.6728	120.9034	199.8594	298.5572	417.0123
	Öz [2]	22.3733	61.6729	120.9039	199.8616	298.5627	–
	Shu and Du [25]	22.3733	61.6728	120.9021	199.9365	299.3886	–
	Abrate [38]	22.3732854	61.672823	120.903392	–	–	–
	Ece et al. [41]	22.37327	61.67281	120.90338	199.85945	298.55552	–
C-S	10	15.4170	49.9216	109.2703	193.8286	193.8286	284.0631
	15	15.4182	49.9648	104.2471	178.4642	273.1126	371.5845
	18	15.4182	49.9649	104.2477	178.2701	271.9534	385.1638
	19	15.4182	49.9649	104.2477	178.2704	272.0350	385.2065
	20	15.4182	49.9649	104.2477	178.2697	272.0364	385.5589
	Öz [2]	15.2752	45.5766	79.3377	133.4684	217.8558	–
	Shu and Du [25]	15.4182	49.9648	104.2471	178.4642	273.1126	–
	Abrate [38]	15.4182	49.9649	104.248	178.270	272.032	385.533
S-S	10	9.8693	39.4174	92.8422	180.1025	180.1025	197.3083
	15	9.8696	39.4784	88.8249	158.0619	248.4716	342.9591
	18	9.8696	39.4784	88.8265	157.9146	246.6789	354.6659
	19	9.8696	39.4784	88.8264	157.9141	246.7488	355.0287
	20	9.8696	39.4784	88.8264	157.9136	246.7441	355.3596
	Öz [2]	9.8695	39.4784	88.8267	157.9147	246.7413	–
	Shu and Du [25]	9.8696	39.4784	88.8249	158.0619	248.4716	–
	Ece et al. [41]	9.86960	39.47841	88.82643	157.91367	246.74011	–
C-F	10	3.5160	22.0346	61.7101	120.1944	174.8618	284.8300
	15	3.5160	22.0345	61.6972	120.9021	199.8443	298.2386
	18	3.5160	22.0345	61.6972	120.9019	199.8594	298.5604
	19	3.5160	22.0345	61.6972	120.9019	199.8595	298.5547
	20	3.5160	22.0345	61.6972	120.9019	199.8595	298.5553
	Ece et al. [41]	3.51602	22.03449	61.69721	120.90191	199.85953	–

The mathematical expression for the natural frequency in these computation is $\Omega = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$. In Tables 7, 8, 9, 10, 11 and 12, the effect of power-law indices (k) on first six natural frequencies of different FG beams under four sets of classical edge supports have been found using all three numerical techniques. Looking into these tabulations, one may easily summarize the following facts:

- In terms of beam constituents, Al/Al_2O_3 FG beam is considered in Tables 7 and 8, Al/ZrO_2 constituents in Tables 9 and 10 and SUS304/Si_3N_4 is assumed in Tables 11 and 12 respectively. Prior to boundary conditions, Tables 7, 9 and 11 considers C-C and C-S edge support based FG beams, whereas Tables 8, 10 and 12 assumes the FG beam under cantilever and S-S edge conditions respectively.
- It may be observed that present results follow descending pattern with increase in power-law indices (k) in case of Al/Al_2O_3 and SUS304/Si_3N_4 beams, whereas a few ambiguities can be seen in Al/ZrO_2 FG beam.
- Among all the classical edge conditions, natural frequencies in all modes of C-C FG beams are always the highest and the least in case of cantilever FG beams.
- In FEM and DQM, 20 element discretizations of domain is being taken for evaluations. On the other hand, 15 number of polynomials involved in transverse displacement have been assumed in RRM. One may easily say that RRM is more efficient than FEM and DQM in terms of convergence criterion, but the computed results at each mode are approximately close to each other.

Table 3 Convergence and comparison of six lowest natural frequencies of isotropic beam ($E_r = \rho_r = 1$) using RRM

BCs	No. of polynomials	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
C-C	6	22.3733	61.6729	121.1306	201.1261	353.4329	531.8278
	8	22.3733	61.6728	120.9055	199.8853	303.2272	430.1219
	10	22.3733	61.6728	120.9034	199.8597	298.7300	417.7781
	13	22.3733	61.6728	120.9034	199.8594	298.5556	417.0128
	15	22.3733	61.6728	120.9034	199.8594	298.5555	416.9911
	Öz [2]	22.3733	61.6729	120.9039	199.8616	298.5627	–
	Shu and Du [25]	22.3733	61.6728	120.9021	199.9365	299.3886	–
	Abrate [38]	22.3732854	61.672823	120.903392	–	–	–
	Ece et al. [41]	22.37327	61.67281	120.90338	199.85945	298.55552	–
C-S	6	15.4182	49.9686	104.4047	189.1063	313.1092	1058.5304
	8	15.4182	49.9649	104.2491	178.7427	275.4075	451.4249
	10	15.4182	49.9649	104.2477	178.2770	272.1555	392.1102
	13	15.4182	49.9649	104.2477	178.2697	272.0318	385.5473
	15	15.4182	49.9649	104.2477	178.2697	272.0310	385.5316
	Öz [2]	15.2752	45.5766	79.3377	133.4684	217.8558	–
	Shu and Du [25]	15.4182	49.9648	104.2471	178.4642	273.1126	–
	Abrate [38]	15.4182	49.9649	104.248	178.270	272.032	385.533
S-S	6	9.8696	39.4791	90.3370	164.5755	508.4307	859.2301
	8	9.8696	39.4784	88.8482	158.1252	266.8736	403.5573
	10	9.8696	39.4784	88.8265	157.9162	247.8587	359.3483
	13	9.8696	39.4784	88.8264	157.9137	246.7404	355.4661
	15	9.8696	39.4784	88.8264	157.9137	246.7401	355.3087
	Öz [2]	9.8695	39.4784	88.8267	157.9147	246.7413	–
	Shu and Du [25]	9.8696	39.4784	88.8249	158.0619	248.4716	–
	Ece et al. [41]	9.86960	39.47841	88.82643	157.91367	246.74011	–
C-F	6	3.5160	22.0348	61.7163	128.3893	223.5514	1006.0133
	8	3.5160	22.0345	61.6973	121.1167	201.0946	355.9771
	10	3.5160	22.0345	61.6972	120.9041	199.8858	303.1623
	13	3.5160	22.0345	61.6972	120.9019	199.8598	298.5585
	15	3.5160	22.0345	61.6972	120.9019	199.8595	298.5556
	Ece et al. [41]	3.51602	22.03449	61.69721	120.90191	199.85953	–

In particular, the eigenvectors corresponding to the natural frequencies of FG beam depicts the mode shapes of the continuous beam domain. In this part, first six two-dimensional (2-D) mode shapes associated with six lowest natural frequencies of SUS304/Si_3N_4 FG beam (k = 10) under different boundary conditions have been demonstrated in Figs. 2, 3, 4 and 5. Moreover, Fig. 2 considers for C-C based FG beam. Similarly, Figs. 3, 4 and 5 are meant for the beam under C-S, cantilever and S-S edge conditions respectively. In view of these, one can easily depict six mode shapes for any given volume fraction of FG constituents.

6 Concluding remarks

The present investigation is associated with free vibration of Euler–Bernoulli functionally graded beams based on finite element, differential quadrature and Rayleigh–Ritz methods. A brief review on related investigations on these methods in handling FG structural problems have also been discussed in detail. Afterwards, a detailed formulations of the given methods to solve the titled problem are clearly organized. Three different FG materials viz. Al/Al_2O_3, Al/ZrO_2 and SUS304/Si_3N_4 are considered to denote the beam constituents. Based on the estimated results, following significant facts may be summarized.

- Effect of slenderness (length-to-thickness) ratio is redundant in case of Euler–Bernoulli beam as it doesn't even occur in the formulation.
- It is interesting to note that natural frequencies follow ascending pattern with increase in k in case of $E_r <$

Table 4 Convergence and comparison of six lowest natural frequencies of S-S FG beam using finite element method

E_r	k	No. of elements	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.5	0	6	2.6424	5.2938	8.0712	10.6512	14.1547	17.4947
		8	2.6419	5.2871	7.9534	10.6119	13.7302	16.3022
		10	2.6418	5.2849	7.9351	10.6035	13.3612	15.9046
		15	2.6418	5.2838	7.9270	10.5743	13.2304	15.9026
		18	2.6418	5.2836	7.9261	10.5704	13.2190	15.8754
		20	2.6418	5.2836	7.9258	10.5692	13.2154	15.8667
		Şimşek and Kocaturk [6]	2.6416	5.2830	7.9237	–	–	–
		Alshorbagy et al. [7]	2.6417	5.2831	7.9238	–	–	–
	0.1	6	2.7159	5.4411	8.2958	10.9476	14.5486	17.9816
		8	2.7154	5.4342	8.1747	10.9072	14.1122	16.7558
		10	2.7153	5.4320	8.1560	10.8986	13.7330	16.3472
		15	2.7153	5.4308	8.1476	10.8686	13.5986	16.3451
		18	2.7153	5.4307	8.1467	10.8646	13.5869	16.3172
		20	2.7153	5.4306	8.1464	10.8633	13.5832	16.3083
		Şimşek and Kocaturk [6]	2.7117	5.4232	8.1339	–	–	–
		Alshorbagy et al. [7]	2.7121	5.4238	8.1349	–	–	–
	2.0	6	2.9718	5.9538	9.0776	11.9792	15.9196	19.6760
		8	2.9713	5.9463	8.9450	11.9350	15.4421	18.3348
		10	2.9712	5.9438	8.9245	11.9256	15.0271	17.8877
		15	2.9711	5.9425	8.9154	11.8927	14.8800	17.8854
		18	2.9711	5.9424	8.9143	11.8884	14.8672	17.8548
		20	2.9711	5.9424	8.9140	11.8870	14.8631	17.8450
		Şimşek and Kocaturk [6]	2.9475	5.8947	8.8411	–	–	–
		Alshorbagy et al. [7]	2.9476	5.8948	8.8413	–	–	–
2.0	0	6	3.7369	7.4866	11.4144	15.0630	20.0178	24.7413
		8	3.7362	7.4771	11.2478	15.0074	19.4174	23.0547
		10	3.7361	7.4740	11.2220	14.9956	18.8956	22.4925
		15	3.7360	7.4724	11.2105	14.9543	18.7106	22.4897
		18	3.7360	7.4722	11.2092	14.9489	18.6945	22.4512
		20	3.7360	7.4721	11.2088	14.9471	18.6894	22.4389
		Şimşek and Kocaturk [6]	3.7359	7.4713	11.2059	–	–	–
		Alshorbagy et al. [7]	3.7359	7.4714	11.206	–	–	–
	0.1	6	3.6815	7.3755	11.2451	14.8396	19.7209	24.3743
		8	3.6808	7.3662	11.0810	14.7848	19.1294	22.7128
		10	3.6807	7.3631	11.0555	14.7732	18.6154	22.1589
		15	3.6806	7.3615	11.0442	14.7325	18.4331	22.1561
		18	3.6806	7.3613	11.0429	14.7271	18.4172	22.1182
		20	3.6806	7.3613	11.0425	14.7254	18.4122	22.1061
		Şimşek and Kocaturk [6]	3.6793	7.3582	11.0362	–	–	–
		Alshorbagy et al. [7]	3.6791	7.3577	11.035	–	–	–
	2.0	6	3.4181	6.8479	10.4407	13.7780	18.3101	22.6306
		8	3.4175	6.8392	10.2883	13.7272	17.7609	21.0880
		10	3.4174	6.8364	10.2647	13.7163	17.2837	20.5737
		15	3.4173	6.8349	10.2542	13.6786	17.1144	20.5711
		18	3.4173	6.8347	10.2529	13.6736	17.0997	20.5359
		20	3.4173	6.8347	10.2526	13.6720	17.0950	20.5247
		Şimşek and Kocaturk [6]	3.3784	6.7563	10.1333	–	–	–
		Alshorbagy et al. [7]	3.3784	6.7564	10.134	–	–	–

Table 5 Convergence and comparison of six lowest natural frequencies of S-S FG beam using differential quadrature method

E_r	k	No. of discretizations	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.5	0	10	2.6417	5.2794	8.1024	11.3823	11.3823	11.8118
		15	2.6418	5.2835	7.9252	10.5720	13.2550	15.5727
		18	2.6418	5.2835	7.9253	10.5670	13.2071	15.8362
		19	2.6418	5.2835	7.9253	10.5670	13.2090	15.8443
		20	2.6418	5.2835	7.9253	10.5670	13.2089	15.8517
		Şimşek and Kocaturk [6]	2.6416	5.2830	7.9237	–	–	–
		Alshorbagy et al. [7]	2.6417	5.2831	7.9238	–	–	–
	0.1	10	2.7152	5.4263	8.3279	11.6990	11.6990	12.1405
		15	2.7153	5.4305	8.1457	10.8662	13.6239	16.0060
		18	2.7153	5.4305	8.1458	10.8611	13.5747	16.2769
		19	2.7153	5.4305	8.1458	10.8611	13.5766	16.2853
		20	2.7153	5.4305	8.1458	10.8611	13.5765	16.2928
		Şimşek and Kocaturk [6]	2.7117	5.4232	8.1339	–	–	–
		Alshorbagy et al. [7]	2.7121	5.4238	8.1349	–	–	–
	2.0	10	2.9711	5.9377	9.1127	12.8014	12.8014	13.2845
		15	2.9711	5.9423	8.9133	11.8901	14.9077	17.5143
		18	2.9711	5.9423	8.9134	11.8846	14.8538	17.8108
		19	2.9711	5.9423	8.9134	11.8846	14.8559	17.8199
		20	2.9711	5.9423	8.9134	11.8845	14.8558	17.8282
		Şimşek and Kocaturk [6]	2.9475	5.8947	8.8411	–	–	–
		Alshorbagy et al. [7]	2.9476	5.8948	8.8413	–	–	–
2.0	0	10	3.7359	7.4662	11.4586	16.0969	16.0969	16.7044
		15	3.7360	7.4720	11.2079	14.9510	18.7454	22.0231
		18	3.7360	7.4720	11.2080	14.9441	18.6777	22.3958
		19	3.7360	7.4720	11.2080	14.9440	18.6804	22.4073
		20	3.7360	7.4720	11.2080	14.9440	18.6802	22.4177
		Şimşek and Kocaturk [6]	3.7359	7.4713	11.2059	–	–	–
		Alshorbagy et al. [7]	3.7359	7.4714	11.206	–	–	–
	0.1	10	3.6805	7.3555	11.2886	15.8582	15.8582	16.4566
		15	3.6806	7.3612	11.0417	14.7293	18.4674	21.6965
		18	3.6806	7.3612	11.0418	14.7224	18.4007	22.0636
		19	3.6806	7.3612	11.0418	14.7224	18.4033	22.0749
		20	3.6806	7.3612	11.0418	14.7224	18.4031	22.0852
		Şimşek and Kocaturk [6]	3.6793	7.3582	11.0362	–	–	–
		Alshorbagy et al. [7]	3.6791	7.3577	11.035	–	–	–
	2.0	10	3.4172	6.8293	10.4810	14.7237	14.7237	15.2793
		15	3.4173	6.8346	10.2518	13.6756	17.1463	20.1443
		18	3.4173	6.8346	10.2519	13.6692	17.0843	20.4853
		19	3.4173	6.8346	10.2519	13.6692	17.0868	20.4957
		20	3.4173	6.8346	10.2519	13.6692	17.0866	20.5053
		Şimşek and Kocaturk [6]	3.3784	6.7563	10.1333	–	–	–
		Alshorbagy et al. [7]	3.3784	6.7564	10.134	–	–	–

Table 6 Convergence and comparison of six lowest natural frequencies of S-S FG beam using RRM

E_r	k	No. of polynomials	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.5	0	6	2.6418	5.2836	7.9924	10.7876	18.9609	24.6489
		8	2.6418	5.2835	7.9262	10.5741	13.7371	16.8925
		10	2.6418	5.2835	7.9253	10.5671	13.2387	15.9404
		13	2.6418	5.2835	7.9253	10.5670	13.2088	15.8541
		15	2.6418	5.2835	7.9253	10.5670	13.2088	15.8506
		Şimşek and Kocaturk [6]	2.6416	5.2830	7.9237	–	–	–
		Alshorbagy et al. [7]	2.6417	5.2831	7.9238	–	–	–
	0.1	6	2.7153	5.4306	8.2148	11.0878	19.4885	25.3348
		8	2.7153	5.4305	8.1468	10.8683	14.1194	17.3626
		10	2.7153	5.4305	8.1458	10.8612	13.6071	16.3840
		13	2.7153	5.4305	8.1458	10.8611	13.5764	16.2953
		15	2.7153	5.4305	8.1458	10.8611	13.5763	16.2917
		Şimşek and Kocaturk [6]	2.7117	5.4232	8.1339	–	–	–
		Alshorbagy et al. [7]	2.7121	5.4238	8.1349	–	–	–
	2.0	6	2.9711	5.9423	8.9889	12.1326	21.3250	27.7222
		8	2.9711	5.9423	8.9145	11.8925	15.4499	18.9988
		10	2.9711	5.9423	8.9134	11.8846	14.8893	17.9279
		13	2.9711	5.9423	8.9134	11.8845	14.8557	17.8308
		15	2.9711	5.9423	8.9134	11.8845	14.8557	17.8269
		Şimşek and Kocaturk [6]	2.9475	5.8947	8.8411	–	–	–
		Alshorbagy et al. [7]	2.9476	5.8948	8.8413	–	–	–
2.0	0	6	3.7360	7.4721	11.3029	15.2560	26.8147	34.8588
		8	3.7360	7.4720	11.2094	14.9540	19.4272	23.8897
		10	3.7360	7.4720	11.2080	14.9441	18.7223	22.5432
		13	3.7360	7.4720	11.2080	14.9440	18.6800	22.4211
		15	3.7360	7.4720	11.2080	14.9440	18.6800	22.4161
		Şimşek and Kocaturk [6]	3.7359	7.4713	11.2059	–	–	–
		Alshorbagy et al. [7]	3.7359	7.4714	11.206	–	–	–
	0.1	6	3.6806	7.3612	11.1353	15.0297	26.4170	34.3417
		8	3.6806	7.3612	11.0431	14.7322	19.1391	23.5353
		10	3.6806	7.3612	11.0418	14.7225	18.4446	22.2088
		13	3.6806	7.3612	11.0418	14.7224	18.4030	22.0885
		15	3.6806	7.3612	11.0418	14.7224	18.4030	22.0836
		Şimşek and Kocaturk [6]	3.6793	7.3582	11.0362	–	–	–
		Alshorbagy et al. [7]	3.6791	7.3577	11.035	–	–	–
	2.0	6	3.4173	6.8346	10.3387	13.9545	24.5272	31.8850
		8	3.4173	6.8346	10.2531	13.6783	17.7699	21.8517
		10	3.4173	6.8346	10.2519	13.6693	17.1251	20.6201
		13	3.4173	6.8346	10.2519	13.6692	17.0865	20.5084
		15	3.4173	6.8346	10.2519	13.6692	17.0865	20.5038
		Şimşek and Kocaturk [6]	3.3784	6.7563	10.1333	–	–	–
		Alshorbagy et al. [7]	3.3784	6.7564	10.134	–	–	–

Table 7 Effect of power-law exponents (k) on Al/Al_2O_3 FG beam under C-C and C-S edge supports using all the assumed methods

BCs	Method	k	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
C-C	FEM	0	13.0305	35.9215	70.4338	116.4803	174.1409	243.5462
		0.1	12.5814	34.6834	68.0062	112.4656	168.1388	235.1519
		0.2	12.2217	33.6918	66.0620	109.2503	163.3318	228.4291
		1	10.9326	30.1380	59.0937	97.7265	146.1034	204.3342
		2	10.4869	28.9095	56.6850	93.7430	140.1481	196.0053
		5	9.8166	27.0615	53.0614	87.7506	131.1892	183.4758
		10	9.0617	24.9806	48.9812	81.0029	121.1013	169.3672
	DQM	0	13.0304	35.9187	70.4150	116.3995	173.8818	242.8708
		0.1	12.5812	34.6807	67.9880	112.3876	167.8887	234.4999
		0.2	12.2216	33.6892	66.0443	109.1745	163.0888	227.7957
		1	10.9324	30.1356	59.0779	97.6587	145.8861	203.7676
		2	10.4868	28.9073	56.6698	93.6780	139.9396	195.4618
		5	9.8164	27.0594	53.0472	87.6897	130.9940	182.9670
		10	9.0616	24.9786	48.9681	80.9467	120.9211	168.8976
	RRM	0	13.0304	35.9187	70.4150	116.3995	173.8808	242.8585
		0.1	12.5812	34.6807	67.9880	112.3876	167.8877	234.4880
		0.2	12.2216	33.6892	66.0443	109.1745	163.0879	227.7841
		1	10.9324	30.1356	59.0779	97.6587	145.8853	203.7572
		2	10.4868	28.9073	56.6698	93.6780	139.9388	195.4519
		5	9.8164	27.0594	53.0472	87.6897	130.9933	182.9577
		10	9.0616	24.9786	48.9681	80.9467	120.9204	168.8890
C-S	FEM	0	8.9797	29.1013	60.7263	103.8821	158.6282	225.0789
		0.1	8.6702	28.0983	58.6333	100.3016	153.1609	217.3212
		0.2	8.4223	27.2950	56.9570	97.4340	148.7821	211.1081
		1	7.5340	24.4159	50.9491	87.1566	133.0884	188.8403
		2	7.2269	23.4206	48.8724	83.6040	127.6636	181.1429
		5	6.7649	21.9235	45.7483	78.2597	119.5028	169.5635
		10	6.2447	20.2377	42.2304	72.2418	110.3135	156.5247
	DQM	0	8.9797	29.0999	60.7146	103.8255	158.4359	224.5522
		0.1	8.6702	28.0969	58.6220	100.2470	152.9751	216.8126
		0.2	8.4223	27.2936	56.9460	97.3810	148.6016	210.6141
		1	7.5339	24.4147	50.9393	87.1092	132.9270	188.3983
		2	7.2268	23.4195	48.8629	83.5585	127.5088	180.7190
		5	6.7648	21.9224	45.7394	78.2171	119.3578	169.1666
		10	6.2447	20.2367	42.2222	72.2025	110.1797	156.1584
	RRM	0	8.9797	29.0999	60.7146	103.8255	158.4327	224.5363
		0.1	8.6702	28.0969	58.6220	100.2470	152.9721	216.7973
		0.2	8.4223	27.2936	56.9460	97.3810	148.5987	210.5992
		1	7.5339	24.4147	50.9393	87.1092	132.9244	188.3850
		2	7.2268	23.4195	48.8629	83.5585	127.5063	180.7062
		5	6.7648	21.9224	45.7394	78.2171	119.3555	169.1547
		10	6.2447	20.2367	42.2222	72.2025	110.1775	156.1474

Table 8 Effect of power-law exponents (k) on Al/Al_2O_3 FG beam under cantilever and S-S edge supports using all the assumed methods

BCs	Method	k	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
C-F	FEM	0	2.0478	12.8332	35.9353	70.4318	116.4777	174.1358
		0.1	1.9772	12.3909	34.6968	68.0043	112.4631	168.1339
		0.2	1.9206	12.0366	33.7048	66.0601	109.2478	163.3270
		1	1.7181	10.7670	30.1496	59.0920	97.7243	146.0992
		2	1.6480	10.3281	28.9207	56.6833	93.7409	140.1440
		5	1.5427	9.6679	27.0719	53.0599	87.7486	131.1854
		10	1.4241	8.9245	24.9902	48.9798	81.0011	121.0977
	DQM	0	2.0478	12.8330	35.9329	70.4141	116.3996	173.8807
		0.1	1.9772	12.3907	34.6944	67.9872	112.3877	167.8876
		0.2	1.9206	12.0365	33.7025	66.0435	109.1746	163.0877
		1	1.7181	10.7669	30.1475	59.0771	97.6587	145.8851
		2	1.6480	10.3280	28.9187	56.6691	93.6781	139.9387
		5	1.5427	9.6678	27.0701	53.0466	87.6898	130.9932
		10	1.4241	8.9244	24.9885	48.9675	80.9468	120.9203
	RRM	0	2.0478	12.8330	35.9329	70.4141	116.3996	173.8808
		0.1	1.9772	12.3907	34.6944	67.9872	112.3877	167.8877
		0.2	1.9206	12.0365	33.7025	66.0435	109.1746	163.0879
		1	1.7181	10.7669	30.1475	59.0771	97.6587	145.8853
		2	1.6480	10.3280	28.9187	56.6691	93.6781	139.9388
		5	1.5427	9.6678	27.0701	53.0466	87.6898	130.9933
		10	1.4241	8.9244	24.9885	48.9675	80.9468	120.9205
S-S	FEM	0	5.7481	22.9931	51.7401	92.0085	143.8476	207.3558
		0.1	5.5500	22.2006	49.9568	88.8373	138.8897	200.2090
		0.2	5.3913	21.5659	48.5285	86.2975	134.9189	194.4851
		1	4.8227	19.2911	43.4097	77.1947	120.6875	173.9707
		2	4.6261	18.5048	41.6403	74.0482	115.7682	166.8795
		5	4.3304	17.3219	38.9785	69.3147	108.3678	156.2118
		10	3.9974	15.9899	35.9812	63.9847	100.0347	144.1997
	DQM	0	5.7481	22.9925	51.7331	91.9700	143.7055	206.9639
		0.1	5.5500	22.2000	49.9501	88.8001	138.7524	199.8306
		0.2	5.3913	21.5653	48.5220	86.2614	134.7855	194.1175
		1	4.8227	19.2906	43.4039	77.1624	120.5683	173.6418
		2	4.6261	18.5043	41.6347	74.0172	115.6538	166.5640
		5	4.3304	17.3214	38.9732	69.2857	108.2607	155.9165
		10	3.9974	15.9895	35.9763	63.9579	99.9359	143.9271
	RRM	0	5.7481	22.9925	51.7331	91.9700	143.7032	206.9342
		0.1	5.5500	22.2000	49.9501	88.8001	138.7502	199.8019
		0.2	5.3913	21.5653	48.5220	86.2614	134.7834	194.0897
		1	4.8227	19.2906	43.4039	77.1624	120.5663	173.6169
		2	4.6261	18.5043	41.6347	74.0172	115.6519	166.5401
		5	4.3304	17.3214	38.9732	69.2857	108.2589	155.8941
		10	3.9974	15.9895	35.9763	63.9579	99.9343	143.9065

Table 9 Effect of power-law exponents (k) on Al/ZrO_2 FG beam under C-C and C-S edge supports using all the assumed methods

BCs	Method	k	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
C-C	FEM	0	7.8765	21.7134	42.5749	70.4085	105.2624	147.2156
		0.1	7.7616	21.3965	41.9537	69.3812	103.7265	145.0676
		0.2	7.6788	21.1682	41.5060	68.6407	102.6195	143.5194
		1	7.5388	20.7822	40.7492	67.3891	100.7483	140.9024
		2	7.6355	21.0488	41.2720	68.2538	102.0410	142.7103
		5	7.6942	21.2107	41.5894	68.7787	102.8258	143.8080
		10	7.5349	20.7716	40.7284	67.3548	100.6970	140.8306
	DQM	0	7.8764	21.7117	42.5635	70.3597	105.1058	146.8074
		0.1	7.7615	21.3949	41.9425	69.3331	103.5722	144.6654
		0.2	7.6787	21.1665	41.4949	68.5931	102.4668	143.1214
		1	7.5386	20.7806	40.7382	67.3424	100.5985	140.5117
		2	7.6354	21.0472	41.2609	68.2064	101.8892	142.3146
		5	7.6941	21.2091	41.5783	68.7310	102.6729	143.4092
		10	7.5348	20.7700	40.7175	67.3080	100.5472	140.4401
	RRM	0	7.8764	21.7117	42.5635	70.3597	105.1052	146.7999
		0.1	7.7615	21.3949	41.9425	69.3331	103.5716	144.6580
		0.2	7.6787	21.1665	41.4949	68.5931	102.4663	143.1142
		1	7.5386	20.7806	40.7382	67.3424	100.5979	140.5046
		2	7.6354	21.0472	41.2609	68.2064	101.8886	142.3074
		5	7.6941	21.2091	41.5783	68.7310	102.6723	143.4019
		10	7.5348	20.7700	40.7175	67.3080	100.5466	140.4330
C-S	FEM	0	5.4279	17.5908	36.7071	62.7933	95.8855	136.0528
		0.1	5.3487	17.3341	36.1715	61.8771	94.4865	134.0677
		0.2	5.2917	17.1491	35.7854	61.2167	93.4781	132.6368
		1	5.1952	16.8364	35.1329	60.1005	91.7736	130.2183
		2	5.2618	17.0524	35.5837	60.8716	92.9511	131.8891
		5	5.3023	17.1836	35.8574	61.3398	93.6660	132.9035
		10	5.1925	16.8278	35.1150	60.0698	91.7268	130.1519
	DQM	0	5.4279	17.5899	36.7000	62.7591	95.7693	135.7344
		0.1	5.3487	17.3333	36.1645	61.8434	94.3719	133.7539
		0.2	5.2916	17.1483	35.7785	61.1834	93.3647	132.3264
		1	5.1951	16.8356	35.1261	60.0678	91.6623	129.9136
		2	5.2618	17.0516	35.5768	60.8385	92.8384	131.5804
		5	5.3023	17.1828	35.8505	61.3064	93.5524	132.5925
		10	5.1925	16.8270	35.1082	60.0371	91.6156	129.8473
	RRM	0	5.4279	17.5899	36.7000	62.7591	95.7674	135.7248
		0.1	5.3487	17.3333	36.1645	61.8434	94.3700	133.7445
		0.2	5.2916	17.1483	35.7785	61.1834	93.3629	132.3171
		1	5.1951	16.8356	35.1261	60.0678	91.6605	129.9044
		2	5.2618	17.0516	35.5768	60.8385	92.8365	131.5711
		5	5.3023	17.1828	35.8505	61.3064	93.5506	132.5831
		10	5.1925	16.8270	35.1082	60.0371	91.6138	129.8382

Table 10 Effect of power-law exponents (k) on Al/ZrO_2 FG beam under cantilever and S-S edge supports using all the assumed methods

BCs	Method	k	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
C-F	FEM	0	1.2378	7.7572	21.7217	42.5737	70.4069	105.2593
		0.1	1.2197	7.6440	21.4048	41.9525	69.3796	103.7235
		0.2	1.2067	7.5625	21.1763	41.5048	68.6392	102.6165
		1	1.1847	7.4246	20.7902	40.7480	67.3876	100.7454
		2	1.1999	7.5198	21.0570	41.2708	68.2522	102.0380
		5	1.2091	7.5777	21.2189	41.5882	68.7772	102.8228
		10	1.1841	7.4208	20.7796	40.7272	67.3532	100.6940
	DQM	0	1.2378	7.7571	21.7202	42.5630	70.3597	105.1051
		0.1	1.2197	7.6440	21.4033	41.9420	69.3331	103.5715
		0.2	1.2067	7.5624	21.1749	41.4944	68.5931	102.4662
		1	1.1847	7.4245	20.7888	40.7377	67.3424	100.5978
		2	1.1999	7.5198	21.0555	41.2604	68.2065	101.8885
		5	1.2091	7.5776	21.2175	41.5778	68.7311	102.6722
		10	1.1841	7.4207	20.7782	40.7170	67.3081	100.5465
	RRM	0	1.2378	7.7571	21.7202	42.5630	70.3597	105.1052
		0.1	1.2197	7.6440	21.4033	41.9420	69.3331	103.5717
		0.2	1.2067	7.5624	21.1749	41.4944	68.5931	102.4663
		1	1.1847	7.4245	20.7888	40.7377	67.3424	100.5979
		2	1.1999	7.5198	21.0555	41.2604	68.2065	101.8886
		5	1.2091	7.5776	21.2175	41.5778	68.7311	102.6723
		10	1.1841	7.4207	20.7782	40.7170	67.3081	100.5466
S-S	FEM	0	3.4746	13.8986	31.2752	55.6161	86.9511	125.3398
		0.1	3.4239	13.6958	30.8188	54.8046	85.6825	123.5110
		0.2	3.3873	13.5496	30.4899	54.2197	84.7680	122.1928
		1	3.3256	13.3026	29.9340	53.2311	83.2223	119.9647
		2	3.3682	13.4732	30.3180	53.9141	84.2901	121.5039
		5	3.3941	13.5769	30.5512	54.3287	84.9385	122.4385
		10	3.3239	13.2958	29.9187	53.2039	83.1799	119.9036
	DQM	0	3.4746	13.8982	31.2710	55.5928	86.8652	125.1028
		0.1	3.4239	13.6954	30.8147	54.7817	85.5978	123.2775
		0.2	3.3873	13.5493	30.4858	54.1970	84.6842	121.9618
		1	3.3256	13.3022	29.9300	53.2088	83.1401	119.7380
		2	3.3682	13.4729	30.3140	53.8915	84.2068	121.2743
		5	3.3941	13.5765	30.5471	54.3060	84.8545	122.2070
		10	3.3239	13.2954	29.9147	53.1817	83.0977	119.6769
	RRM	0	3.4746	13.8982	31.2710	55.5928	86.8638	125.0849
		0.1	3.4239	13.6954	30.8147	54.7817	85.5964	123.2598
		0.2	3.3873	13.5493	30.4858	54.1970	84.6829	121.9443
		1	3.3256	13.3022	29.9300	53.2088	83.1388	119.7208
		2	3.3682	13.4729	30.3140	53.8915	84.2055	121.2569
		5	3.3941	13.5765	30.5471	54.3060	84.8531	122.1895
		10	3.3239	13.2954	29.9147	53.1817	83.0964	119.6598

Table 11 Effect of power-law exponents (k) on SUS304/Si_3N_4 FG beam under C-C and C-S edge supports using all the assumed methods

BCs	Method	k	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
C-C	FEM	0	15.3276	42.2538	82.8501	137.0138	204.8389	286.4791
		0.1	13.5760	37.4251	73.3821	121.3560	181.4301	253.7406
		0.2	12.4426	34.3007	67.2559	111.2248	166.2838	232.5576
		1	9.3265	25.7106	50.4126	83.3701	124.6404	174.3168
		2	8.3871	23.1208	45.3347	74.9724	112.0856	156.7583
		5	7.6011	20.9541	41.0861	67.9464	101.5814	142.0676
		10	7.2205	19.9048	39.0289	64.5442	96.4951	134.9540
	DQM	0	15.3274	42.2505	82.8279	136.9187	204.5341	285.6847
		0.1	13.5758	37.4222	73.3624	121.2718	181.1602	253.0370
		0.2	12.4424	34.2981	67.2379	111.1477	166.0364	231.9128
		1	9.3264	25.7086	50.3991	83.3123	124.4549	173.8334
		2	8.3870	23.1190	45.3225	74.9204	111.9189	156.3236
		5	7.6010	20.9524	41.0751	67.8992	101.4303	141.6736
		10	7.2204	19.9033	39.0184	64.4994	96.3515	134.5798
	RRM	0	15.3274	42.2505	82.8279	136.9188	204.5330	285.6703
		0.1	13.5758	37.4222	73.3624	121.2718	181.1592	253.0241
		0.2	12.4424	34.2981	67.2379	111.1477	166.0355	231.9010
		1	9.3264	25.7086	50.3991	83.3123	124.4542	173.8246
		2	8.3870	23.1190	45.3225	74.9205	111.9182	156.3157
		5	7.6010	20.9524	41.0751	67.8992	101.4298	141.6665
		10	7.2204	19.9033	39.0184	64.4994	96.3510	134.5730
C-S	FEM	0	10.5627	34.2314	71.4313	122.1947	186.5917	264.7564
		0.1	9.3556	30.3194	63.2682	108.2304	165.2681	234.5003
		0.2	8.5746	27.7883	57.9864	99.1950	151.4711	214.9236
		1	6.4272	20.8291	43.4645	74.3530	113.5373	161.0990
		2	5.7798	18.7310	39.0865	66.8636	102.1009	144.8718
		5	5.2381	16.9756	35.4235	60.5974	92.5325	131.2951
		10	4.9759	16.1256	33.6497	57.5632	87.8992	124.7209
	DQM	0	10.5626	34.2297	71.4175	122.1282	186.3654	264.1368
		0.1	9.3555	30.3180	63.2560	108.1715	165.0677	233.9516
		0.2	8.5745	27.7869	57.9752	99.1410	151.2874	214.4206
		1	6.4271	20.8281	43.4561	74.3125	113.3996	160.7220
		2	5.7798	18.7301	39.0789	66.8272	101.9771	144.5328
		5	5.2381	16.9748	35.4166	60.5645	92.4203	130.9878
		10	4.9758	16.1249	33.6432	57.5319	87.7926	124.4291
	RRM	0	10.5626	34.2297	71.4175	122.1282	186.3617	264.1182
		0.1	9.3555	30.3180	63.2560	108.1715	165.0644	233.9350
		0.2	8.5745	27.7869	57.9752	99.1410	151.2844	214.4055
		1	6.4271	20.8281	43.4561	74.3125	113.3973	160.7106
		2	5.7798	18.7301	39.0789	66.8272	101.9751	144.5226
		5	5.2381	16.9748	35.4166	60.5645	92.4184	130.9786
		10	4.9758	16.1249	33.6432	57.5319	87.7909	124.4203

Table 12 Effect of power-law exponents (k) on SUS304/Si_3N_4 FG beam under cantilever and S-S edge supports using all the assumed methods

BCs	Method	k	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
C-F	FEM	0	2.4087	15.0954	42.2701	82.8477	137.0107	204.8329
		0.1	2.1335	13.3703	37.4395	73.3800	121.3532	181.4248
		0.2	1.9554	12.2542	34.3140	67.2540	111.2223	166.2790
		1	1.4657	9.1853	25.7205	50.4112	83.3683	124.6367
		2	1.3180	8.2601	23.1297	45.3334	74.9708	112.0823
		5	1.1945	7.4860	20.9621	41.0849	67.9448	101.5785
		10	1.1347	7.1111	19.9125	39.0277	64.5427	96.4923
	DQM	0	2.4087	15.0953	42.2672	82.8269	136.9188	204.5328
		0.1	2.1335	13.3702	37.4370	73.3615	121.2719	181.1590
		0.2	1.9554	12.2540	34.3116	67.2371	111.1478	166.0354
		1	1.4657	9.1852	25.7188	50.3985	83.3124	124.4541
		2	1.3180	8.2600	23.1282	45.3220	74.9205	111.9181
		5	1.1945	7.4859	20.9607	41.0746	67.8993	101.4297
		10	1.1347	7.1111	19.9112	39.0179	64.4994	96.3509
	RRM	0	2.4087	15.0953	42.2672	82.8269	136.9188	204.5330
		0.1	2.1335	13.3702	37.4370	73.3615	121.2719	181.1592
		0.2	1.9554	12.2540	34.3116	67.2371	111.1478	166.0355
		1	1.4657	9.1852	25.7188	50.3985	83.3124	124.4542
		2	1.3180	8.2600	23.1282	45.3220	74.9205	111.9182
		5	1.1945	7.4859	20.9607	41.0746	67.8993	101.4298
		10	1.1347	7.1111	19.9112	39.0179	64.4994	96.3510
S-S	FEM	0	6.7614	27.0464	60.8610	108.2280	169.2055	243.9091
		0.1	5.9887	23.9556	53.9058	95.8598	149.8688	216.0354
		0.2	5.4888	21.9557	49.4056	87.8572	137.3574	198.0002
		1	4.1142	16.4572	37.0327	65.8546	102.9581	148.4138
		2	3.6998	14.7995	33.3025	59.2212	92.5874	133.4644
		5	3.3531	13.4126	30.1815	53.6712	83.9105	120.9567
		10	3.1852	12.7410	28.6703	50.9838	79.7090	114.9002
	DQM	0	6.7614	27.0457	60.8528	108.1827	169.0382	243.4481
		0.1	5.9887	23.9549	53.8986	95.8197	149.7207	215.6271
		0.2	5.4888	21.9551	49.3990	87.8204	137.2216	197.6259
		1	4.1142	16.4568	37.0277	65.8270	102.8564	148.1333
		2	3.6998	14.7991	33.2980	59.1964	92.4959	133.2121
		5	3.3530	13.4122	30.1774	53.6488	83.8276	120.7281
		10	3.1852	12.7406	28.6664	50.9625	79.6302	114.6830
	RRM	0	6.7614	27.0457	60.8528	108.1827	169.0355	243.4131
		0.1	5.9887	23.9549	53.8986	95.8197	149.7183	215.5961
		0.2	5.4888	21.9551	49.3990	87.8204	137.2194	197.5976
		1	4.1142	16.4568	37.0277	65.8270	102.8547	148.1120
		2	3.6998	14.7991	33.2980	59.1964	92.4944	133.1930
		5	3.3530	13.4122	30.1774	53.6488	83.8262	120.7108
		10	3.1852	12.7406	28.6664	50.9625	79.6289	114.6666

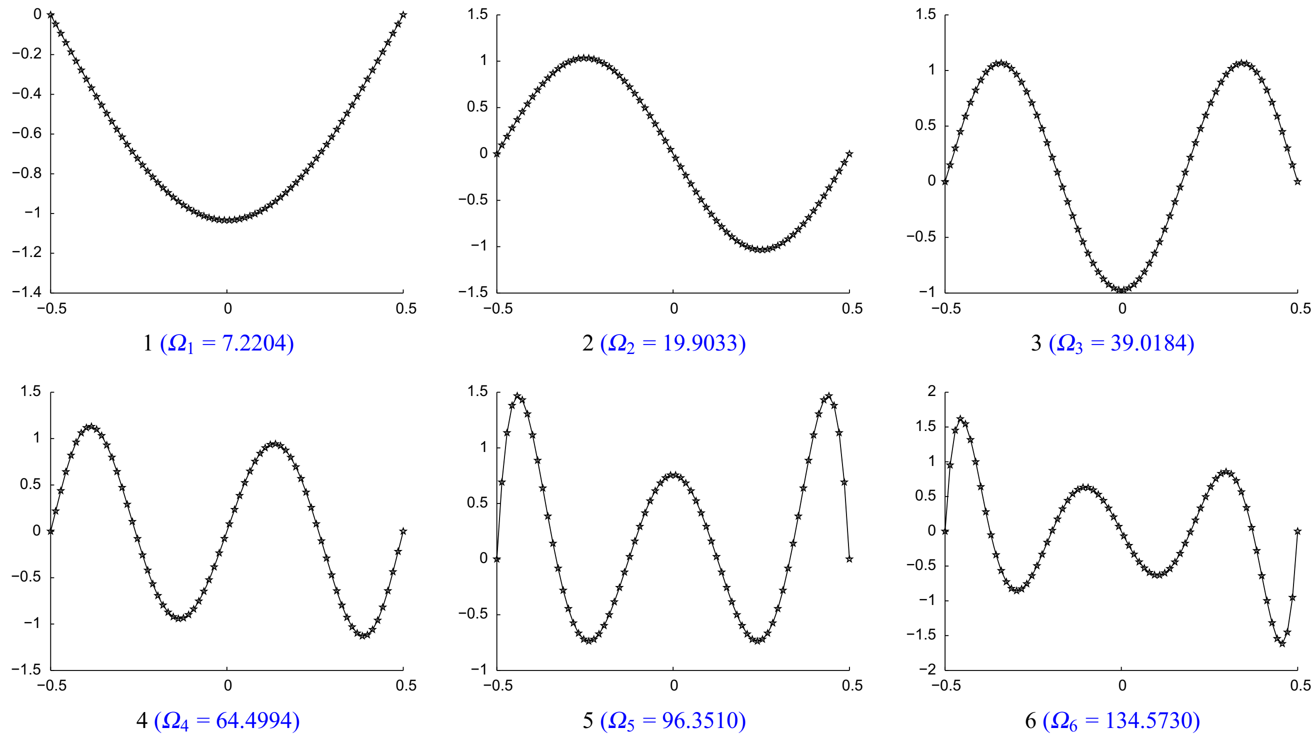


Fig. 2 First six 2-D mode shapes of SUS304/Si_3N_4 C-C FG beam with k = 10

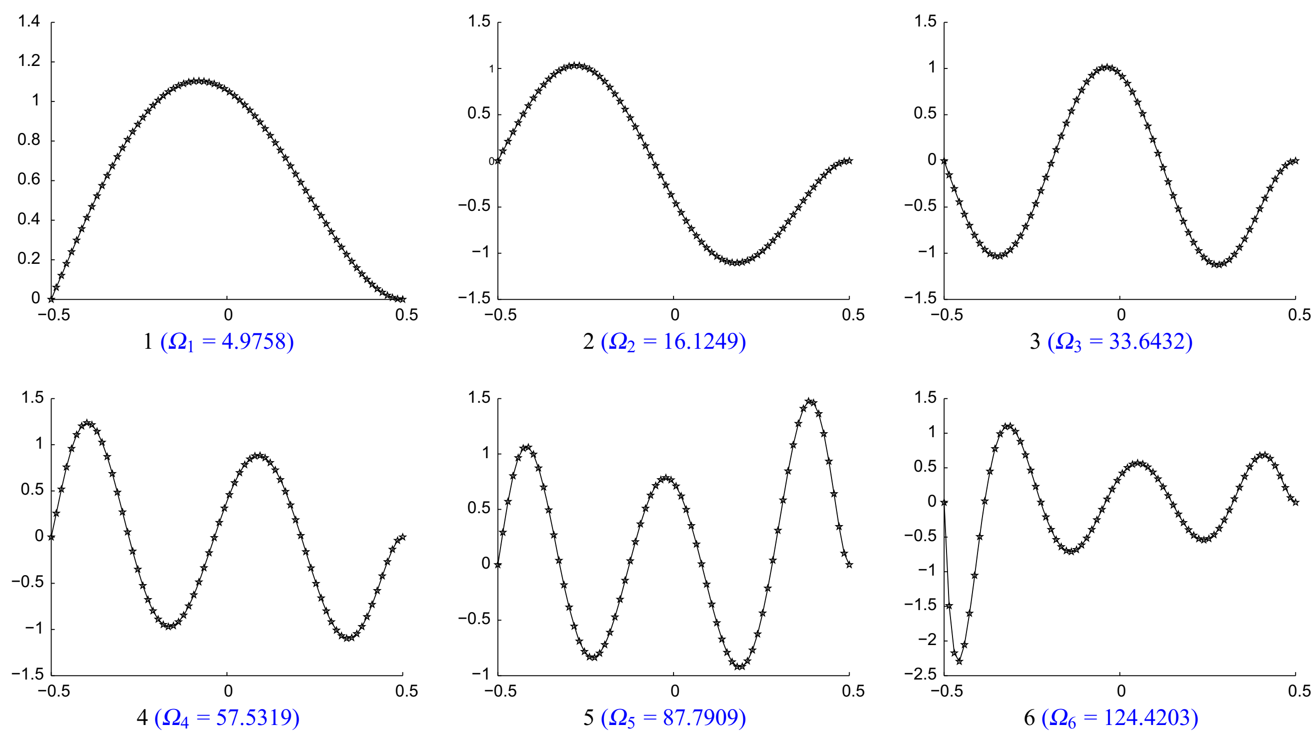


Fig. 3 First six 2-D mode shapes of SUS304/Si_3N_4 C-S FG beam with k = 10

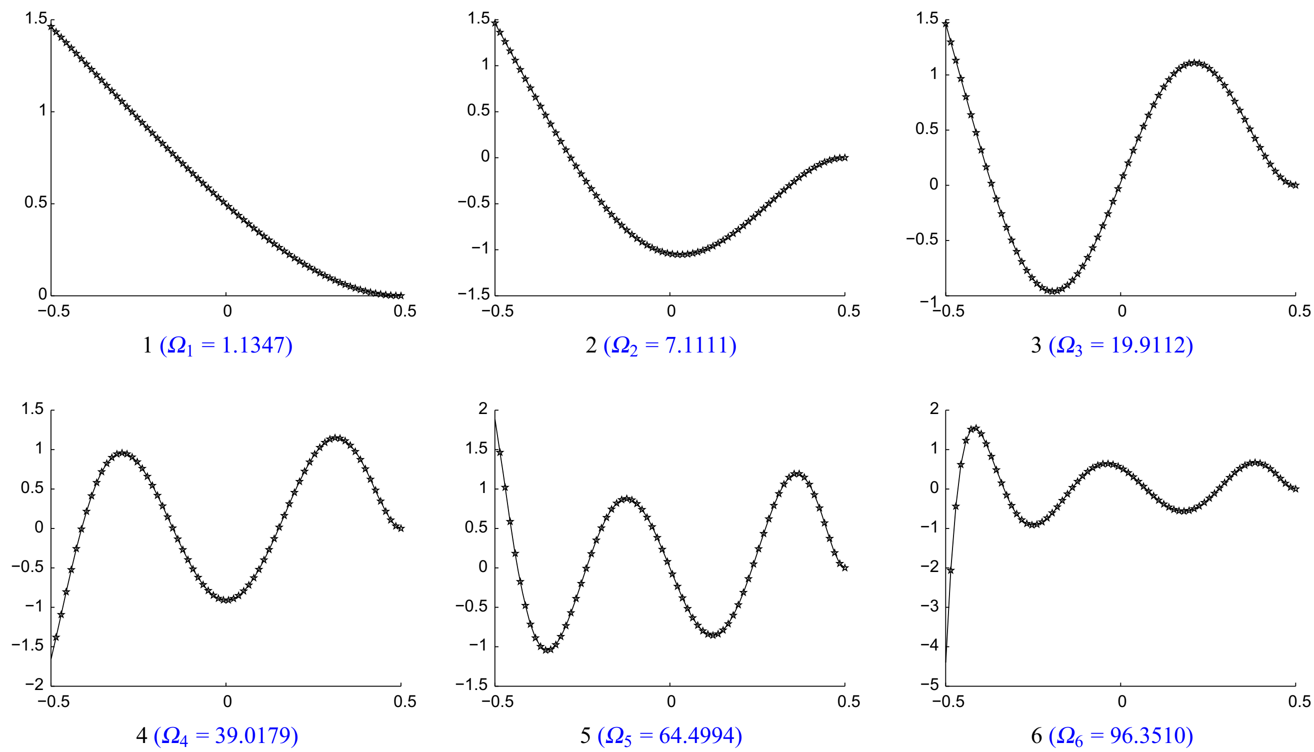


Fig. 4 First six 2-D mode shapes of $SUS304/Si_3N_4$ cantilever FG beam with k = 10

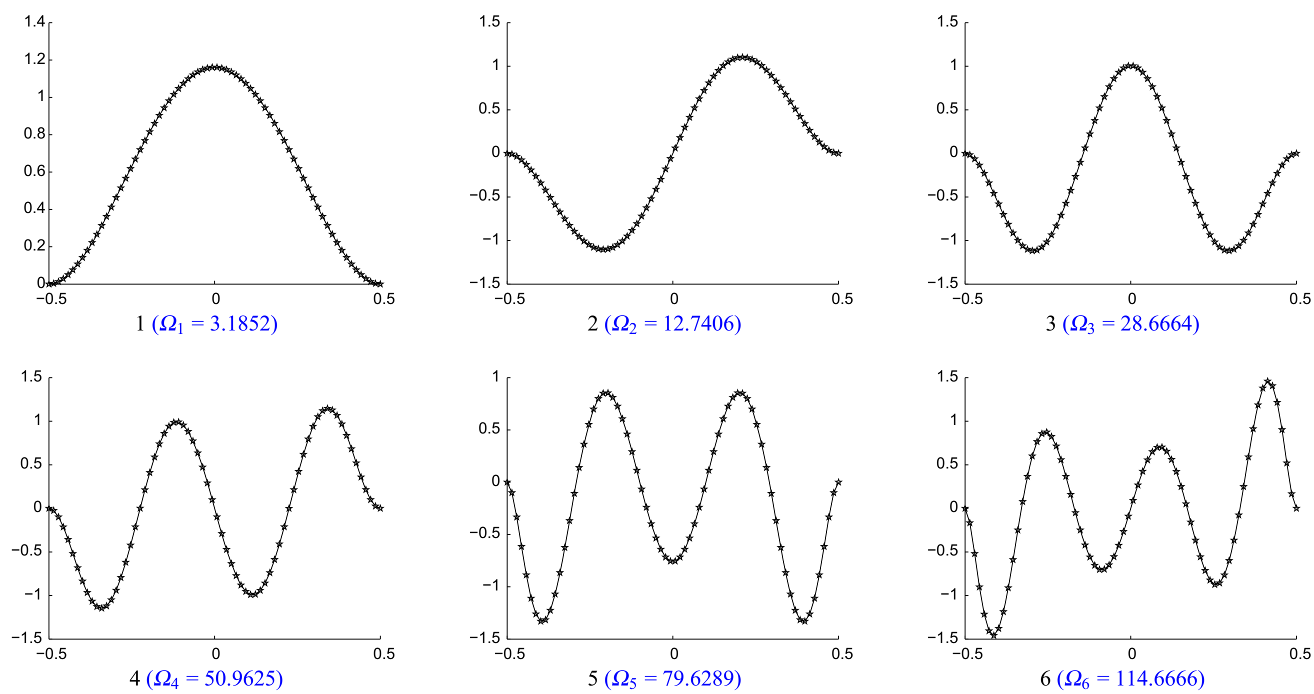


Fig. 5 First six 2-D mode shapes of $SUS304/Si_3N_4$ S-S FG beam with k = 10

1, whereas these may follow descending manner with increase in k while considering $E_r > 1$.

- One may observe that present results follow descending pattern with increase in power-law indices (k) in case of Al/Al_2O_3 and $SUS304/Si_3N_4$ beams, whereas a few ambiguities can be seen in Al/ZrO_2 FG beam.
- Among all the classical edge conditions, natural frequencies in all modes of C-C FG beams are always the highest and the least in case of cantilever FG beams irrespective of the FG constituents considered.
- The convergence of natural frequencies is dependent on various factors with reference to the numerical method assumed. In FEM and DQM, the factor is discretization of beam domain, whereas it is number of polynomials involved in transverse displacement in case of RRM. On contrary, RRM is basically implemented in linear dynamical systems, whereas FEM and DQM can handle both linear and non-linear problems with efficiency. It may be observed that the computed results using three numerical techniques are nearly same, but the convergence is faster in RRM with desired accuracies.

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