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Multi-switching compound synchronization of four different chaotic systems via active backstepping method

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Abstract This manuscript presents a theoretical and numerical analysis to achieve compound synchronization of four non-identical chaotic systems for different multi-switching states. Multi-switching compound synchronization is achieved for three drive systems and one response system via active backstepping technique. By using Lyapunov stability theory, asymptotically stable synchronization states are established. To elaborate the considered scheme an example of Pehlivan system, Liu system, Qi system and Lu system is discussed. The conclusions drawn from computational and analytical approaches are in excellent agreement.

Keywords Multi-switching synchronization · Compound synchronization · Active backstepping Method · Theory of Lyapunov stability

1 Introduction

Chaotic phenomenon exhibits typical and complex behavior of a dynamical system which evolves with time. Due to unpredictable behavior of chaotic and hyperchaotic systems, synchronization has always been an interesting problem for researchers. In last few decades, laser [1], neural network [2], ecology [3], secure communication [4] are some fields witnessing the utility of synchronization.
Although there are several other fields too where chaos synchronization is being used as a tool. But since last three decades secure communication has been one of the most engrossing field for which new synchronization schemes [5,6] have been studied to make the communication more reliable. New methodologies [7–10] have been proposed to attain synchronization. Speedy growth has been seen in the area of chaos synchronization after 1990, when Pecora and Carroll [11] accomplished synchronization for chaotic systems.

After the valuable work of Pecora and Carroll, synchronization has also been achieved for more than one drive and one response systems like combination, combination-combination, compound combination synchronization, compound and double compound synchronization [12–17] etc. Compound synchronization has also been achieved for some integer order and fractional order systems [18].

Combination synchronization and compound synchronization have been studied by the researchers for secure communication [19,20]. The security of information transmission can be enhanced by using more than one drive systems as information signal can be dissevered with different signals of drive systems. Whereas two drive systems are being used in scheme of combination synchronization [19] to enhance security, in compound synchronization [20] a scaling drive system additionally increases grade of security.

In recent years, the field of multi-switching synchronization has attracted attention of researchers. After the outstanding work of Ucar [21], several researches have been done on multi-switching synchronization. Ucar achieved multi-switching synchronization for two identical systems

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via active control while Wang and Sun [22] achieved multi-switching synchronization for two non-identical systems for fully unknown parameters by using adaptive control method.

Multi-switching synchronization has also been achieved by Ajayi et al. [23] for identical chaotic systems. Recently, Vincent et al. [24] has defined a new type of multi-switching synchronization which provides more degree of freedom to state variables to create error vector in different manner. Multi-switching combination synchronization has been investigated by Vincent et al. [24] and Zheng [25] by using different approaches. Complexity of multi-switching synchronization inolving more than one drive systems has its own advantage in secure communication [24,25].

In the work mentioned above, multi-switching synchronization has been achieved either between two chaotic systems or combination of three chaotic systems. Motivated from aforementioned work, the problem of multi-switching compound synchronization of four chaotic systems has been considered in this paper which is still an open problem. In the proposed scheme three systems have been considered as drive systems and one system has been considered as response system. Multi-switching combination synchronization can be achieved as a particular case of the proposed synchronization scheme. Active backstepping method has been applied to achieve the desired synchronization. Amongst different synchronization methods active backstepping technique is one of the most efficient methods to attain synchronization. Numerous different type of synchronizations [15,24,26,27] have been achieved by using this approach.

Although in order to demonstrate the proposed scheme chaotic systems can be chosen arbitrarily, but this approach is explored by using four non-identical chaotic systems, where Pehlivan [28], Liu [29] and Qi [30] systems are considered as drive systems and Lu system [31] is considered as response system. A number of multi-switching states are possible by choosing different combinations of state variables but due to complexity of calculations only three have been discussed in this paper.

This paper is divided in five sections. In Sect. 2 problem of multi-switching compound synchronization is formulated. In Sect. 3 different multi-switching states and construction of controllers are described for the systems considered. In Sect. 4 computational results are given. In Sect. 5 the results achieved by this work and further possible developments are discussed in brief.

2 Problem statement

Suppose first, second and third drive systems are

$$\dot{x}_1 = g_{11}(x), \dot{x}_2 = g_{21}(x), \ldots, \dot{x}_n = g_{n1}(x), \tag{1}$$

$$\dot{y}_1 = g_{12}(y), \dot{y}_2 = g_{22}(y), \ldots, \dot{y}_n = g_{n2}(y), \tag{2}$$

$$\dot{z}_1 = g_{13}(z), \dot{z}_2 = g_{23}(z), \ldots, \dot{z}_n = g_{n3}(z), \tag{3}$$

respectively. The corresponding response system with controller is

$$\begin{aligned} \dot{w}_1 &= g_{14}(w) + V_1, \\ \dot{w}_2 &= g_{24}(w) + V_2, \\ &\vdots \\ \dot{w}_n &= g_{n4}(w) + V_n, \end{aligned} \tag{4}$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$, $z = (z_1, z_2, \ldots, z_n)^T$, $w = (w_1, w_2, \ldots, w_n)^T$ are real state vectors of drive and response systems. $V_i : \mathbb{R}^n \to \mathbb{R}, i = 1, 2, \ldots, n$ are nonlinear control functions and $g_{i1}, g_{i2}, g_{i3}, g_{i4} : \mathbb{R}^n \to \mathbb{R}, i = 1, 2, \ldots, n$ are continuous vector functions.

Let $X = diag(x_1, x_2, \ldots, x_n)$, $Y = diag(y_1, y_2, \ldots, y_n)$, $Z = diag(z_1, z_2, \ldots, z_n)$, $W = diag(w_1, w_2, \ldots, w_n)$ be n dimensional diagonal matrices, we will define compound synchronization as follows:

Definition 1 [17] If there exist four constant diagonal matrices $P, Q, R, S \in \mathbb{R}^n \times \mathbb{R}^n$ and $P \neq 0$ such that the error function E fulfills the following condition:

$$\lim_{t\to\infty} \|E\| = \lim_{t\to\infty} \|PW - QX(RY + SZ)\| = 0, \tag{5}$$

where $\|.\|$ is matrix norm, then the drive systems (1),(2),(3) are realized compound synchronization with the response system (4).

Remark 1 [17] The way X, Y, Z, W are defined, it is obvious that $E = diag(\sigma_{1111}, \sigma_{2222}, \ldots, \sigma_{nnnn})$ is a diagonal matrix. For the error matrix $E = diag(\sigma_{1111}, \sigma_{2222}, \ldots, \sigma_{nnnn}) = PW - QX(RY + SZ)$, right multiplying both sides by a vector $A = [1, 1, \ldots, 1]^T{}_{n\times 1}$, we acquire $E^* = (\sigma_{1111}, \sigma_{2222}, \ldots, \sigma_{nnnn})^T = [PW - QX(RY + SZ)]A$.

Remark 2 If $P = diag(p_1, p_2, \ldots, p_n)$, $Q = diag(q_1, q_2, \ldots, q_n)$, $R= diag(r_1, r_2, \ldots, r_n)$, $S= diag(s_1, s_2, \ldots, s_n)$, then the components of the error vector E^* are obtained as

$$\sigma_{ijkl} = p_i w_i - q_j x_j (r_k y_k + s_l z_l) \tag{6}$$

where indices of the error states are strictly chosen as $i = j = k = l$, and $(i, j, k, l) \in (1, 2, \ldots, n)$.

Definition 2 If the error states in relation to Definition 1 are redefined such that $i = j = k = l$, is not satisfied, where $(i, j, k, l) \in (1, 2, \ldots, n)$ and

$$\lim_{t\to\infty} \|E\| = \lim_{t\to\infty} \|PW - QX(RY + SZ)\| = 0, \tag{7}$$

then the drive systems (1),(2),(3) are said to achieve multi-switching compound synchronization with the response system (4).

In order to formulate the problem in an easy way, suppose (1), (2), (3), (4) are assumed to be three dimensional systems. Defining an arbitrary multi-switching state for three dimensional systems as follows:

$$\begin{aligned}\sigma_{1233} &= p_1 w_1 - q_2 x_2 (r_3 y_3 + s_3 z_3),\\ \sigma_{2312} &= p_2 w_2 - q_3 x_3 (r_1 y_1 + s_2 z_2),\\ \sigma_{3121} &= p_3 w_3 - q_1 x_1 (r_2 y_2 + s_1 z_1).\end{aligned} \tag{8}$$

The error dynamics is

$$\begin{aligned}\dot{\sigma}_{1233} &= p_1 \dot{w}_1 - q_2 \dot{x}_2 (r_3 y_3 + s_3 z_3) - q_2 x_2 (r_3 \dot{y}_3 + s_3 \dot{z}_3),\\ \dot{\sigma}_{2312} &= p_2 \dot{w}_2 - q_3 \dot{x}_3 (r_1 y_1 + s_2 z_2) - q_3 x_3 (r_1 \dot{y}_1 + s_2 \dot{z}_2),\\ \dot{\sigma}_{3121} &= p_3 \dot{w}_3 - q_1 \dot{x}_1 (r_2 y_2 + s_1 z_1) - q_1 x_1 (r_2 \dot{y}_2 + s_1 \dot{z}_1).\end{aligned} \tag{9}$$

On using (1), (2), (3), (4) in previous equation, we get

$$\begin{aligned}\dot{\sigma}_{1233} &= p_1 g_{14}(w) - q_2 g_{21}(x)(r_3 y_3 + s_3 z_3)\\ &\quad - q_2 x_2 (r_3 g_{32}(y) + s_3 g_{33}(z)) + p_1 V_1,\\ \dot{\sigma}_{2312} &= p_2 g_{24}(w) - q_3 g_{31}(x)(r_1 y_1 + s_2 z_2)\\ &\quad - q_3 x_3 (r_1 g_{12}(y) + s_2 g_{23}(z)) + p_2 V_2,\\ \dot{\sigma}_{3121} &= p_3 g_{34}(w) - q_1 g_{11}(x)(r_2 y_2 + s_1 z_1)\\ &\quad - q_1 x_1 (r_2 g_{22}(y) + s_1 g_{13}(z)) + p_3 V_3.\end{aligned} \tag{10}$$

By active backstepping method this system will be transformed into a new system for which (0, 0, 0) equilibrium point will be asymptotically stable. Suppose $l_1 = \sigma_{1233}$ and $\sigma_{2312} = \zeta(l_1)$ is assumed as a virtual controller. Then, differentiating l_1 yields:

$$\dot{l}_1 = G_1(l_1, h_1, V_1), \tag{11}$$

where h_1 is a nonlinear function containing the terms of drive and response systems.

Suppose $K_1 = l_1^2$ is the Lyapunov function for l_1 subsystem (11). Then virtual controller $\zeta(l_1)$ will be designed in such a way that $\dot{K}_1$ will be negative definite and l_1 subsystem will be asymptotically stabilized.

Define the error between σ_{2312} and $\zeta(l_1)$ as $\sigma_{2312} - \zeta(l_1) = l_2$. Now, σ_{3121} will be considered as a virtual controller. Then (l_1, l_2) subsystem will be

$$\begin{aligned}\dot{l}_1 &= G_1(l_1, h_1, V_1),\\ \dot{l}_2 &= G_2(l_1, l_2, \sigma_{3121}, h_2, V_2),\end{aligned} \tag{12}$$

where h_2 is a nonlinear function containing the terms of drive and response systems. The same process will be repeated to stabilize the (l_1, l_2) subsystem, until an asymptotically stable (l_1, l_2, l_3) system will be achieved. At each step a positive definite Lyapunov function with negative definite derivative will ensure the asymptotical stability of each subsystem.

3 Multi-switching compound synchronization of four non-identical systems

Pehlivan system [28] is considered as first drive system (scaling system). Dynamics of Pehlivan system is given below

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1,\\ \dot{x}_2 &= \lambda_1 x_2 - x_1 x_3,\\ \dot{x}_3 &= x_1 x_2 - \mu_1.\end{aligned} \tag{13}$$

Pehlivan system exhibits chaotic behavior for $\lambda_1 = 0.5, \mu_1 = 0.5$. Liu system [29] is considered as second drive system. Dynamics of Liu system is given by

$$\begin{aligned}\dot{y}_1 &= \lambda_2 (y_2 - y_1),\\ \dot{y}_2 &= \mu_2 y_1 - y_1 y_3,\\ \dot{y}_3 &= -\rho_2 y_3 + \delta_2 {y_1}^2.\end{aligned} \tag{14}$$

Liu system exhibits chaotic behavior for $\lambda_2 = 10, \mu_2 = 40, \rho_2 = 2.5, \delta_2 = 4$. Qi system [30] is considered as third drive system given by

$$\begin{aligned}\dot{z}_1 &= \lambda_3 (z_2 - z_1) + \theta_3 z_2 z_3,\\ \dot{z}_2 &= \rho_3 z_1 + \delta_3 z_2 - z_1 z_3,\\ \dot{z}_3 &= z_1 z_2 - \mu_3 z_3.\end{aligned} \tag{15}$$

Qi system exhibits chaotic behavior for $\lambda_3 = 14, \mu_3 = 43, \rho_3 = -1, \delta_3 = 16, \theta_3 = 4$. Lu system [31] is considered as response system. Lu system with controller is

$$\begin{aligned}\dot{w}_1 &= \lambda_4 (w_2 - w_1) + V_1,\\ \dot{w}_2 &= \rho_4 w_2 - w_1 w_3 + V_2,\\ \dot{w}_3 &= w_1 w_2 - \mu_4 w_3 + V_3,\end{aligned} \tag{16}$$

where V_1, V_2, V_3 are the controllers which are designed to achieve the multi-switching compound synchronization. Lu system exhibits chaotic behavior for $\lambda_4 = 36, \mu_4 = 3, \rho_4 = 20$. The following three multi-switching states are considered during the discussion:

Switch One

$$\begin{aligned}\sigma_{1211} &= p_1 w_1 - q_2 x_2 (r_1 y_1 + s_1 z_1),\\ \sigma_{2331} &= p_2 w_2 - q_3 x_3 (r_3 y_3 + s_1 z_1),\\ \sigma_{3121} &= p_3 w_3 - q_1 x_1 (r_2 y_2 + s_1 z_1),\end{aligned} \tag{17}$$

Switch Two

$$\begin{aligned}\sigma_{1122} &= p_1 w_1 - q_1 x_1 (r_2 y_2 + s_2 z_2),\\ \sigma_{2111} &= p_2 w_2 - q_1 x_1 (r_1 y_1 + s_1 z_1),\\ \sigma_{3133} &= p_3 w_3 - q_1 x_1 (r_3 y_3 + s_3 z_3),\end{aligned} \tag{18}$$

Switch Three

$$\begin{aligned}\sigma_{1213} &= p_1 w_1 - q_2 x_2 (r_1 y_1 + s_3 z_3),\\ \sigma_{2313} &= p_2 w_2 - q_3 x_3 (r_1 y_1 + s_3 z_3),\\ \sigma_{3322} &= p_3 w_3 - q_3 x_3 (r_2 y_2 + s_2 z_2).\end{aligned} \tag{19}$$

Now, the controllers will be constructed for switch one.

Theorem 1 *If the control functions are defined as follows:*

$$\begin{aligned}V_1 &= -\frac{1}{p_1}\{(1-\lambda_4)l_1 + h_1\},\\ V_2 &= -\frac{1}{p_2}\left\{(1+\rho_4)l_2 + \frac{p_1}{p_2}\lambda_4 l_1 \right.\\ &\quad \left. -\frac{p_2}{p_1 p_3} l_1 q_1 x_1 (r_2 y_2 + s_1 z_1) + h_2\right\},\\ V_3 &= -\frac{1}{p_3}\left\{(1-\mu_4)l_3 + \left(\frac{p_3}{p_1 p_2} - \frac{p_2}{p_1 p_3}\right) l_1 l_2 \right.\\ &\quad + \frac{p_3}{p_1 p_2}\{l_1 q_3 x_3 (r_3 y_3 + s_1 z_1)\} + \left(\frac{p_3}{p_1 p_2} - \frac{p_2}{p_1 p_3}\right)\\ &\quad \left. \{l_2 q_2 x_2 (r_1 y_1 + s_1 z_1)\} + h_3\right\},\end{aligned} \tag{20}$$

where

$$\begin{aligned}h_1 &= \frac{p_1}{p_2}\lambda_4 q_3 x_3 (r_3 y_3 + s_1 z_1) - \lambda_4 q_2 x_2 (r_1 y_1 + s_1 z_1)\\ &\quad - q_2(\lambda_1 x_2 - x_1 x_3)(r_1 y_1 + s_1 z_1) - q_2 x_2 (r_1 \lambda_2 (y_2 - y_1)\\ &\quad + s_1(\lambda_3 (z_2 - z_1) + \theta_3 z_2 z_3)),\\ h_2 &= \rho_4 q_3 x_3 (r_3 y_3 + s_1 z_1) - \frac{p_2}{p_1 p_3} q_1 q_2 x_1 x_2 (r_1 y_1 + s_1 z_1)\\ &\quad (r_2 y_2 + s_1 z_1) - q_3 (x_1 x_2 - \mu_1)(r_3 y_3 + s_1 z_1)\\ &\quad - q_3 x_3 (r_3 (-\rho_2 y_3 + \delta_2 (y_1)^2))\\ &\quad + s_1(\lambda_3 (z_2 - z_1) + \theta_3 z_2 z_3)),\\ h_3 &= \frac{p_3}{p_1 p_2} q_2 q_3 x_2 x_3 (r_1 y_1 + s_1 z_1)(r_3 y_3 + s_1 z_1)\\ &\quad - \lambda_4 q_1 x_1 (r_2 y_2 + s_1 z_1) - q_1 (x_2 - x_1)(r_2 y_2 + s_1 z_1)\\ &\quad - q_1 x_1 (r_2 (\mu_2 y_1 - y_1 y_3) + s_1 (\lambda_3 (z_2 - z_1) + \theta_3 z_2 z_3)),\end{aligned} \tag{21}$$

and $l_1 = \sigma_{1211}, l_2 = \sigma_{2331}, l_3 = \sigma_{3121}$, *then response system (16) will be in the state of multi-switching compound synchronization with drive systems (13), (14) and (15).*

Proof From Eq. (17), we have

$$\begin{aligned}\dot\sigma_{1211} &= p_1 \dot w_1 - q_2 \dot x_2 (r_1 y_1 + s_1 z_1) - q_2 x_2 (r_1 \dot y_1 + s_1 \dot z_1),\\ \dot\sigma_{2331} &= p_2 \dot w_2 - q_3 \dot x_3 (r_3 y_3 + s_1 z_1) - q_3 x_3 (r_3 \dot y_3 + s_1 \dot z_1),\\ \dot\sigma_{3121} &= p_3 \dot w_3 - q_1 \dot x_1 (r_2 y_2 + s_1 z_1) - q_1 x_1 (r_2 \dot y_2 + s_1 \dot z_1).\end{aligned} \tag{22}$$

By using Eq. (13),(14),(15) and (16) in (22), we get

$$\begin{aligned}\dot\sigma_{1211} &= \frac{p_1}{p_2}\lambda_4 \sigma_{2331} - \lambda_4 \sigma_{1211} + h_1 + p_1 V_1,\\ \dot\sigma_{2331} &= \rho_4 \sigma_{2331} - \frac{p_2}{p_1 p_3}\sigma_{1211}\sigma_{3121}\\ &\quad - \frac{p_2}{p_1 p_3}\sigma_{1211} q_1 x_1 (r_2 y_2 + s_1 z_1)\\ &\quad - \frac{p_2}{p_1 p_3}\sigma_{3121} q_2 x_2 (r_1 y_1 + s_1 z_1) + h_2 + p_2 V_2,\\ \dot\sigma_{3121} &= \frac{p_3}{p_1 p_2}\sigma_{1211}\sigma_{2331} + \frac{p_3}{p_1 p_2}\sigma_{1211} q_3 x_3 (r_3 y_3 + s_1 z_1)\\ &\quad + \frac{p_3}{p_1 p_2}\sigma_{2331} q_2 x_2 (r_1 y_1 + s_1 z_1)\\ &\quad - \mu_4 \sigma_{3121} + h_3 + p_3 V_3.\end{aligned} \tag{23}$$

Let $l_1 = \sigma_{1211}$. Then, from Eq. (23)

$$\dot l_1 = \frac{p_1}{p_2}\lambda_4 \sigma_{2331} - \lambda_4 l_1 + h_1 + p_1 V_1, \tag{24}$$

where $\sigma_{2331} = \zeta(l_1)$ is regarded as the virtual controller. Active backstepping is a recursive feedback procedure, therefore to stabilize the l_1 subsystem we will design the virtual controller $\zeta(l_1)$. Suppose Lyapunov function for l_1 subsystem is defined as

$$K_1 = 0.5 l_1^2. \tag{25}$$

Then derivative of K_1 will be

$$\dot K_1 = l_1 \dot l_1 = l_1 \left\{\lambda_4 \left(\frac{p_1}{p_2}\zeta(l_1) - l_1\right) + h_1 + p_1 V_1\right\}. \tag{26}$$

Suppose $\zeta(l_1) = 0$. Then by using value of V_1 from (20), we get $\dot K_1 = -l_1^2$ which is negative definite. Hence l_1 subsystem will be asymptotically stable. Suppose the error between $\sigma_{2331}$and $\zeta(l_1)$ is $l_2 = \sigma_{2331} - \zeta(l_1)$. Then (l_1, l_2) subsystem will be

$$\begin{aligned}\dot l_1 &= \frac{p_1}{p_2}\lambda_4 l_2 - l_1,\\ \dot l_2 &= \rho_4 l_2 - \frac{p_2}{p_1 p_3} l_1 \sigma_{3121} - \frac{p_2}{p_1 p_3} l_1 q_1 x_1 (r_2 y_2 + s_1 z_1)\\ &\quad - \frac{p_2}{p_1 p_3}\sigma_{3121} q_2 x_2 (r_1 y_1 + s_1 z_1) + h_2 + p_2 V_2,\end{aligned} \tag{27}$$

where $\sigma_{3121} = \zeta(l_1, l_2)$ is assumed as a virtual controller. In order to stabilize (l_1, l_2) subsystem Lyapunov function is defined as

$$K_2 = K_1 + 0.5 l_2^2. \tag{28}$$

Then, derivative of K_2 will be

$$\begin{aligned}\dot{K}_2 &= \dot{K}_1 + l_2 \dot{l}_2 \\ &= -{l_1}^2 + l_2 \Big\{ \rho_4 l_2 - \frac{p_2}{p_1 p_3} l_1 \zeta(l_1, l_2) \\ &\quad - \frac{p_2}{p_1 p_3} l_1 q_1 x_1 (r_2 y_2 + s_1 z_1) \\ &\quad - \frac{p_2}{p_1 p_3} \zeta(l_1, l_2) q_2 x_2 (r_1 y_1 + s_1 z_1) + \frac{p_1}{p_2} \lambda_4 l_1 \\ &\quad + h_2 + p_2 V_2 \Big\}.\end{aligned} \tag{29}$$

By using the value of V_2 from Eq. (20), derivative of K_2 with respect to t will be

$$\dot{K}_2 = -{l_1}^2 - {l_2}^2, \tag{30}$$

which is negative definite which ensures asymptotic stability of (l_1, l_2) subsystem. Suppose error between σ_{3121} and $\zeta(l_1, l_2)$ is defined as $l_3 = \sigma_{3121} - \zeta(l_1, l_2)$. Then

$$\begin{aligned}\dot{l}_3 &= \frac{p_3}{p_1 p_2} l_1 l_2 + \frac{p_3}{p_1 p_2} l_1 q_3 x_3 (r_3 y_3 + s_1 z_1) \\ &\quad + \frac{p_3}{p_1 p_2} l_2 q_2 x_2 (r_1 y_1 + s_1 z_1) - \mu_4 l_3 + h_3 + p_3 V_3.\end{aligned} \tag{31}$$

In order to stabilize the full dimensional (l_1, l_2, l_3) system consider the following Lyapunov function:

$$K_3 = K_2 + 0.5 l_3^2. \tag{32}$$

Then derivative of K_3 will be

$$\begin{aligned}\dot{K}_3 &= \dot{K}_2 + l_3 \dot{l}_3 \\ &= -{l_1}^2 - {l_2}^2 + l_3 \Big\{ (\frac{p_3}{p_1 p_2} - \frac{p_2}{p_1 p_3}) l_1 l_2 \\ &\quad + \frac{p_3}{p_1 p_2} l_1 q_3 x_3 (r_3 y_3 + s_1 z_1) \\ &\quad + \left(\frac{p_3}{p_1 p_2} - \frac{p_2}{p_1 p_3} \right) l_2 q_2 x_2 (r_1 y_1 + s_1 z_1) \\ &\quad - \mu_4 l_3 + h_3 + p_3 V_3) \Big\}.\end{aligned} \tag{33}$$

Using the value of V_3 from Eqs. (20), (33) reduces to

$$\dot{K}_3 = -{l_1}^2 - {l_2}^2 - {l_3}^2, \tag{34}$$

which is negative definite. Hence the equilibrium point (0, 0, 0) of system (l_1, l_2, l_3) given by

$$\begin{aligned}\dot{l}_1 &= -l_1 + \frac{p_1}{p_2} \lambda_4 l_2, \\ \dot{l}_2 &= -l_2 - \frac{p_2}{p_1 p_3} l_1 l_3 - \frac{p_2}{p_1 p_3} l_3 q_2 x_2 (r_1 y_1 + s_1 z_1) - \frac{p_1}{p_2} \lambda_4 l_1, \\ \dot{l}_3 &= \frac{p_2}{p_1 p_3} l_1 l_2 - l_3 + \frac{p_2}{p_1 p_3} l_2 q_2 x_2 (r_1 y_1 + s_1 z_1).\end{aligned} \tag{35}$$

will be asymptotically stable . Thus, asymptotically stable synchronization state is attained by response system and drive systems. □

Corollary 1 *If $x_1 = x_2 = x_3 = \alpha$,then the problem is reduced to multi-switching combination synchronization of systems (14), (15) and (16). The corresponding controllers are*

$$\begin{cases} V_1 = -\frac{1}{p_1} \{(1 - \lambda_4) l_1 + h_1\}, \\ V_2 = -\frac{1}{p_2} \Big\{ (1 + \rho_4) l_2 + \frac{p_1}{p_2} \lambda_4 l_1 \\ \qquad - \frac{p_2}{p_1 p_3} l_1 q_1 \alpha (r_2 y_2 + s_1 z_1) + h_2 \Big\}, \\ V_3 = -\frac{1}{p_3} \Big\{ (1 - \mu_4) l_3 + \left(\frac{p_3}{p_1 p_2} - \frac{p_2}{p_1 p_3} \right) l_1 l_2 \\ \qquad + \frac{p_3}{p_1 p_2} \{ l_1 q_3 \alpha (r_3 y_3 + s_1 z_1) \} \\ \qquad + \left(\frac{p_3}{p_1 p_2} - \frac{p_2}{p_1 p_3} \right) \{ l_2 q_2 \alpha (r_1 y_1 + s_1 z_1) \} + h_3 \Big\}. \end{cases} \tag{36}$$

Corollary 2 *If $x_1 = x_2 = x_3 = \alpha$ and $r_1 = r_2 = r_3 = 0$, then the problem is reduced to multi-switching hybrid projective synchronization of systems (15) and (16). The corresponding controllers are*

$$\begin{cases} V_1 = -\frac{1}{p_1} \{(1 - \lambda_4) l_1 + h_1\}, \\ V_2 = -\frac{1}{p_2} \Big\{ (1 + \rho_4) l_2 + \frac{p_1}{p_2} \lambda_4 l_1 \\ \qquad - \frac{p_2}{p_1 p_3} l_1 q_1 \alpha s_1 z_1 + h_2 \Big\}, \\ V_3 = -\frac{1}{p_3} \Big\{ -(1 - \mu_4) l_3 + \left(\frac{p_3}{p_1 p_2} - \frac{p_2}{p_1 p_3} \right) l_1 l_2 \\ \qquad + \frac{p_3}{p_1 p_2} \{ l_1 q_3 \alpha s_1 z_1 \} + \left(\frac{p_3}{p_1 p_2} - \frac{p_2}{p_1 p_3} \right) \\ \qquad \{ l_2 q_2 \alpha s_1 z_1 \} + h_3 \}. \end{cases} \tag{37}$$

Corollary 3 *If* $x_1 = x_2 = x_3 = \alpha$ *and* $s_1 = s_2 = s_3 = 0$, *then the problem is reduced to multi-switching hybrid projective synchronization of systems* (14) *and* (16). *The corresponding controllers are*

$$\begin{cases} V_1 = -\frac{1}{p_1}\{(1-\lambda_4)l_1 + h_1\}, \\ V_2 = -\frac{1}{p_2}\left\{(1+\rho_4)l_2 + \frac{p_1}{p_2}\lambda_4 l_1 - \frac{p_2}{p_1p_3}l_1q_1\alpha r_2y_2 \right. \\ \quad \left. + h_2\right\}, \\ V_3 = -\frac{1}{p_3}\left\{-(1-\mu_4)l_3 + (\frac{p_3}{p_1p_2} - \frac{p_2}{p_1p_3})l_1l_2 \right. \\ \quad + \frac{p_3}{p_1p_2}\{l_1q_3\alpha r_3y_3\} + \left(\frac{p_3}{p_1p_2} - \frac{p_2}{p_1p_3}\right) \\ \quad \left. \{l_2q_2\alpha r_1y_1\} + h_3\right\}. \end{cases} \tag{38}$$

Corollary 4 *If* $q_1 = q_2 = q_3 = 0$ *or* $r_1 = r_2 = r_3 = s_1 = s_2 = s_3 = 0$, *then the problem of compound synchronization is reduced to the problem of controlling the system* (16). *The controllers are as follows:*

$$\begin{cases} V_1 = -\frac{1}{p_1}\{(1-\lambda_4)l_1 + h_1\}, \\ V_2 = -\frac{1}{p_2}\left\{(1+\rho_4)l_2 + \frac{p_1}{p_2}\lambda_4 l_1 + h_2\right\}, \\ V_3 = -\frac{1}{p_3}\left\{-(1-\mu_4)l_3 + (\frac{p_3}{p_1p_2} - \frac{p_2}{p_1p_3})l_1l_2 + h_3\right\}, \end{cases} \tag{39}$$

for which equilibrium point $(0, 0, 0)$ *will be asymptotically stable.*

Theorem 2 *Response system* (16) *will attain a state of multi-switching compound synchronization with drive system* (13), (14), (15) *for errors defined by* (18), *if the controllers are designed as follows:*

$$\begin{cases} V_1 = -\frac{1}{p_1}\{(1-\lambda_4)l_1 + h_1\}, \\ V_2 = -\frac{1}{p_2}\left\{(1+\rho_4)l_2 + \frac{p_1}{p_2}\lambda_4 l_1 \right. \\ \quad \left. - \frac{p_2}{p_1p_3}l_1q_1x_1(r_3y_3 + s_3z_3) + h_2\right\}, \\ V_3 = -\frac{1}{p_3}\left\{(1-\mu_4)l_3 + \left(\frac{p_3}{p_1p_2} - \frac{p_2}{p_1p_3}\right)l_1l_2 \right. \\ \quad + \frac{p_3}{p_1p_2}\{l_1q_1x_1(r_1y_1 + s_1z_1)\} \\ \quad \left. + \left(\frac{p_3}{p_1p_2} - \frac{p_2}{p_1p_3}\right)\{l_2q_1x_1(r_2y_2 + s_2z_2)\} + h_3\right\}, \end{cases} \tag{40}$$

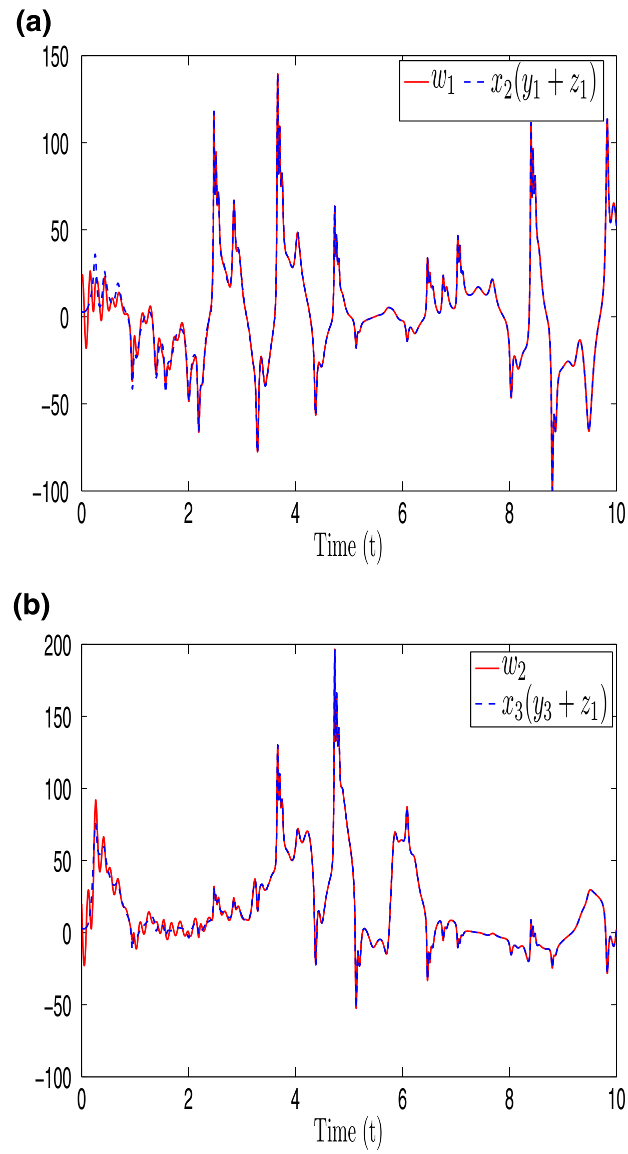

Fig. 1 **a** Synchronization for w_1 and $x_2(y_1 + z_1)$ **b** synchronization for w_2 and $x_3(y_3 + z_1)$ in switch one

where $l_1 = \sigma_{1122}, l_2 = \sigma_{2111}, l_3 = \sigma_{3133}$.

Theorem 3 *Response system* (16) *will attain a state of multi-switching compound synchronization with drive system* (13), (14), (15) *for errors defined by* (19), *if the controllers are designed as follows:*

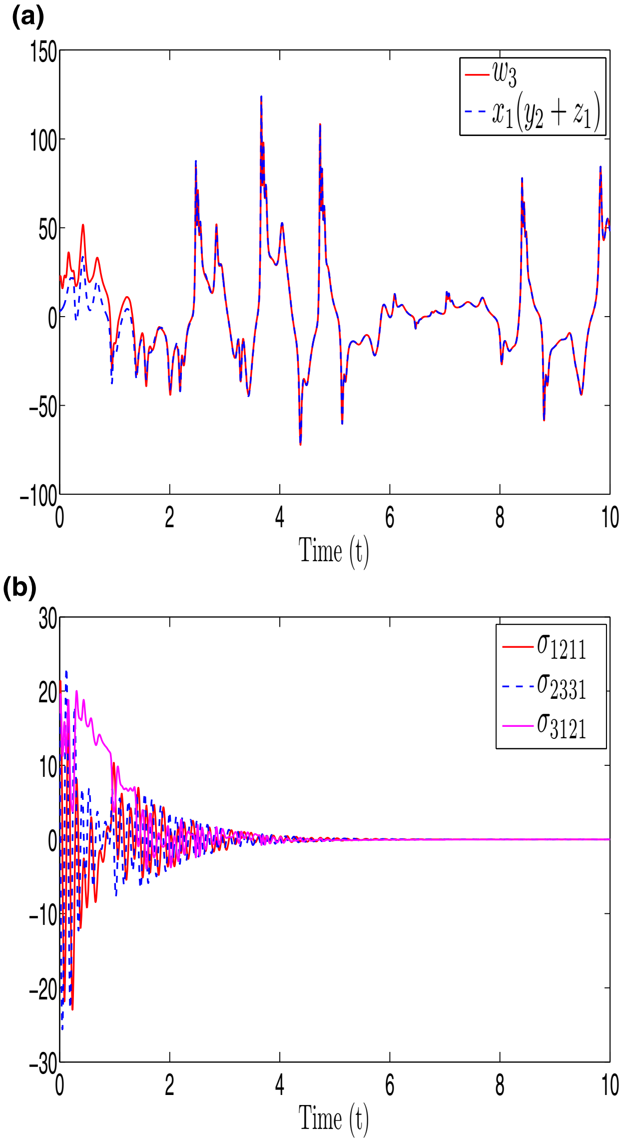

Fig. 2 **a** Synchronization for w_3 and $x_1(y_2+z_1)$ in switch one **b** errors $\sigma_{1211}, \sigma_{2331}, \sigma_{3121}$ converging to zero in switch one

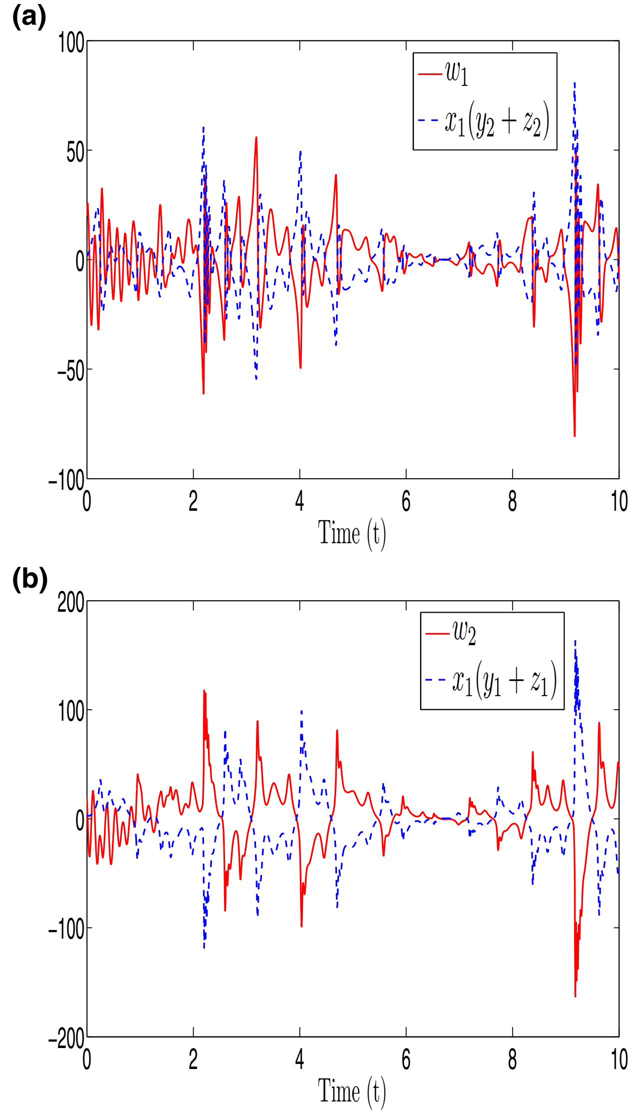

Fig. 3 **a** Anti-synchronization for w_1 and $x_1(y_2+z_2)$ **b** anti-synchronization for w_2 and $x_1(y_1+z_1)$ in switch two

$$\begin{cases} V_1 = -\frac{1}{p_1}\{(1-\lambda_4)l_1 + h_1\}, \\ V_2 = -\frac{1}{p_2}\Big\{(1+\rho_4)l_2 + \frac{p_1}{p_2}\lambda_4 l_1 \\ \quad -\frac{p_2}{p_1 p_3} l_1 q_3 x_3 (r_2 y_2 + s_2 z_2) + h_2\Big\}, \\ V_3 = -\frac{1}{p_3}\Big\{(1-\mu_4)l_3 + \left(\frac{p_3}{p_1 p_2} - \frac{p_2}{p_1 p_3}\right) l_1 l_2 \\ \quad + \frac{p_3}{p_1 p_2}\{l_1 q_3 x_3 (r_1 y_1 + s_3 z_3)\} + \Big(\frac{p_3}{p_1 p_2} \\ \quad -\frac{p_2}{p_1 p_3}\Big)\{l_2 q_2 x_2 (r_1 y_1 + s_3 z_3)\} + h_3\Big\}, \end{cases} \tag{41}$$

where $l_1 = \sigma_{1213}, l_2 = \sigma_{2313}, l_3 = \sigma_{3322}$.

Same corollaries can be obtained for Theorems 2 and 3, but not given here as these can be obtained similarly. Theorems 2 and 3 can also be proved in same manner.

4 Graphical results

Numerical simulations are shown for three switches. Parameters for Pehlivan, Liu and Qi systems are taken as $\lambda_1 = 0.5, \mu_1 = 0.5, \lambda_2 = 10, \mu_2 = 40, \rho_2 = 2.5, \delta_2 = 4, \lambda_3 = 10, \mu_3 = 43, \rho_3 = -1, \delta_3 = 16, \theta_3 = 4$, for which respective systems are chaotic.

Parameters for Lu system are taken as $\lambda_4 = 36, \mu_4 = 3, \rho_4 = 20$. Initial conditions are chosen in the manner of

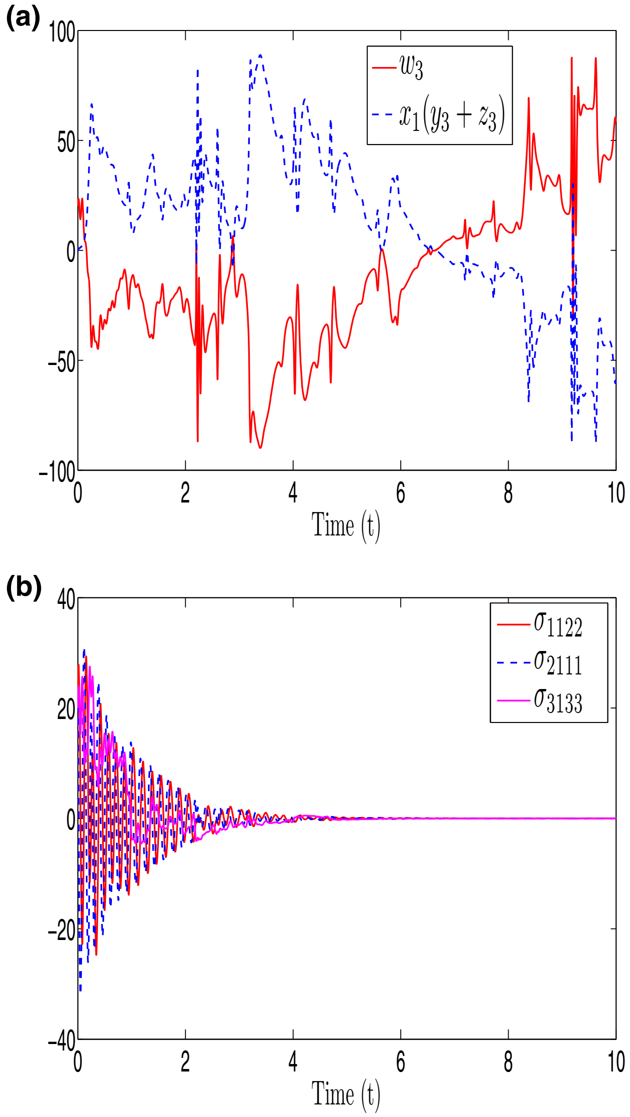

Fig. 4 **a** Anti-synchronization for w_3 and $x_1(y_3+z_3)$ in switch two **b** errors $\sigma_{1122}, \sigma_{2111}, \sigma_{3133}$ converging to zero in switch two

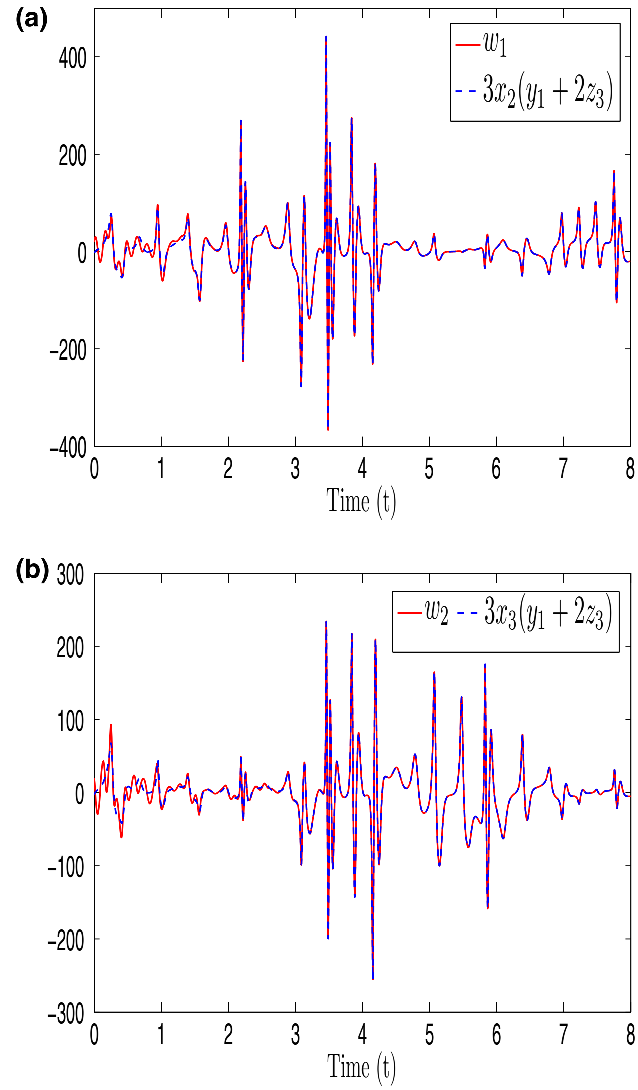

Fig. 5 **a** Synchronization for w_1 and $3x_2(y_1+2z_3)$ **b** synchronization for w_2 and $3x_3(y_1+2z_3)$ in switch three

ensuring the chaotic behavior of drive and response systems. Initial conditions are kept fixed for all four systems throughout the simulations. The initial conditions of the drive and the response systems are arbitrarily chosen as (0.5, 0.5, 0.5), (2, 2, 2), (4, 1, −2) and (20, 20, 20) respectively.

Switch one

In first switch, $p_1 = p_2 = p_3 = 1, q_1 = q_2 = q_3 = 1, r_1 = r_2 = r_3 = 1, s_1 = s_2 = s_3 = 1$ are taken. Hence initial conditions for error dynamical system are (17, 17, 17). Synchronization between the state variables $w_1, x_2(y_1+z_1)$ and $w_2, x_3(y_3+z_1)$ are shown in Figs. 1a, b. Figure 2a, b exhibits synchronization between the state variables $w_3, x_1(y_2+z_1)$ and errors $(\sigma_{1211}, \sigma_{2331}, \sigma_{3121})$ converging to zero.

Switch Two

For second switch, $p_1 = p_2 = p_3 = 1, q_1 = q_2 = q_3 = -1, r_1 = r_2 = r_3 = 1, s_1 = s_2 = s_3 = 1$ are taken which lead to anti-synchronization. Hence initial conditions for error dynamical system are (21.5, 23, 20). Anti-synchronization between the state variables $w_1, x_1(y_2+z_2)$ and $w_2, x_1(y_1+z_1)$ are shown in Fig. 3 a, b. Anti-synchronization between the state variables $w_3, x_1(y_3+z_3)$ and errors $(\sigma_{1122}, \sigma_{2111}, \sigma_{3133})$ converging to zero are shown in Fig. 4a, b.

Switch Three

For third switch, $p_1 = p_2 = p_3 = 1, q_1 = q_2 = q_3 = 3, r_1 = r_2 = r_3 = 1, s_1 = s_2 = s_3 = 2$ are taken. Thus, initial conditions for error dynamical system are (23, 23, 14). Synchronization between the state variables $w_1, 3x_2(y_1+2z_3)$ and $w_2, 3x_3(y_1+2z_3)$ are shown

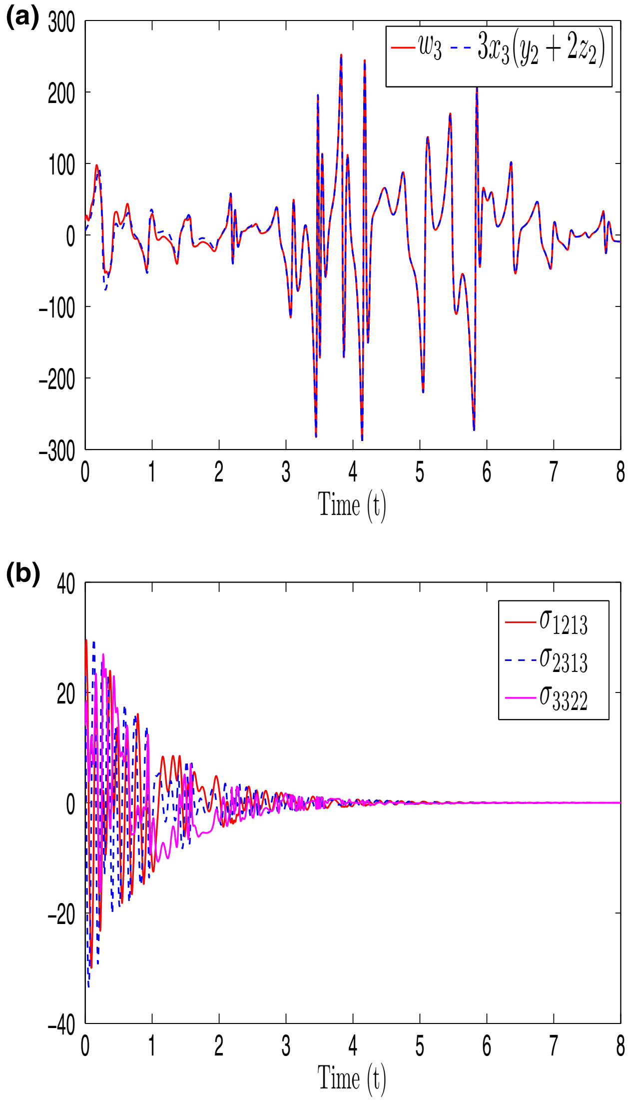

Fig. 6 **a** Synchronization for w_3 and $3x_3(y_2 + 2z_2)$ in switch three **b** errors $\sigma_{1213}, \sigma_{2313}, \sigma_{3322}$ converging to zero in switch three

in Fig. 5a, b. Synchronization between the state variables w_3, $3x_3(y_2 + 2z_2)$ and errors $(\sigma_{1213}, \sigma_{2313}, \sigma_{3322})$ converging to zero are shown in Fig. 6a, b.

5 Conclusion

Multi-switching compound synchronization has been achieved for three drive and one response systems by using active backstepping technique. The feasibility of a recursive method is demonstrated in this manuscript. The method is easy to implement and therefore the approach can be applied on higher dimensional systems too. The effectiveness of the approach is exhibited by numerical simulations. The rigorous analysis and computational approach provide the same results. Further possibility of improvements are still there as the same problem can be considered in case of unknown parameters or for higher dimensional systems which is a problem of further study. The method is efficient for security of communication as its complexity is the main factor which increases grade of security.

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