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On the output controllability of a class of discrete nonlinear distributed systems: a fixed point theorem approach

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Abstract Given a desired signal $y^d = (y_i^d)_{i \in \{0,\ldots,N\}}$, we investigate the optimal control, which applied to nonlinear discrete distributed system $x_{i+1} = Ax_i + Ex_i + Bu_i$, to give a desired output y^d. Techniques based on the fixed point theorems for solving this problem are presented. An example and numerical simulation is also given.

Keywords Output controllability · Optimal control · Nonlinear system · Fixed point theorem

1 Introduction

The research devoted the controllability was started in the 1960s by Kalman and refers to linear dynamical systems. Because the most of practical dynamical systems are nonlinear, that's why, in recent years various controllability problems for different types of nonlinear or semilinear dynamical systems have been considered [1–9].

There are large type of controllability such as completely controllability, small controllability, local controllability, regional controllability, near controllabilitry, null controllability and output controllability [4–6,8–14].

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In the present paper we investigate the output controllability of a class of nonlinear infinite-dimensional discrete systems. More precisely, we consider the nonlinear system whose state is described by the following difference equation

$$(S) \begin{cases} x_{i+1} = Ax_i + Ex_i + Bu_i, & i \in \{0,\ldots,N-1\}, \\ x_0, \end{cases}$$

the corresponding output signal is

$$y_i = Cx_i, \quad \forall i \in \{0,\ldots,N\}.$$

The operator $A : X \longrightarrow X$ is supposed to be bounded on the Hilbert space X (the state space), $E : X \longrightarrow X$ is a nonlinear operator, $B \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(X, Y)$ where the Hilbert space U is the input space and the Hilbert space Y is the output one.

Given a desired output $y^d = (y_i^d)_{i \in \{1,\ldots,N\}}$, we investigate the optimal control $u = (u_i)_{i \in \{0,1,\ldots,N-1\}}$ which minimizes the functional cost

$$J(u) = \| u \|^2$$

over all controls satisfying

$$Cx_i = y_i^d, \quad \forall i \in \{1,\ldots,N\}.$$

To solve this problem and inspired by what was done in [15,16] we use, in the first part, a state space technique to show that the problem of input retrieval can be seen as a problem of optimal control with constraints on the final state [17]. In the second part, we use a technique based on the fixed point theorem (see [2,3,7,18–21]). We establish that the set of admissible controls is completely characterized by

the pseudo inverse corresponding to the linear part of the system and the fixed points of an appropriate mapping. Finally, A numerical example is given to illustrate the obtained results.

Remark 1 The assumption that A is a bounded operator is not so restrictive even in the distributed parameter system. We can see, for example that the discrete system obtained from the evolution equation considered in [22], satisfies this condition.

2 Statement of the problem

We consider the discrete system described by

$$(S) \begin{cases} x_{i+1} = Ax_i + Ex_i + Bu_i, \ i \in \{0, \ldots, N-1\}, \\ x_0 \end{cases} \quad (1)$$

the corresponding output

$$y_i = Cx_i, \quad i \in \{0, \ldots, N\}, \quad (2)$$

where $x_i \in X$ is the state of system (S), $u_i \in U$ is the control variable and $y_i \in Y$ is the output function, $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(X, Y)$. Consider the following control problem. Given a desired trajectory $y^d = (y_1^d, \ldots, y_N^d)$, we try to find the optimal control $u = (u_0, u_1, \ldots, u_{N-1})$ which minimizes the functional cost

$$J(u) = \|u\|^2, \quad (3)$$

over all controls satisfying

$$Cx_i = y_i^d, \quad \forall \ i \in \{1, \ldots, N\}.$$

2.1 An adequate state space approach

In this subsection, we give some technical results which will be used in the sequel. For a finite subset $\sigma_r^s = \{r, r+1, \ldots, s\}$ of $\mathbb{Z}$, with $s \geq r$, let $l^2(\sigma_r^s, X)$ be the space of all sequences $(z_i)_{i \in \sigma_r^s}$, $z_i \in X$.

Remark 2 $l^2(\sigma_r^s, X)$ is a Hilbert space with the usual addition, scalar multiplication and with an inner product defined by

$$< x, y >_{l^2(\sigma_r^s, X)} = \sum_{i=r}^{s} < x_i, y_i >_X .$$

Let L_1 and F be the operators given by

$$L_1 : l^2\left(\sigma_{-N}^{-1}; X\right) \longrightarrow l^2\left(\sigma_{-N}^{-1}; X\right),$$
$$(z_{-N}, \ldots, z_{-1}) \longmapsto (z_{-N+1}, \ldots, z_{-1}, 0),$$

$$F : X \longrightarrow l^2\left(\sigma_{-N}^{-1}; X\right),$$
$$x \longmapsto (0, \ldots, 0, x)$$

and define the variables $z^i \in l^2(\sigma_{-N}^{-1}; Y)$ by

$$z^i = (z_{-N}^i, \ldots, z_{-1}^i),$$
$$z_k^i = \begin{cases} x_{i+k}, & \text{if } i+k \geq 0 \\ x_0, & \text{else,} \end{cases}$$

where $(x_i)_i$ is the solution of system (S). Then the sequence $(z^i)_i$ is the unique solution of the following difference equation

$$\begin{cases} z^{i+1} = L_1 z^i + Fx_i, \ i \in \sigma_0^{N-1}, \\ z^0 = (x_0, x_0, \ldots, x_0). \end{cases}$$

Let $e_i \in X \times l^2(\sigma_{-N}^{-1}; X)$ be the signals defined by $e_i = \begin{pmatrix} x_i \\ z^i \end{pmatrix}$, then we easily establish the following result.

Proposition 1 $(e_i)_{i \in \sigma_0^N}$ *is the unique solution of the difference equation described by*

$$(S_1) \begin{cases} e_{i+1} = \Psi e_i + \Phi e_i + \bar{B} u_i, \ i \in \sigma_0^{N-1}, \\ e_0 = \begin{pmatrix} x_0 \\ (x_0, \ldots, x_0) \end{pmatrix}, \end{cases}$$

where $\Psi = \begin{pmatrix} A & 0 \\ F & L_1 \end{pmatrix}$, $\Phi = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\bar{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$.

Remark 3 The equality

$$e_N = \begin{pmatrix} x_N \\ z^N \end{pmatrix} = \begin{pmatrix} x_N \\ (x_0, \ldots, x_{N-1}) \end{pmatrix},$$

allows us to assimilate the trajectory $(x_0, \ldots, x_{N-1}, x_N)$ of system (S) to the final state e_N of (S_1). This implies that our problem of input retrieval is equivalent to a problem of optimal control with constraints on the final state e_N.

2.2 The optimal control expression

Let's consider the operator Γ defined by

$$\Gamma : X \times l^2\left(\sigma_{-N}^{-1}; X\right) \longrightarrow l^2\left(\sigma_1^N; Y\right) \quad (4)$$
$$(x, (\xi_i)_{-N \leq i \leq 0}) \longmapsto (t_1, \ldots, t_N)$$

with $t_i = C\xi_{i-N}, \quad \forall i \in \sigma_1^{N-1}$ and $t_N = Cx$.

Definition 1 (a) The system (S) is said to be exactly output controllable on σ_1^N if $\forall x_0 \in X$, $\forall y \in l^2(\sigma_1^N; Y)$, $\exists u \in l^2(\sigma_0^{N-1}; U)$ such that $Cx_i = y_i$, $i \in \sigma_1^N$.

(b) The system (S) is said to be weakly output controllable on σ_1^N if $\forall \epsilon > 0$, $\forall x_0 \in X$, $\forall y \in l^2(\sigma_1^N; Y)$, $\exists u$ such that $\| Cx_i - y \|_Y \leq \epsilon$.

Definition 2 (a) The system (S) is said to be Γ-controllable on σ_1^N if $\forall e_0 \in X$, $\forall y^d \in l^2(\sigma_1^N; Y)$, $\exists u \in l^2(\sigma_0^{N-1}; U)$ such that $\Gamma e_N = y^d$.

(b) The system (S) is said to be Γ-weakly controllable on σ_1^N if $\forall \epsilon > 0$, $\forall e_0 \in X$, $\forall y^d \in l^2(\sigma_1^N; Y)$, $\exists u$ such that $\| \Gamma e_N - y^d \|_{l^2(\sigma_1^N; Y)} \leq \epsilon$.

Remark 4 From the above definition, we can easily establish the following results

(i) (S) is exactly output controllable $\sigma_1^N \iff (S_1)$ is Γ-controllable on σ_1^N.
(ii) (S) is weakly output controllable on $\sigma_1^N \iff (S_1)$ is Γ-weakly controllable on σ_1^N.

Proposition 2 *Given a desired output $y^d = (y_1^d, \ldots, y_N^d)$ in $l^2(\sigma_1^N; Y)$, the problem $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ defined as:*

$$(\mathcal{P}_1) \begin{cases} Find\ u^*\ such\ that \\ Cx_i = y_i^d, \quad \forall\ i \in \sigma_1^N & (i) \\ \| u^* \| = inf\{ \| v \| /v\ verify\ (i)\} & (ii) \end{cases}$$

$$(\mathcal{P}_2) \begin{cases} Find\ u^*\ such\ that \\ \Gamma e_N = y^d\ in\ l^2(\sigma_1^N; Y) & (j) \\ \| u^* \| = inf\{ \| v \| /v\ verify\ (j)\} & (jj) \end{cases}$$

have the same solution u^*.

By Proposition 2, the resolution of problem $(\mathcal{P}_1) - (\mathcal{P}_2)$ is equivalent to find the control u^* which ensure the Γ-controllability of system (S_1) and with a minimal cost.

3 Statement of the new problem

We consider the discrete system described by

$$(S_1) \begin{cases} e_{i+1} = \Psi e_i + \Phi e_i + \bar{B} u_i, & i \in \sigma_0^{N-1} \\ e_0 & is\ given \end{cases} \tag{5}$$

where $e_i \in \mathcal{X} = X \times l^2(\sigma_{-N}^{-1}; X)$ is the state of system (S_1), $u_i \in U$ is the control variable, $\Psi \in \mathcal{L}(\mathcal{X})$ and $\bar{B} \in \mathcal{L}(U, X)$. Consider the following control problem. Given a desired trajectory $y^d = (y_1^d, \ldots, y_N^d)$, we find the control u^* which minimizes the functional cost

$$J(u) = \|u\|^2 \tag{6}$$

over all controls satisfying

$$\Gamma e_N = y^d,$$

e_N is the final state of system (S_1) at instant N, and Γ is given by (4). We shall call u^* the wanted control and the solution of system (S_1) is

$$e_i = \Psi^i e_0 + \sum_{j=0}^{i-1} \Psi^j \Phi e_{i-1-j} + \sum_{j=0}^{i-1} \Psi^j \bar{B} u_{i-1-j}, \quad i \in \sigma_1^N. \tag{7}$$

Let L denote the linear operator defined on $T = l^2(\sigma_1^N; \mathcal{X})$ by

$$L : T = l^2(\sigma_1^N; \mathcal{X}) \longrightarrow T$$
$$\xi = (\xi_1, \ldots, \xi_N) \longmapsto L\xi = (L\xi)_{1 \leq i \leq N}$$

where

$$\begin{cases} (L\xi)_i = \Psi^{i-1}\Phi e_0 + \sum_{j=0}^{i-2} \Psi^j \Phi \xi_{i-1-j}; & 2 \leq i \leq N \\ (L\xi)_1 = \Phi e_0 \end{cases}$$

and Let H denote the linear operator defined on $\mathcal{U}$ by

$$H : \mathcal{U} = l^2\left(\sigma_0^{N-1}; U\right) \longrightarrow T$$
$$u = (u_0, \ldots, u_{N-1}) \longmapsto Hu$$

where

$$(Hu)_i = \sum_{j=0}^{i-1} \Psi^j \bar{B} u_{i-1-j}, \quad i \in \sigma_1^N.$$

So, the Eq. (7) can be rewritten as

$$e = (e_1, \ldots, e_N) = \tilde{\Psi} e_0 + Le + Hu \tag{8}$$

where

$$\tilde{\Psi} e_0 = \left(\Psi^i e_0\right)_{1 \leq i \leq N}$$

The operator H is not invertible in general. Introduce then:

$$\tilde{H} : x \in (\ker H)^\perp \to \tilde{H}(x) = H(x) \in Range(H)$$

this operator is invertible and its inverse, which is defined on $Range(H)$ can be extended to $Range(H) \bigoplus Range(H)^\perp$ as follows

$$H^\dagger : x + y \in Range(H) \bigoplus Range(H)^\perp \to \tilde{H}^{-1}(x) \in \mathcal{U}$$

the operator $H^\dagger$ is known as the pseudo inverse operator of H. If $Range(H)$ is closed then $T = Range(H) \bigoplus Range(H)^\perp$

and $H^\dagger$ is defined on all the space T. The mapping $H^\dagger$ satisfies in particular

$$\begin{cases} HH^\dagger x = x, & \forall x \in Range(H) \\ H^\dagger Hy = y, & \forall y \in (\ker H)^\perp. \end{cases}$$

4 Fixed point technique

4.1 Characterization of the set of admissible controls

Let $y^d = (y_1^d, \ldots, y_N^d)$ a predefined output. The aim of this section is to give a characterization of the set of all admissible control in consideration the fixed points of a function appropriately chosen, i.e., We shall characterize the set $\mathcal{U}_{ad}$ of all control which ensure the Γ-controllability.

$$\mathcal{U}_{ad} = \left\{ u \in l^2\left(\sigma_0^{N-1}; U\right) \, / \, \Gamma e_N = y^d \right\}$$

where $(e_0, \ldots, e_N)$ is the trajectory which takes system from the initial state e_0. If $Range(H)$ is closed then $T = Range(H) \bigoplus Range(H)^\perp$ and $H^\dagger$ is defined on all the space T. We suppose that $Range(H)$ is closed. Let $p : T \longrightarrow Range(H)$ be any projection on $Range(H)$ and $\bar{e} \neq 0$ be any fixed element of $Range(H)$, we define

$$f_{\bar{e}} : T \longrightarrow Range(H)$$
$$e \longmapsto f_{\bar{e}}(e) = \begin{cases} 0, & \text{if and only if } \Gamma e_N = y^d \\ \bar{e}, & \text{else} \end{cases}$$

and let

$$\xi : T \longrightarrow T$$
$$e \longmapsto \xi(e) = e - \tilde{\Psi} e_0 - Le \tag{9}$$

and we consider the mapping

$$g : T \longrightarrow T$$
$$e \longmapsto g(e) = \tilde{\Psi} e_0 + Le + p\xi(e) + f_{\bar{e}}(e). \tag{10}$$

Then, we have the following proposition.

Proposition 3 *Let* $P_g = \{e \in T / g(e) = e\}$ *denotes the set of all fixed points of* g. *Then*

$$\mathcal{U}_{ad} = \bigcup_{e \in P_g} H^\dagger \xi(e) + \text{Ker}(H).$$

Proof Let $e^* \in P_g$, we have

$$g(e^*) = \tilde{\Psi} e_0^* + Le^* + p\xi(e^*) + f_{\bar{e}^*}(e) = e^* \tag{11}$$

then

$$e^* - \tilde{\Psi} e_0^* - Le^* = p\xi(e^*) + f_{\bar{e}}(e^*)$$

which implies that

$$\xi(e^*) = p\xi(e^*) + f_{\bar{e}}(e^*) \quad \in Range(H)$$

that means

$$p\xi(e^*) = \xi(e^*)$$

and $f_{\bar{e}}(e^*) = 0$ which carries that $\Gamma e_N^* = y^d$.

Consequently, the Eq. (11) become

$$e^* = \tilde{\Psi} e_0^* + Le^* + \xi(e^*) = \tilde{\Psi} e_0^* + Le^* + HH^\dagger \xi(e^*). \tag{12}$$

Let $u^* = H^\dagger \xi(e^*) + \alpha^*$, with $\alpha^* \in \ker(H)$ and $e^* \in P_g$, then

$$Hu^* = HH^\dagger \xi(e^*) + H(\alpha^*)$$

and from (12), we have

$$Hu^* = HH^\dagger \xi(e^*) = e^* - \tilde{\Psi} e_0^* - Le^*$$

which implies that

$$\begin{cases} e^* &= \tilde{\Psi} e_0^* + Le^* + Hu^* \\ \Gamma e_N^* &= y^d \end{cases}$$

thus

$$u^* \in \mathcal{U}_{ad}.$$

Consequently, $\forall e \in P_g$, we have $H^\dagger \xi(e) + \text{Ker}(H) \subset \mathcal{U}_{ad}$ and

$$\bigcup_{e \in P_g} H^\dagger \xi(e) + \text{Ker}(H) \subset \mathcal{U}_{ad}.$$

Now, we show that $\mathcal{U}_{ad} \subset \bigcup_{e \in P_g} H^\dagger \xi(e) + \text{Ker}(H)$. Let $u^* \in \mathcal{U}_{ad}$ and $(e_1^{u^*}, \ldots, e_{N-1}^{u^*})$ the trajectory of system (S_1) corresponding to control u^*, then we have

$$\begin{cases} e^{u^*} &= \tilde{\Psi} e_0 + Le^{u^*} + Hu^* \\ \Gamma e_N^{u^*} &= y^d \end{cases}$$

and

$$\begin{cases} \xi\left(e^{u^*}\right) &= Hu^* \\ \Gamma e_N^{u^*} &= y^d. \end{cases}$$

Consequently

$$\begin{cases} \xi\left(e^{u^*}\right) &= Hu^* \in Range(H) \\ f_{\bar{e}}\left(e^{u^*}\right) &= 0 \end{cases}$$

and

$$e^{u^*} = \tilde{\Psi} e_0 + L e^{u^*} + p\xi\left(e^{u^*}\right) + f_{\bar{e}}\left(e^{u^*}\right) = g\left(e^{u^*}\right).$$

Then e^{u^*} is a fixed point of the mapping of g, moreover, we can write

$$u^* = H^\dagger \xi\left(e^{u^*}\right) + \left(u^* - H^\dagger \xi\left(e^{u^*}\right)\right)$$

which implies that

$$H\left(u^* - H^\dagger \xi\left(e^{u^*}\right)\right) = Hu^* - HH^\dagger \xi\left(e^{u^*}\right) = 0$$

consequently

$$\left(u^* - H^\dagger \xi\left(e^{u^*}\right)\right) \in \text{Ker}(H)$$

and finally we have

$$\mathcal{U}_{ad} \subset \bigcup_{e \in P_g} H^\dagger \xi(e) + \text{Ker}(H).$$

□

Remark 5 The fixed points of g are independent of the choice of the projection p and the element $\bar{e}$. Indeed, let p_1 and p_2 two projection on $Im\ H$ and $\bar{e}_1$ and $\bar{e}_2$ two any elements not equal to zero of $Im\ H$. Let's consider the applications

$$g_1 : T \longrightarrow T,$$
$$e \longrightarrow g_1(e) = \tilde{\Psi} e_0 + Le + p_1 \xi(e) + f_{\bar{e}_1}(e),$$
$$g_2 : T \longrightarrow T,$$
$$e \longrightarrow g_2(e) = \tilde{\Psi} e_0 + Le + p_2 \xi(e) + f_{\bar{e}_2}(e).$$

Let e a fixed point of g_1, by proof of Proposition 3, we have $\Gamma e_N = y^d$ and $\xi(e) \in Im\ H$, he result that $p_2 \xi(e) = \xi(e)$ and $f_{\bar{e}_1}(e) = 0$, then

$$g_2(e) = \tilde{\Psi} e_0 + Le + \xi(e) = e.$$

What shows that e is a fixed point of g_2. By symmetry, it clear that the fixed points of g_2 are also the fixed points of g_1.

4.2 Problem of minimization

By the above proposition, we can characterize the set of admissible control $\mathcal{U}_{ad}$, among those controls, we allow to determine those with the minimal norm, i.e., we solve the following problem:

$$\bar{\mathcal{P}} : \min_{u \in \mathcal{U}_{ad}} \left(J(u) = \| u \|^2\right).$$

If we suppose that P_g is fini, i.e., $P_g = \{e^1, \ldots, e^q\}$, we have

$$\mathcal{U}_{ad} = \bigcup_{i=1}^q H^\dagger \xi(e^i) + \text{Ker}(H) = \bigcup_{i=1}^q \mathcal{U}_{ad}^i$$

where

$$\mathcal{U}_{ad}^i = H^\dagger \xi(e^i) + \text{Ker}(H)$$

then, we obtain

$$\bar{\mathcal{P}} \Longleftrightarrow \min_{1 \le i \le q} \left(\min_{u \in \mathcal{U}_{ad}^i} \left(J(u) = \| u \|^2\right)\right). \quad (13)$$

Remark 6 Let $u \in \mathcal{U}_{ad}^i$ then $u = H^\dagger \xi(e^i) + v$ with $v \in \text{Ker}(H)$. Thus

$$\begin{aligned}
\| u \|^2 &= <u, u> = <H^\dagger \xi(e^i) + v, H^\dagger \xi(e_i) + v> \\
&= \| H^\dagger \xi(e^i) \|^2 + 2 <H^\dagger \xi(e^i), v> + \| v \|^2 \\
&= \| H^\dagger \xi(e^i) \|^2 + J_i(v)
\end{aligned}$$

finally we have

$$J(u) = \| H^\dagger \xi(e^i) \|^2 + J_i(v),$$

with

$$J_i(v) = 2 <H^\dagger \xi(e^i), v> + \| v \|^2. \quad (14)$$

Lemma 1 *The two following problems are equivalents*

(a) $\begin{cases} \min\limits_{u \in \mathcal{U}_{ad}^i} J(u) = \| u^* \|^2 \\ \text{with } u^* = H^\dagger \xi(e^i) + v^* \end{cases}$

(b) $\begin{cases} \min\limits_{v \in \text{Ker} H} J_i(v) = \| v^* \|^2 \\ \text{with } J_i(v) = 2 <H^\dagger \xi(e^i), v> + \| v \|^2 \end{cases}$

Proof (b) $\Longrightarrow$ (a) Let $w \in \mathcal{U}_{ad}^i$ which implies that $w = H^\dagger \xi(e^i) + \bar{w}$ with $\bar{w} \in \text{Ker}(H)$, then we have

$$J(w) = \| w \|^2 = \| H^\dagger \xi(e_i^*) \|^2 + J_i(\bar{w})$$

thus

$$J(w) \ge \| H^\dagger \xi(e_i^*) \|^2 + J_i(v^*)$$

consequently

$$J(w) \ge \| u^* \|^2 = J(u).$$

So, $\forall w \in \mathcal{U}_{ad}^i$, we have $J(w) \geq J(u^*)$ and

$$\min_{u \in \mathcal{U}_{ad}^i} J(w) = J(u^*) = \| u^* \|^2 .$$

(a) $\Longrightarrow$ (b) Let u^* such that $\| u^* \|^2 = \min_{u \in \mathcal{U}_{ad}^i} (J(u))$, or $\mathcal{U}_{ad}^i$ is closed, then we have $u^* \in \mathcal{U}_{ad}^i$ and there exists $v^* \in \text{Ker}(H)$ such that $u^* = H^\dagger \xi(e_i) + v^*$.

Let $w \in \text{Ker}(H)$ and consider $u = H^\dagger \xi(e_i^*) + w$ then we have

$$\| u \|^2 \geq \| u^* \|^2$$

$\| H^\dagger \xi(e_i^*) \|^2 + 2 < H^\dagger \xi(e_i^*), w > + \|w\|^2 \geq \| H^\dagger \xi(e_i^*) \|^2 + 2 < H^\dagger \xi(e_i^*), v^* > + \| v^* \|^2$

thus, we have

$$2 < H^\dagger \xi(e_i^*), w > + \| w \|^2 \geq 2 < H^\dagger \xi(e_i^*), v^* > + \| v^* \|^2$$

which implies that

$$J_i(w) \geq J_i(v^*), \quad \forall w \in \text{Ker}(H)$$

consequently

$$\min_{v \in \text{Ker} H} J_i(v) = J_i(v^*) = \|v^*\|.$$

$\square$

Theorem 1 *If we suppose that the set P_g is finite, then the optimal control allow to have the Γ-Controllability (then the exactly output controllability of system (S)) is given by*

$$u^* = H^\dagger \xi(e^{i_0}),$$

with e^{i_0} a fixed point of application g given by (10) and which verified

$$\| H^\dagger \xi(e^{i_0}) \|^2 = \inf_{1 \leq i \leq q} \left\{ \| H^\dagger \xi(e^i) \|^2 \right\}.$$

Proof Let's consider $P_g = \{e^1, \ldots, e^q\}$, then by Lemma 1, we have

$$\min_{u \in \mathcal{U}_{ad}} J(u) = \|u\|^2 \quad \text{with} \quad u^* = H^\dagger \xi(e^i) + v^*$$

where v^* is an element of H that achieves the minimum of the functional J given by (14), However, H is a closed subspace, then the minimum of J is reached for $v^* = 0$ and therefore, we have

$$\min_{u \in \mathcal{U}_{ad}} J(u) = \|H^\dagger \xi(e^i)\|^2.$$

While using the equivalence (13), we deduct that

$$\min_{u \in \mathcal{U}_{ad}} (J(u) = \|u\|^2) = \inf_{1 \leq i \leq q} \{\|H^\dagger \xi(e^i)\|^2\} = \|H^\dagger \xi(e^{i_0})\|^2$$

where $e^{i_0} \in \{e^{i_1}, \ldots, e^{i_q}\}$.

$\square$

5 Other application for characterize the set of admissible controls

In this section, a necessary and sufficient condition based on the set of fixed points of an other application appropriately chosen, for that a control is admissible. Indeed, let's consider the operator L, H, Ψ and Γ defined in preceding paragraph and we define the Hilbert spaces $\mathcal{M} = T \times \mathcal{Y}$ with $\mathcal{Y} = l^2(\sigma_1^N; Y)$ and the operators $\mathcal{S}$, $\mathcal{L}$, $\mathcal{H}$ and $\tilde{\xi}$ by

$$\mathcal{S} : \mathcal{X} \longrightarrow \mathcal{M}, \qquad \mathcal{L} \quad T \qquad \longrightarrow \mathcal{M},$$
$$\alpha \longmapsto \begin{pmatrix} \tilde{\Psi}\alpha \\ \Gamma\Psi^N\alpha \end{pmatrix}, \quad x = (x_i)_{i \in \sigma_1^N} \longmapsto \begin{pmatrix} Lx \\ \Gamma(Lx)_N \end{pmatrix},$$
$$\mathcal{H} : \mathcal{U} \longrightarrow \mathcal{M}, \qquad \tilde{\xi} \ \mathcal{M} \longrightarrow \mathcal{M},$$
$$u \longmapsto \begin{pmatrix} Hu \\ \Gamma(Hu)_N \end{pmatrix}, \quad \begin{pmatrix} x \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x \\ z \end{pmatrix} - \mathcal{S}e_0 - \mathcal{L}x.$$

We remind that the solution of system (5) is written in the form

$$e = (e_i)_{i \in \sigma_1^N} = \tilde{\Psi}e_0 + Le + Hu$$

which give

$$\begin{pmatrix} e \\ \Gamma e_N \end{pmatrix} = \mathcal{S}e_0 + \mathcal{L}e + \mathcal{H}u. \tag{15}$$

We suppose that $Im \ \mathcal{H}$ is closed, then the pseudo inverse $\mathcal{H}^\dagger$ of $\mathcal{H}$ is defined on all space $\mathcal{M}$. Let's y^d a fixed element of $\mathcal{Y}$, we define the following application

$$\mathcal{G} : \ \mathcal{M} \longrightarrow \mathcal{M}$$
$$\begin{pmatrix} x \\ z \end{pmatrix} \longmapsto \mathcal{G}\begin{pmatrix} x \\ z \end{pmatrix} = \mathcal{S}e_0 + \mathcal{L}x + p\tilde{\xi}\begin{pmatrix} x \\ y^d \end{pmatrix}$$
$$+ \begin{pmatrix} 0_{\mathbb{R}^N} \\ z - y^d \end{pmatrix}. \tag{16}$$

with $p : T \longrightarrow Im \ \mathcal{H}$ an any projection on $Im(\mathcal{H})$. Then we have the following result.

Lemma 2 *If $\begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{M}$ is a fixed point of $\mathcal{G}$, then $\tilde{\xi}\begin{pmatrix} x \\ y^d \end{pmatrix} \in Im \ \mathcal{H}$.*

Proof If $\begin{pmatrix} x \\ z \end{pmatrix}$ is a fixed point of $\mathcal{G}$, then we have

$$\mathcal{G}\begin{pmatrix} x \\ z \end{pmatrix} = \mathcal{S}e_0 + \mathcal{L}x + p\tilde{\xi}\begin{pmatrix} x \\ y^d \end{pmatrix} + \begin{pmatrix} 0_{\mathbb{R}^N} \\ z - y^d \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix},$$

which implies

$$p\tilde{\xi}\begin{pmatrix} x \\ y^d \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix} - \mathcal{S}e_0 - \mathcal{L}x - \begin{pmatrix} 0_{\mathbb{R}^N} \\ z - y^d \end{pmatrix}$$
$$= \begin{pmatrix} x \\ y^d \end{pmatrix} - \mathcal{S}e_0 - \mathcal{L}x$$
$$= \tilde{\xi}\begin{pmatrix} x \\ y^d \end{pmatrix}$$

which show that $\tilde{\xi}\begin{pmatrix} x \\ y^d \end{pmatrix} \in Im\ \mathcal{H}$. □

Theorem 2 *Let's y^d a desired output, the control $u^* = \mathcal{H}^\dagger\tilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix} + v^*$ ensure the Γ-controllability of system (5) where $x^* \in T$ and $v^* \in Ker\ \mathcal{H}$ if and only if (x^*, y^d) is a fixed point of application $\mathcal{G}$ given by (16).*

Proof Let's consider $x^* \in T$ and $v^* \in Ker\ \mathcal{H}$. If we suppose that $u^* = \mathcal{H}^\dagger\tilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix} + v^* \in \mathcal{U}$ ensure the Γ-controllability of system (S_1), i.e., $\Gamma e_N^{u^*} = y^d$ where $e = (e_1^*, \dots, e_N^*)$ is the trajectory of system (5) corresponding to the control u^*. Then, by Eq. (15), we have

$$\mathcal{H}u^* = \begin{pmatrix} x^* \\ \Gamma e^{u^*} \end{pmatrix} - \mathcal{S}e_0 - \mathcal{L}x^* = \tilde{\xi}\begin{pmatrix} x^* \\ \Gamma e^{u^*} \end{pmatrix} = \tilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix}$$

which implies that $\tilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix} \in Im\ \mathcal{H}$, consequently

$$p\tilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix} = \tilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix}.$$

So, we have

$$\mathcal{G}\begin{pmatrix} x^* \\ y^d \end{pmatrix} = \mathcal{S}e_0 + \mathcal{L}x^* + \tilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix} = \begin{pmatrix} x^* \\ y^d \end{pmatrix}$$

which carries that $(x^*, y^d)^\top$ is a fixed point of $\mathcal{G}$.

Now, if we suppose that $(x^*, y^d)^\top$ is a fixed point of $\mathcal{G}$, then by Lemma 2, we have $\tilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix} \in Im\ \mathcal{H}$. On other hand, we have

$$\begin{pmatrix} x^* \\ y^d \end{pmatrix} = \mathcal{S}e_0 + \mathcal{L}x^* + \tilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix}$$

Let $v^* \in Ker\ \mathcal{H}$, then if we consider the control $u^* = \mathcal{H}^\dagger\tilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix} + v^*$, we have $\mathcal{H}u^* = \tilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix}$ and

$$\begin{pmatrix} x^* \\ y^d \end{pmatrix} = \mathcal{S}e_0 + \mathcal{L}x^* + \mathcal{H}u^*.$$

If we replace $\mathcal{S}$, $\mathcal{L}$ and $\mathcal{H}$ by their expression, we obtain

$$\begin{pmatrix} x^* \\ y^d \end{pmatrix} = \begin{pmatrix} \tilde{\Phi}e_0 + Lx^* + Hu^* \\ \Gamma\Phi^N e_0 + \Gamma(Lx^*)_N + \Gamma(Hu^*)_N. \end{pmatrix}$$

So, $x^* = \tilde{\Phi}e_0 + Lx^* + Hu^*$ implies that x^* is the trajectory of system (5) corresponding to control u^* and

$$y^d = \Gamma(\Phi^N e_0 + (Lx^*)_N + (Hu^*)_N)$$
$$= \Gamma(x_N^*).$$

Consequently the control $u^* = \mathcal{H}^\dagger\tilde{\xi}\begin{pmatrix} x^* \\ y^d \end{pmatrix} + v^*$ ensure the Γ-controllability of system (5), which end the proof. □

6 Example and numerical simulation

Consider the following system

$$\begin{cases} \dot{x}(t) = \Delta x(t) + Mx(t) + Du(t), & 0 < t < 1 \\ x(0) = 0, \end{cases} \quad (17)$$

the output function is given by

$$y(t) = < x(t), \quad \phi_1 > \in \mathbb{R} \quad (18)$$

where $x(t) \in X = L^2(0, 1; \mathbb{R})$, $u(t) \in U = \mathbb{R}$, D and M are respectively the linear and the nonlinear maps defined by

$$D : \mathbb{R} \longrightarrow X$$
$$z \longrightarrow z\phi_1$$

and

$$M : X \longrightarrow X$$
$$x \longrightarrow sin(< x, \phi_1 >)\phi_1$$

where $\phi_n = \sqrt{2}\sin(n\pi.)$, $n \geq 1$. The laplacien Δ is the infinitesimal generator of the strongly-continuous semi-group $(S(t))_{t\geq 0}$ defined by

$$S(t) = \sum_{n=1}^{\infty} e^{-n^2\pi t} < x, \phi_n > \phi_n.$$

The operator M satisfied the Lipshitz condition. Indeed, for all $x, y \in X$, we have

$$\|M(x) - M(y)\| = \|(sin(x_1) - sin(y_1))\phi_1\|$$
$$\leq |sin(x_1) - sin(y_1)|$$
$$\leq 2\left|sin\left(\frac{x_1 - y_1}{2}\right)\right|\left|cos\left(\frac{x_1 + y_1}{2}\right)\right|$$
$$\leq |x_1 - y_1| \leq \|x - y\|.$$

where $x_1 = < x, \phi_1 >$ and $y_1 = < y, \phi_1 >$.

Consequently, the system (17) has a unique mild solution in $L^2(0, 1; X)$ (see Balakrishnan [23]) given by

$$x(t) = \int_0^t S(t-r)Mx(r)dr + \int_0^t S(t-r)Du(r)dr.$$

Let $N \in \mathbb{N}$, $\delta = \frac{1}{N}$ the sampling data, $t_i = i\delta$, $\forall i \in \mathbb{N}$, $A = S(\delta)$, $x_i = x(t_i)$, $u_i = u(t_i)$ and $y_i = y(t_i)$, the discrete version of system (17), (18) is the following

$$\begin{cases} x_{i+1} = Ax_i + Ex_i + Bu_i, & i \geq 0 \\ x_0 = 0, \end{cases} \tag{19}$$

$$y_i = Cx_i, \quad i \geq 0 \tag{20}$$

where E, B, and C are given by
$$E = \int_0^\delta S(r)Mdr, \quad B = \int_0^\delta S(r)Ddr \text{ and}$$
$$Cx = <x, \phi_1 >, \quad \forall x \in X.$$

By a direct calcul, we can verifies that the operator H, L and ξ, the sets $Ker\, H$, $(Ker\, H)^\top$ and $Im\, H$ are given by

$$\begin{cases} H : u \in l^2\left(\sigma_0^{N-1}, \mathbb{R}\right) \longrightarrow Hu \in l^2\left(\sigma_1^N, \mathcal{X}\right) \\ (Hu)_i = \dfrac{1-e^{-\pi^2\delta}}{\pi^2}\left(\left(\underbrace{0, \ldots, 0}_{(N-i+1)-times}, \bar{u}_0, \ldots, \bar{u}_{i-2}\right)\right) \end{cases}$$

where $\bar{u}_i = \sum_{j=0}^i e^{-j\pi^2\delta}u_{i-j}\phi_1$ and $i \in \sigma_1^N$,

$$\begin{cases} L : e \in l^2\left(\sigma_1^N, \mathcal{X}\right) \longrightarrow Le \in l^2\left(\sigma_1^N, \mathcal{X}\right) \\ (Le)_i = \left(\dfrac{1-e^{-\pi^2\delta}}{\pi^2}\right)\left(\left(\underbrace{0, \ldots, 0}_{(N-i+1)-times}, \bar{x}_0, \ldots, \bar{x}_{i-2}\right)\right) \end{cases}$$

where $\bar{x}_i = \sum_{j=0}^i e^{-j\pi^2\delta}\sin(<x_{i-j}, \phi_1>)\phi_1$ and $x_i = Ge_i$, $i \in \sigma_1^N$ with

$$\begin{cases} G : \quad \mathcal{X} \quad \longrightarrow X \\ \quad \left(\begin{matrix} z \\ (z_{-N}, \ldots, z_{-1}) \end{matrix}\right) \longrightarrow z \\ \xi : e \in l^2\left(\sigma_1^N, \mathcal{X}\right) \longrightarrow \xi e \in l^2\left(\sigma_1^N, \mathcal{X}\right) \\ (\xi e)_i = e_i - \dfrac{1-e^{-\pi^2\delta}}{\pi^2}\left(\left(\underbrace{0, \ldots, 0}_{(N-i+1)-times}, \bar{x}_0, \ldots, \bar{x}_{i-2}\right)\right) \end{cases}$$

$Ker\, H = \{0\}$, $(Ker\, H)^\top = \mathcal{U} = l^2\left(\sigma_0^{N-1}; \mathbb{R}\right)$

and

$$Im\, H = \left\{z = (z_i)_{i\in\sigma_1^N} \in l^2\left(\sigma_1^N, \mathcal{X}\right), \; z_i = \left(\left(\underbrace{0, \ldots, 0}_{(N-i+1)-times}, \alpha_1\phi_1, \ldots, \alpha_{i-1}\phi_1\right)\right), i \in \sigma_1^N, \alpha_i \in \mathbb{R}\right\}$$

Let $\tilde{H}$ the operator defined by

$$\tilde{H} : (Ker\, H)^\top = \mathcal{U} \longrightarrow Im\, H$$
$$v \longrightarrow \tilde{H}v = Hv$$

$$\tilde{H}^{-1} : \quad Im\, H \longrightarrow \mathcal{U}$$
$$\left(\left(\underbrace{0, \ldots, 0}_{(N-i+1)-times}, \alpha_1\phi_1, \ldots, \alpha_{i-1}\phi_1\right)\right) \longrightarrow v$$

where

$$\begin{cases} v_0 = \dfrac{\pi^2}{1-e^{-\pi^2\delta}}\alpha_1 \\ v_i = \dfrac{\pi^2}{1-e^{-\pi^2\delta}}[\alpha_{i+1} - e^{-\pi^2\delta}\alpha_i], \quad i \in \sigma_1^{N-1}. \end{cases}$$

Lets $\bar{e} = (b_1, 0, \ldots, 0)^\top$ where $b_1 = \left(\begin{matrix} \phi_1 \\ (0, \ldots, 0) \end{matrix}\right)$ and the projection P

$$P : \quad l^2\left(\sigma_1^N, \mathcal{X}\right) \longrightarrow Im\, H$$
$$z = (z_1, \ldots, z_N) \longrightarrow Pz$$

$$Pz = \left(\left(\underbrace{0, \ldots, 0}_{(N-i+1)-times}, \bar{z}_1, \ldots, \bar{z}_{i-1}\right)\right)_{i\in\sigma_1^N}$$

with $\bar{z}_i = <z_i^1, \phi_1 > \phi_1$ and $z_i^1 = Gz_i$, $i \in \sigma_1^N$.

The map H is given by $H : e \in l^2(\sigma_1^N, \mathcal{X}) \longrightarrow He \in l^2(\sigma_1^N, \mathcal{X})$, where, for every $i \in \sigma_1^N$, we have

$$(He)_i = (f_{\bar{e}}(e))_i + (Le)_i + (P\xi(e))_i$$
$$= (f_{\bar{e}}(e))_i + \left(\left(\underbrace{0, \ldots, 0}_{(N-i+1)-times}, \bar{x}_1, \ldots, \bar{x}_{i-k-2}\right)\right).$$

where $\bar{x}_i = <x_i, \phi_1 > \phi_1$.

Let e a fixe point of H, we have

$$e_i = (f_{\bar{e}}(e))_i + \left(\left(\underbrace{0, \ldots, 0}_{(N-i+1)-times}, \bar{x}_1, \ldots, \bar{x}_{i-k-2}\right)\right)$$

where $x_i = Ge_i$. As $(f_{\bar{e}}(e))_i$, $i \in \sigma_2^N$, we show that $(f_{\bar{e}}(e))_1 = 0$. Indeed, if we suppose that $(f_{\bar{e}}(e))_1 \neq 0$, then

$$(f_{\bar{e}}(e))_1 = \bar{e}_1 = b_1 = \begin{pmatrix} \phi_1 \\ (0, \ldots, 0) \end{pmatrix} \text{ thus}$$

$$< e_i, b_1 >_{X \times l^2(\sigma_{-N}^{-1}, \mathcal{X})} = \|b_1\|^2_{X \times l^2(\sigma_{-N}^{-1}, \mathcal{X})} +$$

$$< \left(\begin{pmatrix} \bar{x}_i \\ \underbrace{0, \ldots, 0}_{(N-i+1)-times}, \bar{x}_1, \ldots, \bar{x}_{i-k-2} \end{pmatrix} \right), b_1 >$$

which implies that

$$< x_i, \phi_1 > = 1 + < x_i, \phi_1 >$$

which is absurd, then $f_{\bar{e}}(e) = 0$ and

$$e_i = \left(\begin{pmatrix} \bar{x}_i \\ \underbrace{0, \ldots, 0}_{(N-i+1)-times}, \bar{x}_1, \ldots, \bar{x}_{i-k-2} \end{pmatrix} \right), \quad \forall i \in \sigma_1^N.$$

$$\Gamma e_N = y^d \iff (< x_1, \phi_1 >, \ldots, < x_N, \phi_1 >) = \\ (y_1^d, \ldots, y_N^d) \\ \iff < x_i, \phi_1 > = y_i^d, \quad i \in \sigma_1^N.$$

Consequently

$$e_i = \left(\begin{pmatrix} y_i^d \phi_1 \\ \underbrace{0, \ldots, 0}_{(N-i+1)-times}, y_1^d \phi_1, \ldots, y_{i-1}^d \phi_1 \end{pmatrix} \right), \quad \forall i \in \sigma_1^N.$$

$$(\xi(e))_i = \left(\begin{pmatrix} y_i^d \phi_1 \\ \underbrace{0, \ldots, 0}_{(N-i+1)-times}, y_1^d \phi_1, \ldots, y_{i-1}^d \phi_1 \end{pmatrix} \right) \\ + \left(\frac{1 - e^{-\pi^2 \delta}}{\pi^2} \right) \left(\begin{pmatrix} \bar{y}_{i-1}^d \\ \underbrace{0, \ldots, 0}_{(N-i+2)-times}, \bar{y}_1^d, \ldots, \bar{y}_{i-2}^d \end{pmatrix} \right)$$

where $\bar{y}_i^d = \sum_{k=0}^{i} e^{-k\pi^2 \delta} y_{i-k}^d \phi_1$ and $\forall i \in \sigma_1^N$.

$$(H^\dagger(\xi(e)))_i = \left(\frac{\pi^2}{1 - e^{-\pi^2 \delta}} \right) \left(y_{i+1}^d - e^{-\pi^2 \delta} y_i^d \right) - \sin \left(y_i^d \right)$$

Consequently

$$\mathcal{U}_{ad} = \left\{ \left(\frac{\pi^2}{1 - e^{-\pi^2 \delta}} \left(y_{i+1}^d - e^{-\pi^2 \delta} y_i^d \right) - \sin \left(y_i^d \right) \right)_{i \in \sigma_0^{N-1}} \right\}$$

Table 1 The values of the exact control and the approximate control

t	Exact control	Approx. control	Error
0.0	0	0	0
0.1	0.36403	0.36403	0
0.2	0.75343	0.74976	3.66591E−03
0.3	1.15265	1.14353	9.12257E−03
0.4	1.54579	1.53043	1.53593E−02
0.5	1.91716	1.89543	2.17311E−02
0.6	2.25198	2.22420	2.77782E−02
0.7	2.53688	2.50371	3.31711E−02
0.8	2.76051	2.72281	3.76965E−02
0.9	2.91395	2.87270	4.12458E−02

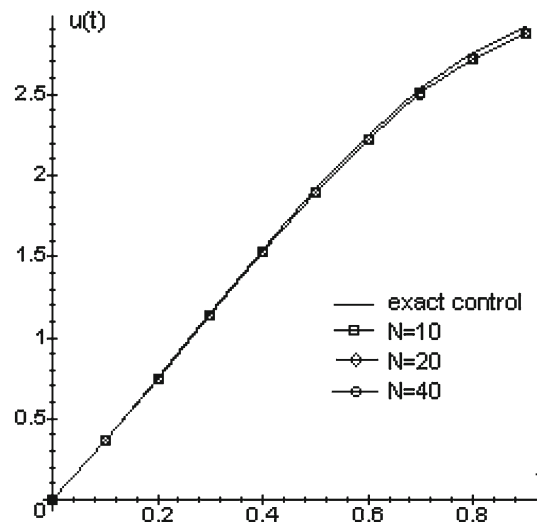**Fig. 1** Graph of the control for different values of N

Numerical simulation For

$$y_i^d = \left(\frac{1 - e^{-\pi^2 \delta}}{\pi^2} \right) \sum_{j=0}^{i-1} \left(2 \sin \left(2j\delta - \frac{\pi}{6} \right) + 1 \right) \left(e^{-\pi^2 \delta} \right. \\ \left. + \left(\frac{1 - e^{-\pi^2 \delta}}{\pi^2} \right) \sin(1) \right)^{i-j-1}, \quad \forall i \in \sigma_1^N,$$

we obtain the numerical results describes in Table 1.

The corresponding optimal cost is $J(u^*) = 0.34819$. Some control trajectories for different values of N are depicted in Fig. 1.

7 Conclusion

In this paper, we investigate the output controllability problem for nonlinear discrete distributed system with energy constraint. We use a technique based on the fixed point theorem and we establish that the set of admissible controls can be characterized by the set of the fixed point of an appropri-

ate mapping. A numerical example is given to illustrate the obtained results.

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