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PI controller tuning for load disturbance rejection using constrained optimization

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Abstract In this paper, a simple and effective PI controller tuning method is presented. To take both performance requirements and robustness issues into consideration, the design technique is based on optimization of load disturbance rejection with a constraint either on the gain margin or phase margin. In addition, a simplified form of the resulting tuning formulae is obtained for first order plus dead time models. To demonstrate the ability of the proposed tuning technique in dealing with a wide range of plants, simulation results for several examples, including integrating, non-minimum phase and long dead time models, are provided.

Keywords Constrained optimization · Gain margin · Load disturbance rejection · Phase margin · PI control

List of symbols

A_m	Desired gain margin
d	Load disturbance signal
FOPDT	First order plus dead time
GM	Gain margin
$G_c(s)$	Controller transfer function
$G_p(s)$	Plant transfer function
IAE	Integral of absolute error
IE	Integral of error
K_c	Proportional gain
K_i	Integral gain
K_p	Gain of FOPDT model
$L(s)$	Loop transfer function
PID	Proportional–integral–derivative
PM	Phase margin
ϕ_m	Desired phase margin
r	Reference signal
SGM	Specified gain margin
SPM	Specified phase margin
T	Time constant of FOPDT model
T_i	Integral time
τ_d	Time delay of FOPDT model
ω	Frequency
y	Output signal

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1 Introduction

In spite of the recent advances in control theory, PID controller is the most widespread form of feedback compensation. This is mainly due to its noticeable effectiveness and simple structure that is conceptually easy to understand. PID is a simple and useful controller, which gives a powerful solution to the control of a huge number of industrial plants. According to the literature, more than 95% of industrial controllers are PID controllers [1–5]. The key reason for this popularity is that a well-designed PID controller can meet most control requirements [6]. In fact, most of the industrial controllers are PI because the derivative action is very often not used. As a result, good PI tuning methods are extremely desirable due to their widespread use.

Since the 1940s, a large number of analytical and numerical methods, which are usually different in complexity and flexibility, have been proposed for tuning of PID controllers

[7–13]. In addition, several well-known control books have chapters on tuning PID controllers [14–17].

Generally, an efficient design method should satisfy the design specifications and be able to deal with a wide range of plants. A satisfactory load disturbance response is often the first goal in control applications. This paper presents a PI tuning method resulting in a set of tuning formulae. To consider performance and robustness requirements, the design objective is the optimization of load disturbance rejection with a constraint either on the gain margin (GM) or the phase margin (PM). As the first order plus dead time (FOPDT) models can approximately model a huge number of industrial plants, the resulting tuning formulae are then applied to these plants to obtain a simple set of tuning formulae.

The paper is organised as follows. In Sect. 2, an analytical method to determine the optimal parameters of PI controllers in terms of minimizing an objective function and satisfying a GM or PM constraint is developed. The method is applied to FOPDT plants in Sect. 3. In Sect. 4, the simplified tuning formulae for FOPDT plants are presented, using dimensional analysis and curve-fitting techniques. In Sect. 5, the resulting tuning formulae are applied to a variety of control examples. Moreover, a comparison between the performance of the proposed tuning formulae and that of one of the most prevalent design methods is given for each example. Finally, the conclusions of the whole study are drawn.

2 Theory

The plant, $G_p(s)$, is controlled by the PI controller in Eq. (1).

$$G_c(s) = K_c + \frac{K_i}{s}. \tag{1}$$

where K_c and K_i are proportional and integral gains, respectively. The aim of control is to reject load disturbance signals, which are the most common and most important disturbances in control that drive systems away from their desired operating points [3]. The output signal of a closed-loop system in the presence of an input load disturbance signal is given by Eq. (2).

$$y = \frac{G_p G_c}{1 + G_p G_c} r + \frac{G_p}{1 + G_p G_c} d. \tag{2}$$

where r, d and y refer to the reference, load disturbance and output signals, respectively. Step disturbances are applied at the input to the plant. A commonly chosen performance metric is the integral of absolute error (IAE). A significant drawback of this criterion is that it is not suitable for analytical approaches, as the evaluation requires computation of time functions [3]. However, the IAE is equivalent to the integral of error (IE) if the error signal is positive. Moreover, the IE may be a good approximation for the IAE for well-damped closed-loop systems. The reason for using IE is that it is appropriate for analytical approaches as its value is directly related to the integral gain, as shown in Eq. (3) [3].

$$IE = \frac{1}{K_i}. \tag{3}$$

In addition, robustness is a key issue in control systems. It is well known that GM and PM are used as measures of robustness. In order to ensure the robustness of the closed-loop system, the optimization problem is constrained so that a desired GM or a required PM is guaranteed. Moreover, the PM acts as a measure of performance as it is related to the damping of the system [18]. Therefore, the design objective is to maximize K_i subject to satisfying the robustness constraint.

2.1 Tuning formulae for a constraint on GM

Assume that the model of the plant is given by Eq. (4).

$$G_p(j\omega) = \alpha(\omega) + j\beta(\omega). \tag{4}$$

where $\alpha(\omega)$ and $\beta(\omega)$ are real and imaginary parts of the plant. The loop transfer function is then written as shown in Eq. (5).

$$L(j\omega) = (\alpha(\omega) + j\beta(\omega)) \left(K_c - j\frac{K_i}{\omega} \right). \tag{5}$$

In order to determine the controller parameters that obtain a desired GM, Eqs. (6) and (7) should be solved.

$$\mathrm{Im}[L(j\omega)] = 0. \tag{6}$$

$$\mathrm{Re}[L(j\omega)] = \frac{-1}{A_m}. \tag{7}$$

where A_m is the value of the desired GM. Inserting Eq. (5) in Eqs. (6) and (7) results in the controller parameters given by Eqs. (8) and (9).

$$K_c = \frac{-\alpha(\omega)}{A_m(\alpha^2(\omega) + \beta^2(\omega))}. \tag{8}$$

$$K_i = \frac{-\omega\beta(\omega)}{A_m(\alpha^2(\omega) + \beta^2(\omega))}. \tag{9}$$

The necessary and sufficient conditions for maximizing K_i and satisfying the GM constraint are given by Eqs. (10) and

(11), respectively.

$$\frac{dK_i}{d\omega} = 0. \tag{10}$$

$$\frac{d^2K_i}{d\omega^2} < 0. \tag{11}$$

Equation (9) can be written as shown in Eq. (12).

$$K_i = \omega f(\omega). \tag{12}$$

where $f(\omega)$ is given by Eq. (13).

$$f(\omega) = \frac{-\beta(\omega)}{A_m(\alpha^2(\omega) + \beta^2(\omega))}. \tag{13}$$

Inserting Eq. (12) into Eq. (10) gives the necessary condition shown in Eq. (14).

$$\frac{dK_i}{d\omega} = f(\omega) + \omega f'(\omega) = 0. \tag{14}$$

where $f'(\omega)$ is the derivative of $f(\omega)$ with respect to ω. ω can be determined by inserting $f(\omega)$ from Eq. (13) and $f'(\omega)$ into Eq. (14), resulting in Eq. (15).

$$\omega = \frac{1}{2\frac{\alpha(\omega)\alpha'(\omega)+\beta(\omega)\beta'(\omega)}{\alpha^2(\omega)+\beta^2(\omega)} - \frac{\beta'(\omega)}{\beta(\omega)}}. \tag{15}$$

where $\alpha'(\omega)$ and $\beta'(\omega)$ are the derivatives of $\alpha(\omega)$ and $\beta(\omega)$ with respect to ω, respectively. Inserting Eq. (14) into Eq. (11), the sufficient condition is obtained as shown in Eq. (16).

$$\frac{d^2K_i}{d\omega^2} = -2f(\omega) + \omega^2 f''(\omega) < 0. \tag{16}$$

The maximizing ω is given by Eq. (15) subject to satisfying Eq. (16). The optimal controller parameters are given by inserting the maximizing ω into Eqs. (8) and (9). This analytical tuning method is referred to as specified gain margin (SGM) because the closed-loop system satisfies a desired GM. An iterative technique, such as the Newton–Raphson method, is required to solve Eq. (15).

2.2 Tuning formulae for a constraint on PM

Assuming the loop transfer function in Eq. (5), Eqs. (17) and (18) should be solved to determine the controller parameters that obtain a desired PM.

$$|L(j\omega)| = 1. \tag{17}$$

$$\pi + \angle L(j\omega) = \phi_m. \tag{18}$$

where ϕ_m is the value of the desired PM. Inserting Eq. (5) into Eqs. (17) and (18) results in Eqs. (19) and (20).

$$K_c^2 + \frac{K_i^2}{\omega^2} = \frac{1}{\alpha^2(\omega) + \beta^2(\omega)}. \tag{19}$$

$$\pi + \tan^{-1}\left(\frac{\beta(\omega)}{\alpha(\omega)}\right) - \tan^{-1}\left(\frac{K_i}{\omega K_c}\right) = \phi_m. \tag{20}$$

Equation (20) can be written as shown in Eq. (21).

$$\omega T_i = \frac{\alpha(\omega)\cos(\phi_m) + \beta(\omega)\sin(\phi_m)}{-\alpha(\omega)\sin(\phi_m) + \beta(\omega)\cos(\phi_m)}. \tag{21}$$

where T_i is given by Eq. (22).

$$T_i = \frac{K_c}{K_i}. \tag{22}$$

Considering Eqs. (19), (21) and (22), PI parameters can be written as shown in Eqs. (23) and (24).

$$K_c = -\frac{\alpha(\omega)\cos(\phi_m) + \beta(\omega)\sin(\phi_m)}{\alpha^2(\omega) + \beta^2(\omega)}. \tag{23}$$

$$K_i = \omega\frac{\alpha(\omega)\sin(\phi_m) - \beta(\omega)\cos(\phi_m)}{\alpha^2(\omega) + \beta^2(\omega)}. \tag{24}$$

Writing Eq. (24) in the form of Eq. (12) with $f(\omega)$ shown in Eq. (25)

$$f(\omega) = \frac{\alpha(\omega)\sin(\phi_m) - \beta(\omega)\cos(\phi_m)}{\alpha^2(\omega) + \beta^2(\omega)}. \tag{25}$$

and applying the necessary condition for maximizing K_i, represented in Eq. (14), to Eq. (25) results in Eq. (26).

$$\omega = \frac{1}{2\frac{\alpha(\omega)\alpha'(\omega)+\beta(\omega)\beta'(\omega)}{\alpha^2(\omega)+\beta^2(\omega)} - \frac{\alpha'(\omega)\sin(\phi_m)-\beta'(\omega)\cos(\phi_m)}{\alpha(\omega)\sin(\phi_m)-\beta(\omega)\cos(\phi_m)}}. \tag{26}$$

whereas the sufficient condition is again given by Eq. (16). If the maximizing ω given by Eq. (26) satisfies Eq. (16), the optimal PI parameters are given by Eqs. (23) and (24). This tuning method is referred to as specified phase margin (SPM).

3 Tuning formulae for FOPDT plants

In this section, the SGM and SPM methods are applied to an important category of industrial plants and simplified versions of Eqs. (8), (9), (15), (23), (24) and (26) are presented.

3.1 SGM tuning formulae for FOPDT plants

A huge number of industrial plants can be modelled by a FOPDT model, shown in Eq. (27).

$$G_p(s) = \frac{K_p e^{-\tau_d s}}{Ts+1}. \tag{27}$$

To design PI controllers for this class of plants, the SGM design method is applied to the FOPDT models. The real and imaginary parts of the plant are given by Eqs. (28) and (29).

$$\alpha(\omega) = \frac{K_p}{1+\omega^2T^2}(\cos(\omega\tau_d) - \omega T\,\sin(\omega\tau_d)). \tag{28}$$

$$\beta(\omega) = \frac{-K_p}{1+\omega^2T^2}(\sin(\omega\tau_d) + \omega T\,\cos(\omega\tau_d)). \tag{29}$$

Inserting Eqs. (28) and (29) into Eqs. (8), (9) and (13) results in Eqs. (30)–(32).

$$K_c = \frac{-\cos(\omega\tau_d) + \omega T\,\sin(\omega\tau_d)}{A_m K_p}. \tag{30}$$

$$K_i = \frac{\omega(\sin(\omega\tau_d) + \omega T\,\cos(\omega\tau_d))}{A_m K_p}. \tag{31}$$

$$f(\omega) = \frac{\sin(\omega\tau_d) + \omega T\,\cos(\omega\tau_d)}{A_m K_p}. \tag{32}$$

Maximizing ω shown in Eq. (33) is given by inserting $f(\omega)$ from Eq. (32) and $f'(\omega)$ into Eq. (14).

$$\omega = \frac{\sin(\omega\tau_d) + \omega T\,\cos(\omega\tau_d)}{-(T+\tau_d)\cos(\omega\tau_d) + \omega T\tau_d\,\sin(\omega\tau_d)}. \tag{33}$$

The sufficient condition for maximizing K_i, shown in Eq. (34), is determined by inserting $f(\omega)$ and $f''(\omega)$ into Eq. (16).

$$A\,\sin(\omega\tau_d) + B\,\cos(\omega\tau_d) > 0. \tag{34}$$

where A and B are given by Eqs. (35) and (36).

$$A = 2 + \omega^2\tau_d(2T + \tau_d). \tag{35}$$

$$B = \omega T(2 + \omega^2\tau_d^2). \tag{36}$$

Finding $\cos(\omega\tau_d)$ from Eq. (33) and substituting it into Eq. (34), the sufficient condition is given by Eq. (37).

$$C\,\sin(\omega\tau_d) > 0. \tag{37}$$

where C is given by Eq. (38).

$$C = (2 + \omega^2\tau_d^2)\frac{\tau_d(1+\omega^2T^2) + T}{2T + \tau_d} + 2\omega^2T\tau_d. \tag{38}$$

Obviously $C > 0$ and it can easily be investigated that $\omega\tau_d < \pi$ holds for the SGM method. As a result, the sufficient condition is always satisfied.

3.2 SPM tuning formulae for FOPDT plants

Substituting Eqs. (28) and (29) into Eqs. (23), (24) and (25) results in Eqs. (39)–(41).

$$K_c = \frac{-\cos(\omega\tau_d + \phi_m) + \omega T\,\sin(\omega\tau_d + \phi_m)}{K_p}. \tag{39}$$

$$K_i = \frac{\omega(\sin(\omega\tau_d + \phi_m) + \omega T\,\cos(\omega\tau_d + \phi_m))}{K_p}. \tag{40}$$

$$f(\omega) = \frac{\sin(\omega\tau_d + \phi_m) + \omega T\,\cos(\omega\tau_d + \phi_m)}{K_p}. \tag{41}$$

Maximizing ω shown in Eq. (42) is given by inserting $f(\omega)$ from Eq. (41) and $f'(\omega)$ into Eq. (14).

$$\omega = \frac{\sin(\omega\tau_d + \phi_m) + \omega T\,\cos(\omega\tau_d + \phi_m)}{-(T+\tau_d)\cos(\omega\tau_d + \phi_m) + \omega T\tau_d\,\sin(\omega\tau_d + \phi_m)}. \tag{42}$$

Inserting $f(\omega)$ and $f''(\omega)$ into Eq. (16), results in the sufficient condition shown in Eq. (43).

$$C\,\sin(\omega\tau_d + \phi_m) > 0. \tag{43}$$

$\omega\tau_d + \phi_m < \pi$ holds for the SPM method, therefore, the sufficient condition is always satisfied.

4 Simplified tuning formulae for FOPDT models

Although simpler versions of Eqs. (15) and (26) for FOPDT plants are presented in Eqs. (33) and (42), an iterative method is still required to solve these nonlinear equations. Using dimensional analysis and curve-fitting methods, simple PI tuning formulae are presented in this section.

The PI controller in Eq. (1) can be written as shown in Eq. (44).

$$G_c(s) = K_c\left(1 + \frac{1}{T_i s}\right). \tag{44}$$

To obtain the optimal PI tuning formulae for a FOPDT model given in Eq. (27), the PI parameters can be defined based on the model parameters, as shown in Eqs. (45) and (46).

$$K_c = f_1(K_p, \tau_d, T). \tag{45}$$

$$T_i = f_2(K_p, \tau_d, T). \tag{46}$$

Functions f_1 and f_2 should be determined to optimize the objective function and satisfy the GM or PM constraint. Obviously, it is a challenging task to obtain these functions as each controller parameter is a function of three model parameters. To cope with this issue, we use dimensional analysis to simplify the procedure for determining f_1 and f_2 [19].

To simplify a problem through reducing the number of its variables to the smallest number of essential variables, dimensional analysis can be employed [20]. Without any change in a given physical system behaviour, relations between variables in the system are defined as relations between dimensionless numbers, using dimensional analysis. A dimensionless number has no physical unit and is formed as a product or ratio of quantities that have units.

Consider a system expressed by Eq. (47)

$$x_1 = f(x_2, x_3, \ldots, x_n). \tag{47}$$

with non-zero $x_1, x_2, \ldots, x_n$. According to Buckingham's pi-theorem [20], this equation can be substituted with Eq. (48)

$$\pi_1 = g(\pi_2, \pi_3, \ldots, \pi_{n-m}). \tag{48}$$

where $\pi_2, \ldots, \pi_{n-m}$ are independent dimensionless numbers and m is the minimum number of $x_2, x_3, \ldots, x_n$, which includes all the units in $x_1, x_2, \ldots, x_n$

For the model given by Eq. (27), the unit of the dead time (τ_d) and the time constant (T) is the second. The unit of the plant gain (K_p) depends on the nature of the system. As the plant gain along with either the dead time or the time constant cover all the units in Eqs. (45) and (46), m is equal to 2. Hence, $\frac{\tau_d}{T}$, named dimensionless dead time, is the only dimensionless number in the model. In the PI controller shown in Eq. (44), the unit of integral time (T_i) is the second. The unit of controller gain is the inverse of the unit of plant gain. Therefore, the remaining dimensionless numbers are dimensionless gain (K_pK_c) and dimensionless integral time ($\frac{T_i}{\tau_d}$ or $\frac{T_i}{T}$). According to Buckingham's pi-theorem, these dimensionless numbers are functions of the dimensionless dead time, as shown in Eqs. (49) and (50) [19].

$$K_pK_c = g_1\left(\frac{\tau_d}{T}\right). \tag{49}$$

$$\frac{T_i}{\tau_d} = g_2\left(\frac{\tau_d}{T}\right). \tag{50}$$

4.1 Simplified SGM tuning formulae for FOPDT models

Having a constraint on GM, the following procedure is proposed for generating formulae for PI controller tuning.

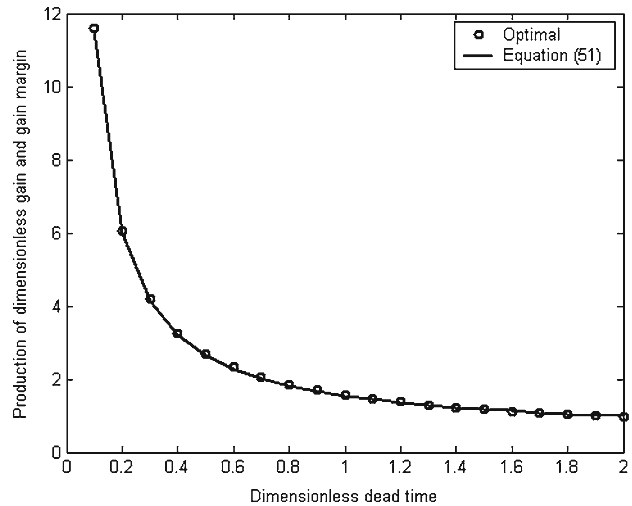

Fig. 1 Optimal values of $K_pK_cA_m$ and the values of $K_pK_cA_m$ given by Eq. (51) versus $\frac{\tau_d}{T}$

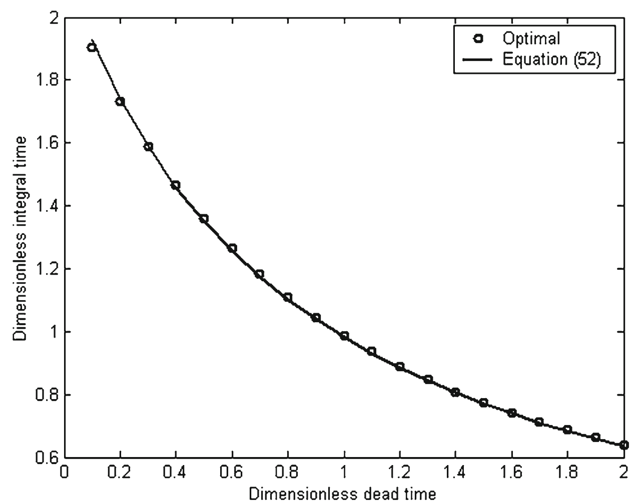

Fig. 2 Optimal values of $\frac{T_i}{\tau_d}$ and the values of $\frac{T_i}{\tau_d}$ given by Eq. (52) versus $\frac{\tau_d}{T}$

Step 1. A range of values of $\frac{\tau_d}{T}$ is selected.

Step 2. Using Eq. (33), ω is determined for each selected value of $\frac{\tau_d}{T}$.

Step 3. For each value of $\frac{\tau_d}{T}$, the optimal values of K_c and T_i are obtained by inserting the resulting ω from step 2 into Eqs. (30), (31) and (22).

Step 4. The optimal values of $K_pK_cA_m$ and $\frac{T_i}{\tau_d}$ versus $\frac{\tau_d}{T}$ are drawn.

Step 5. Functions g_1 and g_2 in Eqs. (49) and (50) are determined by using curve-fitting methods.

Assuming the values of $\frac{\tau_d}{T}$ range from 0.1 to 2, Figs. 1 and 2 show the optimal values of $K_pK_cA_m$ and $\frac{T_i}{\tau_d}$ across the selected values of $\frac{\tau_d}{T}$, respectively. It can be seen from Fig. 1 that $K_p K_c A_m$ is a function of $\frac{\tau_d}{T}$, as shown in Eq. (51).

$$K_p K_c A_m = A_1 + \frac{B_1}{\frac{\tau_d}{T}}. \quad (51)$$

Similarly, the values of $\frac{T_i}{\tau_d}$ are determined from the values of $\frac{\tau_d}{T}$, using Eq. (52).

$$\frac{T_i}{\tau_d} = \frac{A_2}{\frac{\tau_d}{T} + B_2}. \quad (52)$$

Using the least-squares minimization approach, A_1, B_1, A_2 and B_2 are determined for the best match with Figs. 1 and 2. As a result, the optimal values of A_1, B_1, A_2 and B_2 are 0.4331, 1.1191, 1.8095 and 0.8344, respectively.

After simplification, the PI parameters can explicitly be determined using Eqs. (53) and (54).

$$K_p K_c = \frac{1}{A_m}\left(\frac{10T}{9\tau_d} + \frac{3}{7}\right). \quad (53)$$

$$\frac{T_i}{\tau_d} = \frac{\frac{9}{5}}{\frac{\tau_d}{T} + \frac{5}{6}}. \quad (54)$$

4.2 Simplified SPM tuning formulae for FOPDT models

Having a constraint on PM, the procedure mentioned in Sect. 4.1 is used when Eq. (33) in step 2 and Eqs. (30) and (31) in step (3) are substituted by Eqs. (42), (39) and (40), respectively. Also, $K_p K_c A_m$ in step 4 should be replaced by $K_p K_c$. Having obtained the optimal values of $K_p K_c$ and $\frac{T_i}{\tau_d}$ for each value of $\frac{\tau_d}{T}$, the dimensionless gain and dimensionless integral time are given by Eqs. (55) and (56), using cure-fitting techniques.

$$K_p K_c = A_1(\phi_m) + \frac{B_1(\phi_m)}{\frac{\tau_d}{T}}, \quad \frac{\pi}{6} \le \phi_m \le \frac{\pi}{3}. \quad (55)$$

$$\frac{T_i}{\tau_d} = \frac{A_2(\phi_m)\frac{\tau_d}{T} + B_2(\phi_m)}{\frac{\tau_d}{T} + C_2(\phi_m)}, \quad \frac{\pi}{6} \le \phi_m \le \frac{\pi}{3}. \quad (56)$$

where

$$A_1(\phi_m) = \frac{2}{5}\phi_m + \frac{1}{7}. \quad (57)$$

$$B_1(\phi_m) = -\frac{4}{7}\phi_m + \frac{22}{23}. \quad (58)$$

$$A_2(\phi_m) = \frac{5}{6}\phi_m^2 - \frac{8}{11}\phi_m + \frac{3}{7}. \quad (59)$$

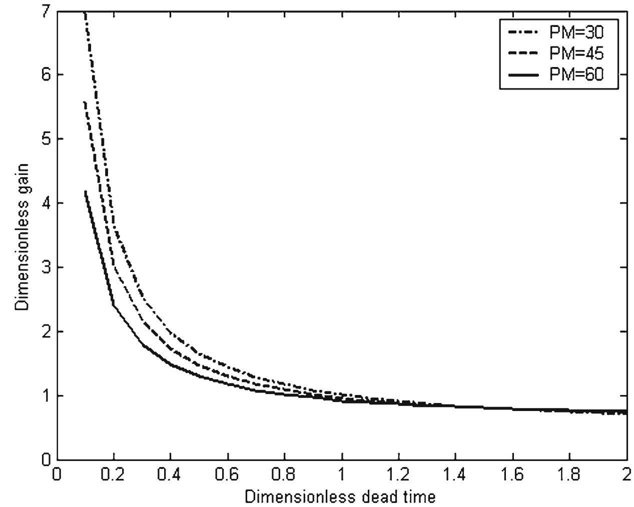

Fig. 3 Values of $K_p K_c$ given by Eq. (55) versus $\frac{\tau_d}{T}$

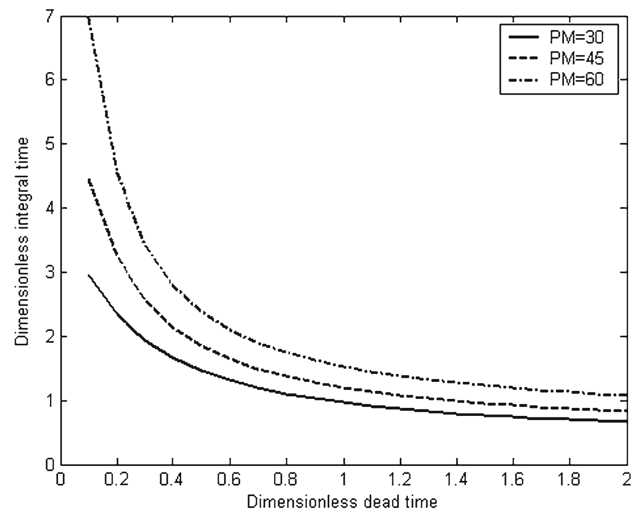

Fig. 4 Values of $\frac{T_i}{\tau_d}$ given by Eq. (56) versus $\frac{\tau_d}{T}$

$$B_2(\phi_m) = -\frac{2}{7}\phi_m^2 + \frac{8}{11}\phi_m + \frac{3}{5}. \quad (60)$$

$$C_2(\phi_m) = -\frac{3}{10}\phi_m + \frac{4}{11}. \quad (61)$$

Figures 3 and 4 show values of $K_p K_c$ and $\frac{T_i}{\tau_d}$ across $\frac{\tau_d}{T}$.

5 Simulation results

Tuning is a trade-off between conflicting design objectives. Both robustness and setpoint regulation are design objectives in conflict with load disturbance rejection [8]. In this section, the SGM and SPM controllers are compared with the Astrom–Panagopoulos–Hagglund (APH) controller [7]. Like the proposed method, the APH technique aims at optimal load disturbance rejection. Similarly, this is done by

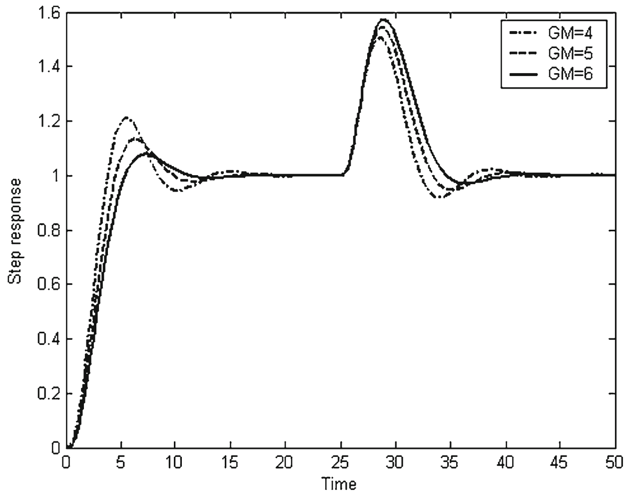

Fig. 5 Closed-loop step responses for different values of GM

Table 1 Comparison results of the SGM controllers to control $G_1(s)$

GM	3.000	4.000	5.000	6.000
K_c	1.167	0.875	0.700	0.583
T_i	1.556			
M_s	2.153	1.783	1.599	1.486
PM	37.45	47.52	54.72	60.01
$\frac{IE}{IAE}$	0.658	0.783	0.870	0.928

minimizing the IE criterion. Robustness is guaranteed by requiring that the maximum sensitivity is less than a specified value.

Example 1

$$G_1(s) = \frac{1}{(s+1)^3}.$$

Inserting $s = j\omega$ into $G_1(s)$ results in

$$G_1(j\omega) = \alpha(\omega) + j\beta(\omega)$$

where

$$\alpha(\omega) = \frac{1-3\omega^2}{(1+\omega^2)^3}$$
$$\beta(\omega) = \frac{\omega(\omega^2-3)}{(1+\omega^2)^3}$$

For a constraint on GM, optimal PI parameters are determined by solving Eq. (15) and inserting the resulting ω into Eqs. (8), (9) and (22). Solving Eq. (15) by a trial and error method results in $\omega = 1.225\frac{r}{s}$. Applying Eq. (62)

Table 2 Comparison results of the SPM controllers to control $G_1(s)$

PM	40	45	50	55	60
ω	0.697	0.650	0.606	0.565	0.523
K_c	1.476	1.374	1.287	1.215	1.154
T_i	2.020	2.123	2.241	2.380	2.541
M_s	2.112	1.947	1.818	1.715	1.633
GM	2.963	3.296	3.646	4.006	4.374
$\frac{d^2K_i}{d\omega^2}$	−5.527	−4.949	−4.442	−3.994	−3.599
$\frac{IE}{IAE}$	0.812	0.894	0.965	1.000	1.000

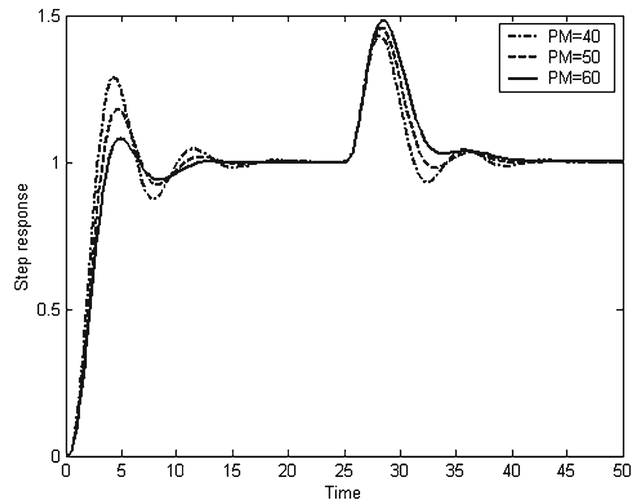

Fig. 6 Closed-loop step responses for different values of PM

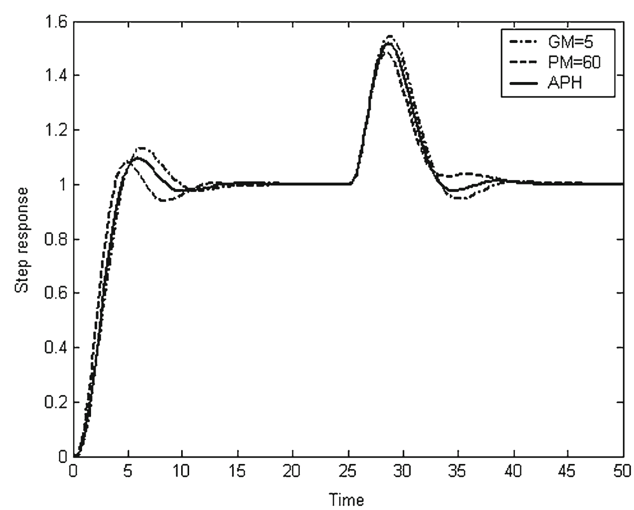

Fig. 7 Closed-loop step responses resulting from the SGM, SPM and APH design methods

$$f''(\omega) = \lim_{\Delta\to 0} \frac{f(\omega+2\Delta) - 2f(\omega+\Delta) + f(\omega)}{\Delta^2}. \quad (62)$$

to $f(\omega)$ in Eq. (13), Eq. (16) gives $\frac{d^2K_i}{d\omega^2} = -14.71$. Hence, the sufficient condition is satisfied. PI parameters are given

Table 3 Comparison results of the SGM, SPM and APH methods to control $G_1(s)$

Method	SGM	SPM	APH
K_c	0.700	1.154	0.862
T_i	1.556	2.541	1.870
b	1.000	1.000	0.930
M_s	1.599	1.633	1.600
GM	5.000	4.374	4.789
PM	54.72	60.00	56.90
$\frac{IE}{IAE}$	0.870	1.000	0.952

by $K_c = \frac{3.5}{A_m}$ and $T_i = \frac{14}{9}$. Closed-loop step responses for different values of GM are shown in Fig. 5. The comparison results are shown in Table 1.

An interesting property of the SGM tuning formulae is that the value of GM can be indicated as a parameter to compromise between performance and robustness. Figure 5 clearly shows that a higher value of GM results in an inferior load disturbance rejection but a better setpoint regulation. It should be noted that higher values of $\frac{IE}{IAE}$ are associated with less oscillatory systems.

For a constraint on PM, optimal PI parameters are determined by solving Eq. (26) and inserting the resulting ω into Eqs. (23), (24) and (22). Considering $f(\omega)$ in Eq. (25) and for $PM = 40^\circ$, the SPM method results in $\omega = 0.697$, $K_c = 1.476$ and $T_i = 2.02$. The sufficient condition in Eq. (16) is also satisfied as $\frac{d^2K_i}{d\omega^2} = -5.527$. Table 2 summarizes the comparison results for different values of PM.

It can be seen from Table 2 that the sufficient condition is satisfied for the selected values of PM. Closed-loop step responses for different values of PM are shown in Fig. 6. Clearly, a better setpoint regulation but an inferior load disturbance rejection is provided by a higher value of PM.

To compare the performance of the SGM, SPM and APH methods, closed-loop step responses are drawn in Fig. 7. A slightly better setpoint regulation is given by the SPM due to a higher value of PM. The setpoint response given by the APH controller is improved using a two-degree of freedom structure. Table 3 shows the comparison results.

An advantage of the SGM and SPM methods is that as soon as ω is determined and subject to satisfying the sufficient condition, the controller parameters are directly given. However, the APH controller parameters cannot be resulted from an explicit set of tuning formulae. They should be computed using a procedure, which may lead to complicated situations [7].

Example 2 In this example, the SGM method is applied to a non-minimum phase plant, a pure time delay unit, a long dead time plant and a plant with complex poles.

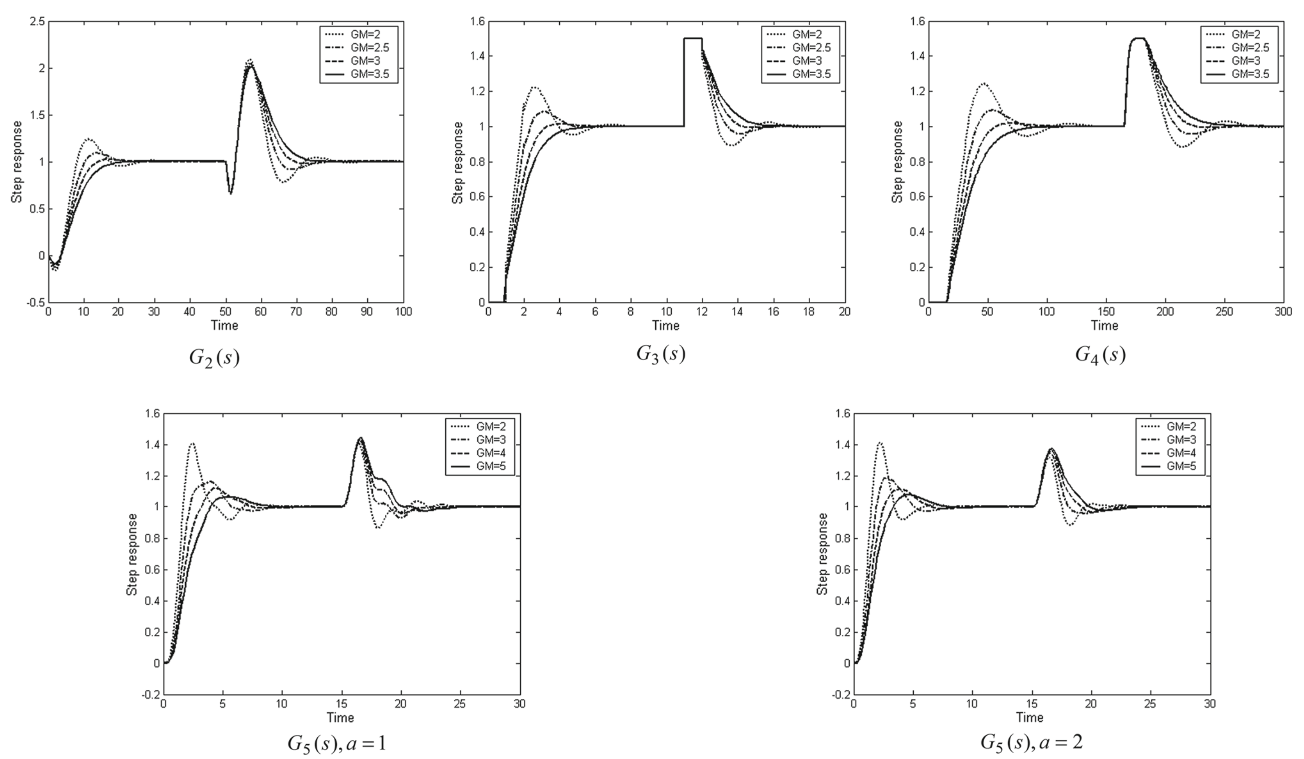

Fig. 8 Closed-loop step responses for different values of GM

Table 4 Comparison results of the SGM controllers

Plant	ω	K_c	T_i	M_s	GM	PM	$\frac{d^2K_i}{d\omega^2}$	$\frac{IE}{IAE}$
$G_2(s)$	0.491	0.268	1.319	2.225	2.000	46.18	−3.368	0.571
		0.214		1.825	2.500	55.19		0.737
		0.179		1.624	3.000	61.12		0.841
		0.153		1.502	3.500	65.31		0.888
$G_3(s)$	2.029	0.177	0.243	1.772	2.500	57.84	−5.482	0.856
		0.147		1.584	3.000	63.35		0.974
		0.126		1.470	3.500	67.23		1.000
		0.111		1.394	4.000	70.12		1.000
$G_4(s)$	0.114	0.231	4.486	2.156	2.000	48.94	−4.933	0.641
		0.185		1.778	2.500	57.48		0.845
		0.154		1.589	3.000	63.05		0.972
		0.132		1.474	3.500	66.98		1.000
$G_5(s), a = 1$	2.236	0.056	0.040	2.090	2.000	37.55	−3.192	0.493
		0.037		1.649	3.000	48.52		0.721
		0.028		1.479	4.000	55.70		0.792
		0.022		1.384	5.000	60.86		0.884
$G_5(s), a = 2$	2.345	0.417	0.248	2.221	2.000	37.04	−14.37	0.521
		0.278		1.671	3.000	47.77		0.705
		0.208		1.489	4.000	54.58		0.799
		0.167		1.391	5.000	59.54		0.858
$G_6(s)$	0.707	0.500	4.000	5.115	2.000	11.81	−2.83	0.324
		0.400		4.203	2.500	14.24		0.352
		0.333		3.788	3.000	15.64		0.357
$G_7(s)$	1.077	0.474	1.726	5.235	2.000	11.19	−3.656	0.223
		0.374		4.832	2.500	12.00		0.221
		0.316		4.744	3.000	12.15		0.214

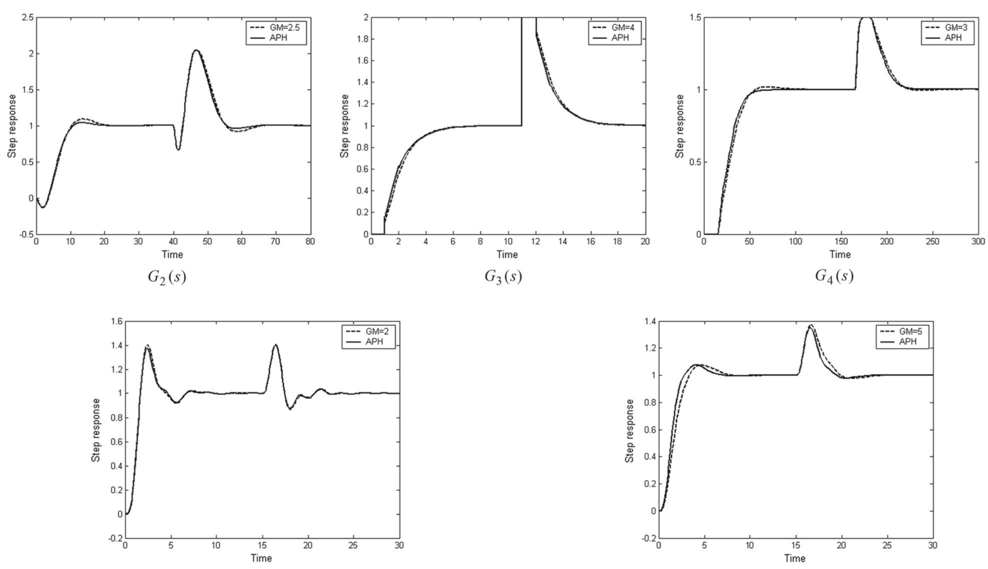

Fig. 9 Closed-loop step responses resulting from the SGM and APH methods

Table 5 Comparison results of the SGM and APH controllers

Plant	Method	SGM	APH
$G_2(s)$	K_c	0.214	0.265
	T_i	1.319	1.640
	b	1.000	0.870
	M_s	1.825	1.797
	GM	2.500	2.476
	PM	55.19	57.93
	$\frac{IE}{IAE}$	0.737	0.798
$G_3(s)$	K_c	0.111	0.158
	T_i	0.243	0.335
	b	1.000	1.000
	M_s	1.394	1.400
	GM	4.000	3.846
	PM	70.12	71.71
	$\frac{IE}{IAE}$	1.000	1.000
$G_4(s)$	K_c	0.154	0.208
	T_i	4.486	5.870
	b	1.000	1.000
	M_s	1.589	1.599
	GM	3.000	2.888
	PM	63.05	64.70
	$\frac{IE}{IAE}$	0.972	1.000
$G_5(s), a=1$	K_c	0.056	0.090
	T_i	0.040	0.065
	b	1.000	1.000
	M_s	2.090	2.002
	GM	2.000	2.005
	PM	37.55	39.24
	$\frac{IE}{IAE}$	0.493	0.510
$G_5(s), a=2$	K_c	0.167	0.313
	T_i	0.248	0.373
	b	1.000	0.880
	M_s	1.391	1.400
	GM	5.000	3.843
	PM	59.54	59.16
	$\frac{IE}{IAE}$	0.858	0.867

$$G_2(s) = \frac{1-2s}{(s+1)^3}. \quad G_3(s) = e^{-s}.$$

$$G_4(s) = \frac{e^{-15s}}{(s+1)^3}. \quad G_5(s) = \frac{9}{(s+1)(s^2+as+9)}.$$

$G_2(s)$ and $G_5(s)$ are not common in control, however, they are included to demonstrate the wide applicability of the proposed tuning method. Closed-loop step responses for different values of GM are shown in Fig. 8. The comparison results are shown in Table 4. Figure 9 shows the fairly similar closed-loop step responses provided by the SGM and APH methods.

Table 6 Comparison results of the SPM controllers

$G_2(s)$	PM	30	45	60
	ω	0.360	0.306	0.258
	K_c	0.560	0.594	0.639
	T_i	1.951	2.506	3.337
	M_s	4.280	3.347	3.090
	GM	1.339	1.457	1.500
	$\frac{d^2K_i}{d\omega^2}$	−2.797	−2.501	−2.202
	$\frac{IE}{IAE}$	0.319	0.519	0.722
$G_3(s)$	PM	30	45	60
	ω	1.605	1.404	1.213
	K_c	0.529	0.580	0.636
	T_i	0.388	0.507	0.680
	M_s	5.042	3.945	3.702
	GM	1.288	1.373	1.392
	$\frac{d^2K_i}{d\omega^2}$	−3.880	−3.234	−2.678
	$\frac{IE}{IAE}$	0.260	0.400	0.541
$G_4(s)$	PM	30	45	60
	ω	0.090	0.078	0.068
	K_c	0.543	0.591	0.644
	T_i	7.086	9.211	12.30
	M_s	4.904	3.800	3.512
	GM	1.284	1.379	1.412
	$\frac{d^2K_i}{d\omega^2}$	−4.727	−4.262	−3.624
	$\frac{IE}{IAE}$	0.283	0.425	0.589
$G_6(s)$	PM	30	40	50
	ω	0.396	0.317	0.246
	K_c	0.439	0.338	0.255
	T_i	8.372	11.93	18.54
	M_s	2.258	1.782	1.505
	GM	3.467	4.925	7.005
	$\frac{d^2K_i}{d\omega^2}$	−0.902	−0.584	−0.354
	$\frac{IE}{IAE}$	1.000	1.000	1.000
$G_7(s)$	PM	30	45	60
	ω	0.707	0.528	0.350
	K_c	0.667	0.510	0.345
	T_i	3.998	7.187	16.30
	M_s	2.429	1.742	1.395
	GM	2.068	2.899	4.464
	$\frac{d^2K_i}{d\omega^2}$	−1.533	−0.845	−0.367
	$\frac{IE}{IAE}$	0.998	1.000	1.000

Results of comparison of the SGM and APH methods are summarized in Table 5. Results of applying the SPM method to $G_2(s)$, $G_3(s)$ and $G_4(s)$ are shown in Table 6. Comparing to each SGM controller, the corresponding SPM controller has a too high gain, resulting in a low gain margin and a high maximum sensitivity.

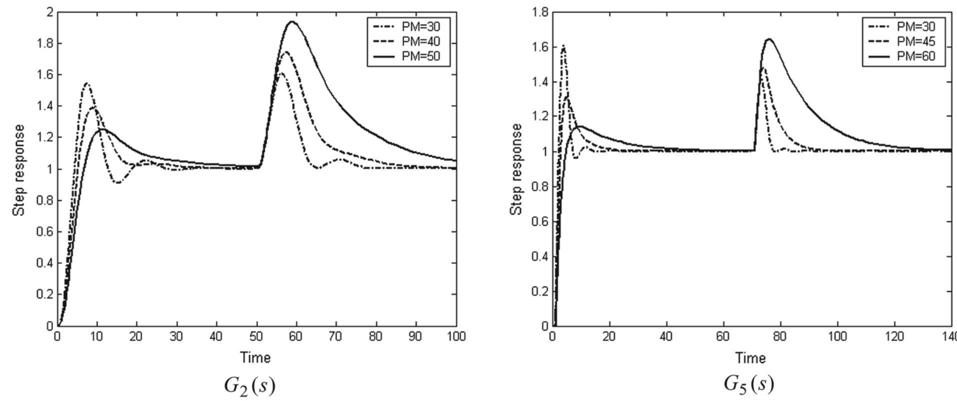

Fig. 10 Closed-loop step responses for different values of PM

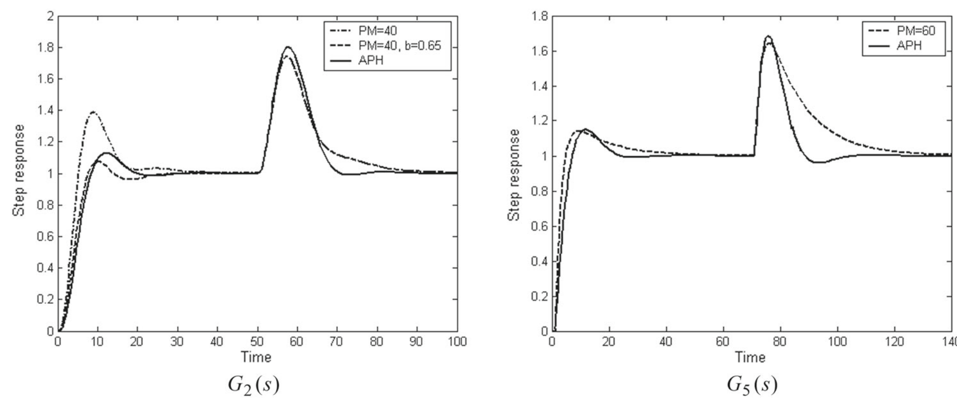

Fig. 11 Closed-loop step responses resulting from the SPM and APH methods

Example 3 In this example, the SPM method is applied to the following integrating plants.

$$G_6(s) = \frac{1}{s(s+1)^2}, \quad G_7(s) = \frac{e^{-s}}{s}.$$

Closed-loop step responses for different values of PM are shown in Fig. 10. The comparison results are shown in Table 6. Figure 11 shows the closed-loop step responses resulting from the SPM and APH methods. As shown in Fig. 11, the setpoint response of the SPM controller can easily be improved using the setpoint weight. For these methods, the comparison results are summarized in Table 7.

The SPM controller for a FOPDT plant is given by solving Eq. (42) and inserting the resulting ω into Eqs. (39), (40) and (22). A plant with dead time and a single pole at origin is a special case of a FOPDT plant when the time constant becomes infinite. Such a plant can be described by Eq. (63).

$$G_{\text{int}}(s) = \lim_{T\to\infty} \frac{K_p e^{-\tau_d s}}{Ts+1} = \frac{K'_p e^{-\tau_d s}}{s}. \tag{63}$$

where K'_p is given by Eq. (64).

$$K'_p = \frac{K_p}{T}. \tag{64}$$

Table 7 Comparison results of the SPM and APH controllers

Plant	Method	SPM	APH
$G_6(s)$	K_c	0.338	0.286
	T_i	11.934	9.000
	b	1.000	0.570
	M_s	1.782	1.801
	GM	4.925	5.436
	PM	40.00	36.92
	$\frac{IE}{IAE}$	1.000	0.989
$G_7(s)$	K_c	0.345	0.282
	T_i	16.30	6.746
	b	1.000	0.660
	M_s	1.395	1.400
	GM	4.464	5.218
	PM	60.00	46.71
	$\frac{IE}{IAE}$	1.000	0.897

For the plant in Eq. (63), Eq. (42) is simplified to Eq. (65).

$$\omega = \frac{2}{\tau_d}\cot(\omega\tau_d + \phi_m). \tag{65}$$

Controller parameters are given by inserting the resulting ω into Eqs. (66) and (67).

$$K_c = \frac{\omega \sin(\omega \tau_d + \phi_m)}{K_p'}. \quad (66)$$

$$T_i = \frac{\tan(\omega \tau_d + \phi_m)}{\omega}. \quad (67)$$

Using Eqs. (65)–(67), results shown in Table 6 for $G_7(s)$ are obtained in a simpler manner.

Results of applying the SGM method to $G_6(s)$ and $G_7(s)$ are shown in Table 4. Comparing to the corresponding SPM controller, the SGM controller does not have a large enough integral time, resulting in a low phase margin and a high maximum sensitivity.

6 Conclusions

To consider both performance and robustness requirements, this paper presented a PI tuning method for the optimization of load disturbance rejection with a constraint either on the GM or on the PM. The design method resulted in the SGM and SPM tuning formulae that could be adapted for the type of system required. Using dimensional analysis and curve-fitting techniques, a simplified form of tuning formulae for FOPDT models was also determined. Simulation results for a variety of examples including integrating, non-minimum phase and long dead time plants showed that the proposed tuning method was effective in dealing with a wide range of plants.

For industrial applications, it is often required that GM and PM specifications fall into desirable ranges. Future research will attempt to minimize the IE criterion subject to simultaneously satisfying predefined constraints on gain and phase margins.

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