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Fractional calculus in economic growth modelling: the Spanish and Portuguese cases

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Abstract This work presents a fractional order approach to model the growth of national economies, namely, their gross domestic products (GDPs). Land area, arable land, population, school attendance, gross capital formation, exports of goods and services, general government final consumption expenditure and money and quasi money are taken as variables to describe GDP. The particular cases of the national economies of Spain and Portugal are studied along the last five decades. Results show that fractional models have a better performance than the other alternatives considered in the literature.

Keywords Fractional calculus · Dynamic models · Gross domestic product · Economic growth

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1 Introduction

Studying the dynamics of finance behaviour and economics is very challenging. Because fractional operators are non-local, they are suitable for constructing models for long series, possessing a memory effect—more so than models using integer derivatives and integrals alone [1]. This is the reason why fractional differential equations possess large advantage in describing economic phenomena over large time periods.

Recently, a variety of fractional order models have been proposed in the literature to account for the behaviour of financial processes from different viewpoints. For example, as diffusion or stochastic processes by means of Lévy models [2–6], continuous time random walks (e.g. [7–12]), or differentiable manifolds [13]. Laskin [14] applied a modified fractality concept to describe the stochastic dynamics of the stock and currency markets. From a macroeconomic point of view, on the one hand, [15,16] proposed a state space model for national economies involving three variables; a similar idea was used in [17] but considering variable orders. On the other hand, [18] and [19] investigated four- and three-variable discrete macroeconomic models, respectively. Likewise, [20–22] discussed financial processes from the chaos systems perspective; results were adapted to estimate the evolution of macroeconomic variables in Japan in [23]. Finally, [24,25] analysed the dynamics of world economies based on pseudo-phase plane and state space analysis.

Thus, many models have been published, among which the classical papers [26,27] on GDP growth. Yet, to the best of our knowledge, no fractional model of GDP as a function of a vector of inputs had yet been found. Given this motivation, the objective of this work is to model the growth of national economies through their gross domestic products (GDPs) by means of a fractional order approach. More precisely, the GDP of national economies is given as function of a vector with nine variables. The particular cases of the national economies of the Iberian peninsula, viz. Spain and Portugal, along the last five decades, are studied. Preliminary results can be found in [28,29]. The sources of the data are given in the "Appendix".

This paper is organised as follows. For reference purposes, Sect. 2 introduces fractional derivatives. Section 3 describes the proposed method to describe the growth of national economies. In Sect. 4, the obtained results after fitting are given for both the Spanish and the Portuguese economies, and then compared. Finally, Sect. 5 draws the concluding remarks and perspectives future works.

2 Fractional calculus

Let us define differential operator D as ${}_cD_t^n f(t) = \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}$ and ${}_cD_t^{-n} f(t) = \underbrace{\int_c^t \cdots \int_c^t}_{|n| \text{ integrations}} f(\tau)\,\mathrm{d}\tau \cdots \mathrm{d}\tau$. It can be shown by mathematical induction that

$$ {}_cD_t^n f(t) = \lim_{h\to 0} \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t-kh)}{h^n}, \quad n \in \mathbb{N} \tag{1}$$

where combinations of a things, b at a time are given by $\binom{a}{b} = \frac{a!}{b!(a-b)!}$. This can be generalised using the Gamma function, which verifies $\Gamma(n) = (n-1)!, n \in \mathbb{N}$ and is defined in $\mathbb{C}\backslash\mathbb{Z}^-$, as

$$ \binom{a}{b} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}, & \text{if } a, b, a-b \notin \mathbb{Z}^- \\ \frac{(-1)^b \Gamma(b-a)}{\Gamma(b+1)\Gamma(-a)}, & \text{if } a \in \mathbb{Z}^- \wedge b \in \mathbb{Z}_0^+ \\ 0, & \text{if } [(b \in \mathbb{Z}^- \vee b-a \in \mathbb{N}) \wedge a \notin \mathbb{Z}^-] \vee \\ & (a, b \in \mathbb{Z}^- \wedge |a| > |b|) \end{cases} \tag{2}$$

Using (2), it is reasonable to generalise (1) for non-integer orders as

$$ {}_cD_t^\alpha f(t) = \lim_{h\to 0^+} \frac{\sum_{k=0}^{\lfloor \frac{t-c}{h} \rfloor} (-1)^k \binom{\alpha}{k} f(t-kh)}{h^\alpha} \tag{3}$$

Values c and t are called terminals. The upper limit of the summation in (3) is diverging to $+\infty$. When $\alpha \in \mathbb{N}$, all terms with $k > \alpha$ will be zero; thus (3) reduces to (1) when $h > 0$. This is the only case in which the summation has a finite number of terms and the result does not depend on terminal c. The upper limit $\lfloor \frac{t-c}{h} \rfloor$ was set so that, if $\alpha = -1, -2, -3, \ldots$, (3) becomes a Riemann integral (calculated from c to t).

For more details on operator D, properties, alternative definitions (e.g. the Caputo definition mentioned below) and Laplace transforms, see e.g. [1,30].

3 Economic growth model

Consider a simple model of a national economy in the following form:

$$ y(t) = f(x_1, x_2 \ldots) \tag{4}$$

where the output model y is the GDP (in 2012 euros) and the x_k are the variables on which the output depends. The inputs considered are the following:

- x_1: land area (km^2);
- x_2: arable land (km^2);
- x_3: population;
- x_4: school attendance (years);
- x_5: gross capital formation (GCF) (in 2012 euros);
- x_6: exports of goods and services (in 2012 euros);
- x_7: general government final consumption expenditure (GGFCE) (in 2012 euros);
- x_8: money and quasi money (M2) (in 2012 euros).

The rationale behind this choice of variables is the following:

- Natural resources are represented by x_1, and their quality by x_2;
- Human resources are represented by x_3, and their quality by x_4;
- Manufactured resources are represented by x_5;
- External impacts in the economy are represented by x_6;
- Internal impacts in the economy are represented by x_7 (budgetary impacts), x_8 (monetary impacts) and also by x_5 (investment). Rather than having x_5 play two roles, we will rather use another variable $x_9 \equiv x_5$ to represent the impact of investment in the economy, bringing the number of inputs up to 9.

This choice of variables joins those traditionally considered in growth accounting [31–33] to those acknowledged by Keynesian models having short-term inputs related to impacts in the economy. The quality of manufactured resources is sometimes translated in a variable such as the number of patents filed each year. We did not use this variable, not so much because of difficulties in finding data, as because this indicator is a poor translation of what it is intended to measure, since nowadays patent systems are increasingly globalised [34].

The integer order model considered is

$$y(t) = C_1x_1(t) + C_2x_2(t) + C_3x_3(t) + C_4x_4(t) + C_5\int_{t_0}^{t} x_5(t)\mathrm{d}t + C_6x_6(t) + C_7x_7(t) + C_8\frac{\mathrm{d}x_8(t)}{\mathrm{d}t} + C_9\frac{\mathrm{d}x_9(t)}{\mathrm{d}t} \quad (5)$$

where C_k are constant weights for each of the variables, and t_0 is the first year considered. Notice that:

– The accumulated gross capital formation $\int_{t_0}^{t} x_5(t)\mathrm{d}t$ is used as a measure of manufactured resources;
– The variation of M2 $\frac{\mathrm{d}x_8(t)}{\mathrm{d}t}$ is used as a measure of the monetary impacts in the economy;
– The variation of the gross capital formation $\frac{\mathrm{d}x_9(t)}{\mathrm{d}t} = \frac{\mathrm{d}x_5(t)}{\mathrm{d}t}$ is used as a measure of the impact of investment in the economy.

Its generalization to non-integer orders is as follows:

$$y(t) = \sum_{k=1}^{9} C_k \mathrm{D}^{\alpha_k} x_k(t), \quad (6)$$

where α_k are the differentiation orders of each variable. The Caputo definition was used [1].

4 Results

As an application of this model, the economies of Spain and Portugal are going to be modelled in the period between 1960 and 2012. This period was considered not only because it is the one for which reliable data can be easily obtained, but also because this is the period where modern economic growth consistently took hold of both these national economies of the Iberian Peninsula. (See economic data in Figs. 1, 2 and also Tables 5, 6 in the "Appendix".)

The goal of the fitting is to calculate the numerical parameters, i.e., orders α_k and coefficients C_k, of the proposed dynamic models (5) and (6) for the two particular national economies. Notice that, for the integer model, the orders are all fixed: that of x_5 is -1, those of x_8 and x_9 are 1, as mentioned in Sect. 3, and all the remaining orders are 0.

The fitting procedure is implemented in MATLAB, using Nelder-Mead's simplex search method as implemented in function *fminsearch*, by minimising the mean square error (MSE) defined as

$$\mathrm{MSE} = \frac{\sum_{j=1}^{N}(y_j - \hat{y}_j)^2}{N}, \quad (7)$$

where N is the number of points, and y_j and $\hat{y}_j$ are the real output and the model output, respectively. In order to evaluate the goodness-of-fit of the obtained models, apart from MSE, the following performance indices will be also calculated:

1. The mean absolute deviation (MAD) as

$$\mathrm{MAD} = \frac{\sum_{j=1}^{N}|y_j - \hat{y}_j|}{N}. \quad (8)$$

2. The coefficient of determination ($\mathrm{R}^2 \in (0, 1)$) defined as

$$\mathrm{R}^2 = 1 - \frac{\sum_{j=1}^{N}(y_j - \hat{y}_j)^2}{\sum_{j=1}^{N}(y_j - \bar{y})^2}, \quad (9)$$

where $\bar{y}$ is the mean of the GDP.
3. The t and p values for each variable.

The calculation of these indices is carried out by means of the MATLAB command *regstats*.

As will be seen below, not all variables turned out to be necessary for the models. This will be evaluated from t and p values for each variable, checking if the performance indexes MAD and R^2 deteriorate significantly when removing one or more variables from the model, and using the Akaike Information Criterion (AIC), given by

$$\mathrm{AIC} = N \log \frac{\sum_{j=1}^{N}\left(y_j - \hat{y}_j\right)^2}{N} + 2K + \frac{2K(K+1)}{N-K-1} \quad (10)$$

where K is the number of parameters of the model. The value of the AIC gives no information about the quality of a model, but comparing the AIC values of different models shows which ones are more likely to be a good model for the data: a lower value indicates a more likely model. Furthermore, if we have M models, the Akaike weight given by

$$w_i = \frac{\exp\left(-\frac{\mathrm{AIC}_i - \min\limits_{M} \mathrm{AIC}}{2}\right)}{\sum_{j=1}^{M} \exp\left(-\frac{\mathrm{AIC}_j - \min\limits_{M} \mathrm{AIC}}{2}\right)} \quad (11)$$

gives the probability of model i being the best among the M models.

4.1 The Spanish economy

The models obtained for the Spanish economy are shown in Fig. 3, with the values of the orders α and the coefficients C given in Table 1. It can be seen that the differentiation orders obtained for x_1, x_2, x_3, x_6 and x_7 of fractional order model

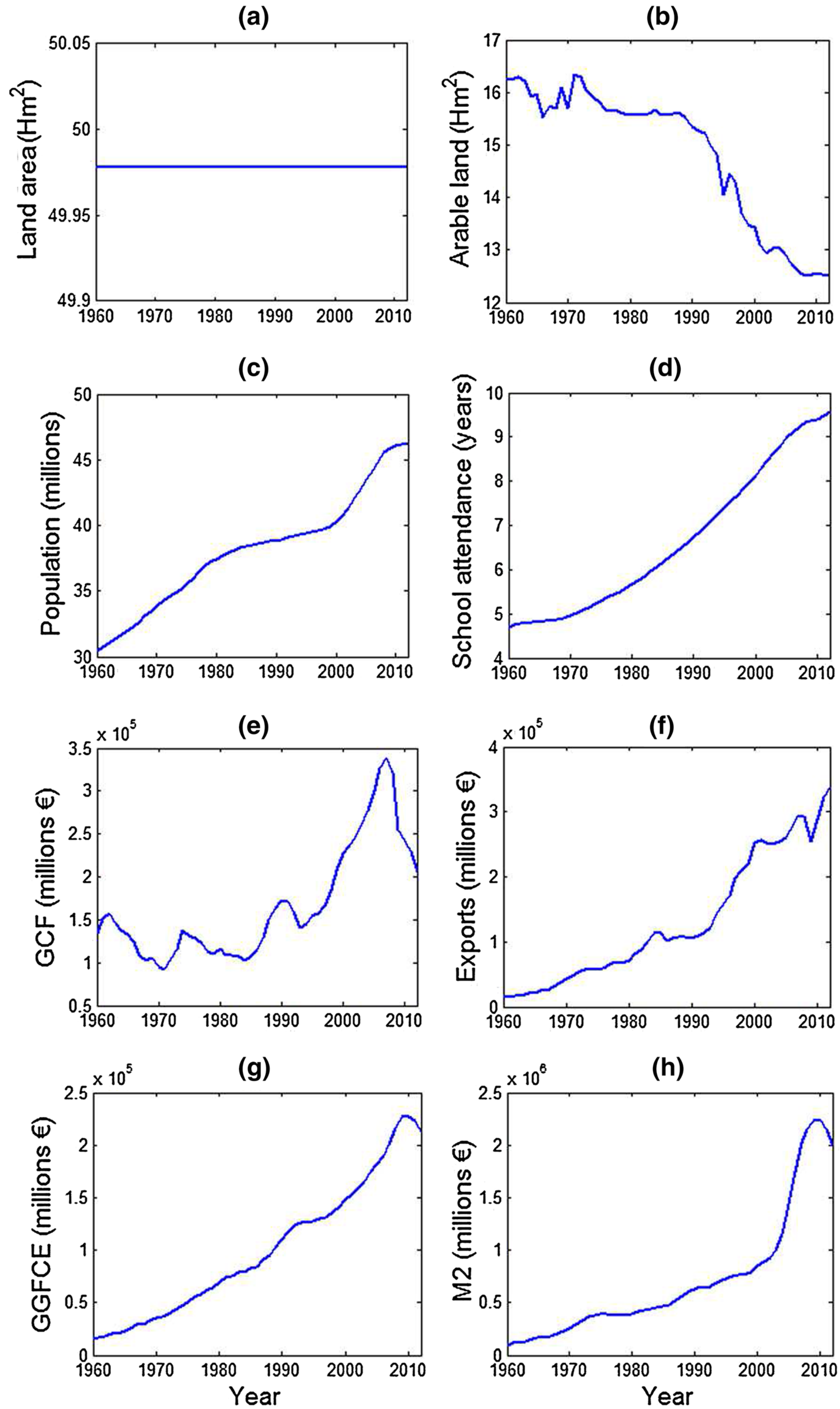
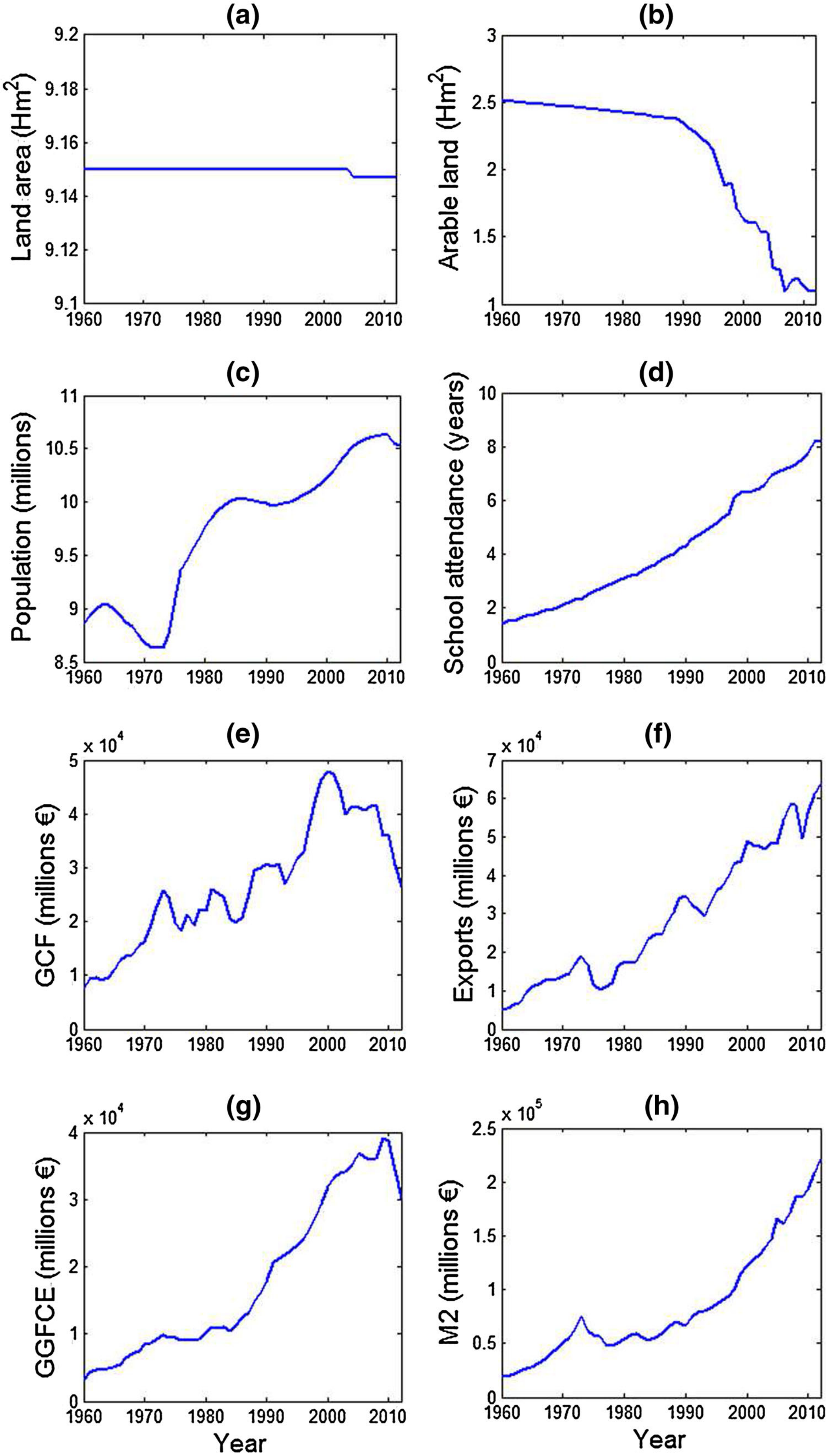


Fig. 1 Data for Spain: **a** land area, **b** arable land, **c** population, **d** school attendance, **e** gross capital formation, **f** exports of good and services, **g** general government final consumption expenditure; **h** Money and quasi money

Fig. 2 Data for Portugal: **a** land area, **b** arable land, **c** population, **d** school attendance, **e** gross capital formation, **f** exports of good and services, **g** general government final consumption expenditure, **h** money and quasi money



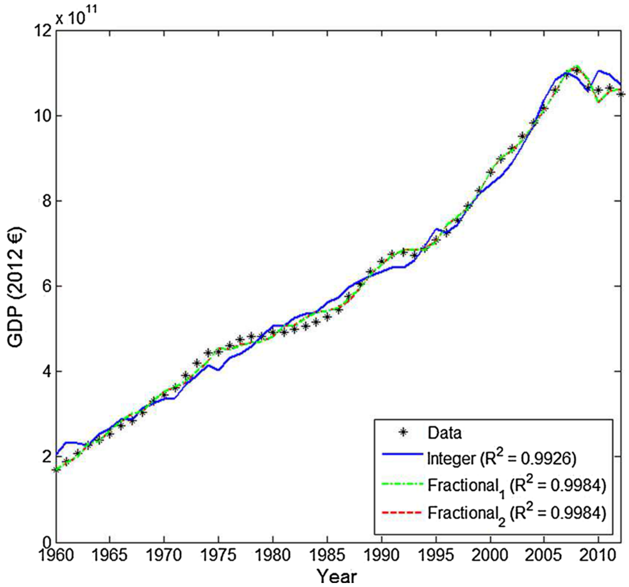


Fig. 3 Fitting results for the Spanish case (models with nine variables): the integer model is given by (5), the fractional$_1$ model by (6), and the fractional$_2$ model by (12)

(6) are zero (or almost zero), which leads us to consider a simpler model, in which only variables x_4, x_5, x_8 and x_9 are assumed to have fractional order influence, as follows:

$$y(t) = \sum_{k=1,2,3,6,7} C_k x_k(t) + \sum_{k=4,5,8,9} C_k\, D^{\alpha_k} x_k(t). \tag{12}$$

What this means is that not all economic indicators have the same influence over time on the GDP: for some (those with $\alpha = 0$) only the current value matters. The results related to this model are also included in Fig. 3 and Table 1.

The performance indices calculated for models (5), (6) and (12) are summarised in Table 2, where t values corresponding to variables which are necessary for the model, assuming a 5 % significance level, are in bold. As observed, the population (x_3) and the variation of GCF (x_9) have a considerable effect on the integer model, whereas the remaining variables have low influence. In contrast, for the fractional models, it is clear that only the arable land (x_2) and the GGFCE (x_7) are variables without much influence in the GDP. This can be confirmed using the AIC: models obtained without one of the variables with a t value higher than the threshold assumed have a 0 % probability of being the best.

The fractional models have a clearly better performance, confirmed by all performance indexes, at the expense of needing more variables (7 against 2 for the integer model).

Taking into account the low influence of variables x_2 and x_7 in the model, let us consider a simpler model with only 7 inputs, with an integer form given by

$$y(t) = C_1 x_1(t) + C_3 x_3(t) + C_4 x_4(t) + C_5 \int_{t_0}^{t} x_5(t)\mathrm{d}t + C_6 x_6(t) + C_8 \frac{\mathrm{d}x_8(t)}{\mathrm{d}t} + C_9 \frac{\mathrm{d}x_9(t)}{\mathrm{d}t}, \tag{13}$$

and a fractional form as

$$y(t) = \sum_{k=1,3,6} C_k x_k(t) + \sum_{k=4,5,8,9} C_k \mathrm{D}^{\alpha_k} x_k(t). \tag{14}$$

The obtained models consisting of seven variables are shown in Fig. 4. The values of the orders α and the coefficients C are also given in Table 1, and the performance indices in Table 2.

There is, of course, a slight deterioration of performance, but the results obtained with fractional model (14) remain highly satisfactory. Actually, the AIC results in this model having the higher probability of being the best among all the five. Again, this is achieved at the expense of the model need-

Table 1 Fitting results for the Spanish economy: orders of the fractional operator and coefficients

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
Integer (5)	0	0	0	0	−1	0	0	1	1
Fractional (6)	0	0	0	0.068	0.860	0	1.250×10^{-4}	−1.020	−0.834
Fractional (12)	0	0	0	0.066	0.855	0	0	−1.016	−0.822
Integer (13)	0	–	0	0	−1	0	–	1	1
Fractional (14)	0	–	0	0.198	−0.809	0	–	−0.995	0.988
	C_1 ($\times 10^5$)	C_2 ($\times 10^6$)	C_3 ($\times 10^4$)	C_4 ($\times 10^{10}$)	C_5 ($\times 10^{-2}$)	C_6 ($\times 10^{-2}$)	C_7 ($\times 10^{-1}$)	C_8 ($\times 10^{-2}$)	C_9 ($\times 10^{-1}$)
Integer (5)	−3.209	−3.137	2.365	2.974	1.385	−4.541	8.841	14.711	10.315
Fractional (6)	8.742	1.619	−1.766	−3.434	645.874	46.990	−5.883	−2.312	4.790
Fractional (12)	8.798	1.595	−1.745	−3.738	646.014	49.866	−3.512	−2.396	4.928
Integer (13)	−12.092	–	1.841	5.111	4.067	13.227	–	20.435	7.070
Fractional (14)	10.302	–	−1.152	−3.994	41.687	38.397	–	−2.098	834.040

Table 2 Performance indices for the Spanish economy

Index/statistic	Variable	Models with nine variables			Models with seven variables	
		Integer (5)	Fractional (6)	Fractional (12)	Integer (13)	Fractional (14)
MSE ($\times 10^{20}$)		5.610	1.228	1.241	6.084	1.320
R^2		0.9926	0.9984	0.9984	0.9920	0.9983
MAD ($\times 10^{10}$)		2.033	0.912	0.920	2.0820	0.9257
t values	x_1	−0.425	**3.953**	**3.831**	−2.150	**5.190**
	x_2	−1.836	2.036	2.044	–	–
	x_3	**3.276**	**−4.117**	**−3.962**	**2.917**	**−3.634**
	x_4	0.724	**−8.277**	**−7.355**	1.879	**−8.121**
	x_5	0.385	**11.977**	**10.731**	1.339	**17.764**
	x_6	−0.113	**4.019**	**4.008**	0.474	**3.669**
	x_7	0.719	−1.489	−0.936	–	–
	x_8	2.237	**−16.560**	**−16.236**	**3.437**	**−15.678**
	x_9	**2.736**	**12.264**	**9.508**	2.093	**12.359**
p values	x_1	0.673	2.762×10^{-4}	4.013×10^{-4}	3.682×10^{-2}	4.634×10^{-6}
	x_2	7.903×10^{-2}	4.777×10^{-2}	4.695×10^{-2}	–	–
	x_3	3.993×10^{-3}	1.662×10^{-4}	2.688×10^{-4}	5.441×10^{-3}	7.001×10^{-4}
	x_4	0.480	1.622×10^{-10}	3.465×10^{-9}	6.657×10^{-2}	1.964×10^{-10}
	x_5	0.720	2×10^{-15}	7.3×10^{-14}	0.187	0
	x_6	0.912	2.252×10^{-4}	2.330×10^{-4}	0.637	6.315×10^{-4}
	x_7	0.502	0.143	0.354	–	–
	x_8	3.334×10^{-2}	0	0	1.257×10^{-3}	0
	x_9	8.941×10^{-3}	1×10^{-15}	3.090×10^{-12}	4.190×10^{-2}	0
AIC		2554.3	2473.8	2474.4	2552.9	2472.0
w		0 %	23 %	18 %	0 %	59 %
AIC wihout one variable	x_1	2551.6	2486.9	2486.8		
	x_2	2555.3	2475.7	2476.3		
	x_3	2563.0	2488.1	2487.8		
	x_4	2552.0	2520.7	2514.0		
	x_5	2551.6	2547.7	2530.8		
	x_6	2551.4	2487.6	2488.3		
	x_7	2552.0	2473.5	2472.5		
	x_8	2557.1	2575.7	2574.5		
	x_9	2559.7	2549.6	2539.6		
w found from the AIC without one variable	x_1	20%	**0 %**	**0 %**		
	x_2	3 %	25 %	13 %		
	x_3	**0 %**	**0 %**	**0 %**		
	x_4	16 %	**0 %**	**0 %**		
	x_5	21 %	**0 %**	**0 %**		
	x_6	22 %	**0 %**	**0 %**		
	x_7	17 %	75 %	87 %		
	x_8	1 %	**0 %**	**0 %**		
	x_9	**0 %**	**0 %**	**0 %**		

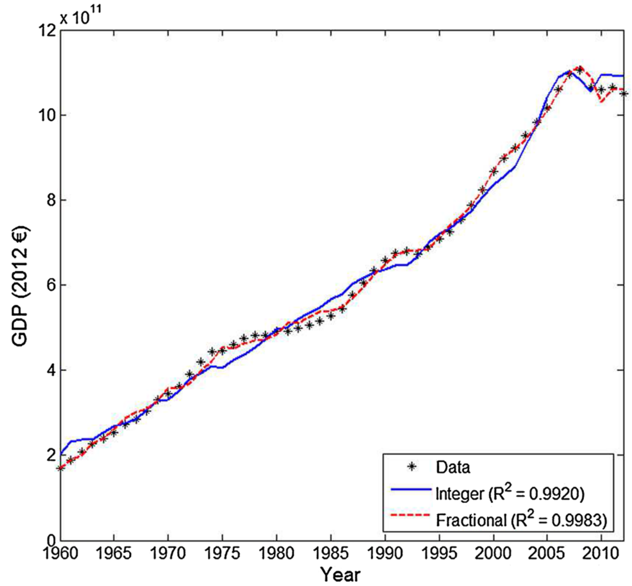


Fig. 4 Fitting results for the Spanish case (models with seven variables): the integer model is given by (13), and the fractional model by (14)

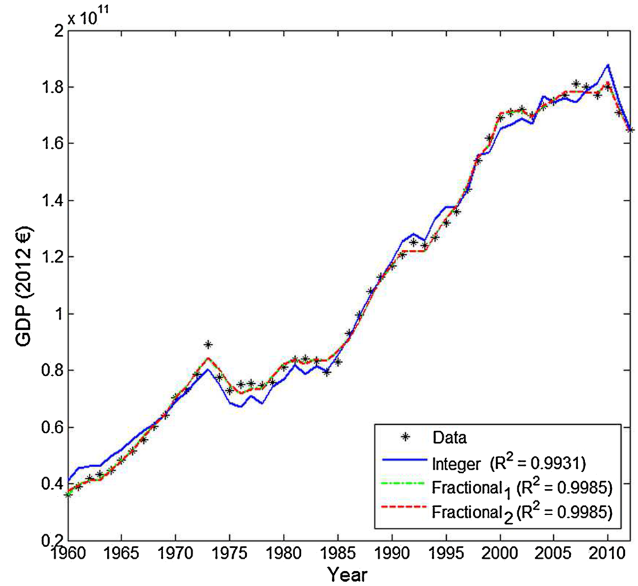


Fig. 5 Fitting results for the Portuguese case (models with nine variables): the integer model is given by (5), the fractional$_1$ model by (6), and the fractional$_2$ model by (15)

ing more independent variables than its integer counterpart, as seen from the t and p values.

Furthermore, fractional orders of x_8 and x_9 appearing in Table 1 are nearly ± 1. It is worth mentioning that the sign of α_8 is different in the integer and fractional models (13) and (14). This is particularly significant since it shows that M2 has an effect over a long time (a derivative of order almost -1 is not a local operator). On the other hand, variables x_1, x_3, x_6 and x_9 turn out to have influence in the present only.

4.2 The Portuguese economy

The models obtained for the Portuguese economy are shown in Fig. 5. The values of the orders α and the coefficients C are given in Table 3.

It can be seen that the differentiation orders for x_1, x_2, x_3, x_6 and x_7 are zero (or almost zero), which leads us to consider a simpler fractional order model, as was done for Spain. In this case, only variables x_4, x_5, x_8 and x_9 are assumed to have fractional order influence, as follows:

$$y(t) = \sum_{k=1,2,3,6,7} C_k x_k(t) + \sum_{k=4,5,8,9} C_k \mathrm{D}^{\alpha_k} x_k(t). \qquad (15)$$

Again, what this means is that only economic indicators for which $\alpha \neq 0$ have an influence over time on the GDP, not limited to the current value. The results related to this model are also included in Fig. 5 and Table 3.

The performance indices are summarised in Table 4. In that table, t values corresponding to variables which are necessary for the model, assuming a 5 % significance level, are in bold. It is clear from the results that the area (x_1) and the population (x_3) are variables without much influence in the GDP; the arable land (x_2) is also unnecessary for the fractional model. The results of the AIC confirm what variables are not needed in the model, even though this time the match is not perfect, but almost.

Furthermore, just as for the model for Spain, the fractional model has a clearly better performance, at the expense of needing more variables (in this case, 6 against 3 for the integer model).

Taking into account the low influence of variables x_1, x_2, and x_3 in the model, let us consider a simpler model with only 6 inputs, with an integer form given by

$$y(t) = C_4 x_4(t) + C_5 \int_{t_0}^{t} x_5(t)\mathrm{d}t + C_6 x_6(t) + C_7 x_7(t) + C_8 \frac{\mathrm{d}x_8(t)}{\mathrm{d}t} + C_9 \frac{\mathrm{d}x_9(t)}{\mathrm{d}t}, \qquad (16)$$

and a fractional form given by

$$y(t) = \sum_{k=6,7} C_k x_k(t) + \sum_{k=4,5,8,9} C_k \mathrm{D}^{\alpha_k} x_k(t). \qquad (17)$$

These new models for the Portuguese economy are shown in Fig. 6. The values of the orders α and the coefficients C, as well as performance indices, are also given in Tables 3 and 4, respectively. Once more, results obtained with fractional model (17) remain highly satisfactory (while w is equal to zero, other indices prove it is likely the best option), at the

Table 3 Fitting results for the Portuguese economy: orders of the fractional operator and coefficients

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
Integer (5)	0	0	0	0	−1	0	0	1	1
Fractional (6)	6.25×10^{-5}	0	1.25×10^{-4}	−2.144	−1.138	0	0	−2.247	−2.435
Fractional (15)	0	0	0	−2.144	−1.138	0	0	−2.221	−2.434
Integer (16)	–	–	–	0	−1	0	0	1	1
Fractional (17)	–	–	–	0	−0.508	0	0	−1.5	−1.552
	C_1 ($\times 10^4$)	C_2 ($\times 10^5$)	C_3 ($\times 10^2$)	C_4 ($\times 10^8$)	C_5 ($\times 10^{-2}$)	C_6 ($\times 10^{-1}$)	C_7	C_8 ($\times 10^{-3}$)	C_9 ($\times 10^{-2}$)
Integer (5)	2.014	31.074	−66.239	44.790	2.149	5.854	3.188	27.642	62.227
Fractional (6)	−6.652	8.534	8.866	9.301	−24.460	9.158	3.037	−9.650	−3.869
Fractional (15)	−8.374	9.191	7.954	9.089	−23.205	9.424	3.069	−9.614	−3.883
Integer (16)	–	–	–	231.964	−9.998	6.960	2.700	−230.754	64.860
Fractional (17)	–	–	–	106.866	15.564	10.823	3.795	20.067	−6.678

Table 4 Performance indices for the Portuguese economy

Index/statistic	Variable	Models with nine variables			Models with six variables	
		Integer (5)	Fractional (6)	Fractional (15)	Integer (16)	Fractional (17)
MSE ($\times 10^{18}$)		18.317	3.884	3.938	6.594	6.594
R^2		0.9931	0.9985	0.9985	0.9879	0.9970
MAD ($\times 10^9$)		3.229	1.421	1.438	4.233	2.600
t values	x_1	0.072	0.043	−0.165	–	–
	x_2	**5.650**	1.429	1.243	–	–
	x_3	−2.101	0.853	0.230	–	–
	x_4	0.870	**11.298**	**6.151**	2.708	**3.218**
	x_5	0.744	**−8.731**	**−4.408**	**−2.336**	**3.307**
	x_6	**2.606**	**9.582**	**8.389**	**2.497**	**8.515**
	x_7	**11.455**	**19.725**	**14.043**	**8.550**	**12.460**
	x_8	0.204	**−6.132**	**−4.147**	−1.337	**4.095**
	x_9	2.166	**−9.294**	**−6.474**	2.172	**−5.264**
p values	x_1	0.943	0.966	0.869	–	–
	x_2	1.100×10^{-6}	0.160	0.221	–	–
	x_3	0.041	0.398	0.819	–	–
	x_4	0.3891	1.4×10^{-14}	2.023×10^{-7}	9.468×10^{-3}	2.366×10^{-3}
	x_5	0.4608	3.690×10^{-11}	6.618×10^{-5}	2.388×10^{-2}	1.837×10^{-3}
	x_6	0.0124	2.454×10^{-12}	1.123×10^{-10}	1.618×10^{-2}	5.205×10^{-11}
	x_7	0	0	0	4.638×10^{-11}	0
	x_8	0.8395	2.155×10^{-7}	1.514×10^{-4}	0.188	1.688×10^{-4}
	x_9	0.0358	6.076×10^{-12}	6.774×10^{-8}	3.503×10^{-2}	3.610×10^{-6}
AIC		2363.1	2280.8	2281.6	2385.9	2310.5
w		0 %	60 %	40 %	0 %	0 %
AIC wihout one variable	x_1	2360.2	2278.0	2278.8		
	x_2	2389.1	2345.4	2280.6		
	x_3	2365.3	2278.1	2278.8		
	x_4	2361.1	2318.6	2311.5		
	x_5	2360.9	2303.7	2298.1		

Table 4 continued

Index/statistic	Variable	Models with nine variables			Models with six variables	
		Integer (5)	Fractional (6)	Fractional (15)	Integer (16)	Fractional (17)
	x_6	2367.8	2340.4	2329.3		
	x_7	2433.4	2385.3	2368.6		
	x_8	2360.2	2313.9	2296.2		
	x_9	2365.6	2315.2	2314.1		
w found from the AIC without one variable	x_1	28 %	51 %	42 %		
	x_2	**0 %**	**0 %**	17 %		
	x_3	2 %	49 %	41 %		
	x_4	18 %	**0 %**	**0 %**		
	x_5	21 %	**0 %**	**0 %**		
	x_6	1 %	**0 %**	**0 %**		
	x_7	**0 %**	**0 %**	**0 %**		
	x_8	28 %	**0 %**	**0 %**		
	x_9	2 %	**0 %**	**0 %**		

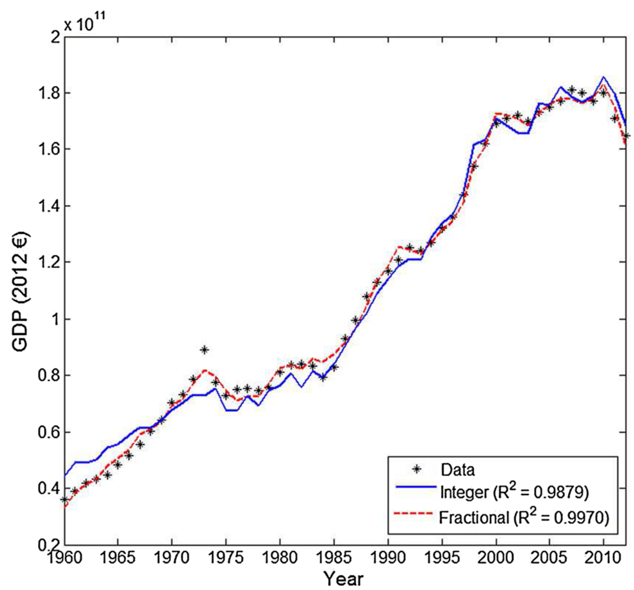


Fig. 6 Fitting results for the Portuguese case (models with six variables): the integer model is given by (16), and the fractional model by (17)

expense of more independent variables than its integer counterpart.

In this case, fractional orders of model (17) appearing in Table 3 are all (nearly) multiples of 1/2, just as in the equations of fractional diffusion (see e.g. [35]). We can thus hypothesise that such diffusion models (useful in areas such as bioengineering or soil dynamics) can also explain how these variables affect the economy; this hypothesis can only be checked when more countries are studied. It is worth mentioning that the sign of x_8 and x_9 is different in the integer and fractional models (16) and (17). This is particularly significant for x_8 (not so much for x_9, since $x_9 = x_5$ and α_5 was already negative), since it shows that M2 has an effect over a long time (a derivative of order -1.5 is not a local operator). So we can say that variables x_5 and x_8 have an influence on the GDP over time similar to that of diffusion processes, while other variables only have influence in the present.

4.3 Comparison: Spanish versus Portuguese economy

The main differences of the models for Portuguese and Spanish economic growth are found in the number of variables which have real influence in economy. In the Portuguese economy, these are school attendance (x_4), GCF (x_5), exports of goods and services (x_6), GGFCE (x_7) and M2 (x_8). In the Spanish case, the area (x_1, which, being a constant, means an independent term) and the population (x_3) are important too, while GGFCE (x_7) is likely not to be needed.

Spain is about five times the size of Portugal in both area and population and is the European Union's fifth largest economy, whereas the Portuguese economy is ranked in the fourteenth position. The results obtained in this work do not allow us to see whether differences in relevant variables are connected with the size of the country and its economy or not.

In the models for both countries, among the variables which appear to be relevant, some have differentiation orders in the fractional model (6) which are zero (or almost zero), leading us to consider a simpler model—namely (12) for the case of Spain, in which only variables x_4, x_5, x_8 and x_9 are assumed to have fractional order influence. These are not the same, however, as in the model for Portugal: x_4 has no fractional order in the Portuguese model; α_9 is nearly 1 in the model for Spain (revealing no long term effects) but is negative and fractional in the model for Portugal (revealing a cumulative influence over time); only α_5 and α_8 seem to denote a comparable effect of their variables for both countries.

5 Conclusions

This paper investigated modelling of national economic growth, namely, the gross domestic product (GDP), from a fractional calculus (FC) point of view. Nine macroeconomic indicators, chosen according to the practice established in the literature, were used to account for the behaviour of this financial process. The particular cases of Spain and Portugal were studied for the period 1960–2012, and results show that fractional models have a better performance than the other alternatives considered and proposed in the literature. In the end, simplified models with six or seven inputs were obtained. External and internal impacts, manufactured resources, and the quality of the natural and human resources are seen to be important factors.

Our future efforts will focus on the study of other economies of the European zone, and how the model could be realistically controlled. This last issue has been addressed, at least for fractional systems, mostly in a purely mathematical way (see e.g. [36]), without considering what might be feasible in practice.

Appendix

The economic data used in this work can be found in Tables 5 and 6. Sources for the economic data in Table 5 are as follows:

- x_1 is taken from [37]. The data concerns what is currently the territory of Spain only, and not what are now Equatorial Guinea and Western Sahara, which were always separate national economies. Slight variations in area, found in the database, which are spurious, since the territory of Spain did not change in the period considered, were discarded. This input is thus constant.
- x_2 and x_3 are taken from [37].
- x_4 is taken from [38]. As the data has a 5-year sampling time (starting in 1960), a third-order spline interpolation was used for intercalary years.
- x_5, x_6 and x_7 are taken from [37], in current euros. The price index mentioned below was used to convert values to 2012 euros.
- x_8 is taken from [39] in current euros in the 1999–2012 period. In the 1962–1968 period, it is taken from [37] also in current euros. These two series are clearly coherent. [40] has data for 1941–1970 in current pesetas; values for 1962–1970 are consistently 60 % of those in [37]: and so for 1960–1961 we used the values of [40] converted to euros and divided by 0.6. The price index mentioned below was used to convert values to 2012 euros.
- The price index mentioned several times above is the one implicit in [37], that for several variables provides values in current euros and in constant euros.

Sources for the economic data in Table 6 are as follows:

- x_1 is taken from [37]. The data concerns what is currently the territory of Portugal only, and not the former colonies, then overseas provinces, granted independence in the 1974–1976 period, and which were always separate national economies. A slight variation in 2004, in all probability spurious, found in the database, was kept; otherwise this input is constant.
- x_2 is taken from [37]. As the series begins in 1961, the value for that year was also assumed to be that of 1960.
- x_3 is taken from [37].
- x_4 is taken from [41] in the 1960–1990 period, when the series ends. In the 1998–2012 period, the value is a weighted average of the percentages of labour force with primary, secondary and tertiary education (to which the weights of 4, 12 and 18 years were assigned, according to the criteria of [41]), taken from [37]. Data in [37] for the 1992–1997 were neglected, as they are clearly inconsistent with figures for the following years (there are abrupt changes in values from 1997 to 1998 that can only result from different criteria used by the source, claimed to be the Eurostat.) The values for 1991–1997 were quadratically interpolated from those in the rest of the series (the resulting fit has a very convincing $R^2 = 0.9964$).
- x_5 is taken from [41] in the 1960–1993 period, in current PTE (Portuguese escudos). In the 1994–2012 period, it is taken from [37], in current euros. Data was converted to euros and the price index mentioned below used to convert values to 2012 euros. [37] has data from 1970 on, and its series coincides notably with that in [41] in the 1970–1993 period, without being precisely equal.
- x_6 and x_7 are taken from [37], in current euros. The price index mentioned below was used to convert values to 2012 euros.
- x_8 is taken from [42] in the 1960–1998 period, when the series ends, in current PTE. Since then Portugal belongs to the Eurozone, making it difficult to build a coherent series. Consequently data for deposits in the 2005–2010 period from [43] was used. These two series were cubically interpolated and extrapolated for 1999–2004 and 2011–2012. All values were converted to euros and the price index mentioned below was used to convert values to 2012 euros.

Table 5 Spanish economic data for years 1960–2012

Year	GDP ($\times 10^{11}$)	x_1	x_2	x_3	x_4	GCF ($\times 10^{10}$)	x_6 ($\times 10^{10}$)	x_7 ($\times 10^{10}$)	x_8 ($\times 10^{10}$)
1960	1.69	499,780	32.51	30,455,000	4.7	13.5	1.41	1.52	9.91
1961	1.89	499,780	32.51	30,739,250	4.74	15.2	1.50	1.66	11.33
1962	2.07	499,780	32.61	31,023,366	4.77	1.58	1.72	1.80	12.34
1963	2.27	499,780	32.42	31,296,651	4.79	14.5	1.75	2.04	13.42
1964	2.39	499,780	31.85	31,609,195	4.81	13.7	2.11	2.10	15.28
1965	2.54	499,780	31.95	31,954,292	4.82	13.3	2.08	2.29	16.46
1966	2.73	499,780	31.03	32,283,194	4.83	12.4	2.43	2.55	17.27
1967	2.85	499,780	31.49	32,682,947	4.85	10.8	2.43	2.87	18.34
1968	3.03	499,780	31.40	33,113,134	4.87	10.2	3.21	2.96	20.48
1969	3.30	499,780	32.18	33,441,054	4.91	10.6	3.74	3.24	23.18
1970	3.45	499,780	31.39	33,814,531	4.95	9.51	4.29	3.49	25.37
1971	3.61	499,780	32.69	34,191,678	5.01	9.15	4.81	3.72	29.16
1972	3.90	499,780	32.59	34,502,705	5.07	10.4	5.34	3.98	32.99
1973	4.20	499,780	32.12	34,817,071	5.15	11.7	5.74	4.28	36.82
1974	4.44	499,780	31.85	35,154,338	5.22	13.8	6.01	4.71	38.07
1975	4.47	499,780	31.66	35,530,725	5.3	13.1	5.67	4.50	38.78
1976	4.61	499,780	31.34	35,939,437	5.37	12.8	5.95	5.58	39.61
1977	4.74	499,780	31.29	36,370,050	5.44	12.2	6.45	5.84	38.18
1978	4.81	499,780	31.31	36,872,506	5.51	11.3	6.85	6.14	37.81
1979	4.82	499,780	31.18	37,201,123	5.58	11.0	6.77	6.40	38.33
1980	4.92	499,780	31.15	37,439,035	5.66	11.7	7.22	6.88	39.55
1981	4.92	499,780	31.17	37,740,556	5.75	10.9	8.21	7.34	41.16
1982	4.98	499,780	31.16	37,942,805	5.85	10.9	8.67	7.52	42.40
1983	5.07	499,780	31.22	38,122,429	5.95	10.7	9.92	7.89	43.74
1984	5.16	499,780	31.34	38,278,575	6.06	10.3	11.27	7.90	45.36
1985	5.28	499,780	31.16	38,418,817	6.17	10.7	11.29	8.28	47.26
1986	5.45	499,780	31.16	38,535,617	6.28	11.5	10.17	8.38	48.38
1987	5.75	499,780	31.20	38,630,820	6.38	13.0	10.45	9.14	52.48
1988	6.04	499,780	31.19	38,715,849	6.49	14.9	10.72	9.51	56.20
1989	6.33	499,780	31.06	38,791,473	6.61	16.4	10.78	10.30	60.39
1990	6.57	499,780	30.70	38,850,435	6.73	17.2	10.60	10.97	62.88
1991	6.74	499,780	30.55	38,939,049	6.86	17.1	10.89	11.71	65.46
1992	6.80	499,780	30.44	39,067,745	6.7	15.9	11.29	12.44	64.49
1993	6.73	499,780	29.99	39,189,400	7.14	14.1	12.23	12.68	67.90
1994	6.89	499,780	29.64	39,294,967	7.28	14.5	14.36	12.57	69.98
1995	7.08	499,780	28.12	39,387,017	7.42	1.55	15.86	12.81	72.81
1996	7.25	499,780	28.93	39,478,186	7.56	15.7	17.14	13.05	75.56
1997	7.53	499,780	28.60	39,582,413	7.69	16.6	19.82	13.17	77.00
1998	7.87	499,780	27.40	39,721,108	7.83	18.5	20.99	13.63	75.94
1999	8.24	499,780	26.96	39,926,268	7.97	20.7	21.99	14.16	79.58
2000	8.66	499,780	26.85	40,263,216	8.13	22.8	25.17	14.85	84.71
2001	8.98	499,780	26.20	40,720,484	8.29	23.7	25.63	15.29	89.47
2002	9.22	499,780	25.87	41,313,973	8.47	24.6	25.20	15.82	92.14
2003	9.51	499,780	26.07	42,004,522	8.64	26.1	25.02	16.46	100.5
2004	9.82	499,780	26.09	42,691,689	8.81	27.8	25.46	17.44	115.3
2005	10.17	499,780	25.87	43,398,143	8.97	30.0	26.10	18.27	143.3

Table 5 continued

Year	GDP ($\times 10^{11}$)	x_1	x_2	x_3	x_4	GCF ($\times 10^{10}$)	x_6 ($\times 10^{10}$)	x_7 ($\times 10^{10}$)	x_8 ($\times 10^{10}$)
2006	10.58	499,780	25.49	44,116,441	9.11	32.7	27.83	19.02	175.1
2007	10.95	499,780	25.22	44,878,945	9.23	33.9	29.46	20.07	202.2
2008	11.05	499,780	25.04	45,555,716	9.32	32.2	29.28	21.54	214.6
2009	10.63	499,780	25.05	45,908,594	9.37	2.55	25.43	22.69	223.2
2010	10.60	499,780	25.12	46,070,971	9.39	2.42	28.82	22.69	224.0
2011	10.65	499,780	25.08	46,174,601	9.47	2.29	32.22	22.30	214.6
2012	10.49	499,780	25.04	46,217,961	9.56	2.06	33.80	21.14	199.5

GDP, x_5, x_6, x_7 and x_8 in 2012 euros, x_1 in km^2, x_2 in % of x_1, x_3 in people and x_4 in years

Table 6 Portuguese economic data for years 1960–2012

Year	GDP ($\times 10^{10}$)	x_1	x_2	x_3	x_4	x_5 ($\times 10^9$)	x_6 ($\times 10^9$)	x_7 ($\times 10^9$)	x_8 ($\times 10^{10}$)
1960	3.62	91,500	27.40	8,857,716	1.4	7.49	5.18	3.32	1.88
1961	3.88	91,500	27.40	8,929,316	1.5	9.25	5.23	4.21	1.93
1962	4.17	91,500	27.35	8,993,985	1.5	9.68	6.37	4.66	2.17
1963	4.33	91,500	27.31	9,030,355	1.6	9.03	6.90	4.71	2.41
1964	4.47	91,500	27.27	9,035,365	1.7	9.66	9.57	4.86	2.68
1965	4.84	91,500	27.22	8,998,595	1.7	10.7	10.9	5.15	2.86
1966	5.17	91,500	27.18	8,930,990	1.8	12.9	11.7	5.57	3.17
1967	5.55	91,500	27.14	8,874,520	1.9	13.5	12.6	6.45	3.52
1968	6.02	91,500	27.09	8,836,650	1.9	13.9	12.6	7.01	3.99
1969	6.43	91,500	27.05	8,757,705	2	15.5	13.2	7.40	4.49
1970	7.03	91,500	27.00	8,680,431	2.1	16.4	13.6	8.47	4.98
1971	7.33	91,500	26.96	8,643,756	2.2	19.6	14.6	8.64	5.56
1972	7.87	91,500	26.92	8,630,430	2.3	22.7	17.0	9.18	6.29
1973	8.90	91,500	26.86	8,633,100	2.3	25.8	18.9	9.93	7.52
1974	7.76	91,500	26.81	8,754,365	2.5	24.6	16.6	9.54	6.20
1975	7.28	91,500	26.75	9,093,470	2.6	19.9	11.8	9.50	5.82
1976	7.50	91,500	26.70	9,355,810	2.7	18.3	10.4	8.98	5.68
1977	7.53	91,500	26.64	9,455,675	2.8	21.2	11.0	9.21	4.97
1978	7.47	91,500	26.59	9,558,250	2.9	19.3	11.9	9.07	4.74
1979	7.59	91,500	26.53	9,661,265	3	22.3	16.3	9.17	5.03
1980	8.11	91,500	26.48	9,766,312	3.1	22.0	17.7	10.3	5.46
1981	8.35	91,500	26.43	9,851,362	3.2	26.0	17.2	11.0	5.83
1982	8.41	91,500	26.37	9,911,771	3.2	25.3	17.7	10.9	5.92
1983	8.33	91,500	26.32	9,957,865	3.4	24.4	20.8	11.0	5.51
1984	7.92	91,500	26.26	9,996,232	3.5	20.5	23.4	10.4	5.34
1985	8.28	91,500	26.21	10,023,613	3.6	19.7	24.6	11.2	5.55
1986	9.29	91,500	26.15	10,032,734	3.8	20.8	24.5	12.4	5.94
1987	9.97	91,500	26.10	10,030,031	3.9	25.3	27.7	13.2	6.37
1988	10.8	91,500	26.04	10,019,610	4	29.4	30.4	14.9	6.85
1989	11.3	91,500	25.99	10,005,000	4.2	30.0	33.9	16.3	6.88
1990	11.7	91,500	25.62	9,983,218	4.3	30.8	34.8	17.9	6.61
1991	12.1	91,500	25.23	9,967,878	4.55	30.1	32.6	20.5	7.43
1992	12.5	91,500	24.86	9,969,953	4.7	30.6	31.1	21.2	7.96
1993	12.4	91,500	24.48	9,982,591	4.85	27.0	29.6	21.7	8.01

Table 6 continued

Year	GDP ($\times 10^{10}$)	x_1	x_2	x_3	x_4	x_5 ($\times 10^9$)	x_6 ($\times 10^9$)	x_7 ($\times 10^9$)	x_8 ($\times 10^{10}$)
1994	12.7	91,500	24.04	10,004,081	5	29.0	32.5	22.4	8.31
1995	13.2	91,500	23.53	10,030,376	5.17	31.7	35.9	23.1	8.64
1996	13.6	91,500	21.86	10,057,861	5.33	32.9	37.0	24.1	9.06
1997	14.4	91,500	20.57	10,091,120	5.49	37.9	40.0	25.5	9.45
1998	15.4	91,500	20.76	10,129,290	6.08	43.2	42.9	27.5	10.0
1999	16.2	91,500	18.74	10,171,949	6.28	46.4	43.8	29.3	11.5
2000	16.9	91,500	17.84	10,225,836	6.31	48.0	48.8	3.20	12.2
2001	17.1	91,500	17.47	10,292,999	6.35	47.3	47.9	3.31	12.7
2002	17.2	91,500	17.61	10,368,403	6.43	44.3	47.6	3.39	13.3
2003	17.0	91,500	16.74	10,441,075	6.6	40.0	47.0	3.41	13.9
2004	17.3	91,500	16.80	10,501,970	6.96	41.5	48.5	3.51	14.7
2005	17.5	91,470	13.97	10,549,424	7.05	41.1	48.3	3.69	16.6
2006	17.7	91,470	13.75	10,584,344	7.16	40.7	54.6	3.63	16.1
2007	18.1	91,470	11.98	10,608,335	7.23	41.4	58.4	3.60	17.2
2008	18.0	91,470	12.83	10,622,413	7.35	41.6	58.4	3.61	18.6
2009	17.7	91,470	13.04	10,632,482	7.54	35.9	49.7	3.92	18.6
2010	18.0	91,470	12.45	10,637,346	7.76	36.2	56.2	3.88	19.4
2011	17.1	91,470	11.96	10,556,999	8.2	30.5	61.2	3.44	20.8
2012	16.5	91,470	11.96	10,526,703	8.2	26.4	64.0	3.02	22.2

GDP, x_5, x_6, x_7 and x_8 in 2012 euros, x_1 in km^2, x_2 in % of x_1, x_3 in people and x_4 in years

– The price index mentioned several times above is taken from [42] for the 1960–2008 period, and extended in the 2009–2012 period using the price index published by the Instituto Nacional de Estatística.

References

1. Valério D, da Costa JS (2013) An introduction to fractional control. IET, Stevenage. ISBN 978-1-84919-545-4
2. Baeumer B, Meerschaert M (2007) Fractional diffusion with two time scales. Phys A Stat Mech Appl 373:237–251
3. Blackledge J (2008) Application of the fractal market hypothesis for modelling macroeconomic time series. ISAST Trans Electron Signal Process 2(1):89–110
4. Blackledge J (2010) Application of the fractional diffusion equation for predicting market behaviour. Int J Appl Math 40(3):130–158
5. Cartea A, del Castillo-Negrete D (2007) Fractional diffusion models of option prices in markets with jumps. Phys A Stat Mech Appl 374(2):749–763
6. Marom O, Momoniat E (2009) A comparison of numerical solutions of fractional diffusion models in finance. Nonlinear Anal Real World Appl 10:3435–3442
7. Gorenflo R, Mainardi F, Scalas E, Raberto M (2001) Mathematical finance trends in mathematics, chap. In: Kohlmann M, Tang S (eds) Fractional calculus and continuous-time finance III: the diffusion limit. Birkhäuser, Basel, pp. 171–180
8. Mainardi F, Raberto M, Gorenflo R, Scalas E (2000) Fractional calculus and continuous-time finance II: the waiting-time distribution. Phys A Stat Mech Appl 287:468–481
9. Meerschaert MM, Scalas E (2006) Coupled continuous time random walks in finance. Phys A Stat Mech Appl 370:114–118
10. Meerschaert MM, Sikorskii A (2012) Stochastic models for fractional calculus. De Gruyter, Berlin
11. Scalas E (2006) The application of continuous-time random walks in finance and economics. Phys A Stat Mech Appl 362:225–239
12. Scalas E, Gorenflo R, Mainardi F (2000) Fractional calculus and continuous-time finance. Phys A Stat Mech Appl 284(1–4):376–384
13. Boleantu M (2008) Fractional dynamical systems and applications in economy. Differ Geom Dyn Syst 10:62–70
14. Laskin N (2000) Fractional market dynamics. Phys A Stat Mech Appl 287:482–492
15. Petrás I, Podlubny I (2007) State space description of national economies: the V4 countries. Computational Stat Data Anal 52(2):1223–1233
16. Skovranek T, Podlubny I, Petrás I (2012) Modeling of the national economies in state-space: a fractional calculus approach. Econ Model 29(4):1322–1327
17. Xu Y, He Z (2013) Synchronization of variable-order fractional financial system via active control method. Cent Eur J Phys 11(6):824–835
18. Hu Z, Tu X, INE (2015) A new discrete economic model involving generalized fractal derivative. Adv Differ Equ 65:1–11
19. Yue Y, He L, Liu G (2013) Modeling and application of a new nonlinear fractional financial model. J Appl Math 2013:1–9
20. Chen WC (2008) Nonlinear dynamics and chaos in a fractional-order financial system. Chaos Solitons Fractals 36(5):1305–1314
21. Dadras S, Momeni HR (2010) Control of a fractional-order economical system via sliding mode. Phys A Stat Mech Appl 389(12):2434–2442
22. Wang Z, Huang X, Shi G (2011) Analysis of nonlinear dynamics and chaos in a fractional order financial system with time delay. Comput Math Appl 62(3):1531–1539

23. Yue Y, He L, Liu G (2013) Modeling and application of a new nonlinear fractional financial model. J Appl Math, p. ID 325050
24. Machado JAT, Mata ME (2015) Pseudo phase plane and fractional calculus modeling of western global economic downturn. Commun Nonlinear Sci Numer Simul 22(1–3):396–406
25. Machado JAT, Mata ME, Lopes AM (2015) Fractional state space analysis of economic systems. Entropy 17:5402–5421
26. Barro RJ (1991) Economic growth in a cross section of countries. Q J Econ 106(2):407–443
27. Sala-I-Martin XX (1997) I just ran two million regressions. Am Econ Rev 87(2):178–183
28. Tejado I, Valério D, Valério N (2014) Fractional calculus in economic growth modeling. The Portuguese case. In: Proceedings of the 2014 international conference on fractional differentiation and its applications (ICFDA14)
29. Tejado I, Valério D, Valério N (2015) CONTROLO'2014 Proceedings of the 11th Portuguese Conference on Automatic Control. In: Lecture notes in electrical engineering, vol. 321, chap. Fractional calculus in economic growth modelling: the Spanish case. Springer, pp 449–458
30. Valério D, Sá da Costa J (2011) An introduction to single-input, single-output fractional control. IET Control Theory Appl 5(8):1033–1057
31. Denison EF (1967) Why growth rates differ. Brooking Institutions, Washington
32. Lucas RE (1988) On the mechanics of economic development. J Monet Econ 22:3–42
33. Maddison A (1994) Explaining the economic performance of nations, 1820–1989. In: Baumol WJ et al (eds) Convergence of productivity. Oxford University Press, Oxford, pp 20–61
34. Archibugi D, Iammarino S (2002) The globalization of technological innovation: definition and evidence. Rev Int Polit Econ 9(1):98–122
35. Magin RL (2004) Fractional calculus in bioengineering. Begell House, Redding
36. Baskonus HM, Mekkaoui T, Hammouch Z, Bulut H (2015) Active control of a chaotic fractional order economic system. Entropy 17:5771–5783
37. World Bank (2013) World development indicators. http://data.worldbank.org/
38. de la Fuente A, Doménech R (2012) Educational attainment in the OECD, 1960–2010. Tech. rep., BBVA
39. Eurostat (2013) Statistics. http://epp.eurostat.ec.europa.eu/portal/page/portal/statistics/themes
40. Argandoña A (1975) La demanda de dinero en España, 1901–1970. Cuad Econ 3(6):3–49
41. Valério N (ed) (2001) Estatísticas Históricas Portuguesas. Instituto Nacional de Estatística, Portugal
42. Mata E, Valério N (1994) História Económica de Portugal: Uma perspectiva global. Editorial Presença, Lisboa
43. INE (2012) Statistical yearbook of Portugal 2011. Instituto Nacional de Estatística, Lisboa