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Singular mean-field optimal control for forward-backward stochastic systems and applications to finance

Mokhtar Hafayed

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Abstract In this paper, we study a class of singular stochastic optimal control problems for systems described by mean-field forward-backward stochastic differential equations, in which the coefficient depend not only on the state process but also its marginal law of the state process through its expected value. Moreover, the cost functional is also of mean-field type. The control variable has two components, the first being absolutely continuous and the second singular control. Necessary conditions for optimal control for this systems in the form of a Pontrygin maximum principle are established by means convex perturbation techniques for both continuous and singular parts. Our stochastic maximum principle differs from the classical one in the sense that here the adjoint equation has a mean-field type. The control domain is assumed to be convex. As an illustration of our results, we consider a mean-variance portfolio selection mixed with a recursive utility functional optimization problem involving singular control. The explicit expression of the optimal portfolio selection strategy is obtained in the state feedback form involving both state process and its marginal distribution, via the solutions of Riccati ordinary differential equations with time-inconsistent solution.

Keywords Singular stochastic control · Mean-field forward-backward stochastic differential equation · Mean-field type maximum principle · Time-inconsistent control problem · McKean–Vlasov systems

M. Hafayed (✉)
Laboratory of Applied Mathematics, Biskra University,
PO Box 145, 07000 Biskra, Algeria
e-mail: hafa.mokh@yahoo.com

1 Introduction

In the present paper, we discuss stochastic singular optimal control problem for a systems governed by controlled nonlinear mean-field forward-backward stochastic differential equations (FBSDEs) of the form:

$$\begin{cases} dx^{u,\xi}(t) = f\left(t, x^{u,\xi}(t), \mathbb{E}(x^{u,\xi}(t)), u(t)\right) dt \\ \quad +\sigma\left(t, x^{u,\xi}(t), \mathbb{E}(x^{u,\xi}(t)), u(t)\right) dW(t) + \mathcal{C}(t)d\xi(t), \\ dy^{u,\xi}(t) = g(t, x^{u,\xi}(t), \mathbb{E}(x^{u,\xi}(t)), y^{u,\xi}(t), \mathbb{E}(y^{u,\xi}(t)), \\ z^{u,\xi}(t), \mathbb{E}(z^{u,\xi}(t)), u(t))dt + z^{u,\xi}(t)dW(t)+\mathcal{D}(t)d\xi(t), \\ x^{u,\xi}(0) = a, \ y^{u,\xi}(T) = h(x^{u,\xi}(T), \mathbb{E}(x^{u,\xi}(T))), \end{cases} \tag{1}$$

where $f, \sigma, g, h, \mathcal{C}$ and $\mathcal{D}$ are given maps and the initial condition a is an $\mathcal{F}_0-$measurable random variable.The special mean-field FBSDEs-(1) which is also called *McKean–Vlasov* systems are obtained as a limit approach, by the mean-square limit, when $n \to +\infty$ of a system of interacting particles of the form

$$\begin{cases} dx_n^j(t) = f(t, x_n^j(t), \frac{1}{n}\sum_{i=1}^n x_n^i(t), u(t))dt \\ +\sigma(t, x_n^j(t), \frac{1}{n}\sum_{i=1}^n x_n^i(t), u(t))dW^j(t) + \mathcal{C}(t)d\xi(t), \\ dy_n^j(t) = -g(t, x_n^j(t), \frac{1}{n}\sum_{i=1}^n x_n^i(t), y_n^j(t), \frac{1}{n}\sum_{i=1}^n y_n^i(t), \\ z_n^j(t), \frac{1}{n}\sum_{i=1}^n z_n^i(t), u(t))dt + z_n^j(t)dW^j(t) + \mathcal{D}(t)d\xi(t), \end{cases}$$

where $(W^j(\cdot) : j \geq 1)$ is a collection of independent Brownian motions, and $\xi(\cdot)$ is the singular part of the control. Noting that mean-field FBSDEs-(1) occur naturally in the probabilistic analysis of financial optimization problems and the optimal control of dynamics of the McKean–Vlasov type. Moreover, the above mathematical mean-field approaches play an important role in different fields of economics, finance, physics, chemistry and game theory.

The expected cost on the time interval $[0, T]$ is defined by

$$J\left(u(\cdot), \xi(\cdot)\right) = \mathbb{E}\left\{\phi\left(x^{u,\xi}(T), \mathbb{E})x^{u,\xi}(T))\right) + \varphi\left(y^{u,\xi}(0), \mathbb{E}\left(y^{u,\xi}(0)\right)\right) + \int_0^T \ell(t, x^{u,\xi}(t), \mathbb{E}(x^{u,\xi}(t)), y^{u,\xi}(t), \mathbb{E}(y^{u,\xi}(t)), z^{u,\xi}(t), \mathbb{E}(z^{u,\xi}(t)), u(t))dt + \int_{[0,T]} \mathcal{L}(t)d\xi(t)\right\}. \quad (2)$$

where ℓ, ϕ, φ and $\mathcal{L}$ are an appropriate functions. This cost functional is also of mean-field type, as the functions ℓ, ϕ, φ depend on the marginal law of the state process through its expected value.

It worth mentioning that since the cost functional J is possibly a nonlinear function of the expected value stands in contrast to the standard formulation of a control problem. This leads to a so-called *time-inconsistent control problem* where the Bellman dynamic programming does not hold. The reason for this is that one cannot apply the law of iterated expectations on the cost functional. Noting that in most cases, the classical singular control problem was studied through dynamic programming principle.

Any admissible control $(u^*(\cdot), \xi^*(\cdot))$ satisfying

$$J\left(u^*(\cdot), \xi^*(\cdot)\right) = \inf_{(u(\cdot),\xi(\cdot))\in\mathcal{U}_1\times\mathcal{U}_2} J\left(u(\cdot), \xi(\cdot)\right), \quad (3)$$

is called an optimal control. The corresponding state processes, solution of mean-field FBSDE-(1), is denoted by $\left(x^*(\cdot), y^*(\cdot), z^*(\cdot)\right) = (x^{u^*,\xi^*}(\cdot), y^{u^*,\xi^*}(\cdot), z^{u^*,\xi^*}(\cdot))$.

The stochastic singular control problems have received considerable research attention in recent years due to wide applicability in a number of different areas, see for instance [1–15] . In most classical cases, the optimal singular control problem was investigated through dynamic programming principle. It was shown in particular that the value function is a solution of some quasi-variational inequalities. Some applications of singular and impulse stochastic control in financial mathematics, a cash management problem, optimal control of an exchange rate, and portfolio optimization under transaction costs have been investigated in Korn [16]. Stochastic maximum principle for optimal control problems of forward backward systems involving impulse controls has been studied in Wu and Zhang [3,12]. The stochastic maximum principle for singular control was considered by many authors, see for instance [1,2,4–10]. The first version of maximum principle for singular stochastic control problems was obtained by Cadenillas and Haussmann [9]. In Dufour and Miller [10], the authors derived stochastic maximum principle where the singular part has a linear form. The necessary and sufficient conditions for near-optimal singular control was obtained by Hafayed Abbas and Veverka [5]. For this type of problem, the reader may consult the papers by Haussmann and Suo [4] and the list of references therein. The necessary and sufficient conditions of near-optimality for singular control for jump diffusion processes have been investigated in Hafayed and Abbas [6]. Necessary and sufficient conditions of near-optimal singular control for mean-field SDE have been established by Hafayed and Abbas [7]. More interestingly, mean-field type stochastic maximum principle for optimal singular control, where the control domain is assumed to be convex, has been studied in Hafayed [8].

The stochastic maximum principle of optimality for classical FBSDEs has been studied by many authors, see e.g. [17–26]. The necessary conditions of optimality for FBSDEs in global form, with uncontrolled diffusions coefficient was derived by Xu [17]. However, Shi and Wu [18] first derived stochastic maximum principle for fully coupled forward-backward stochastic control system in global form. In recent paper by Yong [23], the author completely solved the problem of maximum principle of optimality for fully coupled FBSDEs. He considered an optimal control problem for general coupled FBSDEs with mixed initial-terminal conditions and derived the necessary conditions for optimality when the control variable appears in the diffusion coefficients of the forward equation and the control domain is not necessarily convex. More recently, the general maximum principle for controlled forward-backward stochastic systems has been studied by Wu [22]. This type of maximum principle has broad applications in mathematical finance and economics such as the recursive mean–variance portfolio choice problems. The maximum principles for stochastic recursive optimal control problems under partial information have been investigated in Wang and Wu [20]. Necessary and sufficient conditions for near-optimality for recursive problems have been established by Hui, Huang, Li, and Wang [21]. We refer the readers to [26–28] and the references cited therein, for some other relevant results of optimal controls and FBSDEs. Stochastic forward–backward linear quadratic optimal control problem with delay has been studied in Huang, Li and Shi [24].

The stochastic differential equations of mean-field type was introduced by Kac [34] as a stochastic model for the Vlasov-kinetic equation of plasma and the study of which was initiated by McKean model. Since then, many authors made contributions on mean-field stochastic control and applications, see for instance, [7,8,29–39] . Mean-field stochastic maximum principle of optimality was considered by many authors, see for instance [7,30,33,36–40]. Mean-field singular stochastic control problems have been investigated in Hafayed and Abbas [7]. The mean-field stochastic maximum principle for forward-backward stochastic differential equations with jump poisson processes has been investigated in Hafayed [29]. He derived the necessary conditions for optimality when the control variable not appear in the diffusion coefficient of the forward equation and the control domain is not necessarily convex by means spike variational method. In Lazry and Lions [35] the authors introduced a general mathematical modeling approach for high-dimensional systems of evolution equations corresponding to a large number of particles (or agents). They extended the field of such mean-field approaches also to problems in economics, finance and game theory. In Buckdahn, Djehiche, Li and Peng [32] a general notion of mean-field BSDE associated with a mean-field SDE is obtained in a natural way as a limit of some high dimensional system of FBSDEs governed by a $d-$dimensional Brownian motion, and influenced by positions of a large number of other particles. In Buckdahn, Djehiche and Li [33] a general maximum principle was introduced for a class of stochastic control problems involving SDEs of mean-field type. However, sufficient conditions of optimality for mean-field SDE have been proved in Shi [40]. In Mayer-Brandis, Ø ksendal and Zhou [38] a stochastic maximum principle of optimality for systems governed by controlled Itô-Levy process of mean-field type was proved by using Malliavin calculus. Under the conditions that the control domains are convex, a various local maximum principle have been studied in [36,37,39]. The linear-quadratic optimal control problem for mean-field SDEs has been studied by Shi [40]. The second-order mean-field stochastic maximum principle for jump diffusion processes has been investigated in Hafayed and Abbas [30].

Our main goal in this paper is to establish a stochastic maximum principle for optimal singular stochastic control of mean-field FBSDEs, where the coefficient of the system depend not only on the state process but also its marginal law of the state process through its expected value. The cost functional is also of mean-field type. The mean-field problem under consideration is not simple extension from the mathematical point of view, but also provide interesting models in many applications such as mathematical finance, optimal control for mean-field systems. The proof of our main result is based on convex perturbation method. These necessary conditions are described in terms two adjoint processes, corresponding to the mean-field forward and backward components involving singular controls and a maximum conditions on the Hamiltonian. At the end, as an application to finance, mean-variance portfolio selection mixed with a recursive utility optimization problem is given. To streamline the presentation of this paper, we only study the one dimensional case. This paper extends the results obtained in Wu and Zhang [3] to mean-field singular control problem for systems described by mean-field FBSDEs and generalize the necessary conditions of optimality obtained in Hafayed [8] to mean-field forward-backward systems. This mean-field problem is not simple extension from the mathematical point of view, but also provide interesting models in many applications such as mathematical finance.

The rest of this paper is structured as follows. In Sect. 2 we formulate the mean-field singular stochastic control problem and describe the assumptions of the singular stochastic model. Section 3 is devoted to prove our main result which is a mean-field type stochastic maximum principle for optimal singular control where the system evolves according to mean-field FBSDE. As an illustration, using these results, mean-variance portfolio selection problem: time inconsistent solution is discussed in the last section.

2 Assumptions and problem formulation

We study in this paper stochastic singular optimal control problems of mean-field type of the following kind. Let $T > 0$ be a fixed time horizon and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions on which one-dimensional standard Brownian motion $W(\cdot) = \{W(t) : t \in [0, T]\}$ is defined. We assume $\mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$.

2.1 Notations

For convenience, we will use the following notation in this paper

1. In the sequel, $\mathbb{L}^2_{\mathcal{F}}([0, T]; \mathbb{R})$ denotes the Hilbert space of $\mathcal{F}_t-$adapted processes $(\varkappa(t))_{t\in[0,T]}$ such that $\mathbb{E}\left[\int_0^T |\varkappa(t)|^2 dt\right] < +\infty$.
2. For a differentiable function Φ we denote by $\Phi_\varkappa(t)$ its gradient with respect to the variable $\varkappa$,
3. We set

$$\begin{aligned}
\delta f(t) &= f\left(t, x^*(t), \widetilde{x}^*(t), u(t)\right) - f(t, x^*(t), \widetilde{x}^*(t), u^*(t)).\\
\delta\sigma(t) &= \sigma\left(t, x^*(t), \widetilde{x}^*(t), u(t)\right) - \sigma(t, x^*(t), \widetilde{x}^*(t), u^*(t)).\\
\delta g(t) &= g\left(t, x^*(t), \widetilde{x}^*(t), y^*(t), \widetilde{y}^*(t), z^*(t), \widetilde{z}^*(t), u(t)\right)\\
&\quad -g(t, x^*(t), \widetilde{x}^*(t), y^*(t), \widetilde{y}^*(t), z^*(t), \widetilde{z}^*(t), u^*(t)).\\
\delta\ell(t) &= \ell\left(t, x^*(t), \widetilde{x}^*(t), y^*(t), \widetilde{y}^*(t), z^*(t), \widetilde{z}^*(t), u(t)\right)\\
&\quad -\ell(t, x^*(t), \widetilde{x}^*(t), y^*(t), \widetilde{y}^*(t), z^*(t), \widetilde{z}^*(t), u^*(t)).
\end{aligned}$$

4. In what follows, C represents a generic constants, which can be different from line to line.

Since the objective of this technical note is to study optimal singular stochastic control, we give here the precise definition of the singular part of an admissible control.

Definition 2.1 An admissible control is a pair $(u(\cdot), \xi(\cdot))$ *of measurable* $\mathbb{A}_1 \times \mathbb{A}_2-$valued, $\mathcal{F}_t-$ adapted processes, such that

(i) $\xi(\cdot)$ is of bounded variation, non-decreasing continuous on the left with right limits and $\xi(0) = 0$.
(ii) $\mathbb{E}\left[\sup_{t\in[0,T]} |u(t)|^2 + |\xi(T)|^2\right] < \infty$.

We denote $\mathcal{U}_1 \times \mathcal{U}_2$ the set of all admissible controls. We note that since $d\xi(t)$ may be singular with respect to Lebesgue measure dt, we call $\xi(\cdot)$ the singular part of the control and the process $u(\cdot)$ its absolutely continuous part.

2.2 Assumptions

Throughout this paper, we also assume that the coefficients $f, \sigma, g, \ell, h, \varphi, \phi, \mathcal{C}, \mathcal{D}$ and $\mathcal{L}$ satisfy the following standing assumptions

Assumption (H1) The functions $f, \sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{A}_1 \rightarrow \mathbb{R}$. $g, \ell : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{A}_1 \rightarrow \mathbb{R}$, and $h, \varphi, \phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, are continuous and continuously differentiable in their variables including $(x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, u)$.

Assumption (H2) The derivatives of f, σ, g and h with respect to their variables including $(x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, u)$ are bounded. Further,

1. The derivatives of ϕ with respect to x and $\widetilde{x}$ are bounded by $C\left(1 + |x| + |\widetilde{x}|\right)$.
2. The derivatives of φ with respect to $y, \widetilde{y}$ are bounded by $C\left(1 + |y| + |\widetilde{y}|\right)$.
3. The derivatives of ℓ are bounded by $C(1 + |x| + |\widetilde{x}| + |y| + |\widetilde{y}| + |z| + |\widetilde{z}| + |u|)$.

Assumption (H3) The functions $\mathcal{C} : [0, T] \rightarrow \mathbb{R}$, $\mathcal{D} : [0, T] \rightarrow \mathbb{R}$ and $\mathcal{L} : [0, T] \rightarrow \mathbb{R}^+$ are continuous and bounded.

Under the assumptions (**H1**)–(**H3**), FBSDE-(1) has an unique solution $(x(t), y(t), z(t)) \in \mathbb{L}^2_{\mathcal{F}}([0, T]; \mathbb{R}) \times \mathbb{L}^2_{\mathcal{F}}([0, T]; \mathbb{R}) \times \mathbb{L}^2_{\mathcal{F}}([0, T]; \mathbb{R})$ such that

$$x^{u,\xi}(t) = a + \int_0^t f\left(s, x^{u,\xi}(s), \mathbb{E}(x^{u,\xi}(s)), u(s)\right) ds + \int_0^t \sigma(s, x^{u,\xi}(s), \mathbb{E}(x^{u,\xi}(s)), u(s)) dW(s) + \int_{[0,t]} \mathcal{C}(s) d\xi(s),$$

and for $t \in [0, T]$

$$y^{u,\xi}(t) = y^{u,\xi}(T) - \int_t^T g(s, x^{u,\xi}(s), \mathbb{E}(x^{u,\xi}(s)), y^{u,\xi}(s), \mathbb{E}(y^{u,\xi}(s)), z^{u,\xi}(s), \mathbb{E}(z^{u,\xi}(s)), u(s)) ds + \int_t^T z^{u,\xi}(s) dW(s) + \int_{[t,T]} \mathcal{D}(s) d\xi(s)$$

See Hafayed [29], Buckdahn, Li and Peng [33] for the solutions of mean-field Backward SDEs. See also Wu and Zhang ([3] Proposition 2.1 and Proposition 2.2).

2.3 Adjoint equations of mean-field type

We introduce the new adjoint equations involved in the stochastic maximum principle for our singular mean-field control problem. So for any $(u(\cdot), \xi(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ and the corresponding state trajectory $(x(\cdot), y(\cdot), z(\cdot)) = (x^{u,\xi}(\cdot), y^{u,\xi}(\cdot), z^{u,\xi}(\cdot))$, we consider the following adjoint equations of mean-field type, which are independent to singular control:

$$\begin{cases} d\Psi(t) = -\{f_x(t)\Psi(t) + \mathbb{E}[f_{\widetilde{x}}(t)\Psi(t)] \\ \qquad + \sigma_x(t) Q(t) + \mathbb{E}[\sigma_{\widetilde{x}}(t) Q(t)] + g_x(t) K(t) \\ \qquad + \mathbb{E}(g_{\widetilde{x}}(t) K(t)) + \ell_x(t) + \mathbb{E}(\ell_{\widetilde{x}}(t))\} dt \\ \qquad + Q(t) dW(t) \\ \Psi(T) = -\{h_x(T) K(T) + \mathbb{E}[(h_{\widetilde{x}}(T)) K(T)]\} \\ \qquad + \phi_x(T) + \mathbb{E}(\phi_{\widetilde{x}}(T)), \\ -dK(t) = \left[g_y(t) K(t) + \mathbb{E}\left(g_{\widetilde{y}}(t) K(t)\right) \right. \\ \qquad \left. + \ell_y(t) + \mathbb{E}\left(\ell_{\widetilde{y}}(t)\right)\right] dt \\ \qquad + \left\{g_z(t) K(t) + \mathbb{E}\left[g_{\widetilde{z}}(t) K(t)\right] + \ell_z(t) \right. \\ \qquad \left. + \mathbb{E}(\ell_{\widetilde{z}}(t))\right\} dW(t) \\ K(0) = -\varphi_y(y(0), \mathbb{E}(y(0))) \\ \qquad - \mathbb{E}\left[\varphi_{\widetilde{y}}(y(0), \mathbb{E}(y(0)))\right]. \end{cases} \tag{4}$$

Note that the first adjoint equation (backward) corresponding to the forward component turns out to be a linear mean-field backward SDE, and the second adjoint equation (forward) corresponding to the backward component turns out to be a linear mean-field forward SDE.

We define the Hamiltonian function

$$\mathcal{H} : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{A}_1 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R},$$

associated with the singular stochastic control problem (1)-(2) as follows

$$\begin{aligned} &\mathcal{H}(t, x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, u, \Psi(\cdot), Q(\cdot), K(\cdot)) \\ &:= \Psi(t) f(t, x, \widetilde{x}, u) + Q(t) \sigma(t, x, \widetilde{x}, u) \\ &\quad + K(t) g(t, x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, u) \\ &\quad + \ell(t, x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, u). \end{aligned} \tag{5}$$

If we denote by

$$\mathcal{H}(t) = \mathcal{H}(t, x(t), \widetilde{x}(t), y(t), \widetilde{y}(t), z(t), \widetilde{z}(t), u(t), \Psi(t), Q(t), K(t)),$$

then the adjoint eq. (4) can be rewritten as the following stochastic Hamiltonian system's type

$$\begin{cases} d\Psi(t) = -\{\mathcal{H}_x(t) + \mathbb{E}[\mathcal{H}_{\widetilde{x}}(t)]\}dt + Q(t)dW(t) \\ \Psi(T) = -[h_x(T) + \mathbb{E}(h_{\widetilde{x}}(T))]K(T). \\ -dK(t) = [\mathcal{H}_y(t) + \mathbb{E}(\mathcal{H}_{\widetilde{y}}(t))]dt \\ \qquad + [\mathcal{H}_z(t) + \mathbb{E}(\mathcal{H}_{\widetilde{z}}(t))]dW(t) \\ K(0) = -\varphi_y(y(0), \mathbb{E}(y(0))) \\ \qquad - \mathbb{E}[\varphi_{\widetilde{y}}(y(0), \mathbb{E}(y(0)))]. \end{cases} \tag{6}$$

It is a well known fact that under assumptions (**H1**) and (**H2**), the adjoint eqs. (4), (6) admits a unique solution $(\Psi(t), Q(t), K(t))$ such that

$$(\Psi(t), Q(t), K(t)) \\ \in \mathbb{L}^2_{\mathcal{F}}([0,T];\mathbb{R}) \times \mathbb{L}^2_{\mathcal{F}}([0,T];\mathbb{R}) \times \mathbb{L}^2_{\mathcal{F}}([0,T];\mathbb{R}).$$

Moreover, since the derivatives of $f, \sigma, g, h, \varphi, \phi$ with respect to $(x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z})$ are bounded, we deduce from standard arguments that there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\Psi(t)|^2 + \sup_{t\in[0,T]}|K(t)|^2 + \int_0^T |Q(t)|^2 dt\right] < C. \tag{7}$$

3 Necessary conditions for optimal singular control of mean-field FBSDEs

In this section, we establish a set of necessary conditions of Pontraygin's type for a stochastic singular control to be optimal where the system evolves according to nonlinear controlled mean-field FBSDEs. Convex perturbation techniques for singular and continuous parts are applied to prove our mean-field stochastic maximum principle.

The following theorem constitutes the main contribution of this paper.

Let $(x^*(\cdot), y^*(\cdot), z^*(\cdot))$ be the trajectory of the mean-field FBSDE-(1) and $(\Psi^*(\cdot), Q^*(\cdot), K^*(\cdot))$ be the solution of adjoint eq. (4) corresponding to the optimal singular control $(u^*(\cdot), \xi^*(\cdot))$.

Theorem 3.1 (Maximum principle for optimal singular control of mean-field FBSDEs in integral form). *Let Conditions (**H1**), (**H2**) and (**H3**) hold. If $(u^*(\cdot), \xi^*(\cdot))$ and $(x^*(\cdot), y^*(\cdot), z^*(\cdot))$ is an optimal solution of the mean-field singular control problem (1)–(2). Then the maximum principle holds, that is for all $(u, \xi) \in \mathbb{A}_1 \times \mathbb{A}_2$*

$$0 \le \mathbb{E}\int_0^T \mathcal{H}_u(t, \lambda^*(t), \mathbb{E}(\lambda^*(t)), u^*, \Lambda^*(t))(u - u^*(t))dt \\ + \mathbb{E}\int_{[0,T]} (\mathcal{L}(t) + \mathcal{C}(t)\Psi^*(t) + \mathcal{D}(t)K^*(t))d(\xi - \xi^*)(t), \\ a.e., \ t \in [0,T], \tag{8}$$

where $(\lambda^*(t), \mathbb{E}(\lambda^*(t))) = (x^*(t), \mathbb{E}(x^*(t)), y^*(t), \mathbb{E}(y^*(t)), z^*(t), \mathbb{E}(z^*(t)))$ and $\Lambda^*(t) = (\Psi^*(t), Q^*(t), \times K^*(\cdot))$.

Corollary 3.1 *Under Conditions of* Theorem 3.1. *Then there exists a unique $\mathcal{F}_t$−adapted processes $(\Psi^*(\cdot), Q^*(\cdot), K^*(\cdot))$ solution of mean-field FBSDE-(6) such that for all $(u, \xi) \in \mathbb{A}_1 \times \mathbb{A}_2$:*

$$0 \le \mathcal{H}_u(t, \lambda^*(t), \mathbb{E}(\lambda^*(t))u^*, \Lambda^*(t))(u(t) - u^*(t)) \\ + \mathbb{E}\int_{[0,T]} (\mathcal{L}(t) + \mathcal{C}(t)\Psi^*(t) + \mathcal{D}(t)K^*(t))d(\xi - \xi^*)(t), \\ \mathbb{P}-a.s., \quad a.e.\ t \in [0,T].$$

To prove Theorem 3.1 we need some preliminary results given in the following Lemmas. We derive the variational inequality (8) in several steps, from the fact that

$$J(u^*(\cdot), \xi^*(\cdot)) \le J(u^\epsilon(\cdot), \xi^\epsilon(\cdot)), \tag{9}$$

where $(u^\epsilon(\cdot), \xi^\epsilon(\cdot))$ is the so called convex perturbation of optimal control $(u^*(\cdot), \xi^*(\cdot))$ defined as follows

$$u^\epsilon(t) = u^*(t) + \epsilon(u(t) - u^*(t)) \\ \xi^\epsilon(t) = \xi^*(t) + \epsilon(\xi(t) - \xi^*(t)),$$

where $\epsilon \in [0,1]$ is sufficiently small, $(u(\cdot), \xi(\cdot))$ is an arbitrary element of $\mathcal{F}_t$− measurable random variable with values in $\mathbb{A}_1 \times \mathbb{A}_2$ which we consider as fixed from now on.

We emphasize that the convexity of $\mathbb{A}_1 \times \mathbb{A}_2$ has the consequence that $(u^\epsilon(\cdot), \xi^\epsilon(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ where

$$(u^\epsilon(\cdot), \xi^\epsilon(\cdot)) = (u^*(t), \xi^*(t)) \\ + \epsilon[(u(t), \xi(t)) - (u^*(t), \xi^*(t))].$$

Let $(\lambda^\epsilon(t), \mathbb{E}(\lambda^\epsilon(t))) = (x^\epsilon(t), \mathbb{E}(x^\epsilon(t)), y^\epsilon(t), \mathbb{E}(y^\epsilon(t)), z^\epsilon(t), \mathbb{E}(z^\epsilon(t)))$ be the solution of state eq. (1) and $\Lambda^\epsilon(t) = (\Psi^\epsilon(t), Q^\epsilon(t), K^\epsilon(t))$ be the solution of the adjoint eq. (4) corresponding to perturbed control $(u^\epsilon(\cdot), \xi^\epsilon(\cdot))$.

3.1 Variational equations

Now, we introduce the following variational equations which have a mean-field type. For simplicity of notation, we will still use $f_x(t) = \frac{\partial f}{\partial x}(t, x^*(t), \mathbb{E}(x^*(t)), u^*(t))$ *etc,*. Let $(x_1^\epsilon(\cdot), y_1^\epsilon(\cdot), z_1^\epsilon(\cdot))$ be the solution of the following mean-field FBSDEs:

$$\begin{cases} dx_1^\epsilon(t) = \{f_x(t)x_1^\epsilon(t) + f_{\widetilde{x}}(t)\mathbb{E}(x_1^\epsilon(t)) + f_u(t)u(t)\}dt \\ \qquad + \{\sigma_x(t)x_1^\epsilon(t) + \sigma_{\widetilde{x}}(t)\mathbb{E}(x_1^\epsilon(t)) + \sigma_u(t)u(t)\}dW(t) \\ \qquad + \mathcal{C}(t)d\xi, \qquad x_1^\epsilon(0) = 0, \\ dy_1^\epsilon(t) = \{g_x(t)x_1^\epsilon(t) + g_{\widetilde{x}}(t)\mathbb{E}(x_1^\epsilon(t)) + g_y(t)y_1^\epsilon(t) \\ \qquad + g_{\widetilde{y}}(t)\mathbb{E}(y_1^\epsilon(t)) + g_z(t)z_1^\epsilon(t) + g_{\widetilde{z}}(t)\mathbb{E}(z_1^\epsilon(t)) \\ \qquad + g_u(t)u(t)\}dt + z_1^\epsilon(t)dW(t) + \mathcal{D}(t)d\xi \\ y_1^\epsilon(T) = -[h_x(T) + \mathbb{E}(h_{\widetilde{x}}(T)))]x_1^\epsilon(T). \end{cases} \tag{10}$$

3.2 Duality relations

Our first Lemma below deals with the duality relations between $\Psi^*(t)$, $x_1^\epsilon(t)$ and $K^*(t)$, $y_1^\epsilon(t)$. This Lemma is very important for the proof of *Theorem* 3.1.

Lemma 3.1 *For* $\forall(u(\cdot),\xi(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ *such that* $(u^*(t), \xi^*(t)) + (u(t),\xi(t)) \in \mathcal{U}_1 \times \mathcal{U}_2$, $u^*(t)+u(t) \in \mathcal{U}_1$ *and* $\xi^*(t)+\xi(t) \in \mathcal{U}_2$ *we have*

$$\begin{aligned}
&\mathbb{E}\left(\Psi^*(T)x_1^\epsilon(T)\right)\\
&\quad= \mathbb{E}\int_0^T \left[\Psi^*(t)f_u(t)u(t) + Q^*(t)\sigma_u(t)u(t)\right]dt\\
&\quad -\mathbb{E}\int_0^T \Big\{x_1^\epsilon(t)g_x(t)K(t) + x_1^\epsilon(t)\mathbb{E}(g_{\widetilde{x}}(t)K(t))\\
&\quad + x_1^\epsilon(t)\ell_x(t) + x_1^\epsilon(t)\mathbb{E}(\ell_{\widetilde{x}}(t))\Big\}dt\\
&\quad +\mathbb{E}\int_{[0,T]} \Psi^*(t)\mathcal{C}(t)d\xi,
\end{aligned} \tag{11}$$

similarly,

$$\begin{aligned}
&\mathbb{E}\left[K^*(T)y_1^\epsilon(T)\right]\\
&\quad= -\mathbb{E}\Big\{\left[\varphi_y(0) + \mathbb{E}\left(\varphi_{\widetilde{y}}(0)\right)\right]y_1^\epsilon(0)\Big\}\\
&\quad +\mathbb{E}\int_0^T \Big\{K^*(t)g_x(t)x_1^\epsilon(t) + K^*(t)g_{\widetilde{x}}(t)\mathbb{E}\left(x_1^\epsilon(t)\right)\\
&\quad + K^*(t)g_u(t)u(t) - y_1^\epsilon(t)\ell_y(t) - y_1^\epsilon(t)\mathbb{E}(\ell_{\widetilde{y}}(t))\\
&\quad - z_1^\epsilon(t)\ell_z(t) - z_1^\epsilon(t)\mathbb{E}(\ell_{\widetilde{z}}(t))\Big\}dt + \mathbb{E}\int_{[0,T]} K^*(t)\mathcal{D}(t)d\xi,
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\mathbb{E}\left\{[\phi_x(T) + \mathbb{E}(\phi_{\widetilde{x}}(T))]x_1^\epsilon(T)\right\}\\
&\quad + \mathbb{E}\left\{\left[\varphi_y(0) + \mathbb{E}\left(\varphi_{\widetilde{y}}(0)\right)\right]y_1^\epsilon(0)\right\}\\
&\quad= -\int_0^T \Big\{x_1^\epsilon(t)\ell_x(t) + x_1^\epsilon(t)\mathbb{E}(\ell_{\widetilde{x}}(t)) + y_1^\epsilon(t)\ell_y(t)\\
&\quad + y_1^\epsilon(t)\mathbb{E}(\ell_{\widetilde{y}}(t)) + z_1^\epsilon(t)\ell_z(t) + z_1^\epsilon(t)\mathbb{E}(\ell_{\widetilde{z}}(t))\\
&\quad - \ell_u(t)u(t)\Big\}dt + \mathbb{E}\int_0^T \mathcal{H}_u(t)u(t)dt\\
&\quad + \mathbb{E}\int_{[0,T]} \left[\Psi^*(t)\mathcal{C}(t) + K^*(t)\mathcal{D}(t)\right]d\xi,
\end{aligned} \tag{13}$$

Proof of duality relation (11). By applying integration by parts formula to $\Psi^*(t)x_1^\epsilon(t)$, and since $x_1^\epsilon(0)=0$ we get

$$\begin{aligned}
&\mathbb{E}\left(\Psi^*(T)x_1^\epsilon(T)\right)\\
&\quad= \mathbb{E}\int_0^T \Psi^*(t)dx_1^\epsilon(t) + \mathbb{E}\int_0^T x_1^\epsilon(t)d\Psi^*(t)\\
&\quad + \mathbb{E}\int_0^T Q^*(t)\left[\sigma_x(t)x_1^\epsilon(t) + \sigma_{\widetilde{x}}(t)\mathbb{E}\left(x_1^\epsilon(t)\right) + \sigma_u(t)u(t)\right]dt\\
&\quad= I_1^\epsilon + I_2^\epsilon + I_3^\epsilon.
\end{aligned} \tag{14}$$

A simple computation shows that

$$\begin{aligned}
I_1^\epsilon &= \mathbb{E}\int_0^T \Psi^*(t)dx_1^\epsilon(t)\\
&= \mathbb{E}\int_0^T \Big\{\Psi^*(t)f_x(t)x_1^\epsilon(t) + \Psi^*(t)f_{\widetilde{x}}(t)\mathbb{E}\left(x_1^\epsilon(t)\right)\\
&\quad + \Psi^*(t)f_u(t)u(t)\Big\}dt + \mathbb{E}\int_{[0,T]} \Psi^*(t)\mathcal{C}(t)d\xi,
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
I_2^\epsilon &= \mathbb{E}\int_0^T x_1^\epsilon(t)d\Psi^*(t)\\
&= -\mathbb{E}\int_0^T \Big\{x_1^\epsilon(t)f_x(t)\Psi^*(t) + x_1^\epsilon(t)\mathbb{E}\left[f_{\widetilde{x}}(t)\Psi^*(t)\right]\\
&\quad + x_1^\epsilon(t)\sigma_x(t)Q^*(t) + x_1^\epsilon(t)\mathbb{E}\left[\sigma_{\widetilde{x}}(t)Q^*(t)\right]\\
&\quad + x_1^\epsilon(t)g_x(t)K^*(t) + x_1^\epsilon(t)\mathbb{E}\left[g_{\widetilde{x}}(t)K^*(t)\right]\\
&\quad + x_1^\epsilon(t)\ell_x(t) + x_1^\epsilon(t)\mathbb{E}(\ell_{\widetilde{x}}(t))\Big\}dt.
\end{aligned} \tag{16}$$

By standard arguments we get

$$\begin{aligned}
I_3^\epsilon &= \mathbb{E}\int_0^T Q^*(t)\sigma_x(t)x_1^\epsilon(t)dt\\
&\quad + \mathbb{E}\int_0^T Q^*(t)\sigma_{\widetilde{x}}(t)\mathbb{E}\left(x_1^\epsilon(t)\right)dt\\
&\quad + \mathbb{E}\int_0^T Q^*(t)\sigma_u(t)u(t)dt,
\end{aligned} \tag{17}$$

the duality relation (11) follows immediately from combining (15)–(17) and (14).

Proof of duality relation (12). By applying integration by parts formula to $K^*(t)y_1^\epsilon(t)$ we get

$$
\begin{aligned}
&\mathbb{E}\left(K^*(T)y_1^\epsilon(T)\right)\\
&\quad=\mathbb{E}\left(K^*(0)y_1^\epsilon(0)\right)+\mathbb{E}\int_0^T K^*(t)dy_1^\epsilon(t)\\
&\qquad+\mathbb{E}\int_0^T y_1^\epsilon(t)dK^*(t)\\
&\qquad-\mathbb{E}\int_0^T z_1^\epsilon(t)\Big\{g_z(t)K^*(t)+\mathbb{E}\left[g_{\widetilde{z}}(t)K^*(t)\right]\\
&\qquad+\ell_z(t)+\mathbb{E}\left(\ell_{\widetilde{z}}(t)\right)\Big\}dt\\
&\quad=I_1^\epsilon+I_2^\epsilon+I_3^\epsilon+I_4^\epsilon. \qquad (18)
\end{aligned}
$$

From (11) we obtain

$$
\begin{aligned}
I_2^\epsilon&=\mathbb{E}\int_0^T K^*(t)dy_1^\epsilon(t)\\
&=\mathbb{E}\int_0^T\Big\{K^*(t)g_x(t)x_1^\epsilon(t)+K^*(t)g_{\widetilde{x}}(t)\mathbb{E}\left(x_1^\epsilon(t)\right)\\
&\quad+K^*(t)g_y(t)y_1^\epsilon(t)+K^*(t)g_{\widetilde{y}}(t)\mathbb{E}\left(y_1^\epsilon(t)\right)\\
&\quad+K^*(t)g_z(t)z_1^\epsilon(t)+K^*(t)g_{\widetilde{z}}(t)\mathbb{E}\left(z_1^\epsilon(t)\right)\\
&\quad+K^*(t)g_u(t)u(t)\Big\}dt+\mathbb{E}\int_{[0,T]}K^*(t)\mathcal{D}(t)d\xi, \qquad (19)
\end{aligned}
$$

from (4) we obtain

$$
\begin{aligned}
I_3^\epsilon&=\mathbb{E}\int_0^T y_1^\epsilon(t)dK^*(t)\\
&=-\mathbb{E}\int_0^T\Big\{y_1^\epsilon(t)g_y(t)K^*(t)+y_1^\epsilon(t)\mathbb{E}\left(g_{\widetilde{y}}(t)K^*(t)\right)\\
&\quad+y_1^\epsilon(t)\ell_y(t)+y_1^\epsilon(t)\mathbb{E}\left(\ell_{\widetilde{y}}(t)\right)\Big\}dt, \qquad (20)
\end{aligned}
$$

and

$$
\begin{aligned}
I_4^\epsilon&=-\mathbb{E}\int_0^T[z_1^\epsilon(t)g_z(t)K^*(t)+z_1^\epsilon(t)\mathbb{E}\left(g_{\widetilde{z}}(t)K^*(t)\right)\\
&\quad+z_1^\epsilon(t)\ell_z(t)+z_1^\epsilon(t)\mathbb{E}\left(\ell_{\widetilde{z}}(t)\right)]dt. \qquad (21)
\end{aligned}
$$

Since

$$
\begin{aligned}
I_1^\epsilon&=\mathbb{E}\left(K^*(0)y_1^\epsilon(0)\right)\\
&=-\mathbb{E}\left\{\left[\varphi_y(0)+\mathbb{E}\left(\varphi_{\widetilde{y}}(0)\right)\right]y_1^\epsilon(0)\right\},
\end{aligned}
$$

the duality relation (12) follows immediately from combining (19)–(21) and (18).

Proof of (13). Combining (11) and (12) we get

$$
\begin{aligned}
&\mathbb{E}\left(\Psi^*(T)x_1^\epsilon(T)\right)+\mathbb{E}\left(K^*(T)y_1^\epsilon(T)\right)\\
&\quad=-\mathbb{E}\left\{\left[\varphi_y(0)+\mathbb{E}\left(\varphi_{\widetilde{y}}(0)\right)\right]y_1^\epsilon(0)\right\}\\
&\qquad+\int_0^T\Big\{-x_1^\epsilon(t)\ell_x(t)-x_1^\epsilon(t)\mathbb{E}(\ell_{\widetilde{x}}(t))-y_1^\epsilon(t)\ell_y(t)\\
&\qquad-y_1^\epsilon(t)\mathbb{E}(\ell_{\widetilde{y}}(t))-\ell_u(t)u(t)-z_1^\epsilon(t)\ell_z(t)\\
&\qquad-z_1^\epsilon(t)\mathbb{E}(\ell_{\widetilde{z}}(t))\Big\}dt+\mathbb{E}\int_0^T\mathcal{H}_u(t)u(t)dt\\
&\qquad+\mathbb{E}\int_{[0,T]}\Psi^*(t)\mathcal{C}(t)d\xi+\mathbb{E}\int_{[0,T]}K^*(t)\mathcal{D}(t)d\xi.
\end{aligned}
$$

From (6) and (10) in which $\Psi^*(T)=\phi_x(T)+\mathbb{E}(\phi_{\widetilde{x}}(T)) -\left\{h_x(T)K^*(T)+\mathbb{E}\left[(h_{\widetilde{x}}(T))K^*(T)\right]\right\}$ and $y_1^\epsilon(T)=-[h_x(T)+\mathbb{E}(h_{\widetilde{x}}(T))]x_1^\epsilon(T)$ we get

$$
\begin{aligned}
&\mathbb{E}\left[\Psi^*(T)x_1^\epsilon(T)\right]+\mathbb{E}\left[K^*(T)y_1^\epsilon(T)\right]\\
&\quad=[\phi_x(T)+\mathbb{E}(\phi_{\widetilde{x}}(T))]x_1^\epsilon(T).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\mathbb{E}\int_0^T\{\Psi(t)f_u(t)u(t)+Q(t)\sigma_u(t)u(t)\\
&\qquad+K(t)g_u(t)u(t)+\ell_u(t)u(t)\}dt\\
&\quad=\mathbb{E}\int_0^T\mathcal{H}_u(t)u(t)dt,
\end{aligned}
$$

we get

$$
\begin{aligned}
&\mathbb{E}\left\{[\phi_x(T)+\mathbb{E}(\phi_{\widetilde{x}}(T))]x_1^\epsilon(T)\right\}\\
&\qquad+\mathbb{E}\left\{\left[\varphi_y(0)+\mathbb{E}\left(\varphi_{\widetilde{y}}(0)\right)\right]y_1^\epsilon(0)\right\}\\
&\quad=-\int_0^T\Big\{x_1^\epsilon(t)\ell_x(t)+x_1^\epsilon(t)\mathbb{E}(\ell_{\widetilde{x}}(t))\\
&\qquad+y_1^\epsilon(t)\ell_y(t)+y_1^\epsilon(t)\mathbb{E}(\ell_{\widetilde{y}}(t))+z_1^\epsilon(t)\ell_z(t)\\
&\qquad+z_1^\epsilon(t)\mathbb{E}(\ell_{\widetilde{z}}(t))+\ell_u(t)u(t)\Big\}dt+\mathbb{E}\int_0^T\mathcal{H}_u(t)u(t)dt\\
&\qquad+\mathbb{E}\int_{[0,T]}\left[\Psi^*(t)\mathcal{C}(t)+K^*(t)\mathcal{D}(t)\right]d\xi,
\end{aligned}
$$

This completes the proof of (13). □

To this end we give the following estimations.

3.3 Some prior estimates

The second Lemma present the estimates of the perturbed state process $(x^{\epsilon}(\cdot), y^{\epsilon}(\cdot), z^{\epsilon}(\cdot))$, $x_1^{\epsilon}(t)$, $y_1^{\epsilon}(t)$ and $z_1^{\epsilon}(t)$.

Lemma 3.2 *Under assumptions* (**H1**), (**H2**) *and* (**H3**) *the following estimations holds*

$$\begin{aligned}&\mathbb{E}(\sup_{0\leq t\leq T}\left|x_1^{\epsilon}(t)\right|^2)\to 0,\\&\mathbb{E}(\sup_{0\leq t\leq T}\left|y_1^{\epsilon}(t)\right|^2)+\mathbb{E}\int_0^T\left[\left|z_1^{\epsilon}(s)\right|^2\right]ds\to 0,\end{aligned}\tag{22}$$

as $\epsilon\to 0$,

$$\begin{aligned}&\sup_{0\leq t\leq T}\left|\mathbb{E}\left(x_1^{\epsilon}(t)\right)\right|^2\to 0,\\&\sup_{0\leq t\leq T}\left|\mathbb{E}\left(y_1^{\epsilon}(t)\right)\right|^2+\int_t^T\left|\mathbb{E}\left(z_1^{\epsilon}(s)\right)\right|^2ds\to 0,\end{aligned}\tag{23}$$

as $\epsilon\to 0$,

$$\begin{aligned}&\mathbb{E}(\sup_{0\leq t\leq T}\left|x^{\epsilon}(t)-x^*(t)\right|^2)\to 0,\\&\mathbb{E}(\sup_{0\leq t\leq T}\left|y^{\epsilon}(t)-y^*(t)\right|^2\\&\quad+\mathbb{E}(\int_0^T\left|z^{\epsilon}(t)-z^*(t)\right|^2)dt\to 0,\ as\ \epsilon\to 0,\end{aligned}\tag{24}$$

and

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\frac{1}{\epsilon}\left[x^{\epsilon}(t)-x^*(t)\right]-x_1^{\epsilon}(t)\right|^2\right)\to 0,\\&\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\frac{1}{\epsilon}\left[y^{\epsilon}(t)-y^*(t)\right]-y_1^{\epsilon}(t)\right|^2\right)\to 0,\\&\mathbb{E}\int_0^T\left|\frac{1}{\epsilon}\left[z^{\epsilon}(s)-z^*(s)\right]-z_1^{\epsilon}(s)\right|^2ds\to 0,\end{aligned}\tag{25}$$

as $\epsilon\to 0$.

Let us also point out that the above estimates can be proved by using similar arguments developed in Hafayed ([29] Lemma 3.3 (without jump term)), ([3] Lemma 3.1) or ([19] Lemma 2.2, Lemma 12.3) and ([17] Lemma 1, Lemma 2), so we omit its proofs.

Lemma 3.3 *Let assumptions* (**H1**), (**H2**), *and* (**H3**) *hold. The following variational inequality holds*

$$\begin{aligned}O(\epsilon)\leq\ &\mathbb{E}\int_0^T\Big\{\ell_x(t)x_1^{\epsilon}(t)+\ell_{\widetilde{x}}(t)\mathbb{E}\left(x_1^{\epsilon}(t)\right)+\ell_y(t)y_1^{\epsilon}(t)\\&+\ell_{\widetilde{y}}(t)\mathbb{E}\left(y_1^{\epsilon}(t)\right)+\ell_z(t)z_1^{\epsilon}(t)+\ell_{\widetilde{z}}(t)\mathbb{E}\left(z_1^{\epsilon}(t)\right)\\&+\ell_u(t)u(t)\Big\}dt+\mathbb{E}\left\{\left(\phi_x(T)x_1^{\epsilon}(T)+\phi_{\widetilde{x}}(T)\mathbb{E}\left(x_1^{\epsilon}(T)\right)\right)\right\}\\&+\mathbb{E}\left\{\varphi_y(0)y_1^{\epsilon}(0)+\varphi_{\widetilde{y}}(0)\mathbb{E}\left(y_1^{\epsilon}(0)\right)\right\}\\&+\mathbb{E}\int_{[0,T]}\mathcal{L}(t)d(\xi-\xi^*)(t).\end{aligned}$$

Proof From (9) we have

$$\begin{aligned}&J\left(u(\cdot),\xi(\cdot)\right)-J\left(u^*(\cdot),\xi^*(\cdot)\right)\\&=\mathbb{E}\Big\{\int_0^T[\ell(t,x^{u,\xi}(t),\mathbb{E}(x^{u,\xi}(t)),y^{u,\xi}(t),\mathbb{E}(y^{u,\xi}(t)),\\&\quad z^{u,\xi}(t),\mathbb{E}(z^{u,\xi}(t)),u^{\epsilon}(t))-\ell(t,x^*(t),\mathbb{E}(x^*(t)),\\&\quad y^*(t),\mathbb{E}(y^*(t)),z^*(t),\mathbb{E}(z^*(t)),u^*(t)]dt\\&\quad+\left[\phi\left(x(T),\mathbb{E}(x(T))\right)-\phi\left(x^*(T),\mathbb{E}(x^*(T))\right)\right]\\&\quad+\left[\varphi\left(y(0),\mathbb{E}\left(y(0)\right)\right)-\varphi\left(y^*(0),\mathbb{E}\left(y^*(0)\right)\right)\right]\\&\quad+\int_{[0,T]}\mathcal{L}(t)d\left(\xi(t)-\xi^*(t)\right)\Big\}\geq 0,\end{aligned}\tag{26}$$

by applying Lemma 3.2, we have

$$\begin{aligned}&\frac{1}{\epsilon}\mathbb{E}\left\{\left(\phi(x^{\epsilon}(T),\widetilde{x}^{\epsilon}(T))-\phi(x^*(T),\widetilde{x}^*(T)\right)\right\}\\&\quad=\frac{1}{\epsilon}\mathbb{E}\left\{\int_0^1\phi_x\left(x^*(T)+\lambda\left[x^{\epsilon}(T)-x^*(T)\right],\widetilde{x}^*(T)\right.\right.\\&\quad\quad\left.+\lambda\left[\widetilde{x}^{\epsilon}(T)-\widetilde{x}^*(T)\right]\right)d\lambda\left(x^{\epsilon}(T)-x^*(T)\right)\\&\quad\quad+\int_0^1\phi_{\widetilde{x}}(x^*(T)+\lambda\left[x^{\epsilon}(T)-x^*(T)\right],\widetilde{x}^*(T)\\&\quad\quad\left.+\lambda\left[\widetilde{x}^{\epsilon}(T)-\widetilde{x}^*(T)\right])d\lambda\left(\widetilde{x}^{\epsilon}(T)-\widetilde{x}^*(T)\right)\right\}+O\left(\epsilon\right).\end{aligned}$$

From estimate (25), we get

$$\begin{aligned}&\frac{1}{\epsilon}\mathbb{E}\left\{\left(\phi(x^{\epsilon}(T),\widetilde{x}^{\epsilon}(T))-\phi(x^*(T),\widetilde{x}^*(T)\right)\right\}\\&\quad\to\mathbb{E}\left\{\phi_x(x^*(T),\widetilde{x}^*(T))x_1^{\epsilon}(T)\right.\\&\quad\quad\left.+\phi_{\widetilde{x}}(x^*(T),\widetilde{x}^*(T))\mathbb{E}\left(x_1^{\epsilon}(T)\right)\right\}\\&\quad=\mathbb{E}\left[\left(\phi_x(T)x_1^{\epsilon}(T)+\phi_{\widetilde{x}}(T)\mathbb{E}\left(x_1^{\epsilon}(T)\right)\right)\right]\\&\quad\quad\text{as } O\left(\epsilon\right)\to 0.\end{aligned}\tag{27}$$

Similarly, we have

$$\begin{aligned}&\frac{1}{\epsilon}\mathbb{E}\left\{\left(\varphi(y^{\epsilon}(0),\widetilde{y}^{\epsilon}(0))-\varphi(y^*(0),\widetilde{y}^*(0)\right)\right\}\\&\quad\to\mathbb{E}\left\{\varphi_y(y^*(0),\widetilde{y}^*(0))y_1^{\epsilon}(0)\right.\\&\quad\quad\left.+\varphi_{\widetilde{y}}(y^*(0),\widetilde{y}^*(0))\mathbb{E}\left(y_1^{\epsilon}(0)\right)\right\}\\&\quad=\mathbb{E}\left[\left(\varphi_y(0)y_1^{\epsilon}(0)+\varphi_{\widetilde{y}}(0)\mathbb{E}\left(y_1^{\epsilon}(0)\right)\right)\right],\\&\quad\quad\text{as } O\left(\epsilon\right)\to 0,\end{aligned}\tag{28}$$

and

$$\frac{1}{\epsilon}\mathbb{E}\int_0^T [\ell(t, x^\epsilon(t), \mathbb{E}(x^{u,\xi}(t)), y^\epsilon(t), \mathbb{E}(y^\epsilon(t)),$$
$$z^\epsilon(t), \mathbb{E}(z^\epsilon(t)), u^\epsilon(t)) - \ell(t, x^*(t), \mathbb{E}(x^*(t)), y^*(t),$$
$$\mathbb{E}(y^*(t)), z^*(t), \mathbb{E}(z^*(t)), u^*(t))]dt$$
$$\to \mathbb{E}\int_0^T \left[\ell_x(t)x_1^\epsilon(t) + \ell_{\widetilde{x}}(t)\mathbb{E}\left(x_1^\epsilon(t)\right) + \ell_y(t)y_1^\epsilon(t)\right.$$
$$+\ell_{\widetilde{y}}(t)\mathbb{E}\left(y_1^\epsilon(t)\right) + \ell_z(t)z_1^\epsilon(t) + \ell_{\widetilde{z}}(t)\mathbb{E}\left(z_1^\epsilon(t)\right)$$
$$\left. + \ell_u(t)u(t)\right] dt, \text{ as } O(\epsilon) \to 0. \tag{29}$$

The desired result follows by combining (26)–(29). This complete the proof of Lemma 3.3. □

3.4 Proof of Theorem 3.1.

The desired result follows immediately by combining (13) and Lemma 3.3. □

4 Application: mean-variance portfolio selection with a recursive utility functional involving singular control; time inconsistent problem

The mean-variance portfolio selection theory, which was first introduced by Markowitz [41], is a millstone in mathematical finance and has laid down the foundation of modern finance theory. By using sufficient maximum principle, the authors in Framstad, Øksendal and Sulem [42] gave the expression for the optimal portfolio selection in a jump diffusion market with time consistent solutions. Optimal portfolio and consumption decision problems for a small investor in a market model have been investigated in Jeanblanc–Picqué and Pontier [43]. The continuous time mean-variance portfolio selection problem has been studied in Zhou and Li [44]. The mean-variance portfolio selection problem where the state governed by SDE has been studied by Anderson and Djehiche [37]. Optimal dividend, harvesting rate and optimal portfolio for systems governed by jump diffusion processes have been investigated by Meyer–Brandis, Øksendal and Zhou [38]. Mean-variance portfolio selection problem for mean-field SDEs with jumps has been studied in [39]. Mean-variance portfolio selection problem mixed with a recursive utility functional, where the state process driven by FBSDEs with jumps has been studied by Shi and Wu [25], under the condition that $\mathbb{E}[x(T)] = a$ is a fixed given real positive number. The mean-variance portfolio selection problem mixed with a recursive utility functional "time-consistent approach", where the system governed by classical FBSDEs with singular control has been investigated by Wu and Zhang [3].

In this section, we will apply our mean-field stochastic maximum principle of optimal singular control for a system governed by mean-field FBSDEs to study mean-variance portfolio selection problem mixed with a recursive utility functional optimization in a financial market. Explicit expression of the optimal portfolio selection strategy is obtained in the state feedback form involving both state process $x(\cdot)$ and its marginal distribution $\mathbb{E}(x(\cdot))$, via the solutions of Riccati ordinary differential equations.

Suppose that we are given a mathematical market consisting of two investment possibilities:

(i) Bond: the first asset is a risk-free security whose price $P_0(t)$ evolves according to the ordinary differential equation

$$dP_0(t) = P_0(t)\,\rho_t dt,\ t \in [0, T],\ P_0(0) > 0, \tag{30}$$

where $\rho : [0, T] \to \mathbb{R}_+$ is a locally bounded continuous deterministic function.

(ii) Stock: a risky security where the price $P_1(t)$ at time t is given by

$$\begin{cases} dP_1(t) = \varsigma_t P_1(t)\,dt + \sigma_t P_1(t)\,dW(t), \\ P_1(0) > 0, \end{cases} \tag{31}$$

where $\varsigma : [0, T] \to \mathbb{R}$ and $\sigma : [0, T] \to \mathbb{R}$ are bounded continuous deterministic functions such that $\varsigma_t, \sigma_t \neq 0$ and $\varsigma_t - \rho_t > 0,\ \forall t \in [0, T]$.

The wealth dynamics with singular control: Let $x^{u,\xi}(0) = \zeta > 0$ be an initial wealth and $G \geq 0$. By combining (30) and (31) we introduce the wealth dynamics

$$\begin{cases} dx^{u,\xi}(t) = \left[\rho_t x^{u,\xi}(t) + (\varsigma_t - \rho_t)u(t)\right]dt \\ \qquad\qquad + \sigma_t u(t)dW(t) - Gd\xi(t) \\ x^{u,\xi}(0) = \zeta. \end{cases} \tag{32}$$

Let $\mathbb{A}_1 \times \mathbb{A}_2$ be a compact convex subset of $\mathbb{R} \times \mathbb{R}$. We denote $\mathcal{U}_1 \times \mathcal{U}_2$ the set of admissible $\mathcal{F}_t$−predictable portfolio strategies $(u(\cdot), \xi(\cdot))$ valued in $\mathbb{A}_1 \times \mathbb{A}_2$.

Time inconsistent control problem: We consider the following expected utility functional involving singular control

$$J(u(\cdot), \xi(\cdot)) = \frac{1}{2}Var\left[x^{u,\xi}(T)\right] + \mathbb{E}\left[y^{u,\xi}(0)\right]$$
$$+\mathbb{E}\int_{[0,T]} \mathcal{L}(t)d\xi(t),$$

which implies that

$$J(u(\cdot), \xi(\cdot)) = \mathbb{E}\left\{\frac{1}{2}\left[x^{u,\xi}(T) - \mathbb{E}\left(x^{u,\xi}(T)\right]^2\right.\right.$$
$$\left. - y^{u,\xi}(0) + \int_{[0,T]} \mathcal{L}(t)d\xi(t)\right\}, \tag{33}$$

where $y^{u,\xi}(\cdot)$ is a solution of the Backward SDEs

$$\begin{cases} -dy^{u,\xi}(t) = \left[\rho_t x^{u,\xi}(t) + (\varsigma_t - \rho_t)u(t) - \alpha y^{u,\xi}(t)\right]dt \\ \qquad - z^{u,\xi}(t)dW(t) + \beta d\xi(t),\ y^{u,\xi}(T) = x^{u,\xi}(T). \end{cases} \tag{34}$$

We assume that $\alpha > 0$ and $\beta \geq 0$. We note that this cost functional is nonlinear function of the expected value stands in contrast to the standard formulation of a control problem. This leads to a so called *time inconsistent control problem* in the sense that Bellman's dynamic programming does not hold.

Now, since

$$\begin{aligned} &f\left(t, x^{u,\xi}(t), \mathbb{E}(x^{u,\xi}(t), u(t)\right) \\ &= \rho(t)x^{u,\xi}(t) + (\varsigma_t - \rho_t)u(t), \\ &\sigma\left(t, x^{u,\xi}(t), \mathbb{E}(x^{u,\xi}(t)), u(t)\right) = \sigma_t u(t), \\ &\varphi(y^{u,\xi}(t), \mathbb{E}(y^{u,\xi}(t))) = -y^{u,\xi}(t), \\ &h\left(x^{u,\xi}(t), \mathbb{E}(x^{u,\xi}(t))\right) = x^{u,\xi}(t), \\ &\phi\left(x^{u,\xi}(t), \mathbb{E}(x^{u,\xi}(t))\right) = \frac{1}{2}x^{u,\xi}(t)^2 - x^{u,\xi}(t) \\ &-\frac{1}{2}\left[\mathbb{E}(x^{u,\xi}(t))\right]^2, \\ &g(t, x^{u,\xi}(t), \mathbb{E}(x^{u,\xi}(t)), y^{u,\xi}(t), \mathbb{E}(y^{u,\xi}(t)), \\ &z^{u,\xi}(t), \mathbb{E}(z^{u,\xi}(t)), u(t)) \\ &= -\rho_t x^{u,\xi}(t) - (\varsigma_t - \rho_t)u(t) + \alpha y^{u,\xi}(t), \\ &\ell = 0,\ \mathcal{C}(t) = G,\ \mathcal{D}(t) = \beta, \end{aligned} \tag{35}$$

then the Hamiltonian functional (5) gets the form

$$\begin{aligned} &\mathcal{H}(t, x, \widetilde{x}, y, \widetilde{y}, z, \widetilde{z}, u, \Psi(\cdot), Q(\cdot), K(\cdot)) \\ &= \left[\rho(t)x^{u,\xi}(t) + (\varsigma_t - \rho_t)u(t)\right](\Psi(t) - K(t)) \\ &+ \sigma_t u(t)Q(t) + \alpha y^{u,\xi}(t)K(t). \end{aligned}$$

According to the maximum condition ((8), *Theorem 3.1*), and since $(u^*(\cdot), \xi^*(\cdot))$ is optimal we immediately get

$$(\varsigma_t - \rho_t)\left(\Psi^*(t) - K^*(t)\right) + \sigma_t Q^*(t) = 0. \tag{36}$$

By using (35) where "$h_x(T) = 1$, $h_{\widetilde{x}}(T) = 0$, $\phi_x(T) = x^{u,\xi}(T) - 1$, $\phi_{\widetilde{x}}(T) = -\mathbb{E}(x(T)))$, $g_y(t) = \alpha$, $g_{\widetilde{y}}(t) = g_z(t) = g_{\widetilde{z}}(t) = \ell_z(t) = \ell_{\widetilde{z}}(t) = 0$, $\varphi_y(y(0), \mathbb{E}(y(0))) = -1$, and $\varphi_{\widetilde{y}}(y(0), \mathbb{E}(y(0))) = 0$" the mean-field adjoint eq. (4) being

$$\begin{cases} d\Psi^*(t) = -\rho(t)\left(\Psi^*(t) - K^*(t)\right)dt + Q^*(t)dW(t) \\ \Psi^*(T) = -K^*(T) + x^*(T) - \mathbb{E}(x^*(T)) - 1, \\ -dK^*(t) = \alpha K^*(t)dt, \\ K^*(0) = 1,\ t \in [0, T]. \end{cases} \tag{37}$$

Now, in order to solve the above eq. (37) and to find the expression of optimal portfolio strategy $u^*(\cdot)$ we conjecture a process $\Psi^*(t)$ of the form

$$\Psi^*(t) = \Phi_1(t)x^*(t) + \Phi_2(t)\mathbb{E}\left(x^*(t)\right) + \Phi_3(t), \tag{38}$$

where $\Phi_1(\cdot)$, $\Phi_2(\cdot)$ and $\Phi_3(\cdot)$ are deterministic differentiable functions. (See Hafayed and Abbas [7], Shi and Wu [25], Shi [40], Anderson and Djehiche [37], Framstad, Øksendal and Sulem [43], Ma and Yong [26] and Li [36], for other models of conjecture).

From last eq. in (37), which is a simple *ordinary differential equation* (ODE), we get immediately

$$K^*(t) = \exp(-\alpha t). \tag{39}$$

From (32), we get

$$\begin{aligned} x^{u,\xi}(t) &= \zeta + \int_0^t \left[\rho_s x^{u,\xi}(s) + (\varsigma_s - \rho_s)u(s)\right]ds \\ &\quad + \int_0^t \sigma_s u(s)dW(s) - \int_{[0,t]} G d\xi(s), \\ &= \zeta + \int_0^t \left[\rho_s x^{u,\xi}(s) + (\varsigma_s - \rho_s)u(s)\right]ds \\ &\quad + \int_0^t \sigma_s u(s)dW(s) - G\left[\xi(t) - \xi(0)\right], \end{aligned}$$

by a simple computations and since $\xi(0) = 0$, we get

$$\begin{aligned} d(\mathbb{E}(x^*(t)) &= \left\{\rho(t)\mathbb{E}(x^*(t)) + (\varsigma_t - \rho(t))\mathbb{E}(u^*(t))\right\}dt \\ &\quad - G\mathbb{E}(\xi^*(t)). \end{aligned} \tag{40}$$

Applying Itô's formula to (38) in virtue of SDE- (32) and (40), we get

$$\begin{aligned} d\Psi^*(t) &= \Phi_1(t)\left\{\left[\rho_t x^*(t) + (\varsigma_t - \rho_t)u^*(t)\right]dt \right. \\ &\quad \left. + \sigma_t u^*(t)dW(t)\right\} + x^*(t)\Phi_1'(t)dt \\ &\quad + \Phi_2(t)\left[\rho_t\mathbb{E}(x^*(t)) + (\varsigma_t - \rho_t)\mathbb{E}(u^*(t))\right] \\ &\quad + \mathbb{E}(x^*(t))\Phi_2'(t)dt + \Phi_3'(t)dt, \end{aligned}$$

which implies that

$$\begin{cases} d\Psi^*(t) = \left\{\Phi_1(t)\left[\rho_t x^*(t) + (\varsigma_t - \rho_t)u^*(t)\right] \right. \\ \qquad + x^*(t)\Phi_1'(t) \\ \qquad + \Phi_2(t)\left[\rho_t\mathbb{E}(x^*(t)) + (\varsigma_t - \rho_t)\mathbb{E}(u^*(t))\right] \\ \qquad \left. + \Phi_2'(t)\mathbb{E}\left(x^*(t)\right) + \Phi_3'(t)\right\}dt \\ \qquad + \Phi_1(t)\sigma_t u^*(t)dW(t) \\ \Psi^*(T) = \Phi_1(T)x^*(T) + \Phi_2(T)\mathbb{E}\left(x^*(T)\right) \\ \qquad + \Phi_3(T), \end{cases} \tag{41}$$

where $\Phi_1'(t)$, $\Phi_2'(t)$, and $\Phi_3'(t)$ denotes the derivatives with respect to t.

Next, comparing (41) with (37), we get

$$\begin{aligned}\rho_t\left(K^*(t)-\Psi^*(t)\right) &= \Phi_1(t)\left[\rho_t x^*(t)+(\varsigma_t-\rho_t)u^*(t)\right]+x^*(t)\Phi_1'(t)\\ &\quad+\Phi_2(t)\left[\rho_t\mathbb{E}(x^*(t))+(\varsigma_t-\rho_t)\mathbb{E}(u^*(t))\right]\\ &\quad+\Phi_2'(t)\mathbb{E}\left(x^*(t)\right)+\Phi_3'(t),\end{aligned} \tag{42}$$

$$Q^*(t)=\Phi_1(t)\sigma_t u^*(t). \tag{43}$$

Looking at the terminal condition of $\Psi^*(t)$, in (41) and (37), it is reasonable to get

$$\Phi_1(T)=1,\ \Phi_2(T)=-1,\ \Phi_3(T)=-1-K^*(T). \tag{44}$$

A simple computations, (42) gives

$$\begin{aligned}\Psi^*(t) &= -(\Phi_1(t)+\frac{\Phi_1'(t)}{\rho_t})x^*(t)\\ &\quad-(\Phi_2(t)+\frac{\Phi_2'(t)}{\rho_t})\mathbb{E}(x^*(t))\\ &\quad-\frac{\Phi_1(t)}{\rho_t}(\varsigma_t-\rho_t)u^*(t)\\ &\quad-\frac{\Phi_2(t)}{\rho_t}(\varsigma_t-\rho_t)\mathbb{E}(u^*(t))\\ &\quad+K^*(t)-\frac{\Phi_3'(t)}{\rho_t}.\end{aligned} \tag{45}$$

Comparing (45) and (38), together with (39) we obtain

$$\begin{aligned}\Phi_1(t)&=-\Phi_1(t)-\frac{\Phi_1'(t)}{\rho_t},\\ \Phi_2(t)&=-\Phi_2(t)-\frac{\Phi_2'(t)}{\rho_t},\\ \Phi_3(t)&=\exp(-\alpha t)-\frac{\Phi_3'(t)}{\rho_t},\end{aligned} \tag{46}$$

from (46) and (44) we deduce that $\Phi_1(\cdot)$, $\Phi_2(\cdot)$ and $\Phi_3(\cdot)$ satisfying the following *Riccati ordinary differential equation*

$$\begin{cases}\Phi_1'(t)+2\rho_t\Phi_1(t)=0,\ \Phi_1(T)=1,\\ \Phi_2'(t)+2\rho_t\Phi_2(t)=0,\ \Phi_2(T)=-1,\\ \Phi_3'(t)+\rho_t\Phi_3(t)=\rho_t\exp(-\alpha t),\\ \Phi_3(T)=-\exp(-\alpha T)-1.\end{cases} \tag{47}$$

By solving the first two ODEs in (47) we obtain $t\in[0,T]$

$$\Phi_1(t)=-\Phi_2(t)=\exp(2\int_t^T\rho_s ds). \tag{48}$$

Using the *method of Integrating factors* for the third linear ODE in (47), we get

$$\begin{aligned}\Phi_3(t)&=\frac{1}{\mu(t)}\Big[-\exp(-\alpha T)-1\\ &\quad-\int_t^T\mu(s)\rho_s\exp(-\alpha s)\,ds\Big],\end{aligned} \tag{49}$$

where the integrating factor $\mu(t)$ is given by

$$\mu(t)=\left(exp(2\int_t^T\rho_s ds\right).$$

Combining (36), (43) and (39), we obtain

$$\begin{aligned}u^*(t)&=\frac{(\rho_t-\varsigma_t)}{\Phi_1(t)\sigma_t^2}\Big[\Phi_1(t)\left(x^*(t)-\mathbb{E}(x^*(t))\right)\\ &\quad+\Phi_3(t)-\exp(-\alpha t)\Big],\end{aligned} \tag{50}$$

and by taking expectation we deduce

$$\mathbb{E}(u^*(t))=\frac{(\rho_t-\varsigma_t)}{\Phi_1(t)\sigma_t^2}\left[\Phi_3(t)-\exp(-\alpha t)\right].$$

Let us turn to singular part. Applying to the maximum condition ((8), *Theorem* 3.1), and since $(u^*(\cdot),\xi^*(\cdot))$ is optimal we get

$$\mathbb{E}\int_{[0,T]}(\mathcal{L}(t)+G\Psi^*(t)+\beta K^*(t))d\left(\xi-\xi^*\right)(t)\geq 0.$$

where $(\Psi^*(t),K^*(t))$ is the adjoint processes corresponding to optimal control.

Now, we define a set

$$\Theta=\left\{(w,t)\in\Omega\times[0,T]:\mathcal{L}(t)+G\Psi^*(t)+\beta K^*(t)>0\right\}, \tag{51}$$

and let $\xi(\cdot)\in\mathcal{U}_2$ such that

$$d\xi(t)=\begin{cases}0:\text{if }\mathcal{L}(t)+G\Psi^*(t)+\beta K^*(t)>0\\ d\xi^*(t):\text{otherwise.}\end{cases} \tag{52}$$

By a simple computations it is easy to see that

$$\begin{aligned}0&\leq\mathbb{E}\int_{[0,T]}(\mathcal{L}(t)+G\Psi^*(t)+\beta K^*(t))d\left(\xi(t)-\xi^*(t)\right)\\ &=\mathbb{E}\int_{[0,T]}(\mathcal{L}(t)+G\Psi^*(t)+\beta K^*(t))\mathbf{I}_\Theta(t,w)d\left(-\xi^*\right)(t)\\ &=-\mathbb{E}\int_{[0,T]}(\mathcal{L}(t)+G\Psi^*(t)+\beta K^*(t))\mathbf{I}_\Theta(t,w)d\xi^*(t),\end{aligned}$$

this implies that $\xi^*(\cdot)$ satisfies for any $t\in[0,T]$

$$\mathbb{E}\int_{[0,T]}(\mathcal{L}(t)+G\Psi^*(t)+\beta K^*(t))\mathbf{I}_\Theta(t)d\xi^*(t)=0. \tag{53}$$

Finally, from (51) and (52) we can easy shows that $\xi^*(t)$ has the form:

$$\xi^*(t) = \xi(t) + \int_0^t \mathbf{I}_{\overline{\Theta}}(s, w)ds, \tag{54}$$

which is in $\mathcal{U}_2$, where $\overline{\Theta}$ is the complement of the set Θ.

Theorem 4.1 *The optimal portfolio $(u^*(t), \xi^*(t))$ of our mean-variance portfolio selection problem (33)–(34), when the wealth dynamics evolves according to FBSDE-(32) is given in feedback form by*

$$u^*(t, x^*(t), \mathbb{E}(x^*(t)) = \frac{(\rho_t - \varsigma_t)}{\Phi_1(t)\sigma_t^2}[\Phi_1(t)\left(x^*(t) - \mathbb{E}(x^*(t))\right) + \Phi_3(t) - \exp(-\alpha t)],$$

and

$$\xi^*(t) = \xi(t) + \int_0^t \mathbf{I}_{\overline{\Theta}}(s, w)ds,$$

where Θ, $\Phi_1(t)$ and $\Phi_3(t)$ are given by (51), (48) and (49) respectively.

Concluding Remarks In this paper, we have proved a mean-field type maximum principle for optimal stochastic singular control for systems governed by mean-field forward-backward stochastic differential equations, where the coefficients depend not only on the state process but also its marginal distribution of the state process through it expected value. The cost functional is also of mean field type. Our mean-field maximum principle was applied to a mean-variance portfolio selection mixed with a recursive utility functional optimization problem involving singular control. The optimal portfolio selection strategy was obtained in the state feedback form involving both state process and its marginal distribution, via the solutions of Riccati ordinary differential equations with time-inconsistent solution.

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