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TECHNICAL PAPER

Generalized integral transform solution for free vibration of orthotropic rectangular plates with free edges

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Abstract
Free vibration of orthotropic rectangular thin plates of constant thickness with two opposite edges clamped and one or two edges free is analyzed by generalized integral transform technique. Numerically stable eigenfunctions in exponential function forms of Euler–Bernoulli beams with appropriate boundary conditions are adopted for each direction of the plate. The governing fourth-order partial differential equation for the mode function of free vibration is transformed into a system of linear equations, by integral transform in both directions of the rectangular plate. The boundary conditions at free edges are satisfied exactly by considering the terms generated in the transformed equations by integration by parts, which are absent in the equations by traditional Rayleigh–Ritz method. The natural frequencies of free vibration of orthotropic rectangular thin plates obtained by the proposed integral transform solution are compared with available results in the literature and numerical solutions by finite element analysis, showing excellent agreement and high convergence rate.

Keywords Rectangular thin plates · Orthotropic plates · Free vibration · Integral transform · Exact solutions · Natural frequency

1 Introduction

The present work addresses the free vibration of rectangular plate with one or two free edges, which has received continual attention in the literature [1–11]. It is well known that exact Levy's solutions exist for free vibration of rectangular plates when there is a pair of simply supported opposite edges. The Rayleigh–Ritz method has been used widely to determine free vibration frequency for other combinations of boundary conditions. Leissa [1] presented a comprehensive study of free vibration of rectangular plates and pointed out that among 21 combinations of simply supported, clamped and free boundary conditions at the four edges of a rectangular plate, exact solutions exist for six combinations with at least a pair of simply supported opposite edges (SSSS, SCSC, SCSS, SCSF, SSSF, SFSF), with the four letters indicating the boundary conditions at the edges in counterclockwise order starting from the left edge. Among the remaining 15 combinations of boundary conditions, three combinations do not involve a free edge (CCCC, CCCS, CCSS). For these three cases, the Rayleigh–Ritz method based on corresponding beam functions for the pair of opposite boundary conditions of the rectangular plate generates accurate results for the free vibration as the plate boundary conditions on all edges are exactly satisfied by the trial functions used in the Rayleigh–Ritz method. There are still 12 combinations of boundary conditions that involve at least one free edge. For rectangular plates with at least a free edge, the Rayleigh–Ritz method based on beam functions can only generate approximate results for the natural frequencies of free vibration, as the beam functions do not satisfy the plate boundary conditions at a free edge.

The difficulty with rectangular plate with at least one free edge has received special attention in the literature. A number of methods have been applied to tackle the problem. Bhat and Liew et al. [2, 12] used a set of beam characteristic

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orthogonal polynomials in the Rayleigh–Ritz method to obtain natural frequencies of rectangular plates. The orthogonal polynomials are generated by using a Gram–Schmidt process recursively from the first function that is constructed to satisfy all boundary conditions of the corresponding beam problem associated with the pair of opposite boundary conditions of the plate problem. Dickson and Di Blasio [3] showed that the orthogonal polynomials approach generated excellent results, with simpler first functions compared with Bhat [2] who used order four or five polynomial for the first function. Mizusawa [4] used B-Spline functions in the Rayleigh–Ritz method for isotropic rectangular plates with free edges. Shu and Du [5] proposed a generalized approach to implement general boundary conditions in the GDQ method for free vibration analysis of rectangular plates. Rossi et al. [6] used an optimized Rayleigh–Ritz method and a pseudo-Fourier expansion containing a number of optimization parameters. Kshirsagar and Bhaskar [7] showed that Levy solutions could be used in untruncated infinite series superposition method (UISSM) for any combination of boundary conditions of rectangular plates. Khov et al. [8] presented an exact series method for static and dynamic analyses of orthotropic plates with general boundary conditions, based on the expansion of the displacement function in a 2-D Fourier cosine series supplemented by several 1-D series. Eftekhari and Jafari [9, 10] proposed an alternative Ritz formulation for free vibration of rectangular plates with free edges, based on orthonormal trial functions [2, 3]. Integrations by parts are carried out in both edge directions to implement all general equations on each edge. Xing and Liu [13] obtained exact solutions for free vibrations of orthotropic rectangular thin plates using separation of variables method and presented exact solutions of three configurations (G-G-C-C, S-G-C-C and C-C-C-G) for the first time (where G stands for guided boundary condition). Recently, Banerjee et al. [11] developed the dynamic stiffness matrix of a rectangular plate for general cases to solve the free vibration problem of rectangular plates. Liang et al. [14, 15] proposed a semi-analytical method for the transient response of functionally graded material (FGM) rectangular plates under various boundary conditions, combining the state space method, differential quadrature method and numerical inversion of Laplace transform.

Recently, generalized integral transform technique (GITT) has been applied to obtain analytical solution of bending problem of orthotropic rectangular thin plate with two opposite edges clamped [16]. Among five sets of boundary conditions treated, three sets involve one or two free edges. The boundary conditions at free edges of the rectangular plate were treated exactly by carrying out integral transform of the boundary conditions along the free edge direction. Generalized integral transform technique is a hybrid analytical–numerical method that has been applied successfully in a wide range of flow and heat transfer problems [17–20], as well as in static and dynamic structural analyses [21–34]. In this work, the free vibration of orthotropic thin rectangular plates with a pair of opposite edges clamped and one or two free edges (CSCF, CCCF, CFCF) is studied analytically by using generalized integral transform technique. As the traditional expressions for eigenfunctions of Euler–Bernoulli beams in combinations of hyperbolic functions and trigonometric functions are not suitable for numerical implementations involving high-order modes [35–38], numerically stable expressions for eigenfunctions of Euler–Bernoulli beams with corresponding boundary conditions, in exponential function forms, are adopted as base functions for integral transform. Additional terms due to the difference between the boundary conditions of beam and plate at free edges of the rectangular plate are represented exactly by integral transform. The governing equation and all boundary conditions are integral-transformed into an infinite system of linear algebraic equations for the transformed coefficients. The infinite system of equations is truncated at a sufficiently high order to a finite size linear system of homogeneous equations. The eigenvalues and eigenvectors of the linear system are obtained by using a subroutine in Mathematica [39], thus yielding natural frequencies and mode functions of the free vibration of the rectangular plates. To our best knowledge, it is the first time that integral transform solutions for free vibration of orthotropic rectangular thin plates with free edges are obtained with numerically stable beam functions as base functions, large number of terms in series expansion and exact treatment of the boundary conditions at free edges. The calculated natural frequencies are compared with available results in the literature and a reference finite element solution, showing excellent convergence and agreement with the finite element solution.

2 Mathematical formulation

Let us consider free vibration of an orthotropic rectangular thin plate with a pair of opposite edges clamped and one or two free edges (CSCF, CCCF, CFCF). The plate has length a, width b, constant thickness h and lies in the (x, y) plane when no deflection occurs, as presented in Fig. 1. The notation for the combination of boundary conditions follows that proposed by Leissa [1]. For example, CSCF stands for a rectangular plate with clamped, simply supported, clamped and free boundary conditions at $x = 0$, $y = 0$, $x = a$ and $y = b$ edges, respectively. The governing equation for the free vibration of the orthotropic rectangular thin plate can be written as follows:

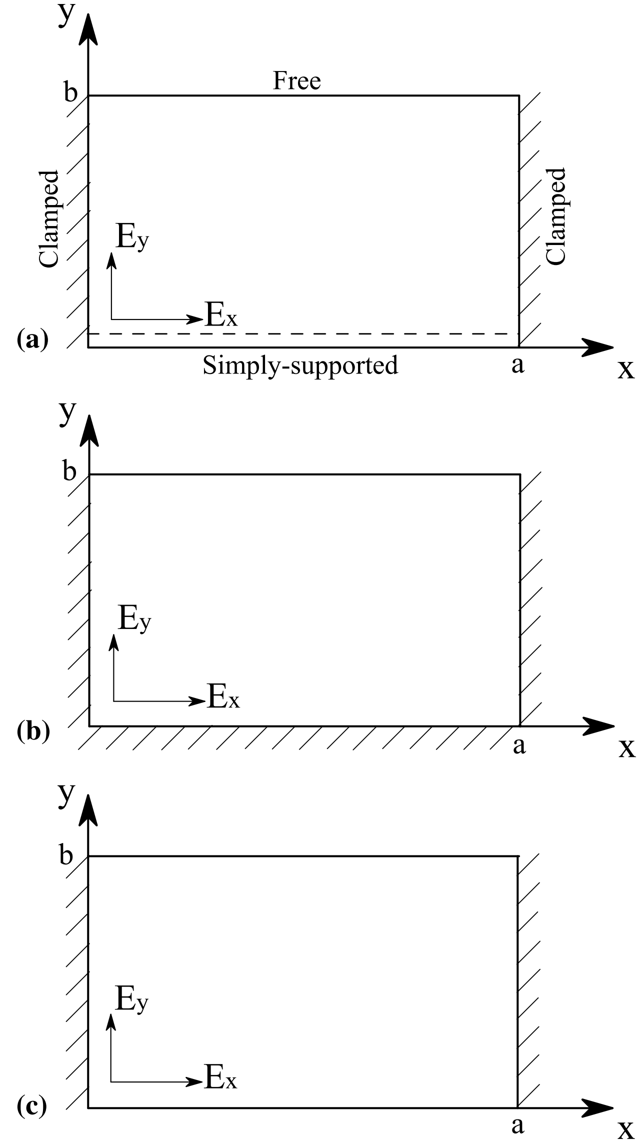

Fig. 1 Orthotropic thin rectangular plates with **a** CSCF, **b** CCCF and **c** CFCF boundary conditions

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + \rho \frac{\partial^2 w}{\partial^2 t} = 0, \tag{1}$$

where $w(x, y, t)$ is the dynamic transverse deflection, ρ is the mass density per unit area, and D_x and D_y are the flexural rigidities in the y and x directions, respectively. $H = D_1 + 2D_{xy}$ is the effective torsional rigidity, in which D_{xy} is called the torsional rigidity and $D_1 = \nu_y D_x = \nu_x D_y$ in terms of the Poisson's ratios ν_x and ν_y. The flexural and torsional rigidities of the plate are defined by

$$D_x = \frac{E_x h^3}{12(1-\nu_x \nu_y)}, \quad D_y = \frac{E_y h^3}{12(1-\nu_x \nu_y)}, \quad D_{xy} = \frac{Gh^3}{12}, \tag{2}$$

where E_x and E_y are the Young's moduli for the principal directions, respectively, and G is the shear modulus.

The following dimensionless variables and parameters are defined:

$$w^* = \frac{w}{w_0}, \quad x^* = \frac{x}{a}, \quad y^* = \frac{y}{b}, \quad t^* = \frac{t}{a^2\sqrt{\rho/D_x}}, \tag{3a-d}$$

$$\xi = \frac{b}{a}, \quad \alpha = \frac{H}{D_x}, \quad \beta = \frac{D_y}{D_x}, \quad \eta = \frac{D_{xy}}{D_x}, \tag{3e-h}$$

where w_0 is a reference deflection.

The governing equation (1) is cast in the following dimensionless form (dropping the superposed asterisks for simplicity):

$$\frac{\partial^4 w}{\partial x^4} + \frac{2\alpha}{\xi^2}\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\beta}{\xi^4}\frac{\partial^4 w}{\partial y^4} + \frac{\partial^2 w}{\partial^2 t} = 0. \tag{4}$$

We consider that a pair of left and right edges is clamped, with the following dimensionless boundary conditions:

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial x} = 0, \quad \text{at} \quad x = 0 \quad \text{and} \quad 1. \tag{5a-b}$$

For the boundary conditions at the bottom and top edges, the following three sets of dimensionless boundary conditions are considered:

(a) Simply supported and free (CSCF):

$$w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2} = 0, \quad \text{at} \quad y = 0, \tag{6a-b}$$

$$\frac{1}{\xi^2}\frac{\partial^2 w}{\partial y^2} + \nu_x \frac{\partial^2 w}{\partial x^2} = 0, \quad \text{at} \quad y = 1, \tag{6c}$$

$$\frac{\beta}{\xi^2}\frac{\partial^3 w}{\partial y^3} + (\alpha + 2\eta)\frac{\partial^3 w}{\partial x^2 \partial y} = 0, \quad \text{at} \quad y = 1; \tag{6d}$$

(b) Clamped and free (CCCF):

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial y} = 0, \quad \text{at} \quad y = 0, \tag{7a-b}$$

$$\frac{1}{\xi^2}\frac{\partial^2 w}{\partial y^2} + \nu_x \frac{\partial^2 w}{\partial x^2} = 0, \quad \text{at} \quad y = 1, \tag{7c}$$

$$\frac{\beta}{\xi^2}\frac{\partial^3 w}{\partial y^3} + (\alpha + 2\eta)\frac{\partial^3 w}{\partial x^2 \partial y} = 0, \quad \text{at} \quad y = 1; \tag{7d}$$

(c) Free and free (CFCF):

$$\frac{1}{\xi^2}\frac{\partial^2 w}{\partial y^2} + \nu_x \frac{\partial^2 w}{\partial x^2} = 0, \quad \text{at} \quad y = 0 \quad \text{and} \quad 1, \tag{8a-b}$$

$$\frac{\beta}{\xi^2}\frac{\partial^3 w}{\partial y^3}+(\alpha+2\eta)\frac{\partial^3 w}{\partial x^2 \partial y}=0, \quad \text{at} \quad y=0 \quad \text{and} \quad 1. \tag{8c-d}$$

Assuming a sinusoidal time response for free vibration,

$$w(x,y,t)=W(x,y)\mathrm{e}^{\mathrm{i}\lambda t}, \tag{9}$$

the governing equation for the mode function becomes

$$\frac{\partial^4 W}{\partial x^4}+\frac{2\alpha}{\xi^2}\frac{\partial^4 W}{\partial x^2 \partial y^2}+\frac{\beta}{\xi^4}\frac{\partial^4 W}{\partial y^4}-\lambda^2 W=0, \tag{10}$$

to be solved with the same boundary conditions.

3 Generalized integral transform solutions

The governing equation (10) combined with boundary conditions (5) with (6), (7) or (8), respectively, for CSCF, CCCF and CFCF is solved by employing the generalized integral transform technique (GITT). The integral transform kernels are based on the appropriate auxiliary eigenvalue problems, namely the fourth-order Sturm–Liouville equation with a complete and orthogonal base. The governing equation is integral-transformed in each coordinate direction through the multiplication of respective eigenfunctions together with the integration in its domain. It should be noted that the integration by parts combined with transformed free boundary terms of plate is employed in the generated terms of the transformed governing equation to incorporate the plate free boundary condition as it is not satisfied by the beam functions. Finally, the truncated transformed equations form a system of homogeneous linear algebraic equations, which is solved through a standard eigensystem algorithm to obtain the eigenvalues and eigenvectors.

3.1 Auxiliary eigenvalue problems

The auxiliary eigenvalue problem in the 'x' coordinate is the same for all three problems above, as follows:

$$\frac{\mathrm{d}^4 X_i(x)}{\mathrm{d}x^4}=\mu_i^4 X_i(x) \quad \text{and} \quad 0<x<1, \tag{11a}$$

$$X_i(0)=\frac{\mathrm{d}X_i(0)}{\mathrm{d}x}=0, \tag{11b-c}$$

$$X_i(1)=\frac{\mathrm{d}X_i(1)}{\mathrm{d}x}=0, \tag{11d-e}$$

and the eigenfunctions $X_i(x)$ and eigenvalues μ_i of problem (11) are given in the Appendix.

For the three sets of homogeneous boundary conditions, the auxiliary eigenvalue problem in the 'y' coordinate is defined by the same governing equation:

$$\frac{\mathrm{d}^4 Y_{kj}(y)}{\mathrm{d}y^4}=\phi_{kj}^4 Y_{kj}(y), \quad 0<y<1, \quad \text{and} \quad k=1,2,3, \tag{12}$$

and different boundary conditions as follows:

(a) CSCF boundary conditions ($k=1$):

$$Y_{1j}(0)=\frac{\mathrm{d}^2 Y_{1j}(0)}{\mathrm{d}y^2}=0, \tag{13a-b}$$

$$\frac{\mathrm{d}^2 Y_{1j}(1)}{\mathrm{d}y^2}=\frac{\mathrm{d}^3 Y_{1j}(1)}{\mathrm{d}y^3}=0; \tag{13c-d}$$

(b) CCCF boundary conditions ($k=2$)*:*

$$Y_{2j}(0)=\frac{\mathrm{d}Y_{2j}(0)}{\mathrm{d}y}=0, \tag{14a-b}$$

$$\frac{\mathrm{d}^2 Y_{2j}(1)}{\mathrm{d}y^2}=\frac{\mathrm{d}^3 Y_{2j}(1)}{\mathrm{d}y^3}=0; \tag{14c-d}$$

(c) CFCF boundary conditions ($k=3$):

$$\frac{\mathrm{d}^2 Y_{3j}(0)}{\mathrm{d}y^2}=\frac{\mathrm{d}^3 Y_{3j}(0)}{\mathrm{d}y^3}=0, \tag{15a-b}$$

$$\frac{\mathrm{d}^2 Y_{3j}(1)}{\mathrm{d}y^2}=\frac{\mathrm{d}^3 Y_{3j}(1)}{\mathrm{d}y^3}=0, \tag{15c-d}$$

where $Y_{kj}(y)$ and ϕ_{kj} are the eigenfunctions and eigenvalues of problem (12) with boundary conditions (13), (14) or (15), and presented in the Appendix for SF, CF and FF boundary conditions, respectively.

3.2 Orthogonality property

The normalized eigenfunctions $X_i(x)$ and $Y_{kj}(y)$ satisfy the following orthogonality property:

$$\int_0^1 X_i(x)X_m(x)\mathrm{d}x=\delta_{im}, \quad \text{and} \quad i,m=1,2,3,\ldots, \tag{16}$$

and

$$\int_0^1 Y_{kj}(y)Y_{kn}(y)\mathrm{d}y=\delta_{jn}, \quad k=1,2,3 \quad \text{and} \quad j,n=1,2,3,\ldots, \tag{17}$$

with δ being the Kronecker delta function.

3.3 Generalized integral transform pairs

Based on the eigenvalue problems, the following generalized integral transform pairs are defined, first in the x direction and second in the y direction:

$$\tilde{W}_m(y) = \int_0^1 W(x,y)X_m(x)\mathrm{d}x, \quad \text{transform,} \tag{18a}$$

$$W(x,y) = \sum_{m=1}^{\infty} X_m(x)\tilde{W}_m(y), \quad \text{inverse,} \tag{18b}$$

$$\bar{W}_{mn} = \int_0^1 \tilde{W}_m(y)Y_{kn}(y)\mathrm{d}y, \quad \text{transform,} \tag{18c}$$

$$\tilde{W}_m(y) = \sum_{n=1}^{\infty} Y_{kn}(y)\bar{W}_{mn}, \quad \text{inverse,} \quad \text{and} \quad m,n = 1,2,3,\ldots \tag{18d}$$

where $Y_{kn}(y)$ represents $Y_{1n}(y)$, $Y_{2n}(y)$ and $Y_{3n}(y)$ for the boundary conditions of CSCF, CCCF and CFCF, respectively.

3.4 Transformed governing equations

The dimensionless governing equation (10) is integral-transformed first in the 'x' direction of the plate, being operated by $\int_0^1 X_i(x)\_\mathrm{d}x$, with application of the inverse formula in Eq. (18):

$$\mu_i^4\tilde{W}_i(y) + \frac{2\alpha}{\xi^2}\sum_{m=1}^{\infty} F_{im}\frac{\partial^2\tilde{W}_m(y)}{\partial y^2} + \frac{\beta}{\xi^4}\frac{\partial^4\tilde{W}_i(y)}{\partial y^4} - \lambda^2\tilde{W}_i(y) = 0, \quad \text{and} \quad i = 1,2,3,\ldots \tag{19}$$

For each set of the CSCF, CCCF and CFCF boundary conditions, Eq. (19) is then operated by $\int_0^1 Y_{ki}(y)\_\mathrm{d}y$, with the inverse formulas in Eq. (18) applied:

$$\mu_i^4\bar{W}_{ij} + \frac{2\alpha}{\xi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{ijmn}\bar{W}_{mn} + \frac{\beta}{\xi^4}\int_0^1 \frac{\partial^4\tilde{W}_i(y)}{\partial y^4}Y_{kj}(y)\mathrm{d}y - \lambda^2\bar{W}_{ij} = 0, \quad k = 1,2,3 \quad \text{and} \quad i,j = 1,2,3,\ldots \tag{20}$$

Integration by parts on the third term in Eq. (20) is carried out:

$$\int_0^1 \frac{\partial^4\tilde{W}_i(y)}{\partial y^4}Y_{kj}(y)\mathrm{d}y = \left[Y_{kj}(y)\frac{\partial^3\tilde{W}_i(y)}{\partial y^3}\right]_0^1 - \left[Y_{kj}'(y)\frac{\partial^2\tilde{W}_i(y)}{\partial y^2}\right]_0^1 + \left[Y_{kj}''(y)\frac{\partial\tilde{W}_i(y)}{\partial y}\right]_0^1 - \left[Y_{kj}'''(y)\tilde{W}_i(y)\right]_0^1 + \bar{W}_{ij}\phi_{kj}^4, \quad \text{and} \quad k = 1,2,3. \tag{21}$$

Applying the generalized integral transform process on the plate free boundary conditions (6, 7 and 8), with the inverse formulas in Eq. (18) applied, we obtain

$$\frac{\partial^2\tilde{W}_i(y)}{\partial y^2} = -\xi^2\nu_x\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\bar{W}_{mn}Y_{kn}(y)F_{im}, \tag{22a}$$

$$\frac{\partial^3\tilde{W}_i(y)}{\partial y^3} = -\frac{\xi^2}{\beta}(\alpha+2\eta)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\bar{W}_{mn}Y_{kn}'(y)F_{im}, \tag{22b}$$

where the coefficient $F_{im} = \int_0^1 X_iX_m''\mathrm{d}x$.

(a) CSCF boundary conditions ($k = 1$)*:*

Equation (21) is simplified by applying the boundary conditions (6a-b) and (13):

$$\int_0^1 \frac{\partial^4\tilde{W}_i(y)}{\partial y^4}Y_{1j}(y)\mathrm{d}y = \left|Y_{1j}(y)\frac{\partial^3\tilde{W}_i(y)}{\partial y^3}\right|_{y=1} - \left|Y_{1j}'(y)\frac{\partial^2\tilde{W}_i(y)}{\partial y^2}\right|_{y=1} + \bar{W}_{ij}\phi_{1j}^4. \tag{23}$$

A system of linear algebraic equations is obtained by introducing Eqs. (23 and 22) into the transformed governing equation (20):

$$\mu_i^4\bar{W}_{ij} + \frac{2\alpha}{\xi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}F_{ijmn}\bar{W}_{mn} + \frac{\beta\nu_x}{\xi^2}Y_{1j}'(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y_{1n}(1)F_{im}\bar{W}_{mn} - \frac{\alpha+2\eta}{\xi^2}Y_{1j}(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y_{1n}'(1)F_{im}\bar{W}_{mn} + \frac{\beta}{\xi^4}\phi_{1j}^4\bar{W}_{ij} - \lambda^2\bar{W}_{ij} = 0. \tag{24}$$

(b) CCCF boundary conditions ($k = 2$)*:*

Similarly, Eq. (21) is simplified by applying boundary conditions (7a-b) and (14):

$$\int_0^1 \frac{\partial^4\tilde{W}_i(y)}{\partial y^4}Y_{2j}(y)\mathrm{d}y = \left|Y_{2j}(y)\frac{\partial^3\tilde{W}_i(y)}{\partial y^3}\right|_{y=1} - \left|Y_{2j}'(y)\frac{\partial^2\tilde{W}_i(y)}{\partial y^2}\right|_{y=1} + \bar{W}_{ij}\phi_{2j}^4. \tag{25}$$

A system of algebraic equations is obtained by introducing Eqs. (25 and 22) into the transformed governing equation (20):

$$\mu_i^4\bar{W}_{ij} + \frac{2\alpha}{\xi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}F_{ijmn}\bar{W}_{mn} + \frac{\beta\nu_x}{\xi^2}Y'_{2j}(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y_{2n}(1)F_{im}\bar{W}_{mn} - \frac{\alpha+2\eta}{\xi^2}Y_{2j}(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y'_{2n}(1)F_{im}\bar{W}_{mn} + \frac{\beta}{\xi^4}\phi_{2j}^4\bar{W}_{ij} - \lambda^2\bar{W}_{ij} = 0. \tag{26}$$

(c) CFCF boundary conditions ($k = 3$)*:*

Finally, Eq. (21) is simplified by applying boundary condition (15):

$$\int_0^1 \frac{\partial^4\tilde{W}_i(y)}{\partial y^4}Y_{3j}(y)\mathrm{d}y = \left[Y_{3j}(y)\frac{\partial^3\tilde{W}_i(y)}{\partial y^3}\right]_0^1 - \left[Y'_{3j}(y)\frac{\partial^2\tilde{W}_i(y)}{\partial y^2}\right]_0^1 + \bar{W}_{ij}\phi_{3j}^4. \tag{27}$$

A similar system of linear algebraic equations is obtained by introducing Eqs. (27 and 22) into the transformed governing equation (20):

$$\mu_i^4\bar{W}_{ij} + \frac{2\alpha}{\xi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}F_{ijmn}\bar{W}_{mn} + \frac{\beta\nu_x}{\xi^2}\Big(Y'_{3j}(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y_{3n}(1)F_{im}\bar{W}_{mn} - Y'_{3j}(0)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y_n(0)F_{im}\bar{W}_{mn}\Big) - \frac{\alpha+2\eta}{\xi^2}\Big(Y_{3j}(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y'_{3n}(1)F_{im}\bar{W}_{mn} - Y_{3j}(0)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y'_n(0)F_{im}\bar{W}_{mn}\Big) + \frac{\beta}{\xi^4}\phi_{3j}^4\bar{W}_{ij} - \lambda^2\bar{W}_{ij} = 0. \tag{28}$$

The coefficients are defined by the following integrals:

$$F_{ijmn} = \int_0^1 X_iX''_m\mathrm{d}x\int_0^1 Y_jY''_n\mathrm{d}y, \tag{29a}$$

$$F_{im} = \int_0^1 X_iX''_m\mathrm{d}x, \tag{29b}$$

which can be calculated analytically from integral formulas [16, 40].

3.5 Solution of the transformed equations

The infinite systems of linear algebraic equations are truncated to a sufficiently large finite order NW in both directions, for computational purposes. The truncated equations (24), (26) and (28) can be written in matrix form as follows:

$$(\mathbf{K} - \lambda^2\mathbf{I})\mathbf{W} = 0, \tag{30}$$

which forms a standard eigenvalue system. The eigenvalues and eigenvectors of the system (30) can be readily obtained by using the *Mathematica* package [39]. The ith mode function $W^{(i)}(x,y)$ corresponding to dimensionless frequency $\lambda^{(i)}$ can be thus constructed by using the inverse defined by Eq. (18):

$$W^{(i)}(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}X_m(x)Y_{kn}(y)\bar{W}_{mn}^{(i)}, \quad \text{and} \quad k = 1,2,3, \tag{31}$$

where $\bar{W}_{mn}^{(i)}$ are the components of the ith eigenvector $\mathbf{W}^{(i)}$.

4 Results and discussion

4.1 Convergence behavior of the solution

Firstly, the convergence behaviors of the first five natural frequencies for the isotropic square plate free vibration with boundary conditions CSCF, CCCF and CFCF are studied by using the GITT approach. The GITT solutions in Eqs. (24, 26 and 28) are examined up to truncation terms NW = 80, in comparison with Leissa solutions [1] and finite element solutions, as shown in Table 1. Three-dimensional linear-elastic finite element analysis is performed by using the Abaqus/Standard 6.14-3 package [41]. To apply the classical thin plate theory in the finite element model, the plate thickness is taken as 1/100th of the edge length of the square plate. The plate is discretized by using 100 *S4R* elements in both x and y directions. The physical parameters for the isotropic square plate are selected as $\rho = 7800$ kg m^{-3}, $E = 210$ GPa, $\nu = 0.3$. Table 1 shows that the first five natural frequencies of free vibration of the three cases considered converge to the third significant digit. The natural frequencies obtained by the converged GITT solutions (NW = 80) agree very well with the reference finite element solutions, to at least third significant digit. There are only small differences at the fourth significant digits. Meanwhile, the natural frequencies obtained by Leissa [1] differ from both our GITT solutions and the reference finite element solutions at the third significant digit. As it

Table 1 Convergence of the natural frequencies ($\lambda = \omega a^2\sqrt{\rho/D}$) for the isotropic square plates with boundary conditions CSCF, CCCF and CFCF

Boundary conditions	Mode	GITT				Leissa [1]	FEM [41]
		NW = 10	NW = 20	NW = 40	NW = 80		
CSCF	1	23.4397	23.4132	23.3940	23.3822	23.460	23.3743
	2	35.6001	35.5915	35.5814	35.5735	35.612	35.5716
	3	63.0683	62.9955	62.9422	62.9090	63.126	62.9180
	4	66.7777	66.7725	66.7692	66.7656	66.808	66.7819
	5	77.4812	77.4499	77.4160	77.3914	77.502	77.4049
CCCF	1	23.9975	23.9687	23.9473	23.9339	24.020	23.9239
	2	40.0281	40.0211	40.0125	40.0054	40.039	40.0004
	3	63.4286	63.3510	63.2941	63.2583	63.493	63.2639
	4	76.7258	76.7194	76.7167	76.7137	76.761	76.7425
	5	80.6862	80.6570	80.6241	80.5992	80.713	80.6019
CFCF	1	22.2412	22.2097	22.1895	22.1780	22.272	22.1719
	2	26.5096	26.4748	26.4451	26.4256	26.529	26.4064
	3	43.6422	43.6335	43.6196	43.6075	43.664	43.5929
	4	61.3837	61.2976	61.2409	61.2075	61.466	61.2201
	5	67.4865	67.3797	67.2918	67.2346	67.549	67.2086

is known that the Rayleigh–Ritz method generates upper bound solutions from the exact solutions and the natural frequencies calculated by the present method have smaller values than Leissa's results, it can be concluded that the integral transform solutions are more accurate than the Rayleigh–Ritz solutions when there is at least a free edge.

4.2 Natural frequencies of the orthotropic plates

The first six natural frequencies of the orthotropic rectangular thin plates with boundary conditions CSCF, CCCF and CFCF obtained by GITT are presented in Table 2. The convergence of the GITT solutions is examined up to truncation term NW = 80, for the aspect ratio a/b = 0.4, 2/3, 1.0, 1.5 and 2.5. To demonstrate the validity and accuracy

Table 2 Natural frequencies ($\lambda = \omega a^2\sqrt{\rho/D_x}$) for the orthotropic rectangular plates with boundary conditions CSCF, CCCF and CFCF (NW = 80)

Boundary conditions	Mode	a/b = 0.4		a/b = 2/3		a/b = 1.0		a/b = 1.5		a/b = 2.5	
		GITT	FEM [41]	GITT	FEM [41]	GITT	FEM [41]	GITT	FEM [41]	GITT	FEM [41]
CSCF	1	35.3492	35.3475	27.7538	27.7553	24.8771	24.8828	23.4976	23.5010	22.7732	22.7773
	2	81.2212	81.2518	69.2144	69.2530	50.9550	50.9655	34.5920	34.5988	26.5353	26.5406
	3	143.380	143.531	88.6143	88.6377	65.0708	65.1117	62.0838	62.0910	35.4645	35.4666
	4	210.895	211.060	129.252	129.416	94.8118	94.8602	63.1742	63.2165	50.7995	50.8045
	5	223.892	224.344	135.107	135.165	116.294	116.335	76.4143	76.4563	62.1983	62.2418
	6	260.009	260.184	200.120	200.292	124.606	124.776	104.721	104.755	66.9305	66.9750
CCCF	1	59.3274	59.3260	33.6733	33.6735	26.6832	26.6853	24.0250	24.0279	22.8811	22.8848
	2	98.3073	98.3300	73.2241	73.2571	61.5696	61.5763	38.4572	38.4595	27.4814	27.4841
	3	156.723	156.856	115.412	115.456	66.2592	66.2980	63.5157	63.5574	37.8688	37.8717
	4	234.695	235.118	132.238	132.399	103.239	103.267	70.7424	70.7546	54.9533	54.9601
	5	290.551	290.932	157.997	158.046	125.465	125.635	79.2484	79.2835	62.2685	62.3115
	6	332.489	333.502	210.947	211.418	137.871	137.945	111.498	111.533	67.5744	67.6149
CFCF	1	22.2626	22.2666	22.2938	22.2977	22.3150	22.3194	22.3329	22.3370	22.3492	22.3532
	2	58.3177	58.3088	39.6678	39.6648	31.3269	31.3259	26.7056	26.7056	23.9921	23.9934
	3	61.3693	61.4126	61.4695	61.5127	61.5341	61.5776	42.3963	42.3959	29.5564	29.5565
	4	120.389	120.562	88.5540	88.5768	67.1428	67.1448	61.5807	61.6243	40.3053	40.3060
	5	122.470	122.483	120.576	120.751	74.6976	74.7248	67.6779	67.7109	57.3157	57.3206
	6	197.303	197.417	122.672	122.693	115.646	115.672	74.3261	74.3325	61.6191	61.6629

of the proposed GITT approach, three-dimensional linear-elastic finite element analysis is performed in the Abaqus/Standard 6.14-3 package [41]. In x and y directions, 100 and $100b/a$ $S4R$ elements are used, respectively. The physical parameters for the orthotropic rectangular plate are selected as $\rho = 7800\ \mathrm{kg\ m^{-3}}$, $E_x = 200$ GPa, $E_y = 800$ GPa, $v_x = 0.075$, $v_y = 0.3$ and $G_{xy} = 174$ GPa. It can be seen in Table 2 that similar to the cases for the isotropic rectangular plate, the integral transform solutions agree very well with the reference finite element solutions, for the first six natural frequencies for orthotropic plates with different aspect ratios and three sets of boundary conditions with one or two free edges.

5 Conclusions

Free vibration of orthotropic rectangular thin plates with a pair of opposite edges clamped and one or two edges free is analyzed by using generalized integral transform technique. Numerically stable expressions for the eigenfunctions of Euler–Bernoulli beams are adopted as base functions along each direction with corresponding boundary conditions, thus overcoming the numerical difficulties with traditional expressions for the eigenfunctions in combinations of hyperbolic and trigonometric functions and allowing expansions to arbitrarily higher orders. More accurate natural frequencies have been obtained in comparison with the classical Leissa solutions obtained by Ritz–Rayleigh method. Examples show that the GITT solutions agree with reference finite element solutions to at least third significant digits for both examples, isotropic square plates and orthotropic rectangular plates with different aspect ratios. It is concluded that by including additional terms generated by integration by parts due to the difference between the plate and beam boundary conditions at a free edge, the integral transform solutions recover the loss of accuracy suffered by using the beam functions in the Rayleigh–Ritz methods when there is a free edge in a rectangular plate, and thus yield accurate natural frequencies of free vibration of orthotropic thin rectangular plates with one or two free edges.

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Appendix

(a) CC (clamped edges):

The eigenfunctions $X_i(x)$ for a pair of clamped edges in the 'x' direction are given by solving problem (11) analytically [35–37]:

$$X_i(x) = (-1)^i \cos(\mu_i x) - \sin(\mu_i x) \cot\left(\frac{\mu_i}{2}\right)^{(-1)^i} - \frac{(-1)^i e^{-\mu_i x}}{1-(-1)^i e^{-\mu_i}} + \frac{e^{-\mu_i(1-x)}}{1-(-1)^i e^{-\mu_i}}. \tag{32}$$

The transcendental equations for the eigenvalues μ_i are given by

$$(-1)^i \tan\left(\frac{\mu_i}{2}\right) = \frac{1-e^{-\mu_i}}{1+e^{-\mu_i}} \quad \text{and} \quad i = 1, 2, 3, \ldots \tag{33}$$

(b) SF (simply supported and free edges):

The eigenfunctions $Y_{1j}(y)$ for a pair of simply supported and free edges in the 'y' direction are given by solving problem (12 and 13) analytically [35–37]:

$$Y_{11}(y) = \sqrt{3}y \quad \text{and}$$
$$Y_{1j}(y) = \sqrt{2}\left(\sin(\phi_{1j}y) - \frac{e^{-\phi_{1j}}\sin(\phi_{1j})}{1-e^{-2\phi_{1j}}} e^{-\phi_{1j}y} + \frac{e^{-\phi_{1j}(1-y)}\sin(\phi_{1j})}{1-e^{-2\phi_{1j}}} \right). \tag{34}$$

The transcendental equations for the eigenvalues ϕ_{1j} are given by

$$\phi_{11} = 0,$$
$$\tan(\phi_{1j}) = \frac{1-e^{-2\phi_{1j}}}{1+e^{-2\phi_{1j}}} \quad \text{and} \quad j = 2, 3, 4, \ldots \tag{35}$$

(c) CF (clamped and free edges):

The eigenfunctions $Y_{2j}(y)$ for a pair of clamped and free edges in the 'y' direction are given by solving problem (12 and 14) analytically [35–37]:

$$Y_{2j}(y) = \cos(\phi_{2j}y) - \frac{1+(-1)^j e^{-\phi_{2j}}}{1-(-1)^j e^{-\phi_{2j}}}\sin(\phi_{2j}y) - \frac{1}{1-(-1)^j e^{-\phi_{2j}}} e^{-\phi_{2j}y} + \frac{(-1)^j}{1-(-1)^j e^{-\phi_{2j}}} e^{-\phi_{2j}(1-y)}. \tag{36}$$

The transcendental equations for the eigenvalues ϕ_{2j} are given by

$$\cos(\phi_{2j}) = \frac{-2e^{-\phi_{2j}}}{1+e^{-2\phi_{2j}}} \quad \text{and} \quad j = 1, 2, 3, \ldots \tag{37}$$

(d) FF (free and free edges):

The eigenfunctions $Y_{3j}(y)$ for a pair of two free edges in the 'y' direction are given by solving problems (12 and 15) given by [35–37]

$$Y_{31}(y) = 1, \quad Y_{32}(y) = \sqrt{3}(1-2y) \quad \text{and}$$
$$Y_{3j}(y) = \cos(\phi_{3j}y) - \frac{1+(-1)^j e^{-\phi_{3j}}}{1-(-1)^j e^{-\phi_{3j}}}\sin(\phi_{3j}y) + \frac{1}{1-(-1)^j e^{-\phi_{3j}}}e^{-\phi_{3j}y} - \frac{(-1)^j}{1-(-1)^j e^{-\phi_{3j}}}e^{-\phi_{3j}(1-y)}. \tag{38}$$

The transcendental equations for the eigenvalues ϕ_{3j} are given by

$$\phi_{31} = \phi_{32} = 0,$$
$$(-1)^j \tan\left(\frac{\phi_{3j}}{2}\right) = \frac{1-e^{-\phi_{3j}}}{1+e^{-\phi_{3j}}} \quad \text{and} \quad j = 3,4,5,\ldots \tag{39}$$

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