Journal of the Brazilian Society of Mechanical Sciences and Engineering (2018) 40:497
https://doi.org/10.1007/s40430-018-1413-0

TECHNICAL PAPER

An efficient finite element formulation for bending, free vibration and stability analysis of Timoshenko beams

Mohammad Karkon[1]

Received: 6 July 2017 / Accepted: 23 September 2018 / Published online: 29 September 2018

Abstract
In this paper, a new three-node element is proposed for analysis of beams with shear deformation effect. In each node of this element, there exist translation and rotation degrees of freedom. The element's formulation is based on the first-order shear deformation theory. For this aim, the displacement field of the element is approximated by a fifth-order polynomial. The shear strain is varied as a quadratic function within the element. It is worth noting that the quadratic function can be used for axial displacement field as well. By employing curvature and shear strain relations of Timoshenko beam theory, the exact and explicit shape functions of the displacement fields are obtained. By utilizing these shape functions, the stiffness matrix and the geometric stiffness matrix of the element are calculated. The mass matrices of the proposed element are derived from kinetic energy relation of the beam. Finally, several numerical tests are performed to assess the robustness of the developed element. The results of the numerical tests prove the absence of the shear locking and demonstrate high accuracy and efficiency of the proposed element for bending, free vibration and stability analysis of Timoshenko beams.

Keywords Finite element · Timoshenko beam · Bending · Free vibration · Stability

1 Introduction

Beams have a wide range of applications in various structures such as buildings and bridges. Two basic theories have been developed for analysis of beams. The Euler–Bernoulli approach neglects shear deformations. This model gives appropriate and acceptable assessments for thin beams, for which shear effects are indeed insignificant. However, as the beam thickness increases the accuracy of the response provided by the Euler–Bernoulli formulation lowers and the obtained results get inadequate. Correspondingly, the effect of shear deformation is formulated in Timoshenko theory. In this approach, the transverse shear strain is assumed to be constant along the thickness. Therefore, this model provides accurate responses for both thin and moderately thick beams.

Up to now, many elements have been proposed based on Timoshenko theory. These elements are classified into two groups which are simple and complex [1–4]. The simple elements consist of two nodes, and at each node, there are two degrees of freedom [1]. Distinctly, a complex element has more than two degrees of freedom at a node or more than two nodes. The first complex element with eight degrees of freedom was proposed by Kapur [5]. Lees and Thomas introduced a hierarchical Timoshenko beam finite element by assuming separate polynomial series for displacement and shear deformation [6, 7]. By using the concept of hierarchical functions, Tessler and Dong [8] presented conforming Timoshenko beam elements which include the effects of transverse shear deformation and rotary inertia. In the recent years, Falsone and Settineri by developing the Euler–Bernoulli beam theory presented a similar formulation for Timoshenko beams. They obtained a single governing differential equation expressed in terms of deflection for Timoshenko beams. It is important to note that the stiffness matrix for these elements with internal nodes is not symmetric [9]. Bouclier et al. developed a new non-uniform rational B-splines (NURBS) finite element for analysis of straight and curved Timoshenko beam problems. To alleviate shear and membrane locking, they utilized the selective reduced integration to evaluate the terms referring to shear and membrane energies [10]. Similarly, Cazzani et al. [11]

Technical Editor: Paulo de Tarso Rocha de Mendonça.

✉ Mohammad Karkon
m.karkon@iaularestan.ac.ir; Karkon443@gmail.com

[1] Civil Engineering Department, Larestan Branch, Islamic Azad University, Larestan, Iran

developed a plane curved Timoshenko beam element based on NURBS interpolation for both geometry and displacements. Based on mixed finite element formulation, Lepe et al. [12] proposed a locking-free element for Timoshenko beams analysis. For analyzing such geomechanics problems as beam-type structures and deep pile foundations, Caillerie et al. [13] proposed a Timoshenko straight beam element with internal degrees of freedom. By using the flexibility and stiffness methods, Khajavi [14] introduced stiffness matrices for both Euler–Bernoulli and Timoshenko tapered and non-prismatic beams.

Free vibration analysis of shear deformable beams is an important stage for the design of skeletal structures. Dawe [15] presented a three-node element for free vibration of Timoshenko beams. Lee and Schultz [16] applied a pseudospectral method using the Chebyshev polynomials as the basis functions, for free vibration analysis of Timoshenko beams and radially symmetric Mindlin plates. By using Lagrange equations, free vibration analysis of Timoshenko beams with different boundary conditions was performed by Kocatürk and Simsek [17]. Ferreira [18] utilized the multiquadric radial basis function method to analyze free vibrations of Timoshenko beams and Mindlin plates. Also, Ferreira and Fasshauer [19], by combining collocation method, radial basis functions and pseudospectral method, presented a new high accuracy numerical scheme for free vibration analysis of Timoshenko beams with various support conditions. Xu and Wang [20] used discrete singular convolution (DSC) method for analyzing the free vibration of Timoshenko beam with various boundary conditions. Lee and Park [21] introduced an isogeometric approach for free vibration analysis of Timoshenko beams. Moallemi-Oreh and Karkon [22] proposed a two-node beam element with two nodal degrees of freedom at each node for stability and free vibration analysis of Timoshenko beams. Hsu [23] carried out free vibration analysis of Timoshenko beam by improving a simple linear two-node C^0 element using two different enrichment formulations.

Similar to the free vibration case, buckling analysis of beam-columns is attractive for the researchers due to their wide application in the design of structures. Kosmatka [24], based upon Hamilton's principle, developed a two-node beam element with four degrees of freedom for stability and free vibration analysis of Timoshenko beams. For stability analysis of deep beams, a one-dimensional higher-order theory has been developed by Matsunaga [25]. Więckowski and Golubiewski [26] by utilizing smoothed functions obtained with the aid of the least square technique improved the accuracy of the finite element method for stability analysis of Euler–Bernoulli and Timoshenko beams. Carrera et al. [27] by employing Carrera unified formulation (CUF) and dynamic stiffness method presented higher-order theories and exact solutions for buckling analysis of beam-columns.

In this article, a new three-node element has been suggested for bending, free vibration and stability analysis of beams based upon the Timoshenko beam theory. The shape functions, the stiffness matrix, the mass matrices and the geometric matrix of the element are explicitly derived. In the following, the accuracy and convergence rate of the presented element is evaluated with several numerical examples and the obtained results are compared with those of other researchers. At first stage, a static analysis of a two-member frame is performed and internal displacements and forces are calculated. In the second stage, the free vibration analyses of Timoshenko beams with different boundary conditions are performed for various values of thickness-to-length ratios. Finally, the stability analysis of simply supported beam is carried out for three different thickness-to-length ratios. The results prove that the suggested element is able to analyze the thick and thin beams with high level of accuracy.

2 Governing equations of Timoshenko beam

The strains and stresses in the Timoshenko beam theory can be written as:

$$\varepsilon = z\frac{\mathrm{d}\theta}{\mathrm{d}x}, \quad \gamma = \frac{\mathrm{d}w}{\mathrm{d}x} - \theta \tag{1}$$

$$\sigma = E\varepsilon, \quad \tau = 2G\gamma, \tag{2}$$

where w and θ are transverse displacement and rotation field, respectively. On the other hand, the internal bending moments M and shear forces V are calculated in the following form:

$$M = \int_A z\sigma \mathrm{d}A = EI\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{3}$$

$$V = k_\mathrm{s}\int_A \tau \mathrm{d}A = GAk_\mathrm{s}\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \theta\right), \tag{4}$$

where k_s, G and A denote the shear correction factor, shear modulus and cross-sectional area of beam, respectively. The equilibrium of moments and shear forces are:

$$V(x) - \frac{\mathrm{d}M}{\mathrm{d}x} = 0 \tag{5}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -q(x). \tag{6}$$

Substituting Eqs. (3) and (4) into Eqs. (5) and (6) transforms the equilibrium equations into:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{GA}k_\mathrm{s}\gamma\right) = q(x) \tag{7}$$

$$\mathrm{GA}k_s\gamma - \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{EI}\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) = 0. \tag{8}$$

3 Finite element formulation

In the finite element method, displacement and rotation fields of the element are related to the nodal degrees of freedom via the shape functions. Figure 1 shows the proposed three-node Timoshenko beam element. In order to compute the shape functions of aforementioned beam element, a fifth-order and a second-order polynomial functions are used for deflection and shear strain field approximations, respectively:

$$\begin{aligned} w &= a_1 + a_2\xi + a_3\xi^2 + a_4\xi^3 + a_5\xi^4 + a_6\xi^5 \\ &= \left[P_w\right]\{a\}, \quad \xi = 2x/l - 1 \end{aligned} \tag{9}$$

$$\gamma = a_7 + a_8\xi + a_9\xi^2. \tag{10}$$

In the FSDT beam theory, the rotation field of the element is calculated as:

$$\theta = \frac{\mathrm{d}w}{\mathrm{d}x} - \gamma = \frac{2}{l}\cdot\frac{\mathrm{d}w}{\mathrm{d}\xi} - \gamma. \tag{11}$$

It is worth emphasizing that the rotation field approximation is of fourth order. According to Eq. (8), the shear strain of the element can be written as follows:

$$\gamma = \lambda l^2 \cdot \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}, \quad \lambda = \frac{\mathrm{EI}}{\mathrm{GA}k_s l^2}. \tag{12}$$

By equating the shear strain of Eqs. (10) and (12), the unknown parameters a_7, a_8 and a_9 are rendered:

$$a_7 = -\frac{48}{l}\left(\lambda a_4 + 80\lambda^2 a_6\right) \tag{13}$$

$$a_8 = -\frac{192}{l}\lambda a_5 \tag{14}$$

$$a_9 = \frac{-480}{l}\lambda a_6. \tag{15}$$

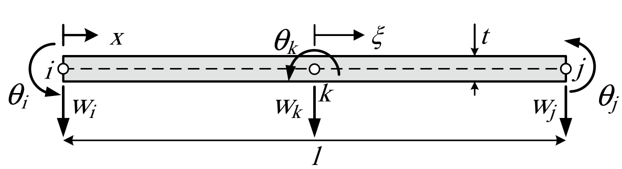

Fig. 1 Three-node Timoshenko beam element

By substituting the parameters a_7, a_8 and a_9 in relation (11), the rotation field is obtained in the following form:

$$\theta = \frac{1}{l}\begin{bmatrix} 0 & 2 & 4\xi & 6\left(\xi^2 + 8\lambda\right) & 8\xi\left(\xi^2 + 24\lambda\right) & 10\left(\xi^4 + 48\lambda\xi^2 + 384\lambda^2\right) \end{bmatrix}\{a\} = \left[P_\theta\right]\{a\}. \tag{16}$$

In order to calculate the shape functions of the element, the indirect method is used. In this strategy, the element's field functions w and θ and nodal displacement vector $\{D\}$ are expressed by the following equations:

$$\begin{Bmatrix} w \\ \theta \end{Bmatrix} = \begin{bmatrix} \left[N_w\right] \\ \left[N_\theta\right] \end{bmatrix}\{D\} = [N]\{D\} \tag{17}$$

$$\{D\} = [G]\{a\} \tag{18}$$

$$\{D\} = \left\{ w_i \ \theta_i \ w_k \ \theta_k \ w_j \ \theta_j \right\}^T. \tag{19}$$

In expression (17), $\left[N_w\right]$ and $\left[N_\theta\right]$ denote the shape functions of the deflection and rotation field of the element. Also, the square matrix $[G]$ in Eq. (18) is dependent on the element geometry, and its rows are formed by substituting the nodes' coordinates or their derivatives in matrices $\left[P_w\right]$ and $\left[P_\theta\right]$:

$$[G] = \frac{1}{l}\begin{bmatrix} l & -l & l & -l & l & -l \\ 0 & 2 & -4 & 6(8\lambda+1) & -8(24\lambda+1) & 10\left(384\lambda^2+48\lambda+1\right) \\ l & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 48\lambda & 0 & 3840\lambda^2 \\ l & l & l & l & l & l \\ 0 & 2 & 4 & 6(8\lambda+1) & 8(24\lambda+1) & 10\left(384\lambda^2+48\lambda+1\right) \end{bmatrix}. \tag{20}$$

On the other hand, the following relation holds for the shape functions $[N]$ and the approximants of the element unknowns:

$$[N] = \begin{bmatrix} \left[N_w\right] \\ \left[N_\theta\right] \end{bmatrix} = \begin{bmatrix} P_w \\ P_\theta \end{bmatrix}[G]^{-1}. \tag{21}$$

Therefore, the explicit forms of the deflection shape functions $\left[N_w\right]$ can be found:

$$\left[N_w\right] = \left[\ N_1 \ N_2 \ N_3 \ N_4 \ N_5 \ N_6\ \right] \tag{22}$$

$$
\begin{aligned}
N_1 &= -\frac{2}{\beta}\xi(\xi-1)\left(11520\lambda^2\xi + 5760\lambda^2 - 144\lambda\xi^3 - 24\lambda\xi^2 + 456\lambda\xi + 120\lambda - 3\xi^3 - \xi^2 + 4\xi\right)\\
N_2 &= \frac{l}{\beta}\xi\left(\xi^2-1\right)\left(92160\lambda^3 - 1152\lambda^2\xi^2 + 768\lambda^2 + 24\lambda\xi^2 - 60\lambda\xi - 24\lambda + \xi^2 - \xi\right)\\
N_3 &= -\frac{8}{\beta}(60\lambda+1)\left(\xi^2-1\right)\left(-\xi^2 + 48\lambda + 1\right)\\
N_4 &= -\frac{4l}{\beta}\xi(48\lambda+1)\left(\xi^2-1\right)\left(960\lambda^2 - 12\lambda\xi^2 + 108\lambda - \xi^2 + 1\right)\\
N_5 &= \frac{2}{\beta}\xi(\xi+1)\left(11520\lambda^2\xi - 5760\lambda^2 - 144\lambda\xi^3 + 24\lambda\xi^2 + 456\lambda\xi - 120\lambda - 3\xi^3 + \xi^2 + 4\xi\right)\\
N_6 &= \frac{l}{\beta}\xi\left(\xi^2-1\right)\left(92160\lambda^3 - 1152\lambda^2\xi^2 + 768\lambda^2 + 24\lambda\xi^2 + 60\lambda\xi - 24\lambda + \xi^2 + \xi\right).
\end{aligned}
\tag{23}
$$

Parameter β in these relations is defined as:

$$\beta = 23040\lambda^2 + 864\lambda + 8. \tag{24}$$

Furthermore, the rotation shape functions $\left[N_\theta\right]$ of the element are, similarly, obtained as:

$$\left[N_\theta\right] = \left[\ N_7\ N_8\ N_9\ \ N_{10}\ N_{11}\ N_{12}\ \right] \tag{25}$$

$$
\begin{aligned}
N_7 &= \frac{4}{\beta l}\xi\left(\xi^2-1\right)(15\xi - 480\lambda + 720\lambda\xi - 8)\\
N_8 &= -\frac{2}{\beta}\xi(\xi-1)\left(5760\lambda^2\xi^2 + 5760\lambda^2\xi - 5760\lambda^2 - 120\lambda\xi^2 + 120\lambda\xi + 24\lambda - 5\xi^2 - \xi + 2\right)\\
N_9 &= \frac{64}{\beta l}\xi(60\lambda+1)\left(\xi^2-1\right)\\
N_{10} &= \frac{8}{\beta}(48\lambda+1)\left(\xi^2-1\right)\left(60\lambda\xi^2 - 60\lambda + 5\xi^2 - 1\right)\\
N_{11} &= -\frac{4}{\beta l}\xi\left(\xi^2-1\right)(15\xi + 480\lambda + 720\lambda\xi + 8)\\
N_{12} &= -\frac{2}{\beta}\xi(\xi+1)\left(5760\lambda^2\xi^2 - 5760\lambda^2\xi - 5760\lambda^2 - 120\lambda\xi^2 - 120\lambda\xi + 24\lambda - 5\xi^2 + \xi + 2\right).
\end{aligned}
\tag{26}
$$

To derive the stiffness matrix, the strain matrix is required. The strains of the Timoshenko beam element are given by:

$$\{\varepsilon\} = \left\{\begin{array}{c}\frac{\mathrm{d}w}{\mathrm{d}x}\\ \frac{\mathrm{d}w}{\mathrm{d}x} - \theta\end{array}\right\} = [B]\{D\} \tag{27}$$

$$[B] = \begin{bmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}x}\\ \frac{\mathrm{d}}{\mathrm{d}x} & -1\end{bmatrix}\left[\begin{bmatrix}N_w\end{bmatrix}\\ \begin{bmatrix}N_\theta\end{bmatrix}\right], \tag{28}$$

where $[B]$ is the strain matrix. The element stiffness matrix is then calculated as an integral:

$$\left[K^e\right] = \int_l [B]^T[C][B]\mathrm{d}x = \frac{l}{2}\int_{-1}^{1}[B]^T[C][B]\mathrm{d}\xi. \tag{29}$$

In the present formula, $[C]$ is the elastic rigidity matrix. For Timoshenko beam element, this matrix has the following shape:

$$[C] = \begin{bmatrix}\mathrm{EI} & 0\\ 0 & \mathrm{GA}k_s\end{bmatrix}. \tag{30}$$

By calculating expression (29), the stiffness matrix of the proposed element is:

$$\left[K^e\right] = \frac{64}{35l^3\beta^2}\begin{bmatrix}
k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16}\\
k_{12} & k_{22} & k_{23} & k_{24} & -k_{16} & k_{26}\\
k_{13} & k_{23} & k_{33} & 0 & k_{13} & -k_{23}\\
k_{14} & k_{24} & 0 & k_{44} & -k_{14} & k_{24}\\
k_{15} & -k_{16} & k_{13} & -k_{14} & k_{11} & -k_{12}\\
k_{16} & k_{26} & -k_{23} & k_{24} & -k_{12} & k_{22}
\end{bmatrix}. \tag{31}$$

The entries of this matrix are given in "Appendix," see expressions (45). Note that the proposed element is free of shear locking. For thin beams, parameter λ will be very small and, therefore, the presented shape functions and stiffness matrix degenerate to the shape functions and stiffness matrix of three-node Euler–Bernoulli element, which can be found in references [28–30]. Also, the nodal force vector of the element is obtained:

$$\left[F^e\right] = \int_l q\left[N_w\right]^T \mathrm{d}x, \tag{32}$$

where q is the load distributed along the element.

4 Mass matrix

The kinetic energy of a Timoshenko beam, including the effects of shear deformation and rotary inertia, can be written as:

$$T = \int_l \rho A \dot{w}^2 \mathrm{d}x + \int_l \rho I \dot{\theta}^2 \mathrm{d}x, \tag{33}$$

where the dot denotes the time derivative. Moreover,ρ, A and I are the mass density of the material, area of the cross section and the second moment of area, respectively. By substituting Eq. (21) into (33) and applying Lagrange's principle, one may arrive to an expression for the mass matrix:

$$\left[M^e\right] = \left[M_1^e\right] + \left[M_2^e\right] = \frac{l}{2}\int_{-1}^{1} \rho A \left[N_w\right]^T \left[N_w\right]\mathrm{d}\xi + \frac{l}{2}\int_{-1}^{1} \rho I \left[N_\theta\right]^T \left[N_\theta\right]^T \mathrm{d}\xi, \tag{34}$$

where the first term is associated with the translational inertia mass matrix and the second part is associated with the rotary inertia mass matrix. The translation mass matrix $\left[M_1^e\right]$ is given by:

$$\left[M_1^e\right] = \frac{64\rho A}{3465\beta^2}\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{12} & m_{17} & m_{18} & m_{19} & -m_{16} & m_{110} \\ m_{13} & m_{18} & m_{111} & 0 & m_{13} & -m_{18} \\ m_{14} & m_{19} & 0 & m_{112} & -m_{14} & m_{19} \\ m_{15} & -m_{16} & m_{13} & -m_{14} & m_{11} & -m_{12} \\ m_{16} & m_{110} & -m_{18} & m_{19} & -m_{12} & m_{17} \end{bmatrix}. \tag{35}$$

In addition, the explicit form of the rotary mass matrix $\left[M_2^e\right]$ can be expressed as:

$$\left[M_2^e\right] = \frac{32\rho I}{315\beta^2}\begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{22} & m_{27} & m_{28} & m_{29} & -m_{26} & m_{210} \\ m_{23} & m_{28} & m_{211} & 0 & m_{23} & -m_{28} \\ m_{24} & m_{29} & 0 & m_{212} & -m_{24} & m_{29} \\ m_{25} & -m_{26} & m_{23} & -m_{24} & m_{21} & -m_{22} \\ m_{26} & m_{210} & -m_{28} & m_{29} & -m_{22} & m_{27} \end{bmatrix}. \tag{36}$$

The nonzero entries of the mass matrices $\left[M_1^e\right]$ and $\left[M_2^e\right]$ are given in formulas (46) and (47) of "Appendix." It is well known that the equation of free vibration can be expressed as follows:

$$\left([K] - \omega_n^2[M]\right)\left\{\phi_n\right\} = 0, \tag{37}$$

where ω_n and ϕ_n represent the natural frequency and the mode shape associated with the nth mode. Also, $[K]$ and $[M]$ are the stiffness matrix and mass matrix of the whole structure, respectively.

5 Geometric stiffness matrix

The concept of the neutral state of equilibrium is used for buckling analysis of beam. As a result, the strain energy associated with axial load P can be expressed as follows:

$$\Delta W = \frac{P}{2}\int_0^l \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \mathrm{d}x. \tag{38}$$

Therefore, the element geometric stiffness matrix of the proposed element can be calculated as [31]:

$$\left[K_g^e\right] = \int_l \left[\frac{\mathrm{d}N_w}{\mathrm{d}x}\right]^T P \left[\frac{\mathrm{d}N_w}{\mathrm{d}x}\right]\mathrm{d}x \tag{39}$$

whose explicit form is:

$$\left[K_g\right] = \frac{32}{315\beta^2}\begin{bmatrix} k_{g1} & k_{g2} & k_{g3} & k_{g4} & k_{g5} & k_{g6} \\ k_{g2} & k_{g7} & k_{g8} & k_{g9} & -k_{g6} & k_{g10} \\ k_{g3} & k_{g8} & k_{g11} & 0 & k_{g3} & -k_{g8} \\ k_{g4} & k_{g9} & 0 & k_{g12} & -k_{g4} & k_{g9} \\ k_{g5} & -k_{g6} & k_{g3} & -k_{g4} & k_{g1} & -k_{g2} \\ k_{g6} & k_{g10} & -k_{g8} & k_{g9} & -k_{g2} & k_{g8} \end{bmatrix}. \tag{40}$$

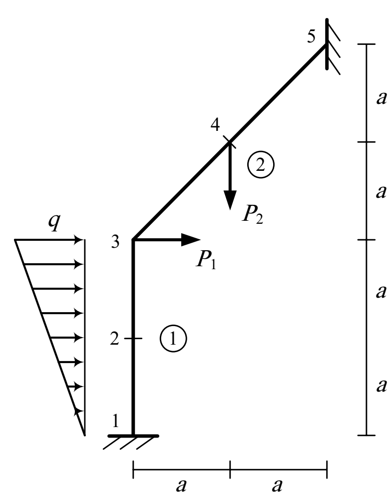

Fig. 2 Geometry and loading of plane frame

The entries of this matrix are given in "Appendix," see expression (48). The equation of stability analysis can be expressed as:

$$\left([K] - P_{\mathrm{cr},i}\left[K_g\right]\right)\{\varphi\} = 0. \tag{41}$$

Similar to free vibration analysis, $P_{\mathrm{cr},i}$ and φ represent the critical load and mode shape associated with the ith mode. Also, $[K]$ and $\left[K_g\right]$ are the stiffness matrix and the geometric stiffness matrix of the whole structure, respectively. The exact value of the beam-column buckling load with shear deformation effect can be written as [32]:

$$P_{\mathrm{cr},i} = \frac{\pi^2 \mathrm{EI}}{L_{\mathrm{eff}}^2}\left(\frac{\left(L_{\mathrm{eff}} i\right)^2 \mathrm{GA}k_s}{L_{\mathrm{eff}}^2 \mathrm{GA}k_s + (\pi i)^2 \mathrm{EI}}\right), \tag{42}$$

where L_{eff} is the effective beam length and i indicates the mode number.

6 Numerical examples

In order to assess the accuracy of the proposed element, some numerical problems have been analyzed and the results are compared with the data available in the literature. At first, static analysis of a two-member plane frame is performed. In subsection 6.2, free vibration analysis of beams with different boundary conditions is considered. Finally, buckling analysis of simply supported beam is carried out for three thickness-to-length ratios.

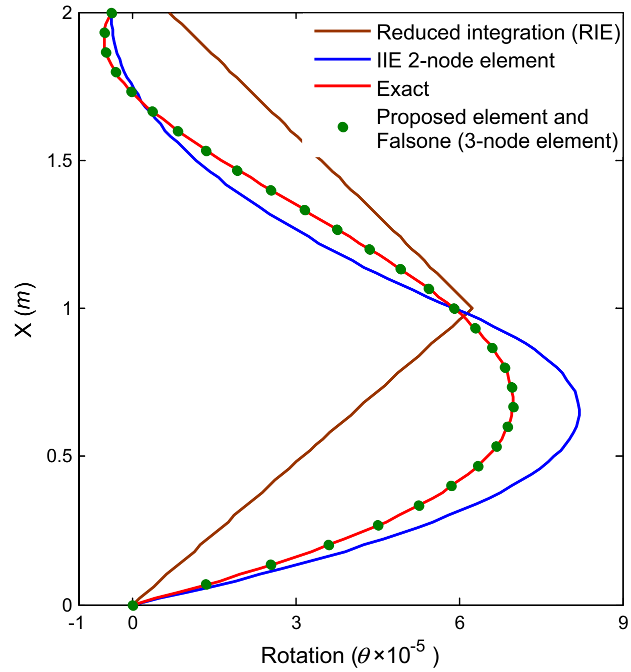

Fig. 4 Rotation along the vertical member

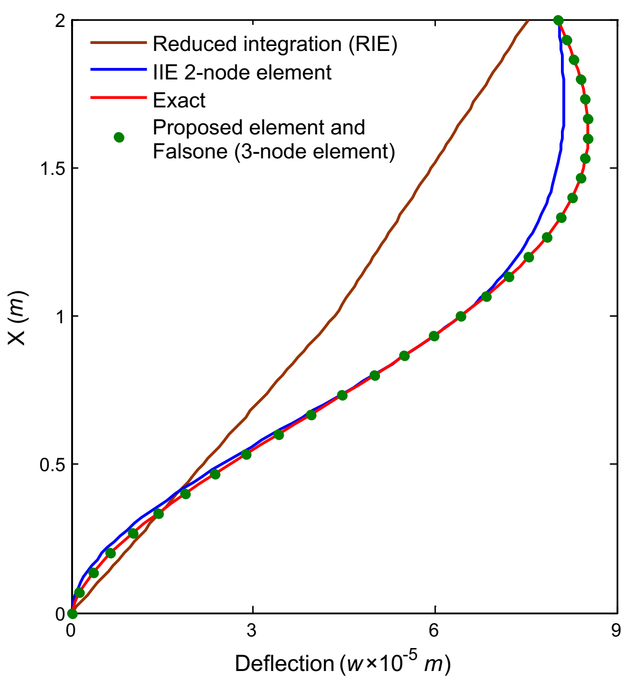

Fig. 3 Transverse deflection along the vertical member

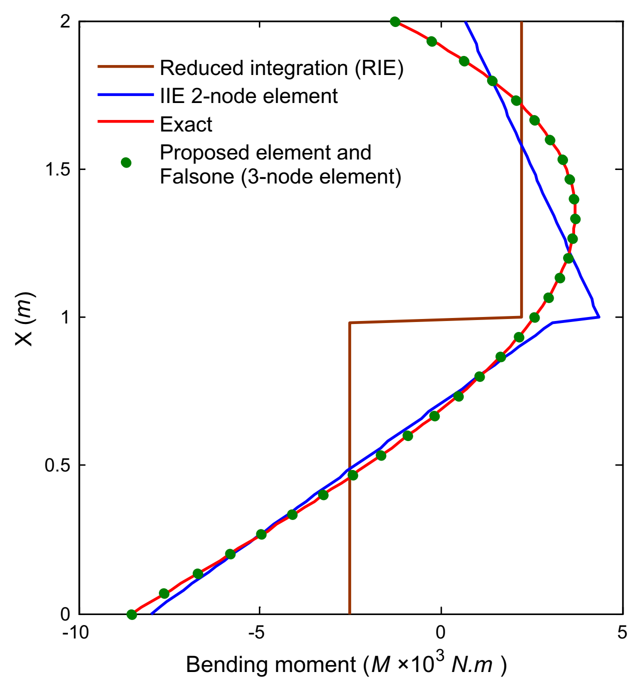

Fig. 5 Internal bending moment along vertical member

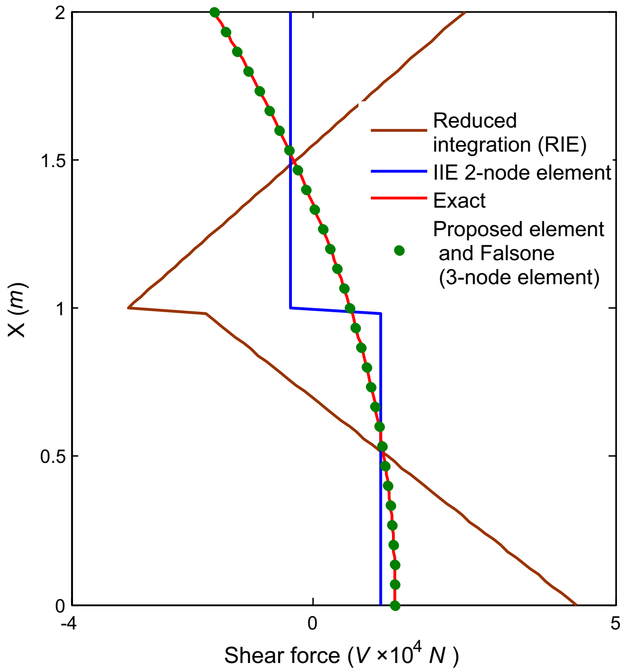

Fig. 6 Internal shear force along vertical member

6.1 Static analysis of a frame

In order to reveal the accuracy and robustness of the proposed element in static analysis, a two-member plane frame is considered. The frame geometry and its loading are shown in Fig. 2. It should be mentioned that the second-order polynomial function is used for the axial displacement approximation. The size of parameter a appearing in this figure is given equal to $a = 1$ m. The material properties of the members are given as: Young's modulus $E = 25 \times 10^9\ \mathrm{N/m^2}$ and Poisson's ratio $\nu = 0.25$. Moreover, the axial and bending stiffnesses of the vertical member of the frame are considered as $\mathrm{EA} = 3 \times 10^9$ N and $\mathrm{EI} = 4 \times 10^7\ \mathrm{Nm^2}$, respectively. These quantities for the angled member are assumed to be $\mathrm{EA} = 7.8125 \times 10^9$ N and $\mathrm{EI} = 3.75 \times 10^7\ \mathrm{Nm^2}$. Note that the shear correction coefficient for the Timoshenko beam element is taken to be $k_s = 5/6$. As shown in Fig. 2, the vertical member is loaded by a triangular distributed load, with maximum value $q = 30$ kN/m. Additionally, two concentrated loads $P = 20$ kN are applied as: a horizontal force at the node 3 and a vertical force at the node 4. The structure is analyzed by using a single element of the proposed type for each member.

The diagrams of the displacements, rotations, internal moment and shear force along the vertical member are plotted in Figs. 3, 4, 5 and 6, respectively. The results obtained with the proposed element are compared with those of reduced integration element (RIE) [3], two-node interdependent interpolation element (IIE) [3], Falsone and

Table 1 Non-dimensional frequency parameter λ_i of the SS thin beam ($t/l = 0.002$) for different meshes

Method	N_{el}^*	Present solution: mode number														
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Proposed Element	1	3.1420	6.2964	11.4811	16.6146	–	–	–	–	–	–	–	–	–	–	–
	2	3.1416	6.2840	9.4384	12.5923	16.3192	22.9628	28.1689	33.2330	–	–	–	–	–	–	–
	4	3.1416	6.2831	9.4246	12.5676	15.7130	18.8753	22.0803	25.1809	28.8834	32.6341	36.6479	45.9309	49.6765	56.3508	63.2055
	8	3.1416	6.2831	9.4245	12.5657	15.7067	18.8474	21.9881	25.1311	28.2719	31.4181	34.5715	37.7376	40.9242	44.1414	47.3975
	16	3.1416	6.2831	9.4245	12.5657	15.7066	18.8473	21.9875	25.1273	28.2666	31.4054	34.5435	37.6811	40.8181	43.9545	47.0904
	32	3.1416	6.2831	9.4245	12.5657	15.7066	18.8473	21.9875	25.1273	28.2666	31.4053	34.5434	37.6807	40.8174	43.9532	47.0881
	64	3.1416	6.2831	9.4245	12.5657	15.7066	18.8473	21.9875	25.1273	28.2666	31.4053	34.5434	37.6807	40.8174	43.9531	47.0880
EBT		3.1416	6.2832	9.4248	12.5664	15.7080	18.8496	21.9911	25.1327	28.2743	31.4159	34.5575	37.6991	40.8407	43.9823	47.1239
Ref. [16]		3.1416	6.2831	9.4245	12.5657	15.7066	18.8473	21.9875	25.1273	28.2666	31.4053	34.5434	37.6807	40.8174	43.9531	47.0880
Ref. [18]		3.1408	6.2862	9.4307	12.5746	15.7136	18.8535	21.9862	25.1167	28.2406	31.3508	34.4613	37.5265	40.6156	43.5748	46.6156
Ref. [23]		3.1413	6.2811	9.4176	12.5494	15.6749	18.7926	21.9010	24.9988	28.0845	31.1568	34.2145	37.2565	40.2815	43.2886	46.2769

N_{el}^*, number of elements

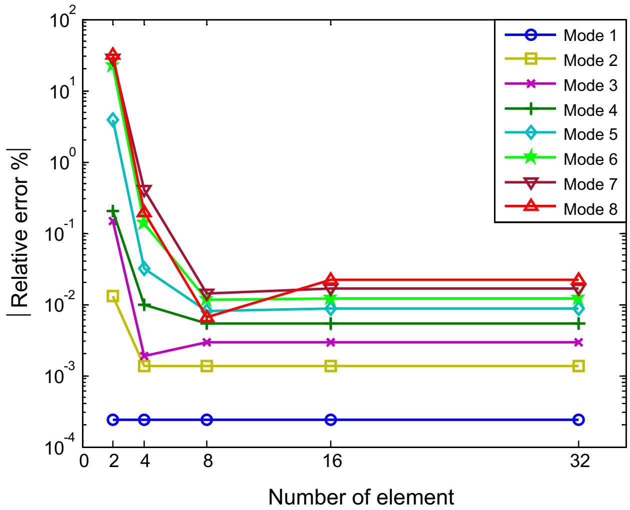

Fig. 7 Error percentage for the first 8 frequencies of the SS beam

Settineri three-node element [9] and exact solution. These figures reveal that the proposed element gives the exact solution in all the cases. It should be mentioned that the exact solution of this problem is a polynomial of fifth order and thus a single element per each member of the frame is sufficient to capture it.

6.2 Free vibration analysis

In order to assess the accuracy of the proposed element for free vibration analysis of shear deformable beams, four types of beams—simply supported (SS) beam, clamped–clamped (CC) beam, free–free (FF) beam and clamped-free beam (CF)—are analyzed and the results are compared with the references available in the literatures. For convenience, the following non-dimensional natural frequencies are introduced:

$$\lambda_i^2 = \omega_i l^2 \sqrt{\frac{\rho A}{\mathrm{EI}}}. \tag{43}$$

To investigate the effect of the number of elements, at first, the free vibration behavior of a simply supported thin beam ($t/l = 0.002$) is analyzed, and first fifteen non-dimensional frequency parameters are obtained. The results of the proposed element are compared with those of Lee and Schultz [16], Ferreira [18], Hsu [23] and Euler–Bernoulli beam theory solution (EBT). Table 1 reveals that the proposed element has high accuracy and a rapid rate of convergence. So that, by using the 32 proposed elements, the results of all fifteen frequencies will be converged. Figure 7 shows the reduction in the relative error for the first 8 frequencies of the simply supported thin beam with $t/l = 0.002$. The corresponding mode shapes are shown in Fig. 8.

In the following examples, frequencies of Timoshenko beams with different boundary conditions and aspect ratios (t/l) are obtained, by utilizing meshes of 32 and 64 elements. The findings are compared with published results of the other researchers. The first fifteen non-dimensional frequency parameters of the SS beam are listed in Table 2. Furthermore, Table 3 lists these parameters for CC beam. The results of the FF beam and the CF beam are presented in Tables 4 and 5, respectively. These tables demonstrate that the proposed element has high accuracy, and the rate of convergence is quite independent of the boundary conditions. It is also observed that the thickness-to-length ratio does not have influence on the convergent rate, and the suggested element is free from the shear locking effect.

6.3 Stability analysis

In this section, the robustness of the proposed element for buckling analysis of shear deformable beams is evaluated. For this aim, a simply supported beam-column (SS) with three thickness-to-length ratios is analyzed and the first five buckling loads are obtained. For simplicity, the critical loads are given in non-dimensional form as follows:

$$\bar{P}_{\mathrm{cr},i} = P_{\mathrm{cr},i} \times \frac{l^2}{\pi^2 \mathrm{EI}}. \tag{44}$$

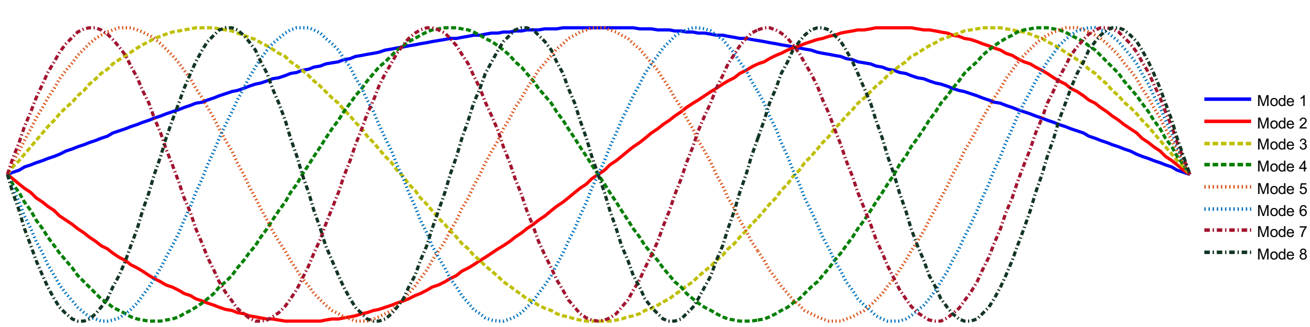

Fig. 8 First 8 mode shapes of the SS beam

Table 2 Non-dimensional frequency parameter λ_i of the SS Timoshenko beam with different thickness-to-length ratios

t/l	Method	Present solution: mode number														
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.002	$N^*_{el}=32$	3.1416	6.2831	9.4245	12.5657	15.7066	18.8473	21.9875	25.1273	28.2666	31.4053	34.5434	37.6807	40.8174	43.9532	47.0881
	$N^*_{el}=64$	3.1416	6.2831	9.4245	12.5657	15.7066	18.8473	21.9875	25.1273	28.2666	31.4053	34.5434	37.6807	40.8174	43.9531	47.0880
	Ref. [16]	3.1416	6.2831	9.4245	12.5657	15.7066	18.8473	21.9875	25.1273	28.2666	31.4053	34.5434	37.6807	40.8174	43.9531	47.0880
	Ref. [18]	3.1408	6.2862	9.4307	12.5746	15.7136	18.8535	21.9862	25.1167	28.2406	31.3508	34.4613	37.5265	40.6156	43.5748	46.6156
	Ref. [17]	3.1416	6.2831	9.4245	12.5656	15.7066	18.8471	21.9878	25.0625	–	–	–	–	–	–	–
0.01	$N^*_{el}=32$	3.1413	6.2811	9.4176	12.5494	15.6749	18.7926	21.9011	24.9988	28.0845	31.1568	34.2146	37.2565	40.2816	43.2888	46.2771
	$N^*_{el}=64$	3.1413	6.2811	9.4176	12.5494	15.6749	18.7926	21.9011	24.9988	28.0845	31.1568	34.2145	37.2565	40.2815	43.2887	46.2769
	Ref. [16]	3.1413	6.2811	9.4176	12.5494	15.6749	18.7926	21.9011	24.9988	28.0845	31.1568	34.2145	37.2565	40.2815	43.2886	46.2769
	Ref. [18]	3.1412	6.2822	9.4179	12.5520	15.6740	18.7932	21.8977	24.9925	28.0759	31.1364	34.1873	37.2114	40.1637	43.2015	45.7626
	Ref. [17]	3.1413	6.2811	9.4176	12.5494	15.6749	18.7925	21.9013	25.0022	–	–	–	–	–	–	–
0.1	$N^*_{el}=32$	3.1157	6.0907	8.8406	11.3433	13.6138	15.6803	17.5728	19.3179	20.9380	22.4514	23.8734	25.2161	26.2638	26.4873	26.4897
	$N^*_{el}=64$	3.1157	6.0907	8.8405	11.3432	13.6133	15.6794	17.5711	19.3152	20.9340	22.4460	23.8664	25.2076	26.1153	26.3338	26.4796
	Ref. [16]	3.1157	6.0907	8.8405	11.3431	13.6132	15.6790	17.5705	19.3142	20.9325	22.4441	23.8639	25.2044	26.0647	26.2814	26.4758
	Ref. [18]	3.1158	6.0908	8.8406	11.3430	13.6129	15.6785	17.5700	19.3140	20.9336	22.4481	23.8756	25.2248	26.0647	26.2814	26.5454
	Ref. [17]	3.1157	6.0907	8.8405	11.3430	13.6131	15.6769	17.5700	19.1928	–	–	–	–	–	–	–
0.2	$N^*_{el}=32$	3.0453	5.6716	7.8397	9.6576	11.2230	12.6038	13.0577	13.4731	13.8455	14.4773	14.9794	15.7259	16.0276	17.0060	17.0447
	$N^*_{el}=64$	3.0453	5.6716	7.8396	9.6572	11.2223	12.6026	13.0387	13.4515	13.8439	14.4477	14.9773	15.6823	16.0250	16.9801	17.0030
	Ref. [16]	3.0453	5.6716	7.8395	9.6571	11.2220	12.6022	13.0323	13.4443	13.8433	14.4378	14.9766	15.6676	16.0241	16.9584	17.0019
	Ref. [18]	3.0454	5.6716	7.8396	9.6570	11.2219	12.6019	13.0323	13.4443	13.8432	14.4378	14.9769	15.6677	16.0269	16.9583	17.0096
	Ref. [17]	3.0453	5.6716	7.8395	9.6569	11.2219	12.5971	13.0323	13.4442	–	–	–	–	–	–	–
EBT		3.1416	6.2832	9.4248	12.5664	15.7080	18.8496	21.9911	25.1327	28.2743	31.4159	34.5575	37.6991	40.8407	43.9823	47.1239

N^*_{el}, number of elements

Table 3 Non-dimensional frequency parameter λ_i of the CC Timoshenko beam with different thickness-to-length ratios

t/l	Method	Present solution: mode number														
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.002	$N^*_{el}=32$	4.7300	7.8530	10.9950	14.1359	17.2766	20.4168	23.5567	26.6960	29.8348	32.9729	36.1103	39.2470	42.3829	45.5178	48.6519
	$N^*_{el}=64$	4.7300	7.8530	10.9950	14.1359	17.2766	20.4168	23.5567	26.6960	29.8348	32.9729	36.1103	39.2470	42.3829	45.5178	48.6519
	Ref. [16]	4.7300	7.8530	10.9950	14.1359	17.2766	20.4168	23.5567	26.6960	29.8348	32.9729	36.1103	39.2470	42.3829	45.5178	48.6519
	Ref. [18]	4.7308	7.8557	10.9988	14.1426	17.2839	20.4277	23.5674	26.7093	29.8455	32.9814	36.1100	39.2292	42.3389	45.4072	48.4564
	Ref. [17]	4.7230	7.8529	10.9949	14.1358	17.2765	20.4166	23.5574	26.7011	–	–	–	–	–	–	–
0.01	$N^*_{el}=32$	4.7284	7.8469	10.9800	14.1062	17.2246	20.3338	23.4325	26.5192	29.5926	32.6515	35.6946	38.7209	41.7294	44.7191	47.6891
	$N^*_{el}=64$	4.7284	7.8469	10.9800	14.1062	17.2246	20.3338	23.4325	26.5192	29.5926	32.6514	35.6946	38.7209	41.7294	44.7190	47.6889
	Ref. [16]	4.7284	7.8469	10.9800	14.1062	17.2246	20.3338	23.4325	26.5192	29.5926	32.6514	35.6946	38.7209	41.7293	44.7189	47.6888
	Ref. [18]	4.7287	7.8478	10.9824	14.1081	17.2293	20.3373	23.4391	26.5248	29.5990	32.6610	35.7003	38.7415	41.7370	44.7835	47.6744
	Ref. [17]	4.7284	7.8469	10.9799	14.1061	17.2244	20.3336	23.4328	26.5242	–	–	–	–	–	–	–
	$N^*_{el}=32$	4.5795	7.3312	9.8562	12.1456	14.2330	16.1501	17.9239	19.5762	21.1242	22.5815	23.9587	25.2627	26.4873	26.4896	27.1353
0.1	$N^*_{el}=64$	4.5795	7.3312	9.8561	12.1454	14.2326	16.1491	17.9221	19.5734	21.1200	22.5756	23.9507	25.2519	26.3351	26.4684	26.9767
	Ref. [16]	4.5796	7.3312	9.8561	12.1454	14.2324	16.1487	17.9215	19.5723	21.1185	22.5735	23.9479	25.2479	26.2831	26.4595	26.9237
	Ref. [18]	4.5796	7.3312	9.8561	12.1453	14.2324	16.1487	17.9217	19.5732	21.1209	22.5805	23.9625	25.2830	26.2837	26.5309	26.9313
	Ref. [17]	4.5795	7.3312	9.8560	12.1453	14.2323	16.1478	17.9214	19.3788	–	–	–	–	–	–	–
	$N^*_{el}=32$	4.2420	6.4180	8.2855	9.9042	11.3499	12.6429	13.4845	13.8159	14.5160	14.9456	15.7522	16.0116	17.0077	17.0414	17.9449
0.2	$N^*_{el}=64$	4.2420	6.4179	8.2854	9.9039	11.3490	12.6409	13.4637	13.8116	14.4894	14.9402	15.7128	16.0060	16.9820	17.0019	17.9381
	Ref. [16]	4.2420	6.4179	8.2853	9.9037	11.3487	12.6402	13.4567	13.8101	14.4806	14.9383	15.6996	16.0040	16.9621	16.9999	17.9357
	Ref. [18]	4.2420	6.4179	8.2853	9.9036	11.3487	12.6401	13.4568	13.8103	14.4806	14.9390	15.6998	16.0072	16.9622	17.0088	17.9598
	Ref. [17]	4.2420	6.4179	8.2853	9.9036	11.3486	12.6357	13.4567	13.8115	–	–	–	–	–	–	–
EBT		4.7300	7.8532	10.9956	14.1372	17.2788	20.4204	23.5619	26.7035	29.8451	32.9867	36.1283	39.2699	42.4115	45.5531	48.6947

N^*_{el}, number of elements

Table 4 Non-dimensional frequency parameter λ_i of the FF Timoshenko beam with different thickness-to-length ratios

t/l	Method	Present solution: mode number														
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.002	$N^*_{el}=32$	4.7300	7.8530	10.9952	14.1362	17.2770	20.4174	23.5575	26.6970	29.8360	32.9744	36.1122	39.2492	42.3854	45.5208	48.6552
	$N^*_{el}=64$	4.7300	7.8530	10.9952	14.1362	17.2770	20.4174	23.5575	26.6970	29.8360	32.9744	36.1122	39.2492	42.3854	45.5208	48.6552
	Ref. [16]	4.7300	7.8530	10.9952	14.1362	17.2770	20.4174	23.5575	26.6970	29.8360	32.9744	36.1122	39.2492	42.3854	45.5207	48.6552
	Ref. [18]	4.4415	7.4609	10.6670	13.6601	16.7730	19.7371	22.8275	25.7490	28.8100	31.6576	34.6559	37.3819	40.2670	42.8805	45.6698
	Ref. [17]	4.7300	7.8530	10.9951	14.1360	17.2768	20.4165	23.5529	26.6751	–	–	–	–	–	–	–
0.01	$N^*_{el}=32$	4.7292	7.8491	10.9843	14.1131	17.2350	20.3483	23.4516	26.5436	29.6229	32.6881	35.7382	38.7719	41.7883	44.7863	47.7650
	$N^*_{el}=64$	4.7292	7.8491	10.9843	14.1131	17.2350	20.3483	23.4516	26.5436	29.6228	32.6881	35.7382	38.7719	41.7882	44.7862	47.7648
	Ref. [16]	4.7292	7.8491	10.9843	14.1131	17.2350	20.3483	23.4516	26.5436	29.6228	32.6881	35.7382	38.7719	41.7882	44.7861	47.7647
	Ref. [18]	4.4503	7.6366	10.6747	13.8503	16.7687	19.9663	22.7863	26.0006	28.6906	31.9119	34.4010	37.6176	39.8584	43.0423	45.2788
	Ref. [17]	4.7292	7.8491	10.9841	14.1129	17.2334	20.3472	23.4402	26.5220	–	–	–	–	–	–	–
	$N^*_{el}=32$	4.6485	7.4972	10.1257	12.5082	14.6697	16.6386	18.4425	20.1038	21.6401	23.0625	24.3730	25.5466	26.4875	26.4887	27.2893
0.1	$N^*_{el}=64$	4.6485	7.4972	10.1255	12.5078	14.6686	16.6365	18.4388	20.0980	21.6314	23.0498	24.3540	25.5130	26.3459	26.4148	27.1730
	Ref. [16]	4.6485	7.4972	10.1255	12.5076	14.6682	16.6358	18.4375	20.0959	21.6283	23.0452	24.3472	25.5006	26.2976	26.3874	27.1340
	Ref. [18]	4.6468	7.4950	10.1236	12.5048	14.6646	16.6314	18.4323	20.0900	21.6225	23.0393	24.3474	25.4939	26.2680	26.3799	28.2519
	Ref. [17]	4.6485	7.4972	10.1254	12.5074	14.6680	16.6352	18.4371	20.0782	–	–	–	–	–	–	–
	$N^*_{el}=32$	4.4496	6.8027	8.7736	10.4112	11.7980	12.8273	13.5795	13.6684	14.7201	14.7592	15.8544	15.9357	17.0148	17.0305	18.0093
0.2	$N^*_{el}=64$	4.4496	6.8026	8.7730	10.4098	11.7951	12.8191	13.5637	13.6562	14.7029	14.7437	15.8279	15.9192	16.9884	16.9977	17.9896
	Ref. [16]	4.4496	6.8026	8.7729	10.4094	11.7942	12.8163	13.5584	13.6520	14.6971	14.7384	15.8190	15.9135	16.9742	16.9918	17.9829
	Ref. [18]	4.4494	6.8023	8.7725	10.4089	11.7934	12.8157	13.5579	13.6515	14.6965	14.7373	15.8183	15.9120	16.9735	16.9909	17.9829
	Ref. [17]	4.4496	6.8026	8.7728	10.4093	11.7940	12.8162	13.5583	13.6517	–	–	–	–	–	–	–
EBT		4.7300	7.8532	10.9956	14.1372	17.2788	20.4204	23.5619	26.7035	29.8451	32.9867	36.1283	39.2699	42.4115	45.5531	48.6947

N^*_{el}, number of elements

Table 5 Non-dimensional frequency parameter λ_i of the CF Timoshenko beam with different thickness-to-length ratios

t/l	Method	Present solution: mode number														
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.002	$N_{el}^*=32$	1.8751	4.6940	7.8545	10.9950	14.1361	17.2768	20.4171	23.5571	26.6965	29.8354	32.9737	36.1113	39.2481	42.3841	45.5193
	$N_{el}^*=64$	1.8751	4.6940	7.8545	10.9950	14.1361	17.2768	20.4171	23.5571	26.6965	29.8354	32.9737	36.1113	39.2481	42.3841	45.5193
	Ref. [21]	1.8751	4.6940	7.8545	10.9950	14.1361	17.2768	20.4171	23.5571	26.6965	29.8354	32.9737	36.1113	39.2481	42.3841	45.5193
	Ref. [20]	1.8783	4.7020	7.8679	11.0138	14.1603	17.3066	20.4525	23.5982	26.7434	29.8882	33.0324	36.1761	39.3192	42.4617	45.6034
	Ref. [17]	1.8751	4.6940	7.8545	10.9949	14.1360	17.2765	20.4168	23.5547	–	–	–	–	–	–	–
0.01	$N_{el}^*=32$	1.8750	4.6928	7.8496	10.9821	14.1097	17.2298	20.3411	23.4421	26.5314	29.6077	32.6698	35.7164	38.7464	41.7588	44.7526
	$N_{el}^*=64$	1.8750	4.6928	7.8496	10.9821	14.1097	17.2298	20.3411	23.4421	26.5314	29.6077	32.6698	35.7164	38.7464	41.7588	44.7525
	Ref. [21]	1.8750	4.6928	7.8496	10.9821	14.1097	17.2298	20.3411	23.4421	26.5314	29.6077	32.6698	35.7163	38.7464	41.7587	44.7525
	Ref. [20]	1.8754	4.6936	7.8509	10.9839	14.1120	17.2327	20.3446	23.4463	26.5363	29.6135	32.6764	35.7241	38.7552	41.7689	44.7641
	Ref. [17]	1.8750	4.6927	7.8495	10.9820	14.1093	17.2294	20.3393	23.4387	–	–	–	–	–	–	–
0.1	$N_{el}^*=32$	1.8677	4.5724	7.4154	9.9875	12.3228	14.4469	16.3904	18.1802	19.8386	21.3826	22.8248	24.1715	25.4165	26.3769	26.6587
	$N_{el}^*=64$	1.8677	4.5724	7.4154	9.9874	12.3225	14.4462	16.3889	18.1776	19.8343	21.3763	22.8157	24.1584	25.3953	26.2618	26.5808
	Ref. [21]	1.8677	4.5724	7.4154	9.9873	12.3224	14.4459	16.3883	18.1766	19.8328	21.3741	22.8125	24.1536	25.3875	26.2187	26.5559
	Ref. [20]	1.8677	4.5729	7.4173	9.9918	12.3303	14.4576	16.4038	18.1956	19.8549	21.3987	22.8392	24.1817	25.4158	26.2239	26.5750
	Ref. [17]	1.8677	4.5724	7.4153	10.5733	12.6524	14.4452	16.1224	16.5083	–	–	–	–	–	–	–
0.2	$N_{el}^*=32$	1.8466	4.2853	6.6114	8.5190	10.1595	11.5746	12.7881	13.3718	13.9678	14.3592	15.1328	15.4949	16.3259	16.5902	17.4488
	$N_{el}^*=64$	1.8466	4.2853	6.6113	8.5187	10.1587	11.5728	12.7838	13.3552	13.9556	14.3433	15.1128	15.4784	16.2968	16.5681	17.4264
	Ref. [21]	1.8466	4.2853	6.6113	8.5186	10.1584	11.5722	12.7824	13.3495	13.9515	14.3379	15.1061	15.4728	16.2866	16.5609	17.4180
	Ref. [20]	1.8466	4.2868	6.6159	8.5269	10.1697	11.5853	12.7954	13.3529	13.9602	14.3461	15.1135	15.4845	16.2939	16.5737	17.4306
	Ref. [17]	1.8465	4.2852	6.6112	10.1580	12.4559	12.7887	13.3540	14.3551	–	–	–	–	–	–	–
EBT		1.8751	4.6936	7.8549	10.9955	14 .1372	17.2788	20.4204	23.5619	26.7035	29.8451	32.9867	36.1283	39.2699	42.4115	45.5531

N_{el}^*, number of elements

Table 6 Non-dimensional critical buckling loads $\bar{P}_{cr,i}$ of the SS Timoshenko beam with different thickness-to-length ratios

t/l	Mode	Proposed element					EBT	TBT	Ref. [25]	Ref. [27]
		$N_{el}=2$	$N_{el}=4$	$N_{el}=8$	$N_{el}=16$	$N_{el}=32$				
0.002	1	1	1	1	1	1	1	1	–	–
	2	4.0021	3.9998	3.9998	3.9998	3.9998	4	3.9998	–	–
	3	9.0459	8.9995	8.9992	8.9992	8.9992	9	8.9992	–	–
	4	16.1137	16.0063	15.9974	15.9974	15.9974	16	15.9974	–	–
	5	28.1248	25.0302	24.9938	24.9936	24.9936	25	24.9936	–	–
0.05	1	0.9936	0.9936	0.9936	0.9936	0.9936	1	0.9936	0.9919	0.992
	2	3.9038	3.8999	3.8999	3.8999	3.8999	4	3.8999	3.8734	3.874
	3	8.5813	8.5098	8.5087	8.5087	8.5087	9	8.5087	8.3875	8.391
	4	14.7092	14.5379	14.5108	14.5106	14.5106	16	14.5106	14.1767	–
	5	25.5057	21.6396	21.5462	21.5447	21.5446	25	21.5446	20.8549	–
0.1	1	0.9750	0.9750	0.9750	0.9750	0.9750	1	0.9750	0.9683	0.9685
	2	3.6345	3.6277	3.6277	3.6276	3.6276	4	3.6276	3.5442	–
	3	7.4112	7.3133	7.3115	7.3114	7.3114	9	7.3114	7.0159	–
	4	11.5619	11.3806	11.3432	11.3429	11.3429	16	11.3429	10.7364	–
	5	18.3804	15.3259	15.2317	15.2298	15.2297	25	15.2297	14.3076	–
0.2	1	0.9069	0.9069	0.9069	0.9069	0.9069	1	0.9069	0.8860	0.8865
	2	2.8451	2.8358	2.8357	2.8357	2.8357	4	2.8357	2.6841	–
	3	4.7524	4.6800	4.6783	4.6783	4.6783	9	4.6783	4.3865	–
	4	6.1553	6.0752	6.0555	6.0553	6.0553	16	6.0553	5.7168	–
	5	7.8371	7.0429	7.0112	7.0105	7.0105	25	7.0105	6.7048	–

The results obtained using the proposed element are compared with those of Matsunaga [25], Carrera [27], the Euler–Bernoulli beam theory solution (EBT) and the Timoshenko beam theory solution (TBT) in Table 6. The data listed in this table confirm the element's high accuracy and rapid convergence. Moreover, it is obvious that the obtained results converge to the TBT solution as the number of elements increases.

7 Conclusion

In this study, a highly efficient three-node element was proposed for static, free vibration and buckling analysis of beams, based on the Timoshenko beam theory. For element formulation, deflection and shear strain fields' approximants are chosen of fifth and second orders, respectively. By employing these fields and classical finite element method relations, the shape functions of the proposed element were calculated in the explicit form. Then, by utilizing these shape functions, the stiffness matrix, the geometric stiffness matrix and the mass matrices of the element were explicitly derived. Finally, the extensive numerical testing was performed to assess the accuracy and efficiency of the author's formulation. The results reveal that the suggested three-node element has a very high accuracy and convergence rate for static, free vibration and buckling analysis of thick and thin beams. Moreover, the numerical experiments prove that the element is free of shear locking for extremely thin beams. In conclusion, the solution accuracies of the famous benchmark problems provide the justification of the suggested element.

Appendix

The nonzero entries of the stiffness matrix $[K^e]$ is:

$$
\begin{aligned}
k_{11} &= 4\mathrm{EI}\left(401587200\lambda^3 + 19440000\lambda^2 + 292140\lambda + 1273\right) \\
k_{12} &= 2\mathrm{EI}l\left(-1857945600\lambda^4 + 76723200\lambda^3 + 6814080\lambda^2 + 119340\lambda + 569\right) \\
k_{13} &= -3584\mathrm{EI}(60\lambda + 1)^3 \\
k_{14} &= 1920\mathrm{EI}l(48\lambda + 1)^2\left(1680\lambda^2 + 180\lambda + 1\right) \\
k_{15} &= -4\mathrm{EI}\left(208051200\lambda^3 + 9763200\lambda^2 + 130860\lambda + 377\right) \\
k_{16} &= 2\mathrm{EI}l\left(-1857945600\lambda^4 - 20044800\lambda^3 + 1975680\lambda^2 + 38700\lambda + 121\right) \\
k_{22} &= 4\mathrm{EI}l^2\left(3715891200\lambda^5 + 6220800\lambda^4 + 15949440\lambda^3 + 967980\lambda^2 + 16605\lambda + 83\right) \\
k_{23} &= -896\mathrm{EI}l(60\lambda + 1)^3 \\
k_{24} &= 320\mathrm{EI}l^2(48\lambda + 1)^2\left(-40320\lambda^3 - 2640\lambda^2 + 156\lambda + 1\right) \\
k_{26} &= 2\mathrm{EI}l^2\left(7431782400\lambda^5 - 277862400\lambda^4 - 14065920\lambda^3 + 116520\lambda^2 + 5490\lambda + 19\right) \\
k_{33} &= 7168\mathrm{EI}(60\lambda + 1)^3 \\
k_{44} &= 1280\mathrm{EI}l^2(48\lambda + 1)^2\left(20160\lambda^3 + 3840\lambda^2 + 192\lambda + 1\right).
\end{aligned}
\tag{45}
$$

The nonzero entries of the translation mass matrix $\left[M_1^e\right]$ is:

$$
\begin{aligned}
m_{11} &= l\left(2189721600\lambda^4 + 179055360\lambda^3 + 5828688\lambda^2 + 88554\lambda + 523\right) \\
m_{12} &= \frac{3}{2}l^2\left(-729907200\lambda^5 + 36495360\lambda^4 + 2871360\lambda^3 + 93060\lambda^2 + 2061\lambda + 19\right) \\
m_{13} &= 44l(60\lambda + 1)^2\left(12096\lambda^2 + 540\lambda + 5\right) \\
m_{14} &= -40l^2(48\lambda + 1)^2\left(-23760\lambda^3 - 792\lambda^2 + 177\lambda + 1\right) \\
m_{15} &= \frac{1}{2}l\left(1368576000\lambda^4 + 141419520\lambda^3 + 4440816\lambda^2 + 49668\lambda + 131\right) \\
m_{16} &= -\frac{1}{4}l^2\left(4379443200\lambda^5 - 218972160\lambda^4 - 120960\lambda^3 + 526680\lambda^2 + 9546\lambda + 29\right) \\
m_{17} &= l^3\left(35035545600\lambda^6 + 291962880\lambda^5 - 11335680\lambda^4 + 193536\lambda^3 + 6774\lambda^2 + 129\lambda + 2\right) \\
m_{18} &= 22l^2(60\lambda + 1)^2(72\lambda + 1) \\
m_{19} &= -3l^3(48\lambda + 1)^2\left(10137600\lambda^4 + 929280\lambda^3 - 9440\lambda^2 + 76\lambda + 1\right) \\
m_{110} &= \frac{3}{4}l^3\left(46714060800\lambda^6 + 389283840\lambda^5 - 15114240\lambda^4 + 258048\lambda^3 - 4168\lambda^2 - 268\lambda - 1\right) \\
m_{111} &= 1408l(60\lambda + 1)^2\left(3024\lambda^2 + 108\lambda + 1\right) \\
m_{112} &= 32l^3(48\lambda + 1)^2\left(1900800\lambda^4 + 411840\lambda^3 + 24960\lambda^2 + 288\lambda + 1\right).
\end{aligned}
\tag{46}
$$

The nonzero entries of the rotary mass matrix $\left[M_2^e\right]$ is:

$$
\begin{aligned}
m_{21} &= \frac{12}{l}\left(403200\lambda^2 + 14880\lambda + 139\right) \\
m_{22} &= -\left(55987200\lambda^3 + 1969920\lambda^2 + 15444\lambda - 39\right) \\
m_{23} &= -\frac{1536}{l}(60\lambda + 1)^2 \\
m_{24} &= -240(48\lambda + 1)^2(60\lambda - 1) \\
m_{25} &= \frac{12}{l}\left(57600\lambda^2 + 480\lambda - 11\right) \\
m_{26} &= -\left(-2073600\lambda^3 + 207360\lambda^2 + 5076\lambda + 9\right) \\
m_{27} &= 4l\left(219801600\lambda^4 + 9711360\lambda^3 + 153144\lambda^2 + 1263\lambda + 7\right) \\
m_{28} &= 48(60\lambda + 1)^2(336\lambda - 1) \\
m_{29} &= -8l(48\lambda + 1)^2\left(-18000\lambda^2 + 600\lambda + 1\right) \\
m_{210} &= l\left(8294400\lambda^4 + 2557440\lambda^3 + 9936\lambda^2 - 924\lambda - 5\right) \\
m_{211} &= \frac{3072}{l}(60\lambda + 1)^2 \\
m_{212} &= 256l(48\lambda + 1)^2\left(3600\lambda^2 + 60\lambda + 1\right).
\end{aligned}
\tag{47}
$$

The nonzero entries of the geometric stiffness matrix $\left[K_g\right]$ is:

$$
\begin{aligned}
k_{g1} &= \frac{12}{l}\left(2409523200\lambda^4 + 137721600\lambda^3 + 2953800\lambda^2 + 30156\lambda + 139\right) \\
k_{g2} &= -3\left(22295347200\lambda^5 + 597196800\lambda^4 + 2695680\lambda^3 + 80640\lambda^2 + 1152\lambda - 13\right) \\
k_{g3} &= -\frac{1536}{l}(60\lambda + 1)^2\left(2520\lambda^2 + 84\lambda + 1\right) \\
k_{g4} &= 240(48\lambda + 1)^2\left(241920\lambda^3 + 26640\lambda^2 + 228\lambda + 1\right) \\
k_{g5} &= -\frac{12}{l}\left(1248307200\lambda^4 + 60307200\lambda^3 + 880200\lambda^2 + 4044\lambda + 11\right) \\
k_{g6} &= -9\left(7431782400\lambda^5 + 199065600\lambda^4 - 714240\lambda^3 - 7680\lambda^2 + 576\lambda + 1\right) \\
k_{g7} &= 4l\left(66886041600\lambda^6 + 398131200\lambda^5 - 17418240\lambda^4 + 483840\lambda^3 + 15174\lambda^2 + 471\lambda + 7\right) \\
k_{g8} &= 48(60\lambda + 1)^2(84\lambda - 1) \\
k_{g9} &= -8l(48\lambda + 1)^2\left(29030400\lambda^4 + 2592000\lambda^3 - 30960\lambda^2 + 420\lambda + 1\right) \\
k_{g10} &= l\left(267544166400\lambda^6 + 1592524800\lambda^5 - 69672960\lambda^4 + 1935360\lambda^3 - 55944\lambda^2 - 2076\lambda - 5\right) \\
k_{g11} &= \frac{3072}{l}(60\lambda + 1)^2\left(2520\lambda^2 + 84\lambda + 1\right) \\
k_{g12} &= 256l(48\lambda + 1)^2\left(1814400\lambda^4 + 388800\lambda^3 + 23040\lambda^2 + 240\lambda + 1\right).
\end{aligned}
\tag{48}
$$

References

1. Thomas D, Wilson J, Wilson R (1973) Timoshenko beam finite elements. J Sound Vib 31:315–330
2. Nickel R, Secor G (1972) Convergence of consistently derived Timoshenko beam finite elements. Int J Numer Meth Eng 5:243–252
3. Reddy J (1997) On locking-free shear deformable beam finite elements. Comput Methods Appl Mech Eng 149:113–132
4. Friedman Z, Kosmatka JB (1993) An improved two-node Timoshenko beam finite element. Comput Struct 47:473–481
5. Kapur KK (1966) Vibrations of a Timoshenko beam, using finite-element approach. J Acoust Soc Am 40:1058–1063
6. Lees A, Thomas D (1982) Unified Timoshenko beam finite element. J Sound Vib 80:355–366
7. Lees A, Thomas D (1985) Modal hierarchical Timoshenko beam finite elements. J Sound Vib 99:455–461
8. Tessler A, Dong S (1981) On a hierarchy of conforming Timoshenko beam elements. Comput Struct 14:335–344
9. Falsone G, Settineri D (2011) An Euler–Bernoulli-like finite element method for Timoshenko beams. Mech Res Commun 38:12–16

10. Bouclier R, Elguedj T, Combescure A (2012) Locking free isogeometric formulations of curved thick beams. Comput Methods Appl Mech Eng 245:144–162
11. Cazzani A, Malagù M, Turco E (2016) Isogeometric analysis of plane-curved beams. Math Mech Solids 21:562–577
12. Lepe F, Mora D, Rodríguez R (2014) Locking-free finite element method for a bending moment formulation of Timoshenko beams. Comput Math Appl 68:118–131
13. Caillerie D, Kotronis P, Cybulski R (2015) A Timoshenko finite element straight beam with internal degrees of freedom. Int J Numer Anal Meth Geomech 39:1753–1773
14. Khajavi R (2016) A novel stiffness/flexibility-based method for Euler–Bernoulli/Timoshenko beams with multiple discontinuities and singularities. Appl Math Model 40(17–18):7627–7655
15. Dawe D (1978) A finite element for the vibration analysis of Timoshenko beams. J Sound Vib 60:11–20
16. Lee J, Schultz W (2004) Eigenvalue analysis of Timoshenko beams and axisymmetric Mindlin plates by the pseudospectral method. J Sound Vib 269:609–621
17. Kocatürk T, Şimşek M (2005) Free vibration analysis of Timoshenko beams under various boundary conditions. Sigma 1:30–44
18. Ferreira A (2005) Free vibration analysis of Timoshenko beams and Mindlin plates by radial basis functions. Int J Comput Methods 2:15–31
19. Ferreira A, Fasshauer G (2006) Computation of natural frequencies of shear deformable beams and plates by an RBF-pseudospectral method. Comput Methods Appl Mech Eng 196:134–146
20. Xu S, Wang X (2011) Free vibration analyses of Timoshenko beams with free edges by using the discrete singular convolution. Adv Eng Softw 42:797–806
21. Lee SJ, Park KS (2013) Vibrations of Timoshenko beams with isogeometric approach. Appl Math Model 37:9174–9190
22. Moallemi-Oreh A, Karkon M (2013) Finite element formulation for stability and free vibration analysis of Timoshenko beam. Adv Acoust Vib 2013:841215. https://doi.org/10.1155/2013/841215
23. Hsu YS (2016) Enriched finite element methods for Timoshenko beam free vibration analysis. Appl Math Model 40:7012–7033
24. Kosmatka J (1995) An improved two-node finite element for stability and natural frequencies of axial-loaded Timoshenko beams. Comput Struct 57:141–149
25. Matsunaga H (1996) Buckling instabilities of thick elastic beams subjected to axial stresses. Comput Struct 59:859–868
26. Więckowski Z, Golubiewski M (2007) Improvement in accuracy of the finite element method in analysis of stability of Euler–Bernoulli and Timoshenko beams. Thin Walled Struct 45:950–954
27. Carrera E, Pagani A, Banerjee JR (2016) Linearized buckling analysis of isotropic and composite beam-columns by Carrera Unified Formulation and dynamic stiffness method. Mech Adv Mater Struct 23:1092–1103
28. Coulter BA, Miller RE (1986) Vibration and buckling of beam-columns subjected to non-uniform axial loads. Int J Numer Meth Eng 23:1739–1755
29. Augarde CE (1998) Generation of shape functions for straight beam elements. Comput Struct 68:555–560
30. Modarakar Haghighi A, Zakeri M, Attarnejad R (2015) 3-node basic displacement functions in analysis of non-prismatic beams. J Comput Appl Mech 46:77–91
31. Ferreira AJ (2008) MATLAB codes for finite element analysis: solids and structures, vol 157. Springer, Berlin
32. Bažant ZP, Cedolin L (2010) Stability of structures: elastic, inelastic, fracture and damage theories. World Scientific, Singapore