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TECHNICAL PAPER

A fast decoupled reliability-based design optimization of structures using B-spline interpolation curves

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Abstract This paper introduces a new method for reliability-based design optimization (RBDO) of structures. In RBDO of structural problems unlike the conventional two-level approaches, sometimes it is not necessary to carry out reliability analysis for each deterministic design. The proposed method may be categorized as a decoupled method for reliable optimum design; however, it is based on the safety factor (SF) concept. To briefly describe the proposed method, a deterministic design optimization (DDO) point is obtained based on an arbitrary SF. The corresponding failure probability (P_{f}) is then determined using Monte Carlo simulation (MCS). The P_{f} is then compared with the targeted P_{f}. If the relative distance error is greater than a desirable tolerance, the cubic B-spline interpolation concept is then employed as a result of which a modified SF is extracted. For the modified SF found, DDO procedure is carried out. The above procedure is iteratively repeated until convergence occurs and the reliable optimum point is found. Finally, the proposed method was applied to solving some structural problems. The obtained results were favourably in accordance with those recorded in the literature while only a fraction of P_{f} computations was necessary.

Keywords Decoupled · Reliability · Optimization · Structural · B-spline · Interpolation

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1 Introduction

Reliability-based design optimization (RBDO) implements structural optimization considering simultaneously the uncertainties in the structural materials properties and/or applied loading. The general structural RBDO problem with both deterministic and probabilistic design constraints can be expressed as:

$$\begin{aligned}
&\text{Min/Max } f(\boldsymbol{d})\\
&\text{Subject to:}\\
&P(G_i(\boldsymbol{d},\boldsymbol{X}) \le 0) \le P_{\mathrm{f}}^{i}, \quad i = 1,\ldots,N_{\mathrm{PC}}\\
&\text{and/or } \sigma_j(\boldsymbol{d}) \le \sigma_{\mathrm{all.}}, \quad j = 1,\ldots,N_m\\
&\text{and/or } u_k(\boldsymbol{d}) \le u_{\mathrm{all.}}, \quad k = 1,\ldots,N_{\mathrm{DOF}}\\
&\text{and/or } \boldsymbol{d}^L \le \boldsymbol{d} \le \boldsymbol{d}^U
\end{aligned} \tag{1}$$

where $\boldsymbol{d} = [d_1, d_2, \ldots, d_n]^{\mathrm{T}}$ is a column vector of n deterministic design variables, $\boldsymbol{X} = [x_1, x_2, \ldots, x_m]^{\mathrm{T}}$ is the m-dimensional vector of random variables, $f(\boldsymbol{d})$ is the objective function, $P(G_i(\boldsymbol{d},\boldsymbol{X}) \le 0)$ denotes the failure probability for the i-th limit state function $G_i(\boldsymbol{d},\boldsymbol{X})$. P_{f}^{i} is the target failure probability of i-th constraint and N_{PC} is the number of probabilistic constraints. In Eq. (1), σ and u are the stress of jth member and the nodal displacement of k-th degree of freedom, respectively. $\sigma_{\mathrm{all.}}$, $u_{\mathrm{all.}}$, $\boldsymbol{d}^{\mathrm{L}}$, $\boldsymbol{d}^{\mathrm{U}}$, N_m and N_{DOF} are respectively allowable member stress, allowable nodal displacement, lower and upper bounds of $\boldsymbol{d}$, total number of members of the structure and total number of degrees-of-freedom. The target failure probability can be expressed in terms of the target reliability index, $P_{\mathrm{f}}^{i} = \Phi(-\beta_{ti})$, as $\Phi(.)$ is the standard normal cumulative distribution function.

The most common routines to solve Eq. 1 include: (1) A two-level or loop-nested approach that introduces a reliability analysis as a probabilistic constraint within a deterministic optimization loop. This approach is namely referred to as the reliability index approach (RIA) based on FORM, or as the performance measure approach (PMA) based on an inverse FORM [1]. (2) A mono-level approach that attempts to solve RBDO problems by avoiding the reliability analysis [2–6]. This method reformulates the reliability constraints and replaces them by utilizing the optimality conditions resulting in a mono DDO loop. This technique expresses inconstancy in some cases since it requires calculation of second-order derivatives [7]. (3) Decoupled approaches where the concept of reliability analysis is carried out whether before or after the DDO procedure [8–10]. Recently many studies have been conducted in this field to decouple the reliability and DDO loops. However, a recent benchmark study [7] shows that such approaches may suffer from non-unicity of the most probable failure point (MPFP) in cases where they rely on the assumption of MPFP.

One of the most popular approaches to reduce computational cost of reliability analysis is the response surface method (RSM) and its adaptive versions (see for example [11–17]). Numerous studies have also been done on the application of RSM to solve structural reliability analysis and structural RBDO problems (see [17–20]). RSM employs polynomial function to approximate the unknown implicit performance function. However, RSM becomes computationally impractical for problems involving a large number of nonlinear random variables, particularly when mixed or statistically dependent random variables are involved [21].

One could refer to the study by Qu and Haftka [22] who proposed the concept similar to safety factors to relate DDO and reliability analysis, namely referred to as the probabilistic sufficiency factor (PSF). In that research PSF is computed by Monte Carlo simulation (MCS) combined with response surface method (RSM) approximations.

In addition, Wu et al. [23] presented a method in 2001 based on safety factors by which the optimization procedure and reliability analysis are employed using a decoupling approach. In fact they substituted random variables with deterministic ones based on safety factors.

The current study attempts to implement a fast method to find reliable optimum solutions, based on the concept of safety factors embedded into the essential formulation of RBDO. The deterministic optimum points are computed in an inverse manner iteratively until the global optimum point is determined whose failure probability, estimated using MCS method, meets that of the target value. The safety factor corresponding to the global optimum point is called the reliable demand factor (RDF). It is worthwhile noting that the dynamic modification of the SFs throughout the proposed approach is performed using the cubic B-spline interpolation concept. It modifies the fitting curve accuracy and reduces the relative error distance with the aim of finding a RDF encountered with the targeted failure probability. The reasons for applying MCS is its versatility for all problems, easy to implement and its capability of computing the probability of failure with the desired precision [24, 25].

The proposed decoupled RBDO approach possesses the following unique advantages: (1) it incorporates safety factors in structural design, a methodology that engineers are familiar with, (2) it identifies solutions which are near the reliable optimum design with a few number of failure probability computations, (3) it avoids computation of optimization and reliability analyses simultaneously, (4) utilization of cubic B-spline interpolation concept is due to its advantageous characteristic to fit modified curves in a piecewise manner without increasing the degree of the polynomial. It finally determines an optimum response corresponding to the target failure probability. This leads to a fast convergence of an optimum SF via a few P_f calculations, and (5) the efficient decoupling approach employed here allows for any type of reliability method to be exploited along with the optimization procedure.

Finally, some structural problems will be attempted containing two and three-dimensional truss structures and the results will be performed.

2 The proposed methodology

2.1 Constraints handling

Considering the deterministic design constraints of Eq. (1), the squared normalized degree of constraints violation V for each design, related to the member stresses and nodal displacements, may be defined as:

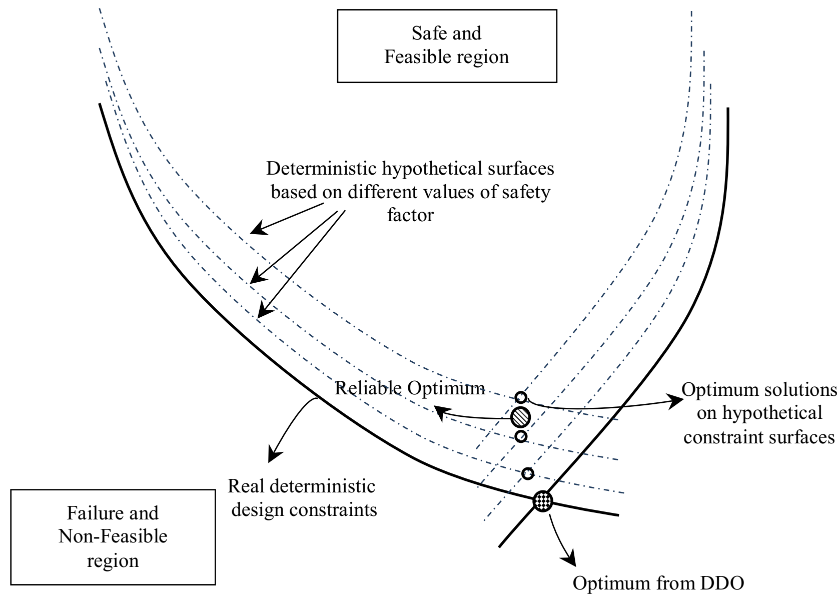
$$V(\boldsymbol{d}) = \sum_{i=1}^{NM} \left[\max\left(0, \left|\frac{\sigma_i(\boldsymbol{d})}{\sigma_{\text{all.}}}\right| - 1\right)\right]^2 + \sum_{j=1}^{ND} \left[\max\left(0, \left|\frac{u_j(\boldsymbol{d})}{u_{\text{all.}}}\right| - 1\right)\right]^2 \tag{2}$$

By incorporating a set of safety factors into Eq. (2), it can be modified as a squared normalized safety factor-based degree of constraints violation V' given by:

$$V'\left(\boldsymbol{d}, \widehat{\boldsymbol{X}}\right) = \sum_{i=1}^{NM} \left[\max\left(0, \left|\frac{\sigma_i(\boldsymbol{d})}{\gamma_s \sigma_{\text{all.}}}\right| - 1\right)\right]^2 + \sum_{j=1}^{ND} \left[\max\left(0, \left|\frac{u_j(\boldsymbol{d})}{\gamma_d u_{\text{all.}}}\right| - 1\right)\right]^2 \tag{3}$$

where, γ_s and γ_d are defined here as safety factors corresponding to stress and displacement, respectively and $\widehat{\boldsymbol{X}}$ denotes the vector of mean values of random variables. Based on the definition of safety factors, γ_s and γ_d are

Fig. 1 Series of deterministic design optimizations based on a sequence variation of safety factors



allowed to vary between 1 and 0. The safety factor that satisfies the target level of reliability is called RDF.

2.2 RBDO formulation based on safety factors

In structural design, due to the stochastic nature of material properties and applied loads and thus design uncertainties, use of stochastic analysis is inevitable. As an alternative, use of safety factors in the process of structural design is a simple way familiar to all engineers. These coefficients are referred to as load or resistance factors. Conventionally in the processes of optimization based on reliability, the deterministic objective function, constrained with the structural failure probability is a repetitive procedure that requires thousands of analyses to be carried out for each design. In fact, the conventional RBDO method utilized in structural optimization, is a nested or a double-loop method which contains two loops; the outer loop that contains optimization procedure and the inner loop, which is related to the reliability analysis. The design vectors of deterministic variables are transferred to the inner loop from the outer one in order to compute the failure probability for each design. Therefore, a gigantic number of analyses should be carried out for the RBDO.

The purpose of the present work is to search for the optimum point on the different safety levels of V' as in Fig. 1 to satisfy the targeted level of reliability. These surfaces are generated by utilizing a series of safety factors. In the proposed approach, use of SF in the objective function simply allows a combined reliability based objective function for optimization. The proposed formulation of decoupled RBDO using SFs is introduced as the following equation:

$$\begin{aligned}&\text{Find}\,\gamma^{*}\\&\text{subject to :}\\&\begin{cases}V'=0\\P_{\mathrm{f}}\left(\boldsymbol{d}^{*},\widehat{X}\right)=P_{\mathrm{f}}^{\mathrm{Target}}\end{cases}\end{aligned}\qquad(4)$$

The objective of Eq. (4) is to find an optimum safety factor γ^* utilized by the cubic B-splines. This will lead to the optimum solution vector $\boldsymbol{d}^*$ under the deterministic safety factor-based constraint V'. After that, the instituted optimum solutions $\boldsymbol{d}^*$ are controlled for the probabilistic constraint as to check the corresponding failure probability over the targeted probability of failure.

2.3 Proposed RBDO procedure using B-spline interpolation concept

The procedure for RBDO of structures using proposed inverse method may be performed according to the flowchart shown in Fig. 2.

Figure 3 indicates the procedure by which the range of response for γ^* is decided upon. The following steps details the whole procedure. In all steps the optimum points are computed for the mean values of the random variables $\widehat{X} = [\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_m}]^{\mathrm{T}}$. P_{f}^{a} means the probability of failure corresponding to $\gamma = a$.

Step 1—Primary check of the design space First for the mean value of safety factor, $\gamma = 0.5$, according to Eq. (3), the corresponding optimum design $\boldsymbol{d}^*$ will be determined. With respect to the type of constraints, γ could represent either γ_s and/or γ_d in most structural problems. The $P_{\mathrm{f}}^{0.5}$ is

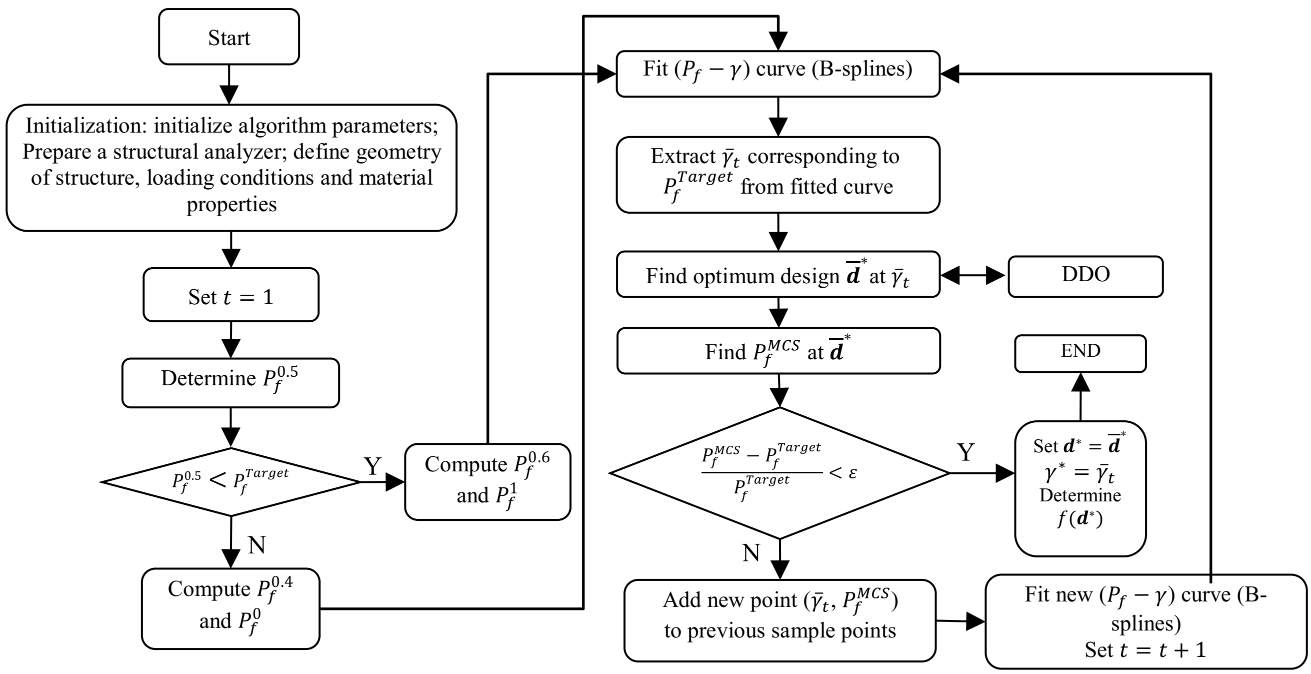


Fig. 2 Flowchart of the proposed method for RBDO of structures

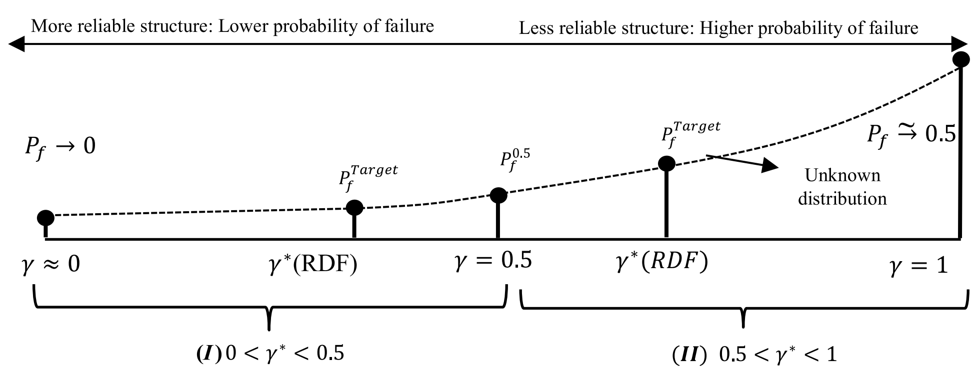


Fig. 3 Structural failure probability against the different values of safety factor

then computed using MCS as described in Sect. 2.4. The reason for this selection is based on the fact that within the range $0 < \gamma < 1$, a mid-point ($\gamma = 0.5$) may be a suitable starting point. As an example, if one uses the medians of $\widehat{X}$, the median-design will have a reliability of approximately 0.5 because the MPP will have a minimum distance of zero resulting in $\Phi(0) = 0.5$ [23].

Step 2—Movement decision for $\boldsymbol{\gamma}$ *towards* γ^* Compare $P_{\mathrm{f}}^{\mathrm{Target}}$, and determine the range of γ^* according to Eq. (5):

$$\begin{cases} \text{(I)} \quad \text{if } P_{\mathrm{f}}^{0.5} > P_{\mathrm{f}}^{\mathrm{Target}} \quad \text{search for } \gamma^* \quad 0 < \gamma < 0.5 \\ \text{(II)} \quad \text{if } P_{\mathrm{f}}^{0.5} < P_{\mathrm{f}}^{\mathrm{Target}} \quad \text{search for } \gamma^* \quad 0.5 < \gamma < 1 \end{cases} \tag{5}$$

In most structural problems, computing $P_{\mathrm{f}}^{0.4}$ for region I and $P_{\mathrm{f}}^{0.6}$ for region II could be very efficient.

Therefore, in case of (I), compute the values of P_{f} for the two auxiliary points $\gamma = 0$ and 0.4. In case of (II), compute the values of P_{f} for the two auxiliary points of two safety factors $\gamma = 0.6$ and 1.

Step 3—Curve fitting In this stage, cubic B-spline interpolation concept is used in order to more accurately fit (P_{f}vs.γ) by having the coordinates of the points found. The authors recommend de Boor, [26] in case further studies about the advancements of B-splines are required.

Step 4—Extracting data from fitted curve Extract the approximate safety factor ($\bar{\gamma}$) corresponding to the value of $P_{\mathrm{f}}^{\mathrm{Target}}$ given, from the fitted curve of step 3.

Step 5—deterministic design optimization (DDO) Use DDO approach to find the optimum design point $\boldsymbol{d}^*$ subject to V' based on the extracted $\bar{\gamma}$ using the mean values of random variables $\widehat{\boldsymbol{X}}$.

Step 6—Reliability analysis Determine the $P_{\mathrm{f}}^{\mathrm{MCS}}$ for the deterministic optimum design $\boldsymbol{d}^*$ of step 5 using MCS.

Step 7—Error estimation Compute the relative distance $\mathrm{error}^{P_{\mathrm{f}}}$ between $P_{\mathrm{f}}^{\mathrm{MCS}}$ and $P_{\mathrm{f}}^{\mathrm{Target}}$ using Eq. (6)

$$\mathrm{error}^{P_{\mathrm{f}}} = \frac{P_{\mathrm{f}}^{\mathrm{MCS}} - P_{\mathrm{f}}^{\mathrm{Target}}}{P_{\mathrm{f}}^{\mathrm{Target}}} \tag{6}$$

Step 8—Check convergence If relative distance $\mathrm{error}^{P_{\mathrm{f}}}$ is less than or equal to a tolerance ε, the $P_{\mathrm{f}}^{\mathrm{MCS}}$ will be assigned as the $P_{\mathrm{f}}^{\mathrm{Target}}$ and the reliability based optimum design and the RDF γ^* is said to being found.

Step 9—Repeating the procedure until convergence If according to Eq. (6) the convergence has not occurred, the $P_{\mathrm{f}}^{\mathrm{MCS}}$ will be considered as another auxiliary point to more accurately fit the curve. Steps 3–9 will be repeated until convergence and the reliable optimum design point $\boldsymbol{d}^*$ will be recorded.

2.4 Monte Carlo simulation

Monte Carlo simulation, named after the casino games of Monte Carlo, Monaco, originates from the research work of Metropolis and Ulam in 1949 [27]. Monte Carlo simulation is known as a simple random sampling method or statistical trial method that makes realizations based on randomly generated sampling sets for uncertain variables. A basic advantage of sampling methods is their direct utilization of experiments to obtain mathematical solutions or probabilistic information concerning problems whose system equations cannot be solved easily by known procedures.

In general, a reliability problem is formulated using a limit state function, $G(\boldsymbol{X})$, where $\boldsymbol{X} = [X_1, X_2, \ldots, X_m]^{\mathrm{T}}$ is a vector with m random variables. Violation of the limit state function or failure is defined by the condition $G(\boldsymbol{X}) \leq 0$ and the probability of failure, P_{f}, is expressed by the following expression (Eq. 7):

$$P_{\mathrm{f}} = P[G(\boldsymbol{X}) \leq 0] = \iint\limits_{G(\boldsymbol{X})\leq 0} \ldots \int J_{X_1X_2\ldots X_m}(\boldsymbol{X})\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_m \tag{7}$$

where $J_{X_1X_2\ldots X_m}(\boldsymbol{X})$ is the joint probability density function. The Monte Carlo method allows the determination of an estimate of the probability of failure, given by equation:

$$P_{\mathrm{f}} \cong \frac{1}{N}\sum_{i=1}^{N} I(\boldsymbol{X}_i) \tag{8}$$

where N is the total number of samples and $I(\boldsymbol{X})$ is a function defined by Eq. (9):

$$I(\boldsymbol{X}) = \left\{ \begin{array}{ll} 1 & \text{if} \quad G(\boldsymbol{X}) \leq 0 \\ 0 & \text{if} \quad G(\boldsymbol{X}) > 0 \end{array} \right\} \tag{9}$$

Equation 9 states that the violation of the ultimate limit state function ($G(\boldsymbol{X}) \leq 0$) occurs when either the strength or serviceability capacity of structure is less than the loading effects. The basic computation procedure of MCS is as follows:

Step 1 Generate N samples for each random variable based on its probability distribution type and construct N sampling set vectors $\boldsymbol{X}$.
Step 2 Compute the value of limit state function for each sample to obtain $G(\boldsymbol{X})$.
Step 3 Check Eq. (9) for the computed $G(\boldsymbol{X})$).
Step 4 Use Eq. (8) to estimate the P_{f}.

To generate a sample value x_i for each random variable, the following general procedure for any type of distribution can be formulated [24, 25]:

Step 1 Generate the random value u_i from a uniformly distributed random variable between 0 and 1.
Step 2 Calculate a sample value x_i from the following equation:

$$x_i = F_X^{-1}(u_i) \tag{10}$$

where F_X^{-1} is the inverse of F_X and F_X is a cumulative distribution function of random variable x.

3 Examples

In this section three benchmark RBDO structural problems were studied: a 10-bar truss with 10 design variables, a 13-bar truss with 7 design variable and a 72-bar spatial truss problem with 16 design variables. The structural P_{f} computation was performed using MCS. As Nowak [24] suggested, the number of samples produced by MCS, with regard to the coefficient of variation for P_{f} as to be less than or equal to ν, is determined by the following equation:

$$N^{\mathrm{MCS}} = \frac{1 - P_{\mathrm{f}}}{\nu^2 P_{\mathrm{f}}} \tag{11}$$

For each problem, ν was set as 0.1 and P_{f} should be equal to the targeted value for that problem.

In all examples, the sequential quadratic programming (SQP) algorithm was used to carry out the optimization procedure. For this purpose, “fmincon” function

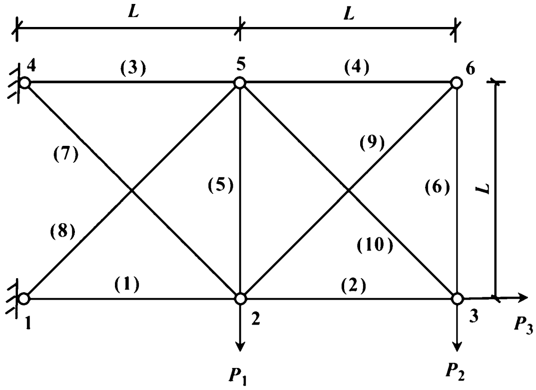


Fig. 4 A 10-bar planar truss

of MATLAB was utilized. Also, the starting point $\boldsymbol{d}^0$ of DDO was considered as the middle of the allowable design bounds, shown as follows:

$$d_j^0 = \frac{d_j^U + d_j^L}{2} \tag{12}$$

where j denotes the design variable number and $\boldsymbol{d}_j^{\mathrm{U}}$ and $\boldsymbol{d}_j^{\mathrm{L}}$ are the upper and lower bounds of j-th variable, respectively. For all three examples, computational expenses will be compared with those from the literature by means of the number of estimations of the failure probability. Also, a package which contains finite element method codes written by the authors in MATLAB software was prepared as an analyser where the linear elastic behaviour of elements was considered.

3.1 A 10-bar truss problem

This benchmark problem is selected to verify the proposed methodology. It was addressed, for example, by Zhao and Qiu [20] as a RBDO problem. The geometry of this 10-bar truss is shown in Fig. 4. The deterministic design variables of the structure are the members cross-sectional areas and the stochastic variables include external loads P_1, P_2 and P_3, module of elasticity E and the length of the horizontal and vertical members L. Their statistical properties are given in Table 1.

The cross-section areas of all bars are design variables and their lower and upper bounds are 0.0001 and 0.002 m^2, respectively. The target reliability index is 2.5, i.e., the target failure probability is 6.21×10^{-3}. The total area of bars is to be minimized. The limit state is defined as nodal maximum vertical displacement of node 3, which should be less than 0.004 m. The modified implicit limit state function G^* is expressed as in Eq. (13) for P_f calculation:

$$G^* = 0.004 - u_{3,y}(\boldsymbol{d}^*, \boldsymbol{X}) \tag{13}$$

Considering Eq. (13), the following formulation is introduced as utilizing the displacement safety factor (γ_d) to obtain optimum deterministic points:

$$V'\left(\boldsymbol{d}, \widehat{\boldsymbol{X}}\right) = \left[\max\left(0, \frac{u_{3,y}\left(\boldsymbol{d}, \widehat{\boldsymbol{X}}\right)}{\gamma_d 0.004} - 1\right)\right]^2 \tag{14}$$

In Eq. (14), the aim is to find the optimum γ_d^* that gives an optimum solution $\boldsymbol{d}^*$, such that $P_\mathrm{f}(\boldsymbol{d},^* \boldsymbol{X})$ is less than the $P_\mathrm{f}^{\mathrm{Target}}$ considering the given tolerance ε.

Having found $P_\mathrm{f}^{0.5}$ and compared the value with $P_\mathrm{f}^{\mathrm{Target}}$, the range $0.5 < \gamma^* < 1$ was nominated. To first fit the curve, the two initial points corresponding to $\gamma = 0.6$ and 1 were computed for P_f. The procedure then was repeated for three iterations until the B-spline interpolation curves were fitted for three auxiliary points (see Fig. 6a, b), as a result of which error$^{P_\mathrm{f}}$ was found less than 1 %. Figure 5a,

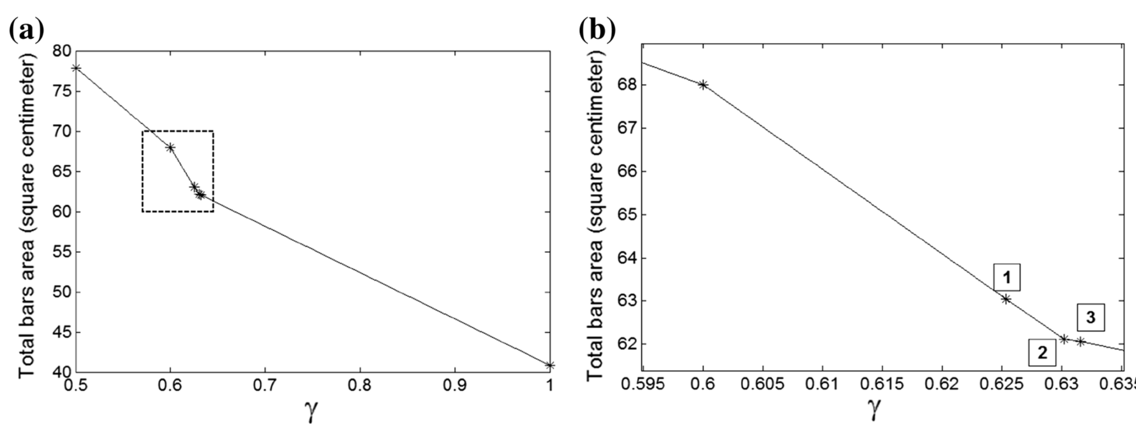


Fig. 5 Convergence histories of objective function, 10-bar truss. **a** For total sample points; and **b** magnified view around the reliable optimum solution with $f = 62.068$ cm^2

Table 1 Statistical properties of random variables, 10-bar truss

Variable	Distribution	Unit	Mean value	Coefficient of variation (CV)
P_1	Normal	kN	60	0.2
P_2	Normal	kN	40	0.2
P_3	Normal	kN	10	0.2
E	Normal	Gpa	200	0.1
L	Normal	m	1	0.05

b illustrates the convergence leading to the optimum design corresponding to the P_f^{Target}. The results are as listed in Table 2. It is worthwhile noting that through the whole process of reliability-based optimization with the proposed technique, the P_f was only computed six times, wherein the method proposed by Zhao and Qiu [20] requires a total of 1904 P_f calculations to find an optimum solution with the objective value of 63.649 m^2 with the corresponding P_f equal to 2.742 × 10^{-3}. One could claim that the proposed

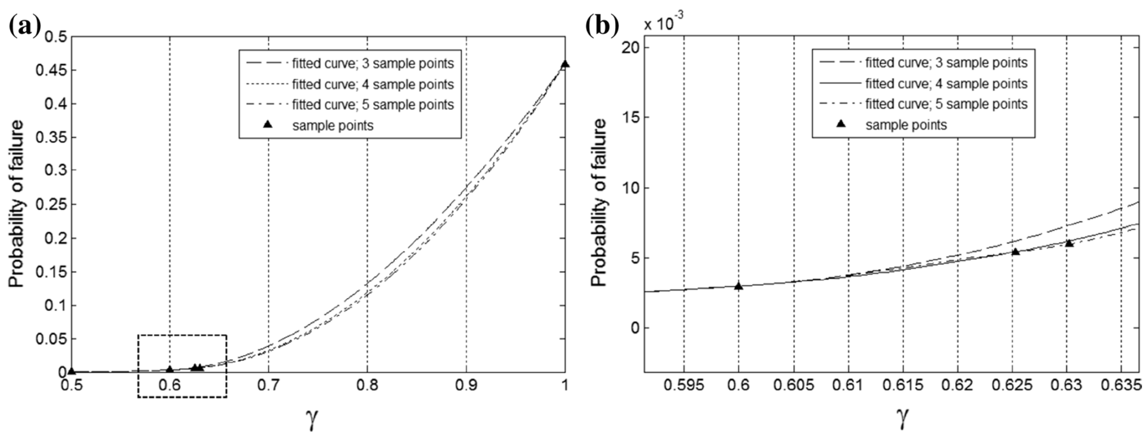


Fig. 6 Fitted cubic B-spline curves for 3, 4 and 5 sample points, 10-bar truss. **a** For 0.5 < FS < 1; and **b** magnified view around the reliable optimum solution with $P_f = 6.15 \times 10^{-3}$

Table 2 RBDO results of 10-Bar truss

Variable group	Bar areas	Zhao and Qiu[a] [20]	Optimal cross-sectional area (m^2) × 10^{-4}					
			Current work					
			Initial sample points			Auxiliary sample points (iterations)		
			$\gamma_d = 0.5$	$\gamma_d = 0.6$	$\gamma_d = 1$	$\bar{\gamma}_{d,1} = 0.6253$	$\bar{\gamma}_{d,2} = 0.6302$	$\gamma_d^* = 0.6316$
1	A_1	10.705	12.500	13.999	6.250	10.000	10.000	10.000
2	A_2	5.914	6.250	10.000	3.125	5.000	5.040	5.042
3	A_3	14.424	17.562	10.000	8.693	14.0465	14.0834	14.0280
4	A_4	1.000	1.000	1.000	1.000	1.000	1.000	1.000
5	A_5	1.000	1.000	1.000	1.000	1.000	1.000	1.000
6	A_6	1.000	1.000	1.000	1.000	1.000	1.000	1.000
7	A_7	5.531	12.500	10.000	6.250	10.000	9.000	9.000
8	A_8	11.853	12.500	10.000	6.250	10.000	10.000	10.000
9	A_9	1.000	1.000	1.000	1.000	1.000	1.000	1.000
10	A_{10}	11.223	12.500	10.000	6.250	10.000	10.000	10.000
Total area (×10^{-4})		63.649	77.812	67.999	40.818	63.0465	62.123	62.068
Exact P_f (MCS)		2.742 × 10^{-3}	0.11 × 10^{-3}	2.97 × 10^{-3}	0.4586	5.433 × 10^{-3}	5.979 × 10^{-3}	6.15 × 10^{-3}
No. of P_f calc.		1904				4	5	6
(error^{P_f} × 100) %						21.51	3.71	0.966

[a] (“fmincon” function of Matlab software for optimization) – (an efficient response surface method for reliability analysis)

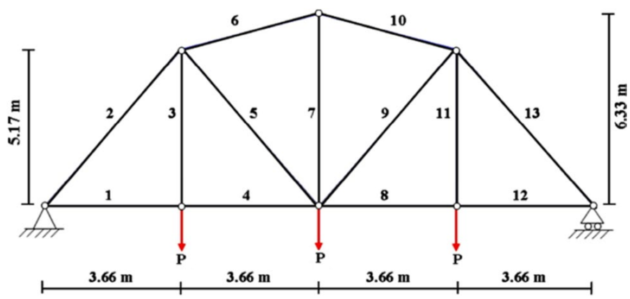


Fig. 7 A 13-bar bridge truss

Table 3 Member grouping details of a 13-bar bridge truss

Group number	Variables
1	A_1, A_{12}
2	A_2, A_{13}
3	A_3, A_{11}
4	A_4, A_8
5	A_5, A_9
6	A_6, A_{10}
7	A_7

Table 4 Statistical properties of random variables, 13-bar bridge truss

Variable	Distribution	Mean value	Coefficient of variation (CV)
Load, P (kN)	Normal	66.726	0.16
Yield stress, F_y (Mpa)	Normal	248.22	0.12

RBDO approach requires less P_f calculations. This optimum solution also possesses a P_f less than the target one with an error less than 1 %.

3.2 A 13-bar bridge truss problem

A 13-bar bridge truss shown in Fig. 7 is considered which has been studied by some authors as a RBDO problem such as Nakib and Frangopol [28] and Ghorbani and Ghasemi [29], previously. Table 3 lists the seven design variables' groups as the cross-sectional areas of members. Their lower and upper bounds are between 6.4516×10^{-5} and 6.4516×10^{-3} m^2, respectively. The material density and modulus of elasticity were considered as 7850 kg/m^3 and 206 GPa, respectively. A load P and material yield stress, F_y are considered as random variables associated with normal distribution with the corresponding parameters defined in Table 4. System failure is assumed to occur when the stress in each member σ_i reaches the yield stress. The modified limit state function G^* of this truss for P_f calculation is therefore defined as:

$$G_i^* = F_y - |\sigma_i(\boldsymbol{d}^*, \boldsymbol{X})| \tag{15}$$

In this example, by embedding the stress safety factor, (γ_s) to Eq. (15), we can reformulate it as the following equation to define V'

$$V'\left(\boldsymbol{d}, \widehat{\boldsymbol{X}}\right) = \sum_{i=1}^{13} \left[\max\left(0, \frac{\sigma_i\left(\boldsymbol{d}, \widehat{\boldsymbol{X}}\right)}{\gamma_s F_y} - 1\right)\right]^2 \tag{16}$$

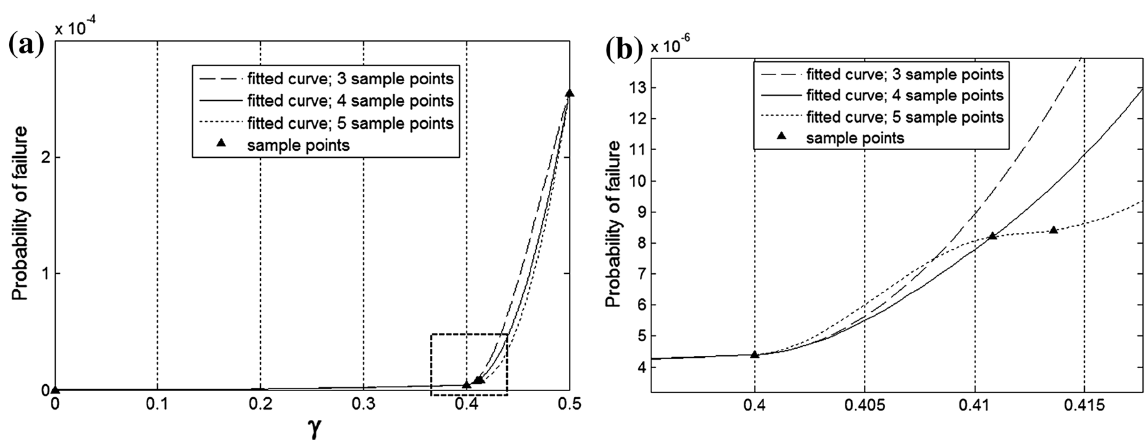


Fig. 8 Fitted cubic B-spline curves for 3, 4 and 5 sample points, 13-bar truss. **a** For 0 < FS < 0.5; and **b** magnified view around the reliable optimum solution with $P_f = 0.99 \times 10^{-5}$

Table 5 RBDO results of 13-bar bridge truss cross-sectional area (m^2) × 10^{-3}

Variable group	Nakib and Frangopol [28][a]	Ghorbani and Ghasemi [29][b]	Current work				
			Initial sample points		Auxiliary sample points (Iterations)		
			$\gamma_s = 0.4$	$\gamma_s = 0.5$	$\bar{\gamma}_{s,2} = 0.4108$	$\bar{\gamma}_{s,2} = 0.4136$	$\gamma_s^* = 0.42$
1	0.746	0. 751	0.800	0.640	0.9545	0.8000	0.7299
2	1.219	1.191	1.2375	0.990	1.2500	1.2125	1.2041
3	0.753	0.726	0.6875	0.550	0.6590	0.6875	0.6723
4	0.745	0.741	0.800	0.640	0.9545	0.8000	0.7299
5	0.227	0.223	0.1125	0.090	0.2650	0.1125	0.1300
6	0.840	0.810	0.900	0.720	0.9545	0.8000	1.0023
7	0.522	0.516	0.5625	0.450	0.6590	0.5625	0.4774
Truss mass (kg)	367.11	359.7	361.8825	289.506	356.6050	353.4226	353.0202
Exact P_f (MCS)	0.64×10^{-5}	0.97×10^{-5}	0.44×10^{-5}	2.548×10^{-4}	0.82×10^{-5}	0.84×10^{-5}	0.99×10^{-5}
No. of P_f calc.	636	230			3	4	5
($\text{error}^{P_f} \times 100$) %	36	3			18	16	1

[a] (Interior penalty function method for optimization) – (Approximate Bounding Techniques for reliability analysis)

[b] Adaptive Neuro Fuzzy Systems and Particle Swarm Optimization for optimization)-(Monte Carlo Simulation for reliability analysis

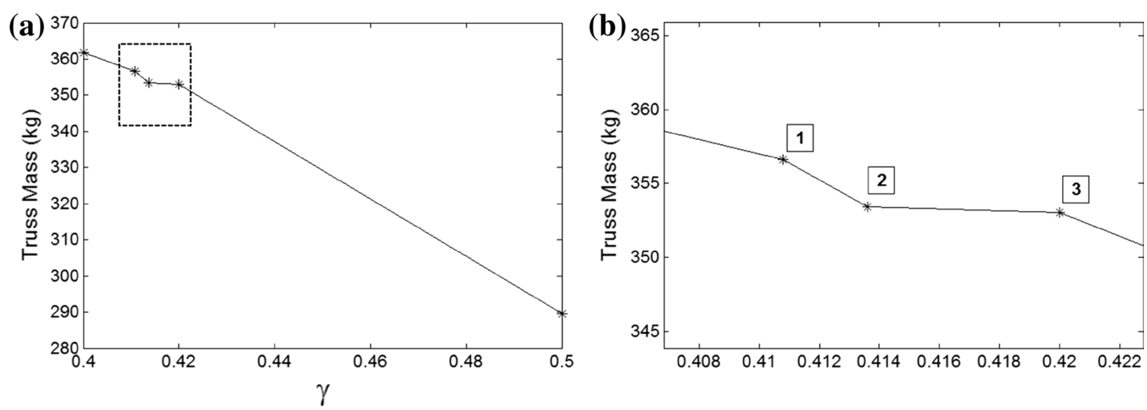


Fig. 9 Convergence histories of objective function, 13-bar truss. **a** For total sample points; and **b** magnified view around the reliable optimum solution with $f = 353.0202$ kg

Having found $P_f^{0.5}$ and compared the value with P_f^{Target}, the range $0 < \gamma^* < 0.5$ was selected. Figure 8a, b shows the converged cubic B-spline-fitted curves of (P_f vs γ) according to the six sample points. At the third iteration where $\bar{\gamma}$ was found equal to 0.42 with the corresponding truss mass of 353.0202 kg, the computed P_f reached the P_f^{Target} with a discrepancy of 1 % in the safe region. Therefore, the authors may claim that the optimum solution is the most reliable since 99 % of the P_f^{Target} was met in the safe region. The results of the proposed RBDO of this problem are listed in Table 5, where the reliable optimum design was slightly lighter than that in the literature.

Figure 9a, b also illustrates the convergence of total truss mass leading to the optimum design corresponding to the P_f^{Target}.

As seen from the results listed in Table 5, the efficiency of the proposed RBDO is the best in terms of the number of P_f evaluations. The proposed RBDO approach requires a total of 5 P_f calculations to find the global optimum solution with the objective value of 353.0202 kg, being 1.7 % less than the best value recorded in the literature.

3.3 A 72-bar spatial truss problem

The 72-bar space truss shown in Fig. 10 was first investigated by Shayanfar et al. [30] in 2014 as a reliability-based design optimization problem. The geometry of the truss with its nodes numbering and grouping of the members are indicated in Fig. 10. In the current study, the design variables were selected as discrete for the cross-sectional areas where

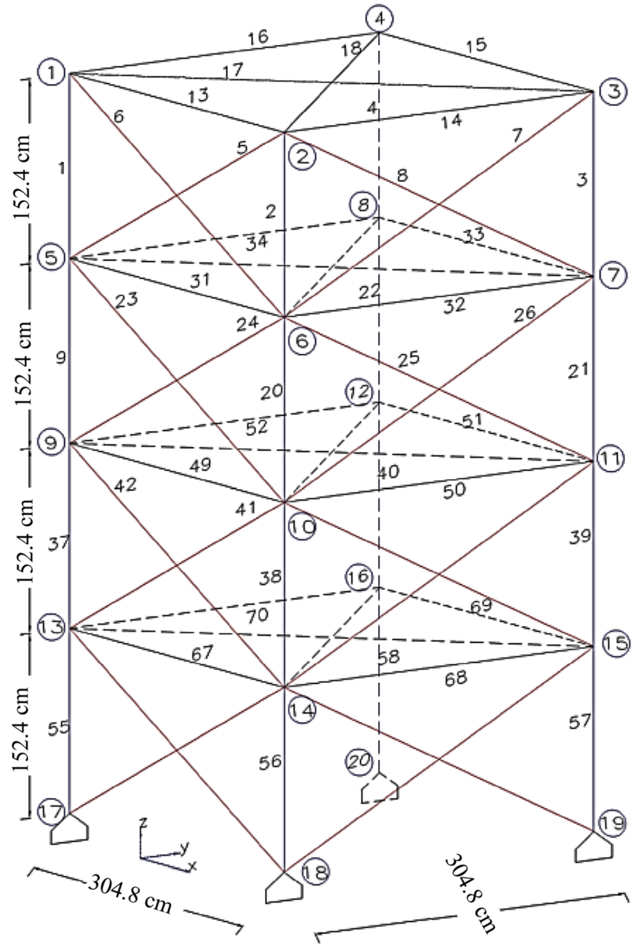


Fig. 10 The 72-bar space truss structure

Table 6 Statistical properties of random variables, 72-bar space truss example

Variable	Distribution	Mean value	Coefficient of variation (CV)
A_i	Normal	To be determined	0.05
E	Lognormal	68.9476 GPa	0.05
P_x	Lognormal	22.2411 kN	0.1
P_y	Lognormal	22.2411 kN	0.1
P_z	Lognormal	−22.2411 kN	0.1

the objective function was kept as to minimize the total truss weight. The density of material is 2767.99 kg/m^3. To enforce symmetry, members of the truss were set into sixteen groups as in Table 7. Loading condition, consists of 22.2411 kN in x, y and −22.2411 kN in z directions at node 1. The lower and upper bounds of members C.S.A. were confined to 6.4516×10^{-5} m^2 and 2.903×10^{-3} m^2, respectively. Properties of the random variables are shown in Table 6. This problem has two reliability constraints to control the lateral displacement of node 1 in x and y directions. The allowable lateral displacement $\Delta_{\text{all.}}$ is considered as 7.62×10^{-3} m in both x and y directions. The modified limit state functions of this space truss for P_f calculation is therefore defined as:

$$G_i^* = \Delta_{\text{all.}} - |\Delta_i(\boldsymbol{d}^*, \boldsymbol{X})| \tag{17}$$

where i is equal to x or y direction of node 1. By embedding the displacement safety factor (γ_d) to Eq. (17), we can reformulate it as the following equation to define the hypothetical constraint surfaces V' as:

$$V'\left(\boldsymbol{d}, \widehat{\boldsymbol{X}}\right) = \sum_{i=1}^{2} \left[\max\left(0, \frac{\Delta_i\left(\boldsymbol{d}, \widehat{\boldsymbol{X}}\right)}{\gamma_d \Delta_{\text{all.}}} - 1\right)\right]^2 \tag{18}$$

As depicted in the results of Table 7, Shayanfar et al. [30] could produce an optimum weight of 243.030 kg for a reliability index of $\beta = 3$ corresponding to a P_f of 0.00135, slightly in the unsafe region. According to the proposed method here, a minimum weight of 243.2706 kg was determined for a targeted P_f of 0.00135.

As listed in Table 7, three iterations as $\bar{\gamma}_d = 0.58$, 0.59 and 0.5913 were done, while the third iteration where $\bar{\gamma}_d$ was found equal to 0.42, the computed P_f reached the P_f^{Target} with a relative distance of less than 1 % in the safe region. Figure 11a, b illustrates the convergence of optimum truss weight leading to the optimum design corresponding to the P_f^{Target}. Figure 12a, b also shows the converged B-spline interpolation curves of (P_f vs γ) according to the 3, 4 and 5 sample points. As seen from the results in Table 7, the proposed method yields high efficiency in terms of the number of P_f evaluations. This is where the proposed method found an optimum weight of 243.2706 kg after a total of 6 P_f evaluations that is less than the total 200 number of P_f evaluations reported by Shayanfar et al. [30].

4 Conclusions

In this article an efficient method based on safety factors and cubic B-spline interpolation concept was introduced to solve RBDO of structures. This technique may be regarded as one in the category of decoupling methods. The proposed method may be distinctive for its ability to highly reduce the number of reliability analyses required for structural reliability-based optimization compared to conventional methods. MCS was utilized for reliability analysis to show the efficiency of the proposed approach. The performance of the proposed technique was verified on three structural problems. The results obtained well state that this method possesses a sufficient speed and accuracy. The

Table 7 RBDO results of 72-bar space truss cross-sectional area (m^2) × 10^{-5}

Variable group	Shayanfar et al.[a]	Current work					
		Initial sample points			Auxiliary sample points (iterations)		
		$\gamma_d = 0.5$	$\gamma_d = 0.6$	$\gamma_d = 1$	$\bar{\gamma}_{d,1} = 0.58$	$\bar{\gamma}_{d,2} = 0.59$	$\gamma_d^* = 0.5913$
1 ($A_{1–4}$)	6.4516	6.4516	6.4516	6.4516	6.4516	6.4516	6.4516
2 ($A_{5–12}$)	58.0644	81.8062	56.6708	41.9547	61.9166	61.20974	69.2966
3 ($A_{13–16}$)	19.3548	22.3096	20.2386	11.3225	20.8696	21.49931	20.4257
4 ($A_{17,18}$)	32.258	34.7934	26.1676	15.1031	29.0463	31.28750	29.3483
5 ($A_{19–22}$)	70.9676	25.0773	20.7806	12.5806	22.1664	20.3135	23.1935
6 ($A2_{3–30}$)	58.0644	65.8708	54.1740	41.6192	65.0218	55.52440	52.3160
7 ($A_{31–34}$)	6.4516	6.4516	6.4516	6.4516	6.4516	6.4516	6.4516
8 ($A_{35,36}$)	6.4516	6.4516	6.4516	6.4516	6.4516	6.4516	6.4516
9 ($A_{37–40}$)	96.774	93.9288	88.7288	41.6192	83.7675	82.0025	94.2449
10 ($A4_{1–48}$)	32.258	53.6902	49.9999	37.4579	55.6192	60.9861	50.8773
11 ($A_{49–52}$)	19.3548	6.4516	6.4516	6.4516	6.4516	6.4516	6.4516
12 ($A_{53,54}$)	19.3548	6.4516	6.4516	6.4516	6.4516	6.4516	6.4516
13 ($A_{55–58}$)	148.3868	193.5092	185.0125	83.2449	163.9028	153.6009	152.0835
14 ($A_{59–66}$)	58.0644	74.4385	58.1482	41.6192	53.2321	52.4543	54.9676
15 ($A_{67–70}$)	6.4516	6.4516	6.4516	6.4516	6.4516	6.4516	6.4516
16 ($A_{71,72}$)	6.4516	6.4516	6.4516	6.4516	6.4516	6.4516	6.4516
Truss weight (kg)	243.030	288.938	240.281	165.606	249.679	243.837	243.2706
P_f^{G1} (MCS)	0.00163	7×10^{-5}	0.00193	0.472	0.00092	0.001275	0.001345
P_f^{G2} (MCS)	0.00148	6×10^{-5}	0.00174	0.48	0.00098	0.00177	0.0013
No. of P_f calc.	200				4	5	6
($\text{error}^{P_f} \times 100$) %					27	5	0.3

[a] (Genetic algorithm for optimization) – (first-order reliability method for reliability analysis)

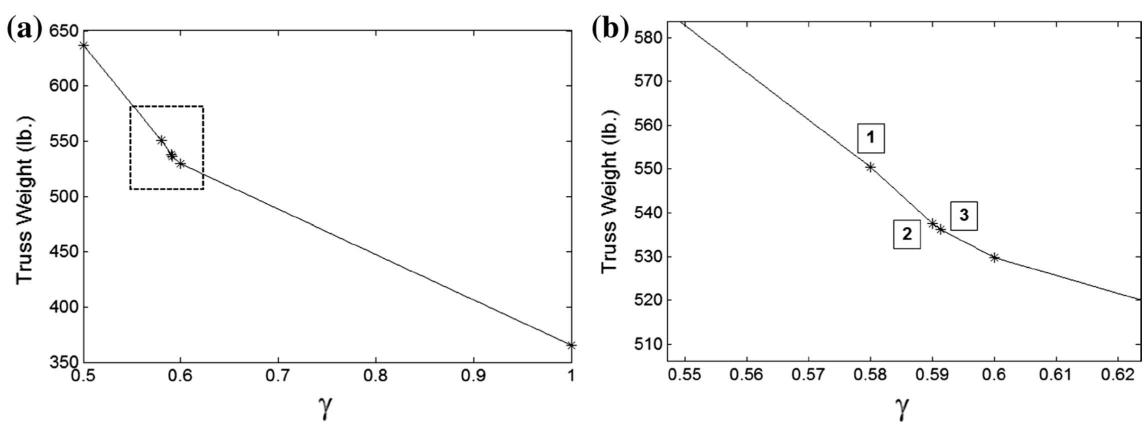


Fig. 11 Convergence histories of objective function, 72-bar truss. **a** For total sample points; and **b** magnified view around the reliable optimum solution with $f = 536.32$ lb

features of the proposed technique may be summarized as follows:

- The proposed RBDO framework contains an efficient decoupling approach that allows for a reasonably high speed of convergence.
- The proposed method may solely require a relatively simple understanding of RBDO together with a low effort of programming.
- This technique could be distinctive with a minimum number of P_f computations required since the P_f determination is carried out only after a global deterministic

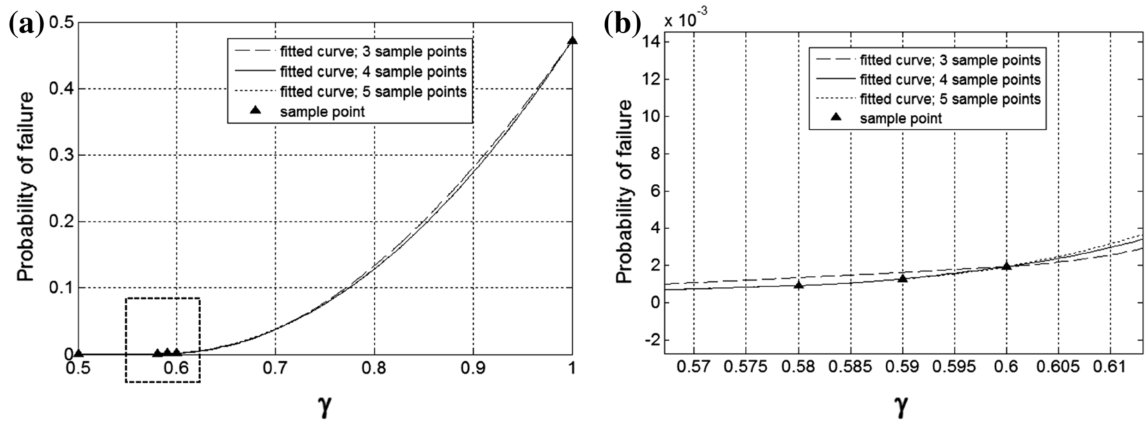


Fig. 12 Fitted cubic B-spline curves for 3, 4 and 5 sample points, 72-bar truss. **a** For 0.5 < FS < 1; and **b** magnified view around the reliable optimum solution with $P_f = 0.001345$

optimum design is found with respect to the mean values of random variables and the intended safety factors.

- B-spline interpolation curves were utilized as a tool to find an optimum safety factor that yields an optimum solution with the corresponding failure probability equal to the targeted P_f. These interpolation curves were modified each time with the newly founded optimum solution.
- One may realizes that the method contains a firm and straightforward framework since in all examples it leads to the global and reliable optimum design that satisfies the targeted P_f with a maximum dispute of 1 % in the safe region.
- In all examples studied, the reliable optimum design point was found only with less than a total of 10 P_f calculations.

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