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Where points came from

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Abstract
The central thesis of this work, argued through references to authors widely separated in time (Pasch, Cantor and Hilbert, but also Plato and Euclid), is that the notion of point as an independent object inevitably produces paradoxes that can only be eliminated if we take points not as entities with an independent, primary reality of their own, but simply as reifications of a methodological choice, namely that of neglecting *in pure theory* any measuring mistake, however small. Thus, for example, we associate *exact* positions to a line's extremes, with no indeterminacy at all, and call these positions *points*.

Keywords Points · Continuum

1 Of the way points are often thought of…

Many mathematicians of the twentieth and twenty-first centuries seem to think of points in abstract, mathematical spaces as primitive objects with an independent being of their own, and quite often as the basic ingredients of all that exists in such spaces. I believe this conception is wrong, and am going to outline a very different one; but first of all, I must describe in some more detail what I take to be today's received view.

I choose to this end an extreme but somehow paradigmatic text, Georg Cantor's *Grundlagen einer allgemeinen Mannigfaltigkeitslehre (Foundations of a General Theory of Manifolds)*, first published in 1883. I am writing a century and a third later, and of course many changes have occurred in mathematics and philosophy of mathematics since then; moreover, Cantor is quite extreme, and other, more recent authors would certainly change details here and there, but I believe nevertheless his position is somehow typical, even today.

Now, in § 10 of *Grundlagen* we read:

> "…in my opinion, one cannot start from the so-called *intuition form* of *space* to get clear about *space* itself, for it is only with the help of an already conceptually *given* continuum that *space* and the figures we think in it can achieve a content making them an object not just of aesthetical reflections, philosophical subtleties or imprecise comparisons, but of a rigorously exact mathematical research.
> It only remains then to introduce, with the aid of the concept of real number defined in § 9,[1] a **purely arithmetical**, as general as possible concept of a **point continuum**. I am using as a foundation—nor could it be otherwise—the flat *arithmetical* n-dimensional space G_n, i.e., the manifold of all value systems
>
> $$\left(x_1/x_2/\ldots/x_n\right)$$
>
> where every x can take, independently of all others, *all real* numerical values. I call each of these value systems an *arithmetical* point of G_n [1, p. 192; italics in the original, my boldface and translation]".

I translate as *point continuum* Cantor's term *Punktkontinuum;* this word obviously stands for a manifold whose elements are points. Such a manifold is a continuum if and only if it is *perfect*, i.e., all its elements also are accumulation points of it, and *connected (zusammenhängend)*, i.e., such that for any two x, y belonging to it and any real number α, however small, there is a path from x to y entirely lying in M and consisting of a finite number of straight segments not exceeding α in length [1, p. 194]. Thus, according to Cantor a n-dimensional continuum has points as elements, and points are in turn ordered n-tuples of real numbers.

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[1] Cantor uses convergent sequences of rational numbers to introduce real ones, but details would be out of place here.

Of course, after Cantor many other authors have written about the continuum, offering viewpoints that differ (sometimes very widely so) from his, but the idea that continua (including the biggest one, space) are manifolds whose elements are points and the latter are ontologically more basic than the continua containing them prevails, nevertheless, even today.[2] In this Cantor is more extreme (and clear) than many other authors, but not atypical; he dose not stand alone.

2 ...and of some well known difficulties in this view

A warning is in order, before going on: I will only discuss "ordinary", i.e., three-dimensional, Euclidean, Archimedean space. I am doing so not because I don't find other spaces interesting (I do), but simply in order to be reasonably brief and simple.

To begin with, sets have a peculiarity that was already well known to Bolzano around 1840, long before Cantor, though it was explicitly stated as an axiom only in 1908 by Zermelo: *extensionality*, i.e., the fact that a set is given once its elements are, so that two manifolds with the same elements, whatever their structural differences, are one and the same *set*. Now, three-dimensional, Euclidean, Archimedean space is *no* set in this ordinary sense of the term; nor does Cantor believe it is, since he writes, immediately after the passage quoted in § 1, that

> The distance between any two points is defined by the expression
>
> $$\sqrt{\left(x_1{}' - x_1\right)^2 + \left(x_2{}' - x_2\right)^2 + \cdots + \left(x_n' - x_n\right)^2},$$
>
> which entails that *any* point manifold of this sort is Euclidean, i.e., very richly structured [1, p. 194].

There are, however, some well known difficulties in this view, I am going to briefly recall them.

The first difficulty is that you have no figures in space thus described unless you associate points in them with certain well defined sets of ordered triplets of real numbers, *exactly those triplets and no others*, and to do that you must know *independently* where the triplet $\langle 0, 0, 0\rangle$ is, the direction of the x-, y- and z-axes, and the length of a unitary segment (that must, in turn, be somewhere). Now position, direction and length are spatial concepts, and if you define or construct space using spatial notions you are obviously moving in a vicious circle.

One might answer to this that all "Euclidean" systems differing only in zero-point's position, axes' orientation and unit of measure are isomorphic, and Euclidean space is none of them, but their equivalence class under isomorphism, i.e., an abstract structure: for an abstract structure is, of course, just an equivalence class of concrete structures under isomorphism. A Euclidean system with a given origin, orientation and unity of measure is a concrete structure, all such structures are isomorphic, and Euclidean space simply is their equivalence class under isomorphism. But this answer fails, for in such an abstract "Euclidean space" we have no means to locate anything, and so no figure at all.

I am afraid the only way out of this blind alley is the conclusion (though we tend to forget about it) that in geometry numbers alone are not enough, for we also need answers to some where-questions: *Where* is the origin of the axes? *Whither* do they go? *Which* is the unit-segment, and *where* are its extremes? But all this can be summed up in one sentence: there is no reducibility of space to numbers alone.[3]

3 Cantor's dust

By the way, Cantor himself stumbled upon something extremely difficult to reconcile with his vision of a *purely arithmetical* continuum. In endnote 10 of *Grundlagen* we read:

> As an example of a perfect point set being everywhere dense in no interval, however small, I introduce the aggregate of all real numbers contained in the formula
>
> $$z = c_1/3 + c_2/3^2 + \cdots + c_\nu/3^\nu + \cdots,$$
>
> where the c_is can take as value either 0 or 2 and the sequence can have either a finite or an infinite number of members.[4]

To visualise this construction we can imagine an initial segment AB, divide it into three equal parts, erase the central one, leaving the other two momentarily untouched, then proceed in the same way with the left and right sub-segments and so on, to infinity. If we imagine this process completed, we must conclude no segment is left, for if one is still there (and no longer one is) after n steps, in step $(n+1)$ its central third will be eliminated. The resulting point set is known as "Cantor's dust", so called because its elements can fill up no length at all, and yet they are just as many as those binary

[2] Anyway, such prevalence is no unanimity; for example, the intuitionistic view is very different. I grant it is not entirely clear to me, for I find its central notion, that of *spread*, quite blurred; it is obvious, however, that according to intuitionists points are things to look for and construct, not primitive objects serving as foundations for all the rest, and in this – for all their lack of clarity – I feel a greater affinity with them than with Cantor.

[3] I am not claiming this position is new. I know it is not.

[4] *Gesammelte Werke* cit., p. 207. My translation.

sequences of 0s and 1s entirely filling, in standard theory, *any* segment, including of course AB.

So far, so good. But why, given a segment AB, can we find a one-to-one correspondence between *all* of its points and binary (finite or infinite) sequences of 0s and 1s, while binary sequences of 0s and 2s only correspond to a subset of AB such that no length, however small, can be associated to it? And yet there is a simple, almost trivial bijection between the two sets of sequences: that associating to any 0–2 one, call it x, the y we get from it substituting all its 2s with 1s—and vice versa.

But it is not difficult to see the reason why Cantor's dust is what it is. When we associate points in a segment AB with binary sequences of 0s and 1s, at each step we divide a subsegment s of AB's into two halves and *these halves entirely exhaust* s, so that there is nothing left in it over and above them. As a consequence, no point in AB is left out; there is a perfect bijection between such points and binary sequences. But with binary sequences of 0s and 2s things are different, for at each step p 1/3 of the points remaining just before p is left out…of course: but that means we handle *space* in different ways, we leave no void in one case while we do in the other, and this entirely changes things. We completely cover a continuum with a certain binary procedure, we don't with another, also binary, and the difference is one between filling and not filling: an unreducibly spatial, unreducibly visualisable difference. A continuum *can* come out of a certain work with *numbers, provided there also is a spatial, non-numerical aspect in such work*. Cantor did not perceive this because he was—like so many after him—something that can be described as an arithmetical fanatic.

4 A different approach

In Cantor's by now classical approach points are independent objects in a very strong sense, for an ordered triplet of real numbers needs neither space at large, nor any other triplet to be there. Only three real numbers and their ordering are needed; and points, thus understood, are ontologically prior to space.

I believe anyway another view is not simply possible, but more adequate: namely, that points *presuppose* continua, and cannot be thought of sensibly and correctly without them. Actually, we systematically use *this* view, for we cannot think of a point without locating it somewhere, but doing this and at the same time thinking of space as the set of points is circular. Anyway, to introduce this alternative viewpoint I need to take a long, long step back—as far back as Plato.

5 "If the one is not"

One of Plato's most difficult dialogues is *Parmenides*, written according to specialists around 360 B.C., about three-quarters of which consist of a long, difficult and quite often paradoxical discussion about "the one", a term Plato never defines. This discussion is "dialectical" i.e., in question-and-answer form, but with a strong asymmetry between questioner and answerer, for the former is Parmenides, the famous philosopher, and the latter a lad of 15. Now, towards the end of the dialogue, with the two characters asymmetrically discussing the hypothesis "that the one is not" (while in an earlier part they had discussed the opposite alternative, namely "that the one is"), we find a passage making perfect sense if *there* (not in the whole discussion) "the one" means "the point" (while another term, "heap", means something like "object having extension"), and no sense at all otherwise.

If the one is not, says Plato, there only are "the others",

> …but the heap each of them is is infinite in multitude, and if one takes that which seems to be the smallest, suddenly, as in a dream during sleep, instead of one it will seem to be many, and instead of very small immensely great, with respect to its fragments.—Quite right.[5]

And just a few lines further he adds:

> …and will not a heap appear to be limited with respect to another heap although it has, with respect to itself, neither beginning, nor middle, nor end?—How is it that it has none?—Because when you want to grasp in thought one of these as something that is, another beginning always appears before the beginning, and another end always remains after the end, and in the middle there will be other things more central than the middle, but smaller, because in none of these cases is it possible to take anything as one, given that the one is not.—Very true.—And every being that we take in thought must necessarily be splintered into fragments, for it would always be a heap without a one.—Absolutely.—And will such a thing appear to be one to somebody looking at it from a distance and not sharply, while it will necessarily seem infinite in multitude to those who know it nearby and sharply, since it is deprived of the one, which is not?—That is most necessary.[6]

This deserves a word by word comment, but the one I have in mind is only possible if, I repeat, by "the one" we

[5] Plato, *Parmenides*, 164 d, my translation.

[6] *Op. cit*., 165 a-c, my translation.

mean "point" and by "heap" "object having extension". I rewrite both passages with these substitution, getting first

> …but the *object having extension* each of them is is infinite in multitude, and if one takes that which seems to be the smallest, suddenly, as in a dream during sleep, instead of one it will seem to be many, and instead of very small immensely great, with respect to its fragments.—Quite right.

and then

> …and will not an *object having extension* appear to be limited with respect to another such object although it has, with respect to itself, neither beginning, nor middle, nor end?—How is it that it has none?—Because when you want to grasp in thought one of these as something that is, another beginning always appears before the beginning, and another end always remains after the end, and in the middle there will be other things more central than the middle, but smaller, because in none of these cases is it possible to take anything as one, given that *there are no points*.—Very true.—And every being that we take in thought must necessarily be splintered into fragments, for it would always be an *object having extension* without *any points*.—Absolutely.—And will such a thing appear to be one to somebody looking at it from a distance and not sharply, while it will necessarily seem infinite in multitude to those who know it nearby and sharply, since it is deprived of *points*, which are not?—That is most necessary.

Here I am doing an *abduction*, i.e., proposing a hypothesis unproved, but such that if you grant it, then something (in this case, both quoted passages) becomes comprehensible and entirely plausible, while it would make no sense at all otherwise. This does not prove my abduction is *true*, but abductions do not require to *be proved*, they serve to *prove* something else; they do their job when they achieve this end, and if they do, there is nothing more to require of them.

This said, I am going to give an almost word by word comment, driven by my abduction, of both these Platonic passages:

> I "the *object having extension* each of them is is infinite in multitude".

"Each of them": what are "they"? The context leaves no doubt: "the others", i.e., heaps, extended objects—not points, for there are none ("the one is not"). But what does "infinite in multitude" mean? Well, take any extended object: it is no "one" but an assemblage of parts and each of these is in turn an assemblage of parts, to infinity.

> II "if one takes that which seems to be the smallest, suddenly, as in a dream during sleep, instead of one it will seem to be many, and instead of very small immensely great, with respect to its fragments".

Take for example a *visible* circle, with its *visible centre. That* "centre", being visible, is no point, no *one*, but simply a central area; and it is true that such an area must have outer and inner portions, but even supposing one of these portions is innermost (whatever that may mean), it will be in turn no point, but an area with a central portion of its own—and so on, to infinity. But then the visible "centre" has in fact parts with respect to which it is immensely great.

All this presupposes, of course, infinite divisibility of extension—but also makes sense if and only if we grant it.

Let us turn now to the second passage:

> III "will not an *object having extension* appear to be limited with respect to another such object although it has, with respect to itself, neither beginning, nor middle, nor end?—How is it that it has none?—Because when you want to grasp in thought one of these as something that is, another beginning always appears before the beginning, and another end always remains after the end, and in the middle there will be other things more central than the middle, but smaller, because in none of these cases is it possible to take anything as one, given that *there are no points*"

I believe the best way to discuss this passage is by means of examples. Let us take, then, the extremities of a segment and suppose one of them is initial ("the beginning") and one final ("the end"). Now, if the beginning is no point but just an initial portion of the segment, it will have in its turn an initial part of its own, that will also be no point, and so on to infinity ("another beginning always appears before the beginning"); and there will also be, for the same reason, not just an end but also an end of the end, an end of the end of the end and so on ("another end always remains after the end"). As for the reason why "in the middle there will be other things more central than the middle, but smaller", it has already been explained in point II.

> IV "And will such a thing appear to be one to somebody looking at it from a distance and not sharply, while it will necessarily seem infinite in multitude to those who know it nearby and sharply, since it is deprived of the one, which is not?—That is most necessary".

Consider first someone who looks at a figure intuitively and acritically ("from a distance and not sharply"), and then someone who examines it closely and critically ("nearby and sharply"). *If the one is not* (if there are no points) the first will say, e.g., "Yes, this is a square. All its sides are equal, all its angles are right"; but the second, the knower, the sharp one, must say "This is *approximately* a square, its sides are *approximately* equal and its angles *approximately* right", and will never be able to drop that "approximately", for he could only if there were points, and there are none.

6 Equality and unequality

Let us forget for a moment about Plato and develop an independent reflection. If we believed there are no points, we could not consistently say, e.g., "a circumference is a closed line whose *points* have all the same distance from one not on it, the centre", and not simply because the word "point" means nothing to us. We would also have to give up the concept of *sameness* or *equality* ("same distance") as we know it. But here we have just stumbled upon something both fundamental and complicated, requiring a new analysis and a new, long step back.

Let us take three girls, Mary, Jane and Lucy. We see Mary and Jane side by side, and they are the same height (please let us forget, at the moment, about the possibility of higher or lower heels). Then we see Jane and Lucy together, and they are also the same height; but next we see Mary and Lucy side by side, and—Mary is visibly taller. Not much of course, but there is no doubt.

We have an explanation for such occurrences: there must be differences too small, *individually*, for us to perceive, but that can sum up so that their sum reaches the threshold of perception; and this does not go just for girls, but for all visible objects. We might as well take three *visible* lines, and there would be no reason to exclude we can have, with them, the same problem we had with Mary, Jane and Lucy—and then we could solve it in exactly the same way.

So inconsistency of *observed* equality is accommodated invoking the imperfection of our measuring instruments. This is standard practice—but for engineers and physicists (or at least experimental ones), not for mathematicians. For mathematicians, when they do geometry, all sides of an equilateral triangle, all rays of a circle, all right angles are *equal*: not approximately, but strictly. If the equalities they work with were approximate, cases of the Mary-Jane-Lucy sort would exist not just in actual practice, but in theory too, and *there* they are not allowed, for if they were, everything would be enormously more complicated. To give just one example, we do not say the ratio between the circumference and the diameter is *approximately* π, we simply say it is π. More generally, a reasonably manageable geometry is only possible after a methodological choice: that of neglecting all measuring errors, however small—which can be done, of course, just in pure mathematics and not in its practical applications. But this is only possible if we have standard, i.e., strictly unextended, points.

Not that *all* pure mathematicians assume that points, thus understood, exist: there is at least one important exception, Moritz Pasch, who right at the beginning of his 1882 *Vorlesungen über neuere Geometrie* wrote "… diejenigen Körper, deren Theilung sich mit den Beobachtungsgrenzen nicht verträgt, werden Punkte genannt…" (those bodies whose division is incompatible with the limits of observation are called points) [3, p. 3].

This is an obviously empiricist view, and empiricism is an extremely serious philosophy; but I am afraid it is inadequate as a foundation of geometry, for it makes rescue from such cases as those seen at the beginning of this section impossible. Let us suppose, with Pasch (and with average kids of 10 or 11), that a point is something *very* small, and more exactly *the smallest possible thing*: then accommodating cases, that *do* occur, such as "Line p is as long as line q, line q is as long as line r, and line r is longer than line p" becomes impossible, for we cannot invoke differences *smaller than the smallest possible visible thing*, a point, but whose sum reaches visibility. There are none. And this is why we *must* have strictly unextended points. Nothing would be exact, without them. Moreover such exactness, once its need is understood, is a permanent necessity. Giving it up would be giving up geometry as a science. And we have seen there are reasons to believe Plato had understood this perfectly well.

7 Sixty years later, Euclid

This is a work going back and forth between *ancient* and *timeless* geometry, and now it's the ancient polarity that surfaces again.

Euclid's *Elements* were probably published around 300 B.C., ie., a couple of generations after *Parmenides*, and supposing Euclid had never read Plato would be most unreasonable. Now, the *Elements* begin with twenty-three propositions we usually call *definitions* today (although the Greek for "definition" is *diorismos*, while Euclid uses *horoi*, terms), and the first three of these propositions are

I. A point is that which has no part.
II. A line is breadthless length.
III. The extremities of a line are points [2, p. 153].

A point then is no "heap", for a heap, however small, *must* have parts; further, the intersection x of two lines p and q *must* be a point for it can have no part, since (supposing for simplicity p is horizontal and q vertical) there are neither a right nor a left side in it, q having none, and neither an upper nor a lower side, p having none; *and the extremities of a line are exact positions with no approximation margin at all, however small*.[7] We are in a domain where exactness is perfect, and points are no "building blocks" all the rest can be constructed with, but the reification of a methodological choice not to take the inexactness, however small, always occurring in geometrical *practice* into account.

This is, however, just a part of how things really went—and go. Before Euclid, the Greeks used to call a point either a *stigme*, puncture—a very small hole left by something sharp—or a *semeion*, sign; then Euclid regularised language, dropped *stigme*, and from that moment only *semeion* was used.[8]

8 Back to Cantor

However, the naïve vision according to which points are somehow things existing on their own, with at least some independence from "space", as we say today, or from all "places" and "figures", as the ancient Greeks used to say,[9] is surprisingly vital. It can take very sophisticated forms, one of which is just Cantor's, with his vision of points as ordered n-tuples of real numbers (with $n = 3$ in ordinary space); for such an object exists independently, once its elements and their ordering are given.

There are some effectivity problems here, and although Cantor could not know about them in 1883, when *Grundlagen* were written, *today* we can't forget about them. The biggest one is that Cantor insists again and again that the elements of a well defined set must be given *nach einem Gesetz*, according to a law, and though he knows that the power of the continuum is more than denumerable (actually, *he* discovered it was), he seems to ignore that that of the set of possible finite expressions, so *a fortiori* of possible laws, is denumerable; and of course he also ignored what Turing would discover only in 1936, namely that (1) the property "computable", as applied to real numbers, is undecidable, and (2) there are real numbers definable, but not computable; so there *must* be n-tuples of real numbers with some non-computable element, but then—where and what is *that* point?

9 Conclusions

But these are marginal difficulties. The central one is quite different, and to discuss it I must first say something very general.

The late nineteenth and early twentieth centuries saw a flourishing of axiomatisations of geometry. The most famous one, of course, is Hilbert's. The aim of all these axiomatisations was twofold, full rigour and formalisation; the former was driven, essentially, by the need of correcting some increasingly obvious slips in Euclid's old system and the latter by that of finding a place for the rapidly increasing "non-evident" (e.g., non-Euclidean, non-Archimedean, four-dimensional etc.) theories. But what was never challenged was the very old (Aristotelean, to be exact) one-directional idea of geometry, and science more generally: namely, that it must be founded (1) on a finite list of primitive terms on whose basis all others had to be defined, with no feedback of *definienda* upon *definientia*, and (2) on a finite list of axioms, with no feedback of consequences upon them.

Now, this view misses something essential, namely that feedback of defined terms upon primitive ones (and of consequences upon axioms) *is* there, as we can clearly see in the case of the most fundamental term of all, point. What's the use of points? Why have they been introduced? Why is it impossible, in practice, to dispense with them? We are unable to answer any of these questions unless we have geometry as a whole in mind; but if we do, it isn't just that the answer becomes obvious, but the whole nature of the enterprise is also seen in a different light. Axiomatisation *is* useful and, indeed, necessary; formalisation also gives new, precious insights; but on their own, they miss an essential aspect of geometry (and I am not saying of it alone), its two-directionality, the *inter*dependence of primitive and defined terms, of axioms and theorems, the fact that primitive terms

[7] By the way, *III* does not explain a term's *meaning* but ascribes a *quality*, point-ness, to certain things, so it could never be taken as a definition (but Euclid does not say it is one: for him it is a *horos*, not a *diorismos)*.

[8] By the way, after the substitution of Greek with Latin as the international language in Western Europe around the fifth century A.D., this was reversed, for the Latin word *punctum*, the ancestor of *point-Punkt-punto*, is an inflexion of the verb *pungere*, to sting. I don't exactly know when or by whom this reversal of Euclid's old choice was done, but there might be specialists who do.This means, of course, that the analogy of *points*, or *signs*, with very small bodies was also alive among ancient mathematicians (and still is today), but their (and our) *subjective* ambiguities should not be confused with the *objective* status of "points" (or "signs").

[9] It must be pointed out that nothing like the modern notion of space existed in Antiquity. It is true the English word *space* (just like the Italian *spazio* or the French *espace*) comes from the Latin *spatium*, but *spatium* was used to denote distance, and not something infinite, isotropic and (leaving complications of general relativity aside) uniform where everything having physical reality in the case of physics, and every possible geometrical figure in that of mathematics, is nested. The Romans would say, for example, that there was a *spatium* of a hundred miles between two towns, but had nothing like our modern usage; and the same is true of the Greeks. Strange as it may seem, we do not find anything like our present notion of space before Newton.

are what they are because as they are they serve to define the notions we need, and axioms are what they are because as they are they serve to prove the theorems we are interested in.[10]

So, reasoning along these lines, what is the use of the notion of point? Well, it simply allows us to neglect measuring mistakes, however small, and so introduce manageable and comparatively simple *further* definitions and prove a long, open-ended row of theorems in a comparatively quick and easy way. Moreover, if we see this notion in such a light we also get a natural and simple solution of an old paradox working mathematicians happily ignore, but that *is* there anyway, for all their ignoring it: namely, that if we see points as things with an extension, then all of geometry becomes hopelessly difficult and inexact, but if we suppose they have none, they are nothing at all, for how can anything having no extension *be there*, in the realm of extended things? Anyway, if they are no independent objects, but simply the reification of the methodological choice of ignoring all measuring mistakes, however small (and there is no manageable geometry without this choice), inconsistencies disappear. And maybe somebody might find my thesis that old Plato had already understood this perfectly well too strong: but the text *is* there, and I know of no better way of interpreting it.

Translated from the Italian by Daniele A. Gewurz.

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[10] I am not saying they are *perfect* for this twofold end, which is never completely achieved (and this is why revisions or integrations, especially of axioms, are comparatively frequent): but we know from the history of mathematics that a system of primitive terms and axioms *must* be reasonably efficient in this double pursuit to be successful.