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D for Dimension

A guide to imagination

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Abstract This article presents a bird's-eye view of the different aspects of the notion of dimension in mathematics.

Keywords Dimension · n-Dimensional spaces · Flatland · Fractals · Koch snowflake

In mathematics, the word "dimension" identifies a concept with two very different characteristics: on the one hand, it appears in different areas, often making it possible to reach a deeper understanding of the issues we are investigating, on the other hand, it has entered the "narrative" that mathematics tells about itself to those outside it.

With one important exception—fractional (or fractal) dimensions—the presentation model of the notion of dimension is still the one given in the 1941 essay *What Is Mathematics?* by Richard Courant and Herbert Robbins [2]. A specific paragraph (albeit one that can be omitted without losing the thread of the reasoning) is devoted to dimension, which is presented as a characteristic that is not very difficult to determine as long as we deal with simple geometric shapes (points, lines, triangles, polyhedra), but that requires a more precise definition when trying to extend this notion to more general classes of points. Poincaré in 1912 had already suggested the need for a deep analysis and had observed that a precise definition can be obtained recursively: a space is *n-dimensional* if any two of its points can be separated by removing a subclass of points of dimension $n-1$. After all, a definition by induction can be found even in Euclid's *Elements*, where a 1-dimensional figure is something whose boundary consists of points, a figure is 2-dimensional if it is bordered by curves, while a 3-dimensional figure has its boundary formed by surfaces. But our adventure cannot stop at 3-dimensions: in the mathematics of the early twentieth century, the one reported about by Courant and Robbins, the word "space" was used to refer to any "system of objects for which a notion of 'distance' or 'neighborhood' is defined" [2, p. 250] and these spaces can of course have more than 3-dimensions. We need only think of the example represented by the *n*-dimensional space whose points are the ordered *n*-tuples of real numbers and in which distance is defined in a similar way to that in the Euclidean plane. In fact, the definition of dimension extends in a natural way to these spaces and identifies a topological feature: there are no two spaces of different dimensions that can be topologically "equal".

Here, then, a large part of the narrative that we can construct today about the meaning of dimension seems to be already written: we need only grasp and develop the concise clues we have seen (and learn how to present them outside the classrooms too). But it did not happen this way and something different was added along the way.

As remarked, among others, by Thomas F. Banchoff in his *Beyond the Third Dimension* [1], mathematicians receive from common language the word "dimension" already loaded with many meanings. We speak of dimensions when we describe an object as being large or small, or through its length, width and depth, but also when we describe it in terms of different properties such as its weight, capacity, brightness, accuracy of the measurement, measured speed, image resolution and so on. A good example is when we are dealing with configuration spaces: when studying the mechanics of a robotic arm, we often act on the angles formed by the various parts of the arm; what is important is not only the location of the object in the

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robot's "hand" but also the different angles formed by every joint. The questions "How can the robot's arm move, and as a function of how many parameters?" Naturally leads to the question "How many dimensions does the configuration space have?" The operating constraints of the arm become constraints on the representations in space.

We are also confronted with dimensions when we are lined up behind a group of curious, dawdling tourists in a salt mine and, overcome by the desire to get out of the tunnel, we begin to dream of being able to fly over their heads, or perhaps we simply regret not having chosen another time for our visit. We are also, in a very broad sense, referring to dimension when we speak of a "new dimension" to say that we are looking at the problem we are interested in from a new point of view.

However, if you ask an ordinary person the meaning of "dimension", you can be sure that the winners will be the spaces of science fiction or, for the more discerning ones, the spacetime of relativity, or perhaps the dimensions of the universe, which can be "seven, but also eleven, but also…". This is because, in fact, even among mathematicians, geometers are those mainly considered to be involved with dimensions.

One, two, three… the three-dimensions are those of the world in which we live. Then, from an early age, we learned at school to pretend that there are objects with no thickness (our drawings) and that there are 2-dimensional spaces that we have the chance to see from the outside as if we were Gulliver meeting the inhabitants of Lilliput, or at least with the same broad and sympathetic gaze.

It took the imagination of Edwin A. Abbott, in 1884, to help us "see" the plane from within and not from the outside. *Flatland: A Romance of Many Dimensions* was probably intended primarily to show the miseries of Victorian society transferring its features to a "different" world and thus providing today's teachers of mathematics with a nice opportunity to educate without boring. What Abbott says about the social structure and especially about women now brings a smile, even though it is very clear to us that the same could be still said, *mutatis mutandis*, of our societies and not just Western ones, unfortunately. To be clear, even in Flatland woman is the dangerous sex since, being a straight line, she "is a needle; being, so to speak, all point, at least at the two extremities"; as such, she is able to literally pierce a man. But the reason why we are interested in Abbott's text here is because it provides us with a suggestion about how to imagine a space that is different from the one in which we live.

To help the Square to understand the nature of Spaceland, the country of three-dimensions, Abbott asks him to begin by imagining how he himself would see Lineland, the land of lines, the space having a single dimension. For instance, he may see at a glance the whole world, something impossible even to the king of that country, a long line segment who does not know that the Square can touch his inside without passing through his endpoints… So, if we want to train ourselves to understand what a four-dimensional space looks like, we can begin by studying how to move from two-dimensional space to the one we live in. How do they live in Flatland? What does an inhabitant of Flatland see of three-dimensional objects? How can he picture them? Which traces does a sphere passing through Flatland leave? Only after answering questions like these, we can begin, by analogy, to make some assumptions.

In this way, Abbott helps us once more to appreciate that fundamental tool for mathematics, analogy, and to observe more carefully the passage that allows us to "export" statements, from the world we know to worlds we do not know.

Analogy is probably the dominant idea in the history of the notion of dimension, and began to be used long before Abbott's Square. From Plato to Galilei, confidence grows ever stronger. Very soon we realize that if we wholly understand a statement about plane geometry, we can probably find by analogy interesting statements about solid geometry.

From squares to cubes, from circles to spheres: shouldn't these transitions suggest other natural steps to higher dimensions? What do we see in Fig. 1? A progression of

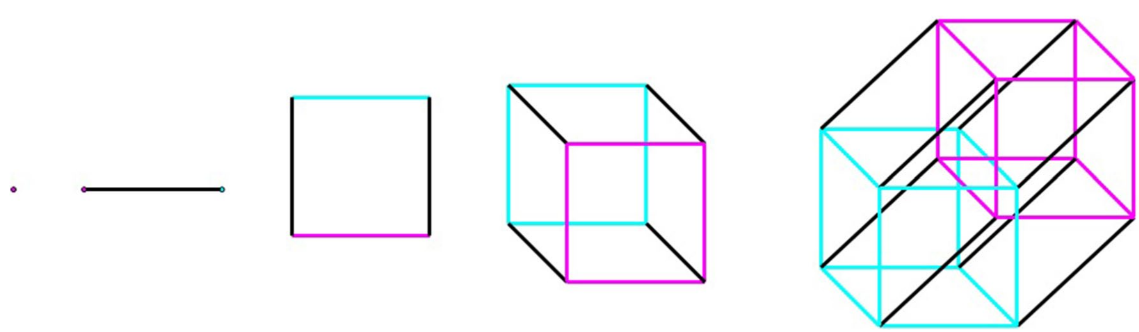


Fig. 1 Dimension 1, 2, 3, 4

spaces that we could have started even another step further back.

A point, a 0-dimensional figure, with no degrees of freedom; a line, a 1-dimensional figure, with one degree of freedom; a square, a two-dimensional shape with two degrees of freedom; a cube, a three-dimensional figure, with three degrees of freedom; a hypercube, a four-dimensional figure… and so on (Figs. 2, 3).

How else can we describe them? If we consider the vertices, for example, we pass from the single vertex of the point to the 2 vertices of the segment, to the 4 vertices of the square, to the 8 vertices of the cube, to the 16 vertices of the hypercube, and so forth; if we consider the boundary, the numbers of the figures are given by the 2 vertices of the segment, the 4 sides of the square, the 6 squares of the cube, the 8 cubes of the hypercube, and so on. In this way we may conjecture that an n-hypercube, namely a cube in dimension n, has 2^n vertices and $2n$ faces of dimension $n-1$, which in turn are $(n-1)$-hypercubes. This conjecture becomes a proof when we clarify the underlying inductive construction: the Cartesian product by an interval of a suitable length, orthogonal to the space of dimension $n-1$ identified by the $(n-1)$-hypercube.

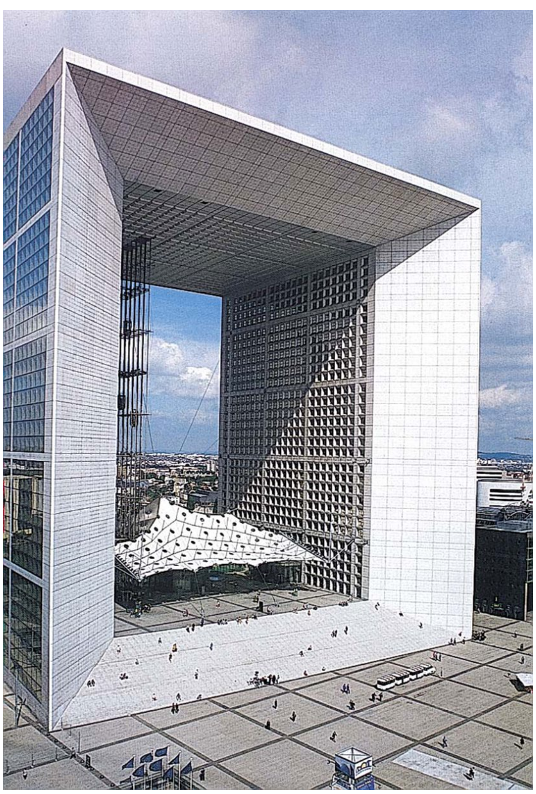


Fig. 2 The structure of the Grande Arche de la Défense in Paris suggests the structure of a hypercube (© matematita)

Fig. 3 A building in Madrid that recalls the Schlegel diagram of a hypercube (© matematita)

	Vertices	Edges	2-faces	3-faces	4-faces	5-faces	…
Point	1						
Segment	2	1					
Square	4	4	1				
Cube	8	12	6	1			
Hyper-cube	16	32	24	8	1		
5-hyper-cube	32	80	80	40	10	1	
…							

Considering these shapes in sequence according to their dimensions tells us something more. We may connect them, for instance, to the exponent that dictates their behaviour when we have to measure something. If we double the sides of a square table to be covered, we need a square that, with respect to the original one, has the side doubled (2) and the area quadrupled (2^2). If instead we double the measurements of a cubic container, we need a cube that, with respect to the starting one, has edges with double length and faces with quadruple surface and, to fill it, a volume of material that corresponds to eight times that of departure, that is to say 2^3 times.

The patterns (here's another key word while doing mathematics) we meet in these descriptions often have correspondences in higher-dimension geometries, and this suggests to us that we can go further: in dimension n the volume of a n-cube having side length m is m^n.

For many people, the backdrop to these remarks is the coordinate structure. If we know the length, width and height of a brick, we can reproduce it because these three numbers perfectly identify its shape. And for many applications of mathematics to everyday life, only bricks of this type are needed, and these three dimensions are sufficient.

But once we have built $\mathbb{R}^2$ and $\mathbb{R}^3$, where is the difficulty in "thinking" $\mathbb{R}^n$ with $n > 3$ and grasping the analogies and suggestions that the list of coordinates suggests to us? The synergy of different methods and tools will give us clearer solutions, simpler proofs, case by case. There are no major obstacles even in imagining infinite-dimensional spaces. The first examples are for everyone: the set of polynomials in one indeterminate with real coefficients, when we assign no limit to the degree of the polynomial, is a space for which there is no finite basis and therefore has infinite dimension; the same holds for the set of continuous functions defined on a closed bounded interval. These are not trivial examples.

However, dimension does not say everything we are interested in about the space for which we compute it, since it gives us essentially "local" information. Figures of two-dimensions include not only the plane, but also the sphere, the "doughnut" and more in general the tori of genus greater than 1. Dimension does not even tell us how many points there are in the space. Does a line segment have fewer points than a square? No, Cantor's remark to Dedekind, "I see it but I cannot believe it", shows that dimension, which is indeed an invariant under homeomorphisms, is not so under transformations that are bijective but not continuous (Fig. 4).

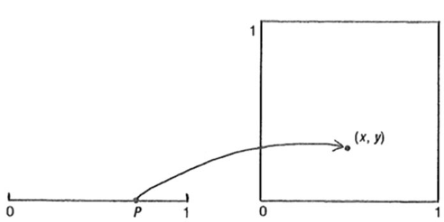
So we are confronted with infinity, one of the greatest achievements of mathematics.

The suggestions from Courant and Robbins that we have chosen as a guide stop here, but geometers were not satisfied to have spaces for every natural number. Dimension may even be an irrational number!

As Benoît Mandelbrot wrote:

> Why is geometry often described as "cold" and "dry?" One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles [3, p. 1].

In doing so, he opened a new chapter in the definition of dimension and gave way to a beautiful, new intellectual adventure that affects not only our most obvious assumptions but also our imagination. And with good reason, especially because, since the middle of the last century, computer graphics have allowed us to "see" realities that we could only speculate about before (Figs. 5, 6).



P = 0.72357646...; X = 0.7374..., Y = 0.2566...

Fig. 4 Correspondence between a unit segment and a square

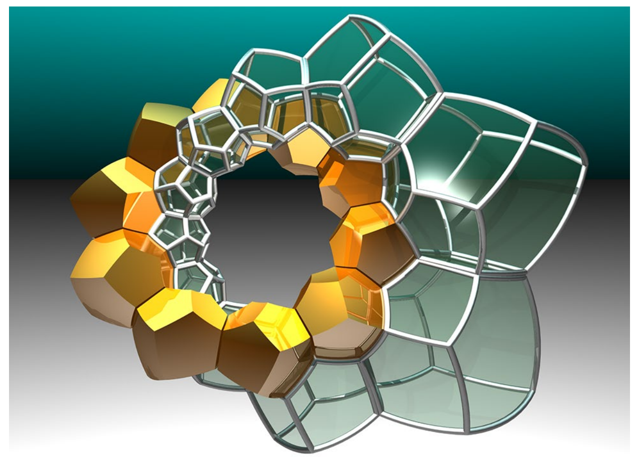


Fig. 5 Two rings of dodecahedra in a 120-cell, which is a regular polytope in 4-dimensional space (© matematita, image by Gian Marco Todesco)

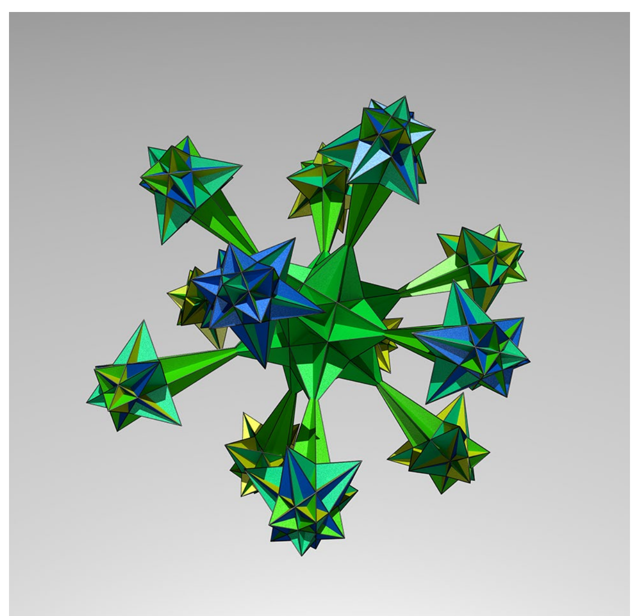


Fig. 6 A stellated polytope passes through a 3-dimensional space (© matematita, image by Gian Marco Todesco)

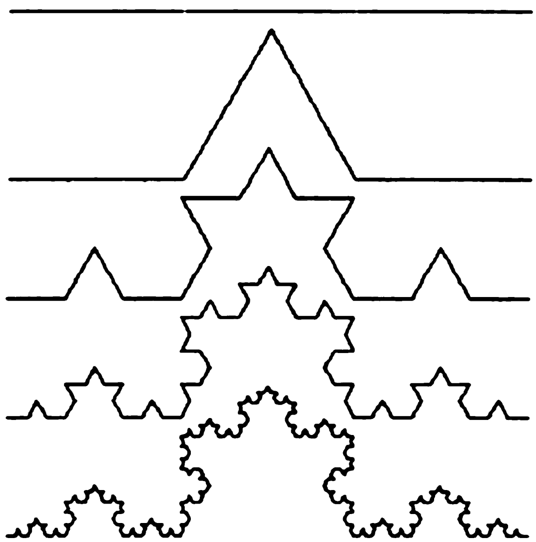


Fig. 7 Koch snowflake

One of the curves that appears to best describe the geometry of nature is the so-called Koch snowflake. The first steps of its construction are shown in Fig. 7. It can be shown that the process that governs those steps leads to a limit figure that is precisely the snowflake. At each step, the curve consists of four copies of the previous step, each contracted by a factor of 3.

What do we mean by the dimension of the snowflake? We said above that the dimension (whether by that we mean length, area, volume or other) describes what happens to the size of a figure F when it is expanded by a factor k in every direction. In one dimension the length is multiplied by k, in two dimensions the area is multiplied by k^2, the volume in three dimensions by k^3. This happens because, if the figure is dilated by a factor k, the result can be divided into $k^{\dim F}$ copies of the starting figure. Thus, the size of the snowflake must be that number d such that 3^d equals 4, that is, the number $d = \log_3 4$, between 1 and 2, which is not an integer and is approximately equal to 1.2618595.

Dimension as we have described it so far allows mathematicians to provide technicians with the necessary tools when they must decide how much fibre is needed in a renovation, the amount of paint needed to decorate some wall, what capacity a heating system is required to have if we want to triple the cubic volume of the cafeteria, how many images in sequence can describe the state of the bone structure of a volleyball player after an accident, or what are the best options to buy shares in the financial market. The concept has pervaded the internal practice of mathematics too: determining what elements and how many of them are sufficient to describe a structure is an operation that leads to new definitions of dimension. Even without mentioning the dimension of vector spaces (which seem created too specifically to explain the n-dimensional spaces that we mentioned earlier), we can recall the dimension of field extensions, which have given the most complete form to the insights on the solutions of an algebraic equation, or the Krull dimension of a ring, which provides information about the affine varieties related to it. We can also recall the progress in the study of knots derived from studying the topology of their complement space, thus showing the inevitability of codimension 2. This crosses transversally into abstract algebra, raising many other interesting questions, but that cannot be covered in this bird's-eye view of dimension.

Translated from the Italian by Daniele A. Gewurz

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