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The Euclidean right triangle: remarks on the regular pentagon and the golden ratio

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Abstract The article contains an original construction of the Euclidean triangle and its proof, from which a simple graphical technique for the construction of the regular pentagon is derived.

Keywords Euclid · Ptolemy · Descriptive geometry · Golden ratio · Regular pentagon construction

While attempting a proof of the well-known theorem "The sides of a regular decagon and a regular hexagon inscribed in the same circle are the legs of a right triangle whose hypotenuse is the side of the pentagon inscribed in the same circle" (Proposition 10 of Book XIII of Euclid's *Elements*), I stumbled upon another, novel method to construct the regular pentagon, which I believe could be of some interest for teachers.

There are several methods to construct the regular pentagon, devised along the centuries by celebrated scholars, starting with Euclid himself and later Claudius Ptolemy. However, the geometrical contents of those constructions notwithstanding, I believe that the reasons and proofs of those methods are rarely shown and explained systematically to students, even in science-oriented secondary schools.

In particular, both Euclid (in the *Elements*) and Ptolemy (in the *Almagest*) construct the pentagon starting from the circle in which it will end up being inscribed. Later, procedures were developed to construct this regular polygon starting from a side. In any case, the existence of a "golden triangle"—formed by two diagonals originating in the same vertex of the pentagon and the opposite side—turns out to be crucial.

Another geometrical figure that appears more or less explicitly in most known constructions is the rectangle having one side whose length is double that of the other (or the right triangle with one leg twice the length of the other, as in the "Carlyle circle" construction, or in the three-circle construction), in which a segment of the longer side is found to be in the golden section ratio to the length of the whole side. The starting figure is actually, as in Euclid, a square, but the essential element is basically that rectangle (or, more simply, the triangle with legs in a ratio 1:2).

In Proposition 11 of Book II of the *Elements* Euclid constructs with ruler and compasses the "golden rectangle", starting from a square ABCD (Fig. 1), whose side will correspond to the shorter leg of the triangle. Having found the midpoint E of the side AD, we take it as the centre of a circle with radius EC and draw an arc that will intersect the extension of side AD at G. Hence, AG is the longer side of the golden rectangle ABHG.

Note that, since AD (or HG) is in the golden ratio with AG, it would be easy to prove that DG is in the golden ratio with DC, so showing that the triangle ECD, whose leg DC is twice the other leg ED, is interesting too.

The procedure to find the golden ratio, as is well known, derives from the Ptolemaic construction involving the chords of a circle, as shown in Fig. 2.

Given a line segment AB, draw from B the perpendicular to AB itself and find on it the point O such that AB is twice BO. Draw the circle having centre in O and OB as radius, which is as a consequence tangent to AB at B. On the line AO call C and B the intersections with the circle. Let E on AB be such that AE = AC. The segments AE and AB are in the golden ratio, that is, AB : AE = AE : EB.

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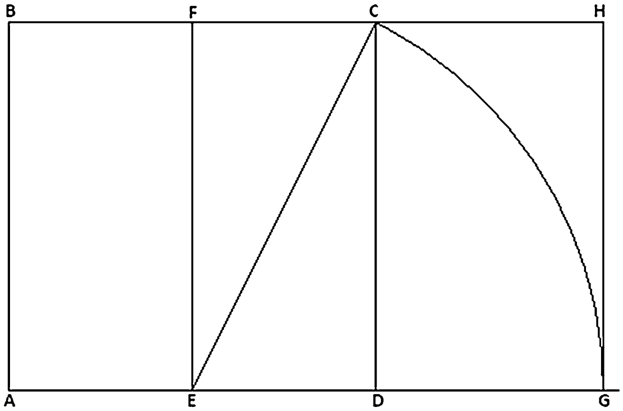

Fig. 1 Construction of the golden rectangle

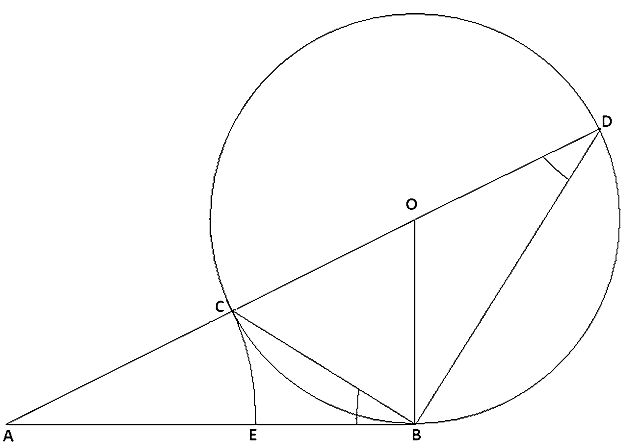

Fig. 2 Construction of the golden ratio

Indeed, consider the power of the point A with respect to a circle, we have $AB^2 = AC \cdot AD = AC \cdot (AC + AB)$, and hence $AB \cdot (AB - AC) = AC^2$.

If we were to draw a circle arc with centre O and radius OA until intercepting a point F on the extension of OB downwards, the line segment BF (just like AC or AE) would be in the golden ratio with AB. Here again, we started from the rectangle with the side AB twice the side BO (of which in Fig. 2 just one-half, AOB, appears). In conclusion, the triangle AOB is analogous to the triangle ECD of Fig. 1.

The construction brought about by the search for a "new" proof of Proposition 10 also derives from that ratio 1:2 on a right angle, but its goal, unlike the method mentioned previously, is the search for the centre of a regular pentagon, given its side. Figure 3 shows that this is a very simple construction, suitable for teaching.

Given the segment AB—the side of the pentagon—and its midpoint C, draw the line segment BD, orthogonal to AB and of the same length. Connect C and D, and find on CD the point E at a distance from C equal to the length of CB. The point O, the intersection of the vertical line above C and the arc of a circle withe centre in A and radius AE, is the centre of the circle circumscribing the pentagon and having radius OA.

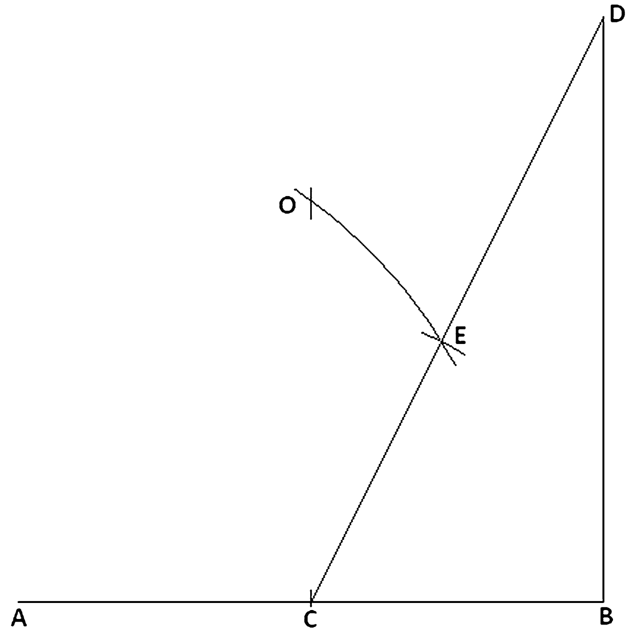

Fig. 3 Construction of the centre of a pentagon of given side

This construction was found, unintentionally, while trying to prove Proposition 10 of Euclid's Book XIII, that is, the existence of a right triangle (which I shall call "Euclidean right triangle") whose legs' lengths are those of a regular hexagon and a regular decagon, both inscribed in circles with equal radii (and equal to the length of the side of the hexagon), while the hypotenuse is the side of a regular pentagon, again inscribed in the same circle.

The Euclidean right triangle can be easily obtained with the procedure shown in Fig. 4.

Draw a half-circle having centre in O, the midpoint of AB, and AB itself as a diameter. From B draw a line perpendicular to AB and from A draw a line forming a 45° angle with AB. Call C the intersection point of the two segments, join it with the centre O, so obtaining the median of the triangle ACB with respect to AB. Denote by D the intersection of the half-circle and the segment OC. Joining D with the endpoints A and B of the starting segment, we get a right triangle ADB whose shorter leg DB is in the golden ratio with the longer leg AD. To prove this relation, let us add a few elements to the construction in Fig. 4, in which, for the time being, the legs of the right triangle have been removed (Figs. 5, 6).

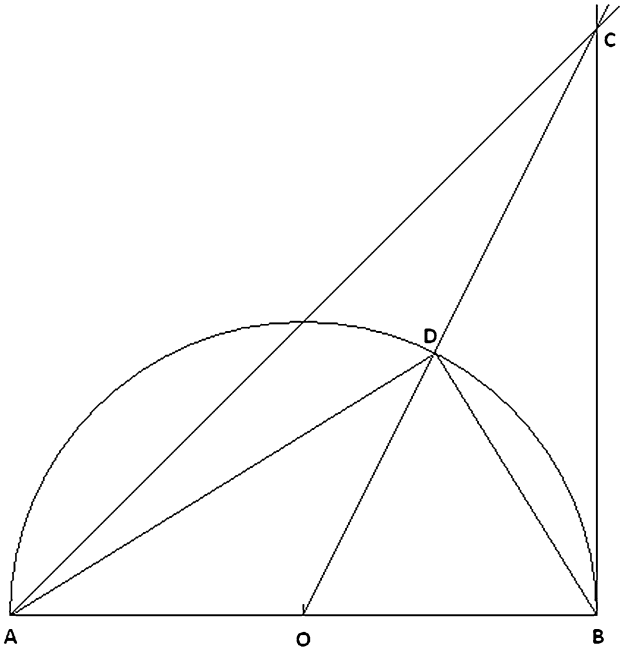

Fig. 4 Construction of the Euclidean right triangle

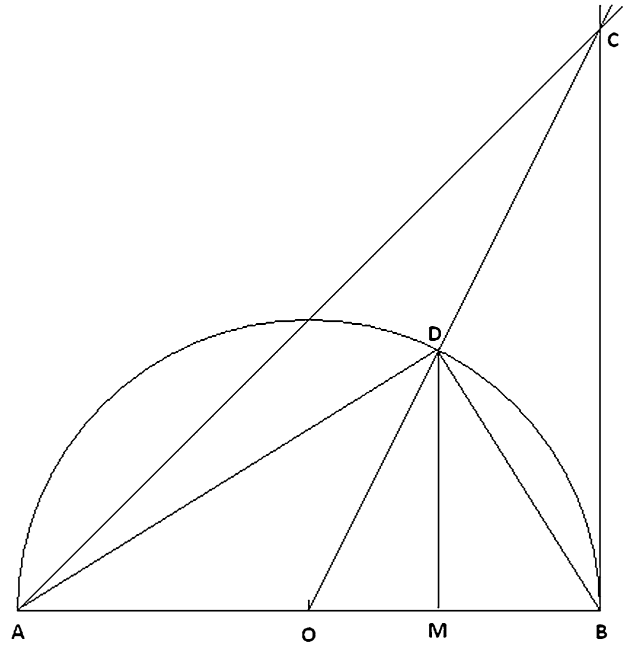

Fig. 6 Proof of the Euclidean right triangle (2)

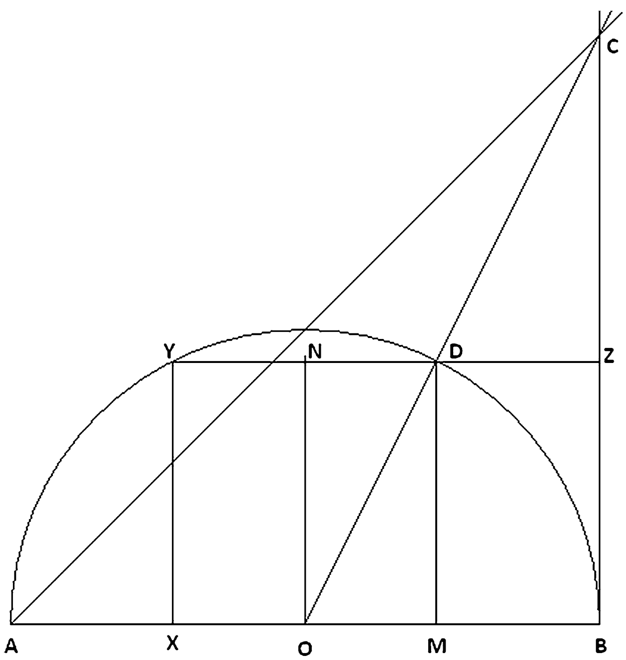

Fig. 5 Proof of the Euclidean right triangle (1)

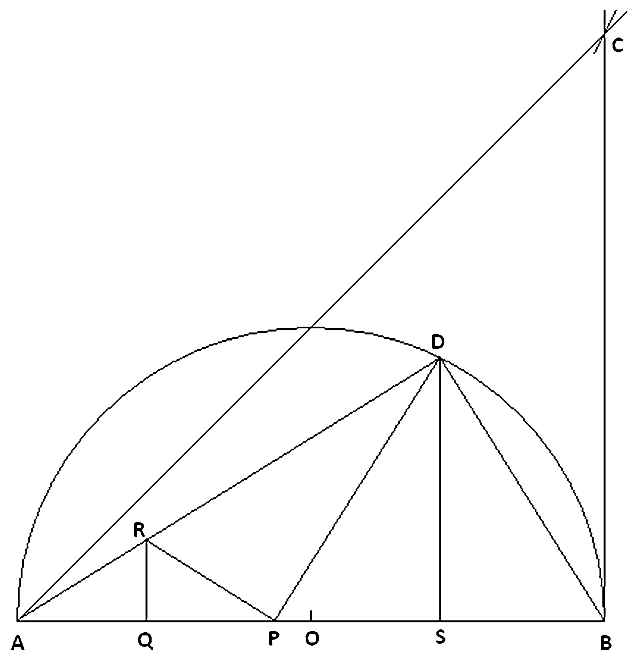

Fig. 7 The Euclidean right triangle and Ptolemy's chords (1)

It is obvious that MD is twice OM, XYDM is a square and the rectangle XYZB is a "golden rectangle" in which YD and YZ are in the golden ratio.

Form this construction it can be easily deduced that

MD:MB = MB:(MD − MB).

This means that MB is in the golden ratio with MD.

Hence, the triangle MDB has its legs in the golden ratio. Since ADB is similar to it, DB is in the golden ratio with AD too.

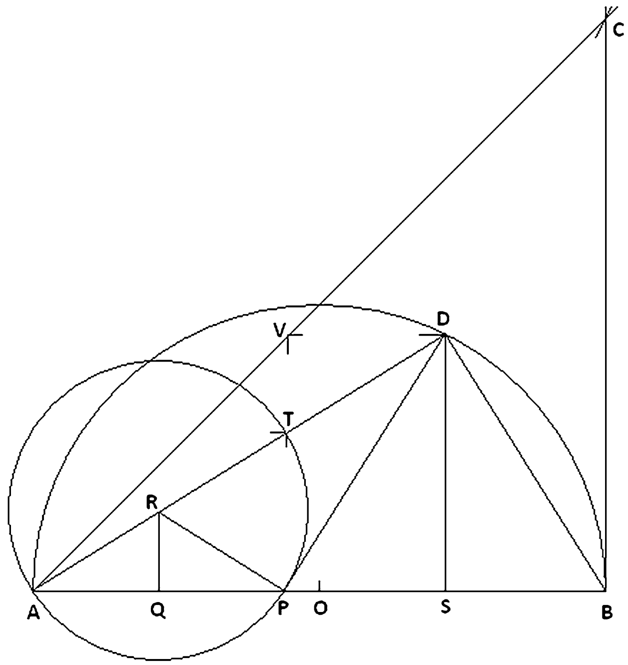

Fig. 8 The Euclidean right triangle and Ptolemy's chords (2)

For the sake of historical elegance, it is useful to correlate this with the studies on Ptolemy's chords. Thus, the construction in Figs. 7 and 8 holds too.

Project the point D on the base of the triangle ABD, obtaining DS, the altitude with respect to AB, and find P, the symmetrical point of B with respect to S. If Q is the midpoint of AP, find on its vertical the point R on AD.

Draw the circle having centre R and radius RP.

The right triangle ATP is congruent to the triangle SDB, since they have each a leg with the same length (indeed, DS = VP = AP, by construction) and congruent angles (BAD = SDB since they are both complements of DBS).

Moreover, RPQ = BAD, RPQ = SDB = PDS, RPQ + DPR = PDS + 90°; from this it follows that DPR = 90°.

DP is tangent at P to the circle with centre in R, so it is the mean proportional of DA and DT,

DA:DP = DP:DT

but, since DP = AT, we also get

DA:DP = DP:(DA – DP);

thus, DB = DP is in the golden ratio with DA.

If we consider the leg AD as the side of a hexagon and the radius of the circle that circumscribes it, then DB is the side of a decagon inscribed in the same circle. Indeed, the side of the decagon forms, with two adjacent radii, a golden triangle having vertex angle, at the centre of the circle, of 36° and congruent base angles of 72°. The side of the decagon is in the golden ratio with the radius and thus with the side of the inscribed hexagon of the same circle.

It remains for us to prove that the hypotenuse AB is congruent with the side of a pentagon inscribed in the same circle of the other two polygons. To this end, let us draw a circular arc with centre in B and radius BD. It will intersect the segment ON in the point L and the side BC in the point E (Fig. 9).

Presently, our goal is to prove that the angle LBO is 18° wide. But studying the illustration it may be interesting to notice that this angular width, once proven, is characteristic of the golden rectangle, as seen in Fig. 10.

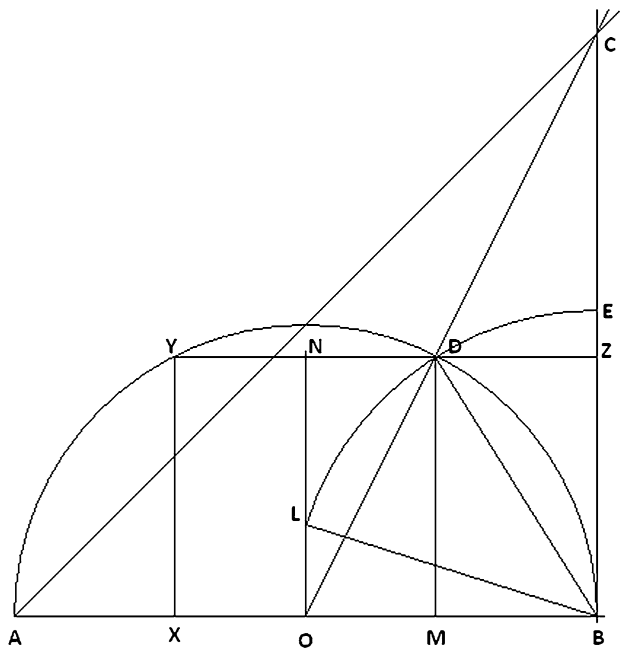

Fig. 9 Checking the pentagon

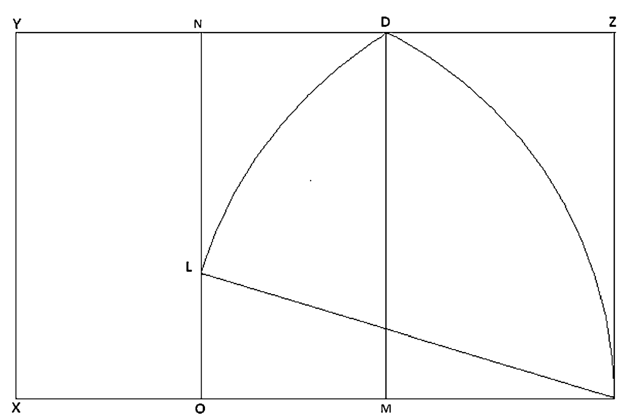

Fig. 10 Construction of the golden rectangle

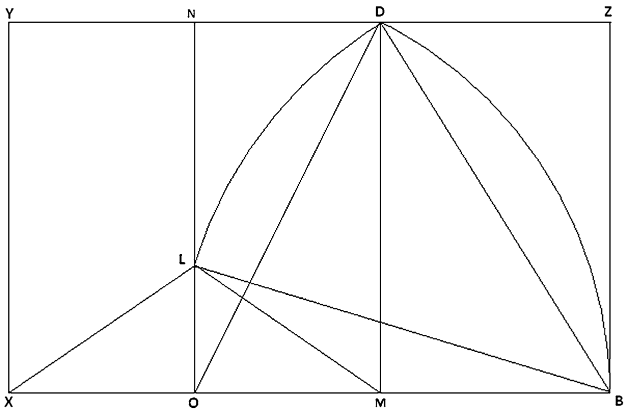

Fig. 11 Enlarged detail of Fig. 9

We have added to the usual Euclidean construction (*Elements*, Book II, Proposition 11) the arc DL, centred in B, and the segment LB.

Going back to Fig. 9, enlarging the part we are interested in (Fig. 11), we notice the following relations:

Application of:

Pythagorean Theorem to MDB (keeping in mind that MD = 2OM):

$MB^2 = DB^2 - MD^2$

(a) $MB^2 = LB^2 - 4OM^2$

Golden relation to MDB

(XYZB is a golden rectangle)

(b) $MB^2 = 2OM(2OM - MB)$

Pythagorean Theorem to OLM

(c) $LM^2 = LO^2 + OM^2$

Pythagorean Theorem to OLB

(d) $LO^2 = LB^2 - (OM + MB)^2$

Consequences

from (d) $LO^2 = LB^2 - OM^2 - MB^2 - 2OM \cdot MB$

from (c) $LM^2 = LB^2 - OM^2 - MB^2 - 2OM \cdot MB + OM^2$

$LM^2 = LB^2 - MB^2 - 2OM \cdot MB$

from (b) $LM^2 = LB^2 - 2OM(2OM - MB) - 2OM \cdot MB$

$LM^2 = LB^2 - 4OM^2 + 2OM \cdot MB - 2OM \cdot MB$

(e) $LM^2 = LB^2 - 4OM^2$

Conclusion: LM = MB

(comparing (a) with (e)).

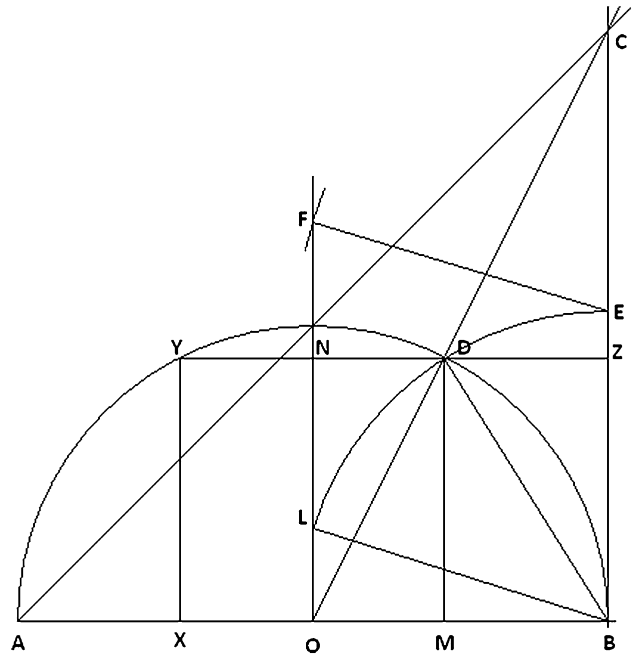

Fig. 12 Final part of the proof

Having constructed the triangle XLM with X symmetric to M with respect to ON, it is isosceles and with the side LM (congruent with MB) in the golden ratio with the base XM. Thus, it is a "golden triangle" with base angles MXL and LMX of 36° and vertex angle XLM of 108°.

But, if LMX = 36°, then MLB and LBM (having ascertained that the triangle LMB is isosceles) are 18° wide.

In particular, LBO = 18°, as required.

Determining this angle was essential to complete the proof but also enriches our knowledge of the golden rectangle, as we have seen.

By studying Fig. 12, obtained by adding some further elements (we have prolonged the vertical line above the centre O of the half-circle and found the point F, the intersection of this line and the arc having centre E and radius BD; basically, the leg BD of our "Euclidean triangle" has been copied to E and then to F), it is easy to recognise that FEBL is a parallelogram having angle in E equal to 90° + 18° = 108°.

The triangle FEB is isosceles and its base angles (FBE and EFB) are of 36°. Thus, FE = EB = BD is in the golden ratio with FB (not drawn). But, since BD is in the golden ratio with the longer leg AD (not drawn), it follows that AD = FB.

We may conclude by pointing out that a circle arc with centre in A and radius AD would intersect the vertical above O in the point F, describing the isosceles triangle AFB, whose base angles are equal to 54° (in the vertex B we have 90° − 36° = 54°) and vertex angle equal to 72°.

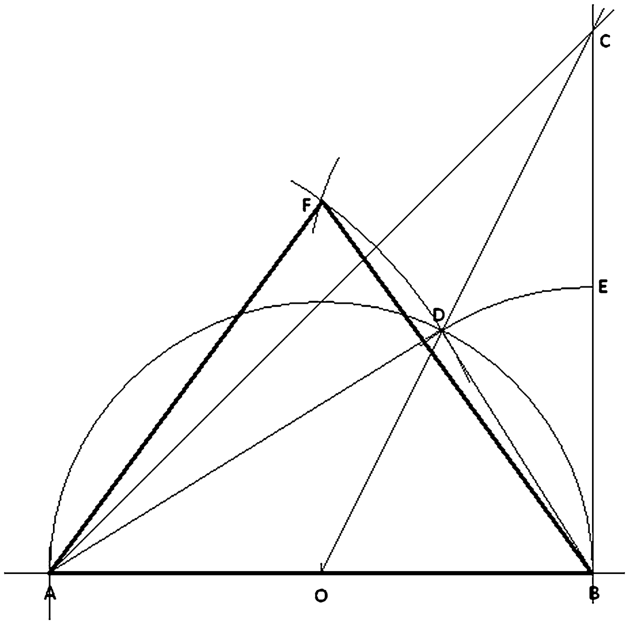

Fig. 13 Conclusion of the proof

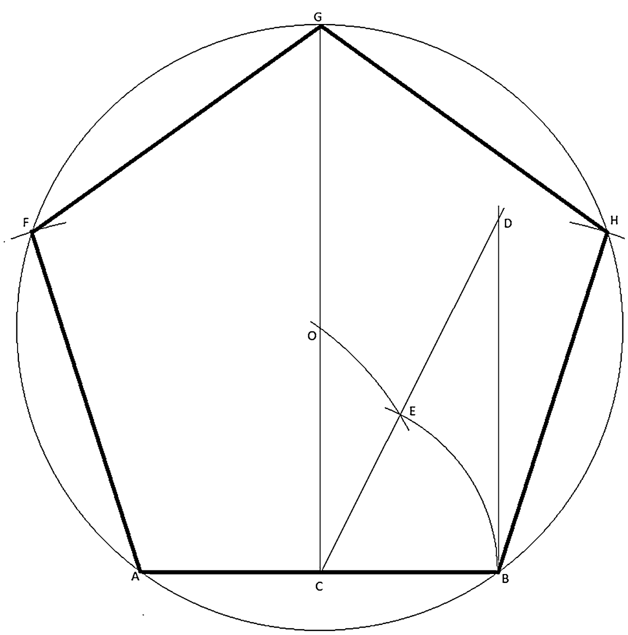

Fig. 14 Construction of the regular pentagon

Hence it is a one-fifth “wedge” of the regular pentagon (Fig. 13).

Thus, AB, the hypotenuse of the right triangle ADB, is the side of the regular pentagon inscribed in the circle in which are also inscribed the regular hexagon and decagon whose sides are the legs of the same triangle. The side of the hexagon is the radius of the circle too.

Appendix: Complete (and new) construction of the regular pentagon, given its side AB

Given the side AB, from the endpoint B draw the line BD, perpendicular to it (Fig. 14). Having found the point D such that BD has the same length as AB, join C, the midpoint of the side AB, to D. With centre in C and radius CB, draw an arc of circle that intersects CD in the point E.

The distance AE is the radius of the circumscribed circle of the regular pentagon having side AB. Thus, the arc centred in A with radius AE intersects the line perpendicular to AB and passing through C in the point O, centre of this circle (the point O may be found, in a more direct way, by intersecting the last-mentioned arc with another one, with the same radius and centre in B). Having drawn the circle, the vertices of the regular pentagon are determined, one by the line CO (vertex G), and the other two by the circle arc centred, respectively in A and B and radius AB (vertices F and H).

Translated from the Italian by Daniele A. Gewurz.

Giuseppe Sansò after completing two years of engineering, has been certified as a teacher and teaches technology and computer science in secondary school. His research interests include combinatorics, astronomical exploration, photography and writing.