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Irregular functions and fractal objects: from Weierstrass to Mandelbrot

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Abstract The aim of this paper is to show the origins of fractal geometry in the mathematical work of Weierstrass, Peano, Julia and Hausdorff, as well as from input from studies on Brownian motion and turbulence in physics by Richardson, Perrin and Kolmogorov. It concludes with a brief review of some of the many applications of fractal geometry in science and technology.

Keywords Fractals · Fractal geometry · Fractal dimension · Fractal · Self-similarity · Brownian motion · Turbulence

1 Introduction

> Philosophy is written in this grand book—I mean the universe—which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering around in a dark labyrinth. [10, pp. 237–238]

This passage by Galilei can be seen as the manifesto of a programme to describe physical reality mathematically, started by Galileo and Newton and developed by the great mathematicians of the eighteenth century. A central role in this approach was played by differential equations, both ordinary and partial. Thus, it is not surprising that the study of "regular" functions (differentiable, or at least differentiable except for some isolated points) was favoured.

Of course, it is easy enough to see that "clouds are not spheres, mountains are not cones" [11, p. 1], but on the other hand it is not that easy to bring oneself to abandon good old geometry and the analysis of differentiable functions.

There were exceptions studied by Weierstrass, Peano, Hausdorff and Julia in mathematics and by Perrin and Richardson in physics, which were long considered to be pathologies. These studies remained somewhat on the fringe: mathematicians saw them as a kind of elegant constructions of monsters, while physicists as pathologies not representative of the phenomena (erroneously) considered to be really important.

The man who recognised the widespread presence and importance of such behaviour in natural sciences, and who created the term "fractal" itself is Benoît Mandelbrot. Starting in the 1960s, inspired by the work of the precursors already mentioned, Mandelbrot developed a new form of geometry, introducing the neologism "fractal". Fractal geometry has become popular and well-known among the general public due to its graphical and aesthetic aspects which, interesting as they are, are not especially significant from a scientific viewpoint. We are going to briefly survey the development of the theory of fractals and its applications to physics.

2 The dimension of an object

We say that a regular curve has dimension one, since one variable (a curvilinear coordinate) is sufficient to determine one point on it; analogously, the surface of a sphere has

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dimension two since a point on it is given by two coordinates (latitude and longitude).

However, we might proceed in a different way. Let us consider a regular curve and ask ourselves how to measure its length. We can approximate the curve with a broken line consisting of segments of length ℓ and then let ℓ become smaller and smaller. Denoting by $N(\ell)$ the number of segments, if ℓ is small enough then the length L is about $N(\ell)\ell$; thus, $N(\ell)$ is proportional to $1/\ell$. In the case of a surface, we may "tile" it with $N(\ell)$ small squares of side ℓ; the area A is approximated by $N(\ell)\ell^2$, and hence $N(\ell)$ is proportional to $1/\ell^2$. Analogously, to fill a three-dimensional object, a number of small cubes of side ℓ that is proportional to $1/\ell^3$ will be required.

From this remark comes the idea of generalising the notion of dimension: an object is said to have fractal dimension D_F if the number of small (hyper)cubes having side ℓ required to cover the object behaves like

$$N(\ell) \sim \frac{1}{\ell^{D_F}} \tag{1}$$

Of course, for a regular object, the fractal dimension is simply the usual dimension. Are there objects for which D_F is not an integer?

The answer is yes. An object is said to be a fractal if D_F is not integer. Typically, a fractal has a self-similar structure: the part is similar to the whole, and looking at a figure with a given resolution is not possible to say what scale we are looking at. The truly important point is that this kind of behaviour is not a by-product of pathological mathematical models. On the contrary, this kind of "roughness" is very common, and can be found in the attractors of the dissipative chaotic dynamical systems and in many natural phenomena, such as turbulence and the large-scale structures of galaxies.

3 A little history

Here follows a brief historical survey—far from exhaustive—of the studies that anticipated Mandelbrot's work.

3.1 Mathematicians

3.1.1 Weierstrass

In his work on the theory of functions and Fourier series, Weierstrass studied functions that were continuous but not differentiable. An example is

$$f(x) = \sum_{n=1}^{\infty} A^{-n} \cos\left(2\pi B^{n-1} x\right) \tag{2}$$

where B is an integer and $A < B$; this function is extremely irregular and its graph, shown in Fig. 1, has an infinite length; it has a fractal dimension $D_F = 2 - \ln A/\ln B$, with values between 1 and 2.

3.1.2 Von Koch

Another example of an irregular mathematical object is the Koch curve, due to Helge von Koch, whose construction is shown in Fig. 2: take a line segment of unit length and divide it in three equal parts. The central part is removed and substituted by two segment of its same length. Repeat this procedure on the four elements so obtained, then on the sixteen of the next generation, and so on for infinitely many times. After n iterations we get 4^n segments of length 3^{-n}; thus, the length increases as $(4/3)^n$, and this corresponds to a fractal dimension $\ln 4/\ln 3 \simeq 1.2618$.

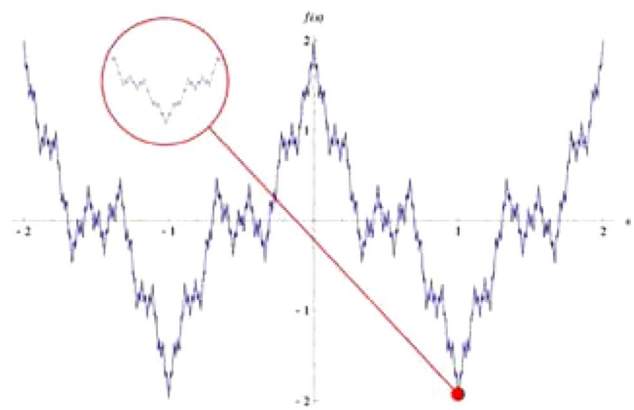


Fig. 1 Weierstrass function; the *enlarged inset* shows the self-similar structure

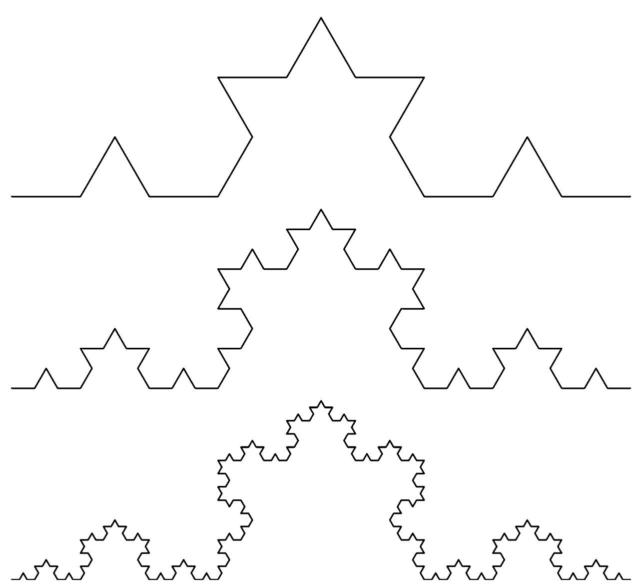


Fig. 2 Scheme for the construction of Koch curve

3.1.3 Peano

The most famous case of a "monster" curve is perhaps Peano's: a continuous curve filling a square. The iterative scheme for its construction is shown in Fig. 3.

3.1.4 Julia

Consider the following iterative rule (describing a discrete-time dynamical system):

$$z_{t+1} = z_t^2$$

where z is a complex variable; it is easy to see that if z_0 lies within the unit circle then $z_t \to 0$ when $t \to \infty$, while if $|z_0| > 1$ then z_t goes to infinity for $t \to \infty$. Thus, the unit circle $|z| < 1$ is the basin of attraction of 0.

Here is now an apparently harmless variant:

$$z_{t+1} = z_t^2 + c. \tag{3}$$

We ask ourselves about the shape of the finite-value basin of attraction. Unlike what we might expect, we do not get a simple regular deformation of a circle, but an object whose boundary is all but smooth: see Fig. 4.

3.1.5 Hausdorff

Consider a regular measure $\mu(\mathbf{x})$, that is, uniformly continuous with respect to Lebesgue measure, with $\mathbf{x} \in R^d$, where d is the dimension of the space; we have.

$$d\mu(\mathbf{x}) = p(\mathbf{x})d\mathbf{x},$$

where $p(\mathbf{x})$ is a non-negative function; considering a sphere having radius ℓ and centre in $\mathbf{x}$, if ℓ is small enough we have:

$$\int_{|x-y|<\ell} d\mu(y) \simeq \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} p(x)\,\ell^d. \tag{4}$$

The quantity $\int_{|x-y|<\ell} d\mu(y)$, which can be seen as the mass within the sphere, increases as ℓ^d.

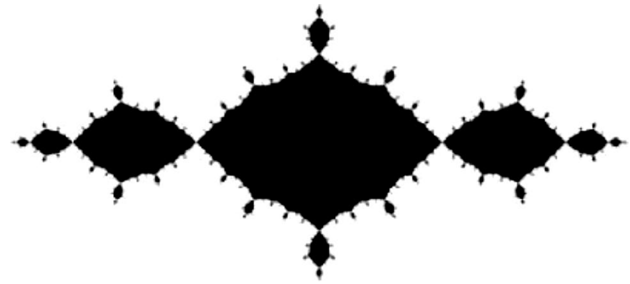


Fig. 4 An example of finite-value basin of attraction for $c \neq 0$

Hausdorff, in his studies in measure theory, generalised the notion of dimension: if the measure is singular with respect to Lebesgue measure, that is, if (4) does not hold, then

$$\int_{|x-y|<\ell} d\mu(y) \sim \ell^{D_H},$$

where D_H is called the Hausdorff dimension. As we shall see later on, D_H can depend on $\mathbf{x}$.

3.2 Brownian motion

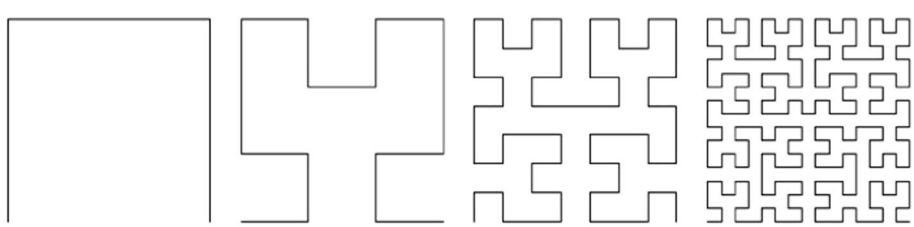
In 1827, Robert Brown, a botanist, discovered the phenomenon now called Brownian motion: a colloidal particle (that is, an object with a size on the order of one micron, and hence macroscopically small but microscopically large), immersed in a liquid, has a very irregular, zigzagging motion. At the turn of the twentieth century, thanks to the work of Albert Einstein and Marian Smoluchowski, the phenomenon was interpreted in terms of statistical mechanics. If we observe one of its spatial components long enough, we find

$$\left\langle [x(t) - x(0)]^2 \right\rangle \simeq 2Dt,$$

where $\langle\rangle$ denotes the average, for instance on many particles, and

$$D = \frac{TR}{6\pi\eta a N_A}$$

Fig. 3 Scheme for the construction of Peano curve



is the diffusion coefficient, with R, η, T, a and N_A, respectively, the universal gas constant, the viscosity of the fluid, its temperature, the radius of the colloidal particle (for instance, a pollen grain) and Avogadro's number.

Brownian motion might appear to be just an oddity, but it is instead a wonderful magnifying glass into the microscopic world and makes it possible to find a relation between macroscopic quantities (which are observable experimentally), such as D, T, R, η and a, and a quantity related to the microscopic world, such as Avogadro's number N_A.

In his experimental studies on Brownian motion, Jean Baptiste Perrin was among the first ones to realise the importance of self-similar systems. Here is Perrin's interesting remark, perhaps the first statement regarding self-similar phenomena:

> Consider, for instance, one of the white flakes that are obtained by salting a soap solution. At a distance its contour may appear sharply defined, but as soon as we draw nearer its sharpness disappears…. The use of magnifying glass or microscope leaves us just as uncertain, for every time we increase the magnification we find fresh irregularities appearing, and we never succeed in getting a sharp, smooth impression, such as that given, for example, by a steel ball. [15, p. ix]

Figure 5 shows an example of Brownian motion: it is apparent that a magnification of a part has the same properties as the whole.

In the 1920s, Norbert Wiener introduced what is now called a Wiener process, a continuous Gaussian stochastic process with the following properties:

$$x(0) = 0, \langle x(t) \rangle = 0, \langle x(t_1)x(t_2) \rangle = \min\{t_1, t_2\}.$$

This process is not differentiable, and we have $\Delta x(\Delta t) = x(t + \Delta t) - x(t) \sim \sqrt{\Delta t}$; in more formal terms, the variable $\Delta x(\lambda \Delta t) = x(t + \lambda \Delta t) - x(t)$ has the same properties as $\sqrt{\lambda}\Delta x(\Delta t)$.

If we apply to Brownian motion the procedure for computing the length D_F of a graph, we easily find $N(\Delta t) \sim \Delta t^{-3/2}$ and hence $D_F = 3/2$. In general, for a curve with Hölder exponent $0 < h \leq 1$, that is, $\Delta x(\Delta t) \sim \Delta t^h$, one has that $D_F = 2 - h \in (1, 2)$; of course, when $x(t)$ is differentiable, that is, when $h = 1$, then $D_F = 1$.

For more on Brownian motion, see [2, 5].

3.3 Richardson

The English scientist Lewis Fry Richardson (1881–1953), though undeservedly little known, played a fundamental, often posthumous, role in twentieth-century science. Although his name is associated with many important topics

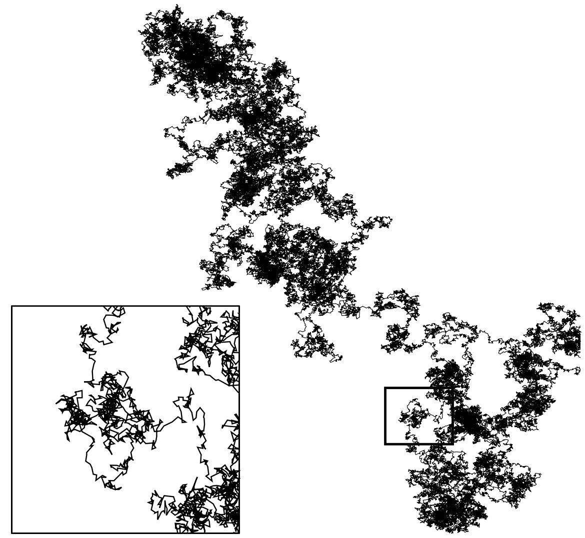


Fig. 5 An example of Brownian motion; the image in the inset is an enlarged detail

in fluid dynamics, meteorology and numerical analysis (it is sufficient to consider his criterion for the stability of flows, the idea of a scale-dependent diffusion coefficient and the algorithm bearing his name, still used to integrate differential equations), he is unfamiliar even to physicists and mathematicians.

Besides his important contributions to meteorology, numerical analysis and fluid dynamics, Richardson was the first to attempt a mathematical description of war; he also pioneered the study of self-similar systems and was one of the fathers of fractals (for more on Richardson, see [1, 18]).

The first to ask "how long is the coast of Britain?" was not Mandelbrot (who is considered to be the father of fractals) but Richardson. Among the papers found after his death, there are log–log graphs where he plotted, as a function of resolution ℓ, the length $L(\ell)$ of the coastlines of Great Britain, of the land boundary of Germany, of the Spain-Portugal boundary, of the coastlines of Australia and South Africa. Rather than a convergence to a constant value, Richardson observed a behaviour of the form $L(\ell) \sim \ell^{-\alpha}$, where α is basically zero for the coastlines of South Africa, while in the other cases it is positive and increases with the roughness of the line; in modern terms, $\alpha = D_F - 1$ (where D_F is the fractal dimension).

In his studies of fluid dynamics Richardson realised that natural phenomena cannot always be described by regular functions. For instance, he observed that in turbulence, rather than a typical scenario with small variations around an average value and some rare, never especially large (say on the order of one standard deviation), fluctuations, we have long intervals of quiescence, when the signal has a regular course and stays close to its mean value, broken off by short irregular periods with huge fluctuations. On the contrary, in the case of Gaussian variables, there is a limit to the size of the fluctuations.

These observations led Richardson to seriously consider the apparently absurd question "Does wind has a speed?" From few empirical data, he guessed the self-similar structure of turbulence; here is how he summarised his insight in a short poem (inspired by a parody of Swift):

Big whorls have little whorls.
That feed on their velocity;
And little whorls have lesser whorls.
And so on to viscosity.
(In the molecular sense). [16, p. 66]

Figure 6 shows the time evolution of the energy dissipated in a turbulent fluid; an alternation of long intervals with small fluctuations around the mean value and short, strong, irregular excursions can be observed.

The mathematical formalisation of this idea was due, in the 1940s, to Andrey Kolmogorov, who showed that, in the so-called inertial range, the velocity field is very rough, quite different from the ordinary functions we are used to: the velocity difference $\delta v(\ell)$ between two points at a distance ℓ is not proportional to ℓ; on the contrary, we have a non-analytical behaviour $\delta v(\ell) \sim \ell^{1/3}$, with enormous (or infinite when Reynolds numbers go to infinity) velocity gradients.

3.4 Mandelbrot

Besides Perrin and Richardson, fractals and self-similar structures had already been studied at the turn of the twentieth century by Weierstrass, Hausdorff and Julia, as mentioned. Then came Mandelbrot.

A Frenchman of Polish–Lithuanian origins, Mandelbrot studied at the École Polytechnique in Paris. Impatient with the too formal climate that the Bourbaki group had imposed on French mathematics, he moved to the United States, where he earned a master's degree in aeronautics. After obtaining his doctorate in mathematics in Paris in 1952, he spent time in France and Switzerland and in 1957 he finally moved back to the US, working at IBM and at Yale University.

His scientific path was characterised by very varied interests: linguistics, information theory, finance, turbulence, cosmology and more. The diversity of his research is well described by the professorships he had: in mathematics, economics, physiology, engineering.

By studying very disparate phenomena, from the trend of the prices of cotton to the frequency of transmission errors in telephone lines, Mandelbrot realised the omnipresence of self-similar phenomena, which, far from being an oddity, are often the norm. Even though, as Galileo wrote, the book of Nature is written in the language of mathematics, Mandelbrot became convinced of the necessity to go beyond usual geometry; indeed, as quoted above, "clouds are not spheres, mountains are not cones". The introduction of fractal geometry and the development of mathematical techniques inspired by information theory and probability theory made it possible to study very diverse phenomena: ecosystems, data on earthquakes, turbulent fluids, large-scale structures of galaxy clusters, processes of clustering and percolation in petroleum prospecting.

For an introduction to fractals, see [14]; for more on their mathematical theory, see [6, 7].

4 Fractals and self-similar structures: not just images for T-shirts

4.1 Fractals and chaos

Consider a dynamical system, given for instance by a differential equation

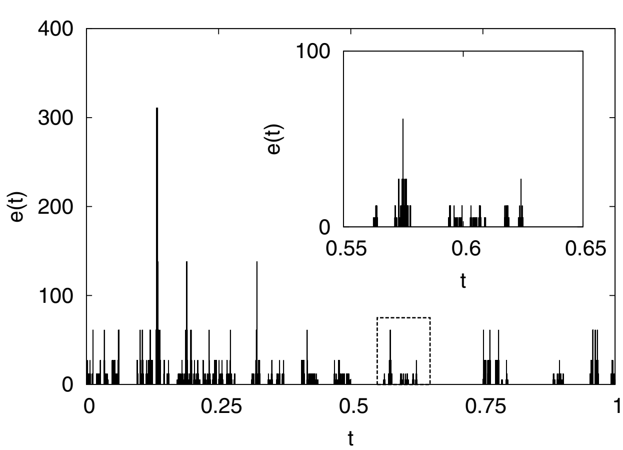


Fig. 6 The energy dissipated in a turbulent fluid as a function of time; notice how the enlargement of one part is equal to the whole graph

Benoît Mandelbrot, on the occasion of the awarding of the Légion d'honneur on 11 September 2006 at École Polytechnique (© DAVID Monniaux, www.wikipedia.org)

$$\frac{dx}{dt} = f(x),$$

or a discrete-time map

$$x(t+1) = g(x(t)),$$

confining ourselves to the dissipative case where the volume in the phase space contracts. Time-asymptotically, the system will get closer and closer to its attractor. If the motion is not chaotic, the corresponding attractor is said to be regular: it can be a point, a closed curve, a surface or a hypersurface.

For instance, in the case of a pendulum with friction, the attractor is a point: for the van der Pol system

$$\frac{dx}{dt} = y, \frac{dy}{dt} = -\omega^2 x + \mu(1-x^2)y,$$

if $\mu < 0$ the attractor is the fixed point (0, 0), while if $\mu > 0$ the attractor is a closed curve (the limit cycle).

In a system with strong dependence on initial conditions, we have an attractor that is not regular, but has a generally very complex structure, typically of a fractal nature. In this case, we call it a chaotic attractor or, in the terminology introduced by David Ruelle, a strange attractor.

As an example, consider Hénon map:

$$x_{t+1} = 1 - ax_t^2 + y_t,\ y_{t+1} = bx_t;$$

its attractor is shown in Fig. 7.

In general, the measure $\mu(\mathbf{x})$ on the attractor is not homogenous; thus, a single dimension does not suffice to describe it. The Hungarian mathematician Alfréd Rényi introduced a way to characterise non-homogeneous singular measures. Divide the phase space (that is, the region where $\mu(\mathbf{x}) \neq 0$) in cells $\{\Lambda_i(\ell)\}$ having side ℓ, and define the probability of being in the i-th cell:

$$P_i(\ell) = \int_{\Lambda_i \ell} d\mu(x).$$

We can now define the Rényi dimensions d_q (where q is a real number)

$$\sum_i P_i(\ell)^q \sim \ell^{(q-1)d_q},$$

or in a more formal way

$$d_q = \lim_{\ell \to 0} \frac{1}{q-1} \frac{\ln \sum P_i(\ell)^q}{\ln \ell}.$$

It can be shown that the function d_q is non-increasing; for $q = 0$, we have $d_0 = D_F$. Obviously in the case of a uniform distribution on the fractal we have $d_q = D_F$, while in the (typical) case where this does not happen the measure is said to be multifractal. The dimension d_1 is called the information dimension and is the most important one, since for almost all points x in the attractor we have

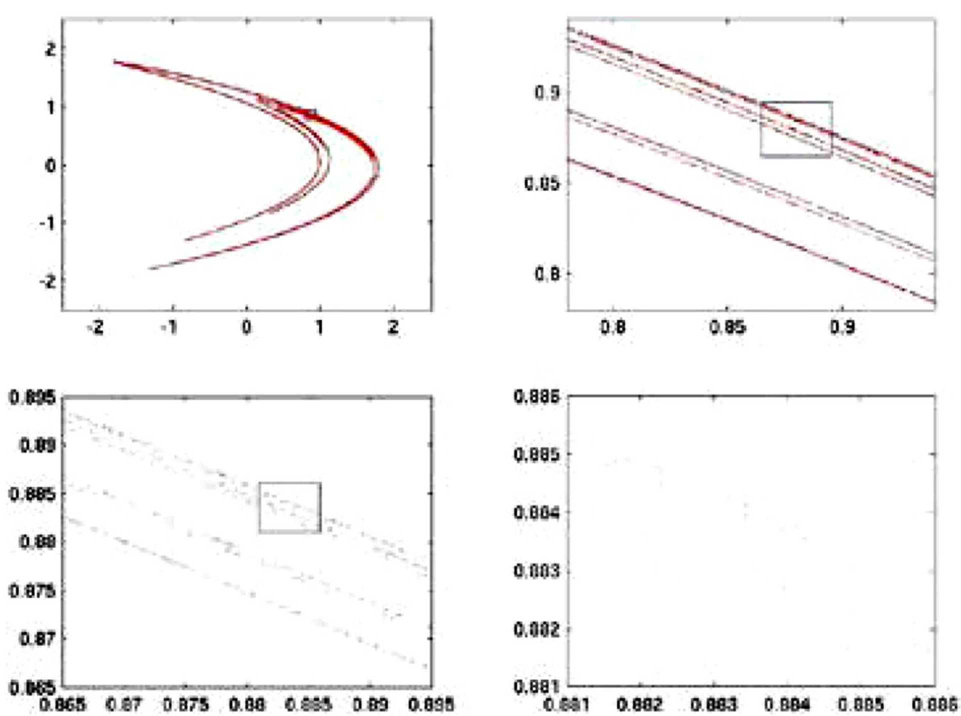


Fig. 7 Hénon attractor (for $a = 1.4$ and $b = 0.3$), obtained for a very long trajectory and with some enlargements; the lower right image contains little detail since it includes few points

$$\int_{|x-y|<\ell} d\mu(y) \sim \ell^{d_1}.$$

The farther d_q is from the horizontal line D_F, the more dishomogeneous the measure is. For the Hénon map having $a=1.4$ and $b=0.3$ we have $d_2 \simeq 1.20$, $d_0=D_F\simeq 1.26$.

4.2 Fractals in turbulence

Developed turbulence, that is, the irregular behaviour of a fluid at high Reynolds numbers $R_e=UL/v$ (where U and L are the characteristic speed and length of the velocity field, v is the kinematic viscosity), is a phenomenon of great importance and very difficult to explain. We shall briefly discuss only the aspects related to self-similarity.

In 1941 Kolmogorov showed that in the inertial range (that is, $\eta \ll \ell \ll L$, where $\eta = LR_e^{-3/4}$, is the length at which dissipation becomes significant) the following relation holds:

$$\left\langle \delta v(\ell)^3 \right\rangle = -\frac{4}{5}\langle \epsilon \rangle \ell \qquad (5)$$

where $\delta v(\ell)$ is the difference between the longitudinal velocities of two points at a distance ℓ, and $\langle \epsilon \rangle$ is the mean density of dissipated energy.

From (5) it is natural to conjecture that $\delta v(\ell)$ is a process with Hölder exponent $h=1/3$. Experimental measurements, and more recently numerical simulations, show a more complex scenario: rather than a power law for a process with Hölder exponent 1/3, that is, $\left|\delta v(\ell)^p\right| \sim \ell^{p/3}$, we have

$$\left|\delta v(\ell)^p\right| \sim l^{\zeta_p},$$

with $\zeta_p \neq p/3$.

We are witnessing the so-called anomalous scaling, in which a single exponent does not suffice to describe the statistical properties. This is a situation somehow similar to multifractal measures. For the developed turbulence, it has been shown that the multiaffine structure of the velocity field is associated to the multifractalness of the dissipated energy density of the space $\epsilon(\mathbf{x})$, which is concentrated in very small zones. Under suitable hypotheses, it is possible to show that Rényi dimensions d_p of the measure $\mu(\mathbf{x})=C\epsilon(\mathbf{x})$, where C is determined by the normalisation $\int d\mu(\mathbf{x})=1$, determine the exponents ζ_p:

$$\zeta_p = \frac{p}{3} + \left(\frac{p}{3}-1\right)\left(d_{\frac{p}{3}} - 3\right).$$

The experimental data show that the dissipated energy has a multifractal structure and is localised on a fractal of dimension between 2.8 and 2.9.

For more on fractals and turbulence, see [4, 8, 12].

4.3 Fractals more or less everywhere

Let us conclude with a brief overview of the use of fractal geometry.

Astrophysics Let us recall quickly an old paradox (due to Heinrich Wilhelm Olbers in the nineteenth century): in a static universe in which mass distribution, on a sufficiently large scale, is uniform, the intensity of light is necessarily infinite at each point. Indeed, the number of stars at a distance between R and $R+dR$ from a given point is proportional to R^2dR; since the luminosity of a single star is proportional to R^{-2}, we have that the contribution given to the total luminosity by the stars from R to $R+dR$ is proportional to dR; so, after integrating, we get an infinite value! Currently, astronomers believe that the solution lies in the expansion of universe.

Nevertheless, it is to be remarked that the paradox would not hold in a static universe where the star distribution is fractal with $D<2$, either. It is easy to repeat the computation: the number of stars between R and $R+dR$ is proportional to $R^{D-1}dR$, hence the contribution to luminosity is proportional to $R^{D-3}dR$, and for $D<2$ the integral converges.

In the 1970s, it was observed experimentally by the French astrophysicist Gérard de Vaucouleurs that the density within a sphere of radius R is proportional to $R^{-\alpha}$ with $\alpha \simeq 1.8$; in the language of fractal geometry, the galaxies are distributed on a fractal with dimension $D=3-\alpha \simeq 1.2$. Recent data on more accurate catalogues show that $D \simeq 2$. Experts are not in complete agreement about the fact that fractal distribution holds at all scales, but the fractal scaling applies at least up to R of the order of 20 megaparsec (that is, about 6.5×10^7 light years). See a survey in [9].

Medicine In mammals, blood vessels and the bronchial tree show a self-similar, fractal structure. It has been conjectured that such a structure optimises the distribution of blood in arteries and of oxygen in the bronchi. Fractal models for blood vessels are now systematically used in numerical simulations for haemodynamics. See a survey in [19].

Geophysics In the dynamics of the atmosphere and the oceans several kinds of turbulent phenomena (with features technically more complex than those previously discussed) play a fundamental role, so it is not surprising to find fractal and multifractal distributions.

Fractal structures are present in seismic faults, drainage basins and coastlines. Studying these problems from first

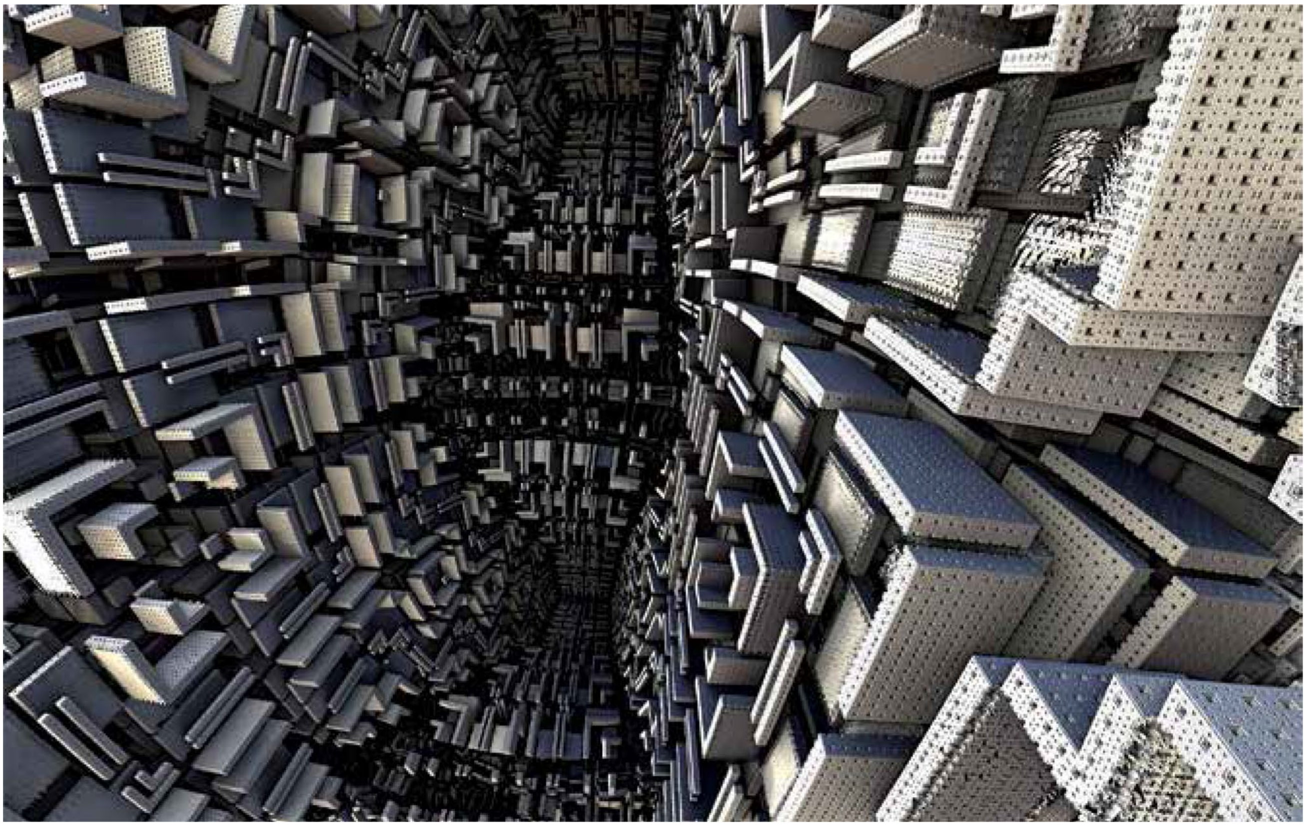


Fig. 8 An example of an image of obtained with IFS: a fantasy architecture

principles has a forbidding difficulty. Fortunately, in some cases (for instance, for coastlines and basins), it is possible to use mathematical models able to reproduce, even quantitatively, the observed data; this makes it possible to detect the fundamental mechanisms underlying the formation of structures. See a survey in [17].

Fractals in Hollywood The imaging techniques based on the iterated function system (IFS), which can generate self-similar images, are widely used in film industry. Many of the images we see in films (especially in science-fiction and fantasy ones), such as landscapes, clouds, aurorae, fires, are computer-generated.

4.3.1 Iterated function system

Let us close with a brief look at the IFS method to generate fractal images. This very powerful technique is also used to compress images.

Consider a vector **x** in D dimensions and a linear transformation $\mathbf{W}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where A is a $D \times D$ matrix and **b** a D-dimensional vector. Assume that we have N possible linear transformations, that is, $(A_1, \mathbf{b}_1)$ with probability p_1, $(A_2, \mathbf{b}_2)$ with probability p_2, and so on until $(A_N, \mathbf{b}_N)$ with probability p_N; moreover, for each i we have

$$|W_i(x) - W_i(y)| < s_i\,|x - y|$$

If the following inequality holds

$$s_1^{p_1} s_2^{p_2} \ldots s_N^{p_N} < 1$$

then, by iterating a large number of times the algorithm

$$x_{t+1} = W_{i(t)}(x_t)$$

where $i(t)$ is chosen independently of $i(t-1)$ and equals 1 with probability p_1, 2 with probability p_2, and so on, we obtain a well-defined image. Obviously, the shape of the image depends on the transformations $\{\mathbf{W}_i\}$, while the density of the points depends on the probability $\{p_i\}$. It is possible to obtain colour images by suitably associating a colour shade to a given point density.

Even with just a few linear transformations, for instance $N = 4$, it is possible to generate beautiful, realistic images; of course, some practice is necessary to use this method. Figure 8 shows an example of image obtained with IFSs.

For more on fractals and imaging, see [3, 13].

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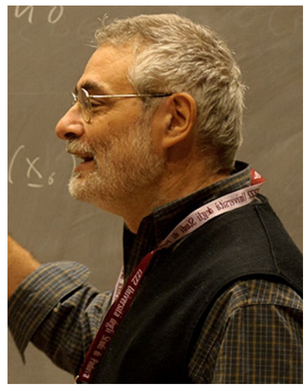

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