SeMA Journal (2023) 80:527–547
https://doi.org/10.1007/s40324-022-00316-y

Finite difference schemes for the parabolic p-Laplace equation

Félix del Teso[1] · **Erik Lindgren**[2]

Received: 26 May 2022 / Accepted: 8 October 2022 / Published online: 21 October 2022

Abstract

We propose a new finite difference scheme for the degenerate parabolic equation

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f, \quad p \geq 2.$$

Under the assumption that the data is Hölder continuous, we establish the convergence of the explicit-in-time scheme for the Cauchy problem provided a suitable stability type CFL-condition. An important advantage of our approach, is that the CFL-condition makes use of the regularity provided by the scheme to reduce the computational cost. In particular, for Lipschitz data, the CFL-condition is of the same order as for the heat equation and independent of p.

Keywords p-Laplacian · Mean value property · Viscosity solutions · Finite differences · Explicit scheme

Mathematics Subject Classification 35K10 · 35K55 · 35K92 · 35K67 · 35D40 · 35B05 · 49L20

1 Introduction

Recently, a new monotone finite difference discretization of the p-Laplacian was introduced by the authors in [9]. It is based on the mean value property presented in [4, 8]. The aim of this paper is to propose an explicit-in-time finite difference numerical scheme for the following Cauchy problem

✉ Félix del Teso
felix.delteso@uam.es
https://sites.google.com/view/felixdelteso

Erik Lindgren
eriklin@math.kth.se

1 Departamento de Matematicas, Universidad Autónoma de Madrid (UAM), Campus de Cantoblanco, 28049 Madrid, Spain

2 Department of Mathematics, KTH-Royal Institute of Technology, 100 44 Stockholm, Sweden

$$\begin{cases} \partial_t u(x,t) - \Delta_p u(x,t) = f(x), & x \in \mathbb{R}^d \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

and study its convergence. Here, $p \geq 2$ and Δ_p is the p-Laplace operator,

$$\Delta_p \psi = \mathrm{div}(|\nabla \psi|^{p-2} \nabla \psi).$$

The main result is the pointwise convergence of our scheme given Hölder continuous data (f and u_0) and a stability type CFL-condition. See Theorem 2.2 for the precise statement and (CFL) for the CFL-condition. One of the advantages of our approach is that the CFL-condition makes use of the regularity provided by the scheme. As a consequence, for Lipschitz continuous data, the CFL-condition is of the same order as the one for the heat equation. In general, the order of the CFL-condition depends on p and on the regularity of the data.

1.1 Related literature

Equation (1.1) has attracted much attention in the last decades. We refer to [11, 13] for the theory for weak solutions of this equation and to [23] for the relation between viscosity solutions and weak solutions. To the best of our knowledge, the best regularity results known are $C^{1,\alpha}$−regularity in space for some $\alpha > 0$ (see [11, Chapter IX]) and $C^{0,1/2}$−regularity in time (see [3, Theorem 2.3]).

The literature regarding finite difference schemes for parabolic problems involving the p-Laplacian is quite scarce. One reason for that is naturally that, since the p-Laplacian is in divergence form, it is very well suited for methods based on finite elements, see for instance [1, 2, 14, 19, 21] for related results.

In the stationary setting, there has been some development of finite difference methods the past 20 years. Section 1.1 in [28] provides an accurate overview of such results, we will only mention a few. In [5, 10, 18, 28], finite difference schemes for the p-Laplace equation based on the mean value formula for the *normalized* p-Laplacian (cf. [25]) are considered. Since the corresponding parabolic equation for the normalized p-Laplacian is completely different in nature (see [15, 22]), these methods do not seem very well suited to be used for the parabolic equation considered in this paper. In [9], the authors of the present paper studied a monotone finite difference discretization of the p-Laplacian based on the mean value property presented in [4, 8]. We also seize the opportunity to mention [27], where difference schemes for degenerate elliptic and parabolic equations (but not for equation (1.1)) are discussed.

It is noteworthy that, in dimension $d = 1$, the spatial derivative of a solution of (1.1) is a solution of the Porous Medium Equation (PME). See [29, 30] for a general presentation of the PME, and [20] for a proof of this fact. Finite difference schemes for the PME are well known, see [6, 7, 12, 16, 26].

2 Assumptions and main results

In this section, we introduce a general form of finite difference discretizations of Δ_p and the associated numerical scheme for (1.1). This is followed by our assumptions, the notion of solutions for (1.1) and the formulation of our main result.

2.1 Discretization and scheme

In order to treat (1.1), we consider a general discretization of Δ_p of the form

$$D_p^h\psi(x) = \sum_{y_\beta\in\mathcal{G}_h} J_p(\psi(x+y_\beta)-\psi(x))\omega_\beta, \tag{2.1}$$

where

$$J_p(\xi)=|\xi|^{p-2}\xi, \quad \xi\in\mathbb{R}, \quad \mathcal{G}_h := h\mathbb{Z}^d = \{y_\beta := h\beta \,:\, \beta\in\mathbb{Z}^d\}$$

and ω_β are certain weights $\omega_\beta=\omega_\beta(h)$ satisfying $\omega_\beta=\omega_{-\beta}\geq 0$.

We also need to introduce a time discretization. We will employ an explicit and uniform-in-time discretization. Let $N\in\mathbb{N}$ and consider a discretization parameter $\tau>0$ given by $\tau=T/N$. Consider also the sequence of times $\{t_j\}_{j=0}^N$ defined by $t_0=0$ and $t_j=t_{j-1}+\tau=j\tau$. The time grid, $\mathcal{T}_\tau$, is given by

$$\mathcal{T}_\tau = \bigcup_{j=0}^{N}\{t_j\}.$$

Then, our general form of an explicit finite difference scheme of (1.1) is given by

$$\begin{cases} U_\alpha^j = U_\alpha^{j-1} + \tau\left(D_p^h U_\alpha^{j-1} + f_\alpha\right), & \alpha\in\mathbb{Z}^d,\ j=1,\ldots,N,\\ U_\alpha^0 = (u_0)_\alpha & \alpha\in\mathbb{Z}^d, \end{cases} \tag{2.2}$$

where $f_\alpha := f(x_\alpha)$, $(u_0)_\alpha = u_0(x_\alpha)$ and D_p^h is given by (2.1).

2.2 Assumptions

In order to ensure convergence of the scheme (2.2), we impose the following hypotheses on the data and the discretization parameters. This entails a regularity assumption on the data, some assumptions on the discretization and a nonlinear CFL-condition on the parameters, as is customary for explicit schemes.

Hypothesis on the data. We assume that

$$u_0, f:\mathbb{R}^d\to\mathbb{R} \text{ are bounded and globally Hölder continuous functions for some } a\in(0,1]. \tag{$\mathrm{A}_{u_0,f}$}$$

More precisely,

$$|u_0(x)-u_0(y)|\leq L_{u_0}|x-y|^a \quad\text{and}\quad |f(x)-f(y)|\leq L_f|x-y|^a, \qquad \text{for all}\quad x,y\in\mathbb{R}^d,$$

for some constants $L_{u_0}, L_f\geq 0$. Sometimes we will write $\Lambda_{u_0}(\delta):=L_{u_0}\delta^a$ and $\Lambda_f(\delta):=L_f\delta^a$ to simplify the presentation.

Hypothesis on the spatial discretization. For the discretization, we assume the following type of monotonicity and boundedness:

$$\omega_\beta=\omega_{-\beta}\geq 0,\ w_\beta=0 \text{ for } y_\beta\notin B_r \text{ for some } r>0, \text{ and } \sum_{y_\beta\in\mathcal{G}_h}\omega_\beta\leq Mr^{-p} \tag{A_ω}$$

Here $M = M(p,d) > 0$. In addition, we assume the following consistency for the discretization:

$$\text{For } \psi \in C_b^2(\mathbb{R}^d \times [0,T]), \text{ we have that } D_p^h \psi = \Delta_p \psi + o_h(1) \text{ as } h \to 0^+ \text{ uniformly in } (x,t). \tag{A$_c$}$$

Examples of discretizations satisfying these properties can be found in Sect. 5.

Hypothesis on the discretization parameters. We assume the following stability condition on the numerical parameters:

$$h = o_r(1) \quad \text{and} \quad \tau \leq C r^{2+(1-a)(p-2)} \tag{CFL}$$

with

$$C = \min\left\{1, \frac{1}{M(p-1)\left(L_{u_0} + TL_f + 3\tilde{K} + 1\right)^{p-2}}\right\}$$

and $\tilde{K}$ a constant given in (3.4), depending on p, the modulus of continuity in time of the discretized solution and some universal constants coming from a mollifier.

Remark 2.1 For Lipschitz data u_0 and f, the condition (CFL) reads $\tau \leq Cr^2$ for a certain constant $C = C(u_0, f, d, p, T) > 0$. We note that, regardless of the constant C, the relation between τ and r is always quadratic (as in the linear case $p = 2$) and independent of p. It is important to mention that this is computationally very relevant, especially if we want to deal with problems related to large p.

2.3 Main result

We now state our main result regarding the convergence of the scheme. Several other properties of the scheme are also obtained, but we will state them later.

Theorem 2.2 *Let $p \in [2, \infty)$ and assume (A$_{u_0,f}$) and (A$_\omega$). Then for every $h, \tau > 0$, there exists a unique solution $U \in \ell^\infty(\mathcal{G}_h \times \mathcal{T}_\tau)$ of (2.2). If the hypotheses (CFL) and (A$_c$) additionally hold, then*

$$\max_{(x_\alpha, t_j) \in \mathcal{G}_h \times \mathcal{T}_\tau} |U_\alpha^j - u(x_\alpha, t_j)| \to 0 \quad \text{as} \quad h \to 0^+,$$

where u is the unique viscosity solution of (1.1).

2.4 Viscosity solutions

Throughout the paper, we will use the notion of viscosity solutions. For completeness, we define the concept of viscosity solutions of (1.1), adopting the definition in [23].

Definition 2.1 Assume (A$_{u_0,f}$). We say that a bounded lower (resp. upper) semicontinuous function u in $\mathbb{R}^d \times [0,T]$ is a viscosity supersolution (resp. subsolution) of (1.1) if

(a) $u(x,0) \geq u_0(x)$ (resp. $u(x,0) \leq u_0(x)$);

(b) whenever $(x_0, t_0) \in \mathbb{R}^d \times (0, T)$ and $\varphi \in C_b^2(B_R(x_0) \times (t_0 - R, t_0 + R))$ for some $R > 0$ are such that $\varphi(x_0, t_0) = u(x_0, t_0)$ and $\varphi(x, t) < u(x, t)$ (resp. $\varphi(x, t) > u(x, t)$) for $(x, t) \in B_R(x_0) \times (t_0 - R, t_0)$, then we have

$$\varphi_t(x_0, t_0) - \Delta_p \varphi(x_0, t_0) \geq f(x_0) \quad (\text{resp.} \quad \varphi_t(x_0, t_0) - \Delta_p \varphi(x_0, t_0) \leq f(x_0)).$$

A viscosity solution of (1.1) is a bounded continuous function u being both a viscosity supersolution and a viscosity subsolution (1.1).

Remark 2.3 We remark that it is not necessary to require strict inequality in the definition above. It is enough to require $\varphi(x, t) \leq u(x, t)$ (resp. $\varphi(x, t) \geq u(x, t)$) for $(x, t) \in B_R(x_0) \times (t_0 - R, t_0)$.

We also state a necessary uniqueness result that will ensure convergence of the scheme. Without such a result, we would only be able to establish convergence up to a subsequence. The theorem below is a consequence of the fact that viscosity solutions are weak solutions (see Corollary 4.7 in [23]) and that bounded weak solutions are unique (see Theorem 6.1 in [11]).

Theorem 2.4 *Assume* ($\mathrm{A}_{u_0,f}$)*. Then there is a unique solution of* (1.1).

3 Properties of the numerical scheme

In this section we will study properties of the numerical scheme (2.2). More precisely, we establish existence and uniqueness for the numerical solution, stability in maximum norm, as well as conservation of the modulus of continuity of the data.

3.1 Existence and uniqueness

We have the following existence and uniqueness result for the numerical scheme.

Proposition 3.1 *Assume* ($\mathrm{A}_{u_0,f}$)*,* (A_ω)*, $p \geq 2$ and $r, h, \tau > 0$. Then there exists a unique solution $U \in \ell^\infty(\mathcal{G}_h \times \mathcal{T}_\tau)$ of the scheme* (2.2).

Proof First we note that, for a function $\psi \in \ell^\infty(\mathcal{G}_h)$, we have that

$$|D_p^h \psi_\alpha| \leq \sum_{y_\beta \in \mathcal{G}_h} J_p(\psi(x_\alpha + y_\beta) - \psi(x_\alpha))\omega_\beta \leq (2\|\psi\|_{\ell^\infty(\mathcal{G}_h)})^{p-1} \sum_{y_\beta \in \mathcal{G}_h} \omega_\beta < +\infty.$$

Then, for each $\alpha \in \mathbb{Z}^d$, U_α^j is defined recursively using the values of U_β^{j-1} for $\beta \in \mathbb{Z}^d$, and we have that

$$\sup_{y_\alpha \in \mathcal{G}_h} |U_\alpha^j| = \sup_{y_\alpha \in \mathcal{G}_h} |U_\alpha^{j-1}| + \tau \left(\left(2 \sup_{y_\alpha \in \mathcal{G}_h} |U_\alpha^{j-1}| \right)^{p-1} \sum_{y_\beta \in \mathcal{G}_h} \omega_\beta + \sup_{y_\alpha \in \mathcal{G}_h} |f_\alpha| \right).$$

The conclusion follows since

$$\sup_{y_\alpha \in \mathcal{G}_h} |f_\alpha| \leq \|f\|_{L^\infty(\mathbb{R}^d)} \quad \text{and} \quad \sup_{y_\alpha \in \mathcal{G}_h} |U_\alpha^0| = \sup_{y_\alpha \in \mathcal{G}_h} |u_0(y_\alpha)| \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}.$$

□

3.2 Stability and preservation of the modulus of continuity in space

First we will prove that the scheme preserves the regularity of the data.

Proposition 3.2 *Assume* ($A_{u_0,f}$)*,* (A_ω)*,* $p \geq 2$*,* $r, h, \tau > 0$ *and* (CFL)*. Let* U *be the solution of* (2.2)*. For every* $j = 0, \dots, N$*, we have*

$$|U_\alpha^j - U_\gamma^j| \leq \Lambda_{u_0}(|x_\alpha - x_\gamma|) + t_j \Lambda_f(|x_\alpha - x_\gamma|), \quad \text{for all} \quad x_\alpha, x_\gamma \in \mathcal{G}_h.$$

Proof By assumption ($A_{u_0,f}$), for any given $x_\alpha, x_\gamma \in \mathcal{G}_h$, we have that

$$|U_\alpha^0 - U_\gamma^0| = |u_0(x_\alpha) - u_0(x_\gamma)| \leq \Lambda_{u_0}(|x_\alpha - x_\gamma|).$$

Assume by induction that

$$|U_\alpha^j - U_\gamma^j| \leq \Lambda_{u_0}(|x_\alpha - x_\gamma|) + t_j \Lambda_f(|x_\alpha - x_\gamma|).$$

Using the scheme at x_α and x_γ we get

$$U_\alpha^{j+1} - U_\gamma^{j+1} = U_\alpha^j - U_\gamma^j + \tau \sum_{y_\beta \in \mathcal{G}_h} \Big(J_p(U_{\alpha+\beta}^j - U_\alpha^j) - J_p(U_{\gamma+\beta}^j - U_\gamma^j) \Big) \omega_\beta + \tau(f_\alpha - f_\gamma).$$

Now, since $p \geq 2$, we have, by Taylor expansion, that

$$J_p(U_{\alpha+\beta}^j - U_\alpha^j) - J_p(U_{\gamma+\beta}^j - U_\gamma^j) = (p-1)|\eta_\beta|^{p-2} \Big((U_{\alpha+\beta}^j - U_{\gamma+\beta}^j) - (U_\alpha^j - U_\gamma^j) \Big),$$

for some $\eta_\beta \in \mathbb{R}$ between $(U_{\alpha+\beta}^j - U_\alpha^j)$ and $(U_{\gamma+\beta}^j - U_\gamma^j)$. Thus,

$$\begin{aligned} U_\alpha^{j+1} - U_\gamma^{j+1} =& (U_\alpha^j - U_\gamma^j) \left(1 - \tau(p-1) \sum_{y_\beta \in \mathcal{G}_h} |\eta_\beta|^{p-2} \omega_\beta \right) \\ &+ \tau(p-1) \sum_{y_\beta \in \mathcal{G}_h} |\eta_\beta|^{p-2} (U_{\alpha+\beta}^j - U_{\gamma+\beta}^j) \omega_\beta + \tau(f_\alpha - f_\gamma). \end{aligned} \tag{3.1}$$

Now observe that, by the induction assumption, we have

$$\begin{aligned} |\eta_\beta| &\leq \sup_{y_\alpha \in \mathcal{G}_h} \{|U_{\alpha+\beta}^j - U_\alpha^j|\} \leq \sup_{y_\alpha \in \mathcal{G}_h} \{\Lambda_{u_0}(|x_{\alpha+\beta} - x_\alpha|) + t_j \Lambda_f(|x_{\alpha+\beta} - x_\alpha|)\} \\ &= \Lambda_{u_0}(|x_\beta|) + t_j \Lambda_f(|x_\beta|). \end{aligned}$$

By (A_ω), we have $w_\beta = 0$ for $y_\beta \notin B_r$ for some $r > 0$, and we deduce that

$$\sum_{y_\beta \in \mathcal{G}_h} |\eta_\beta|^{p-2} \omega_\beta \leq \big(\Lambda_{u_0}(r) + t_j \Lambda_f(r) \big)^{p-2} \sum_{y_\beta \in \mathcal{G}_h} \omega_\beta \leq \frac{(L_{u_0} + t_j L_f)^{p-2} M}{r^{2+(1-a)(p-2)}}.$$

Thus, by (CFL), we get

$$\tau(p-1) \sum_{y_\beta \in \mathcal{G}_h} |\eta_\beta|^{p-2} \omega_\beta \leq 1.$$

Using the above estimate and the induction hypothesis in (3.1), we get that

$$\begin{aligned}
|U_\alpha^{j+1} - U_\gamma^{j+1}| \leq & |U_\alpha^j - U_\gamma^j| \left(1 - \tau(p-1) \sum_{y_\beta \in \mathcal{G}_h} |\eta_\beta|^{p-2}\omega_\beta\right) \\
& + \tau(p-1) \sum_{y_\beta \in \mathcal{G}_h} |\eta_\beta|^{p-2} |U_{\alpha+\beta}^j - U_{\gamma+\beta}^j|\omega_\beta + \tau|f_\alpha - f_\gamma| \\
\leq & \left(\Lambda_{u_0}(|x_\alpha - x_\gamma|) + t_j\Lambda_f(|x_\alpha - x_\gamma|)\right)\left(1 - \tau(p-1) \sum_{y_\beta \in \mathcal{G}_h} |\eta_\beta|^{p-2}\omega_\beta\right) \\
& + \tau(p-1) \sum_{y_\beta \in \mathcal{G}_h} |\eta_\beta|^{p-2}\Big(\Lambda_{u_0}(|x_{\alpha+\beta} - x_{\gamma+\beta}|) \\
& + t_j\Lambda_f(|x_{\alpha+\beta} - x_{\gamma+\beta}|)\Big)\omega_\beta + \tau\Lambda_f(|x_\alpha - x_\gamma|) \\
\leq & \left(\Lambda_{u_0}(|x_\alpha - x_\gamma|) + t_j\Lambda_f(|x_\alpha - x_\gamma|)\right)\left(1 - \tau(p-1) \sum_{y_\beta \in \mathcal{G}_h} |\eta_\beta|^{p-2}\omega_\beta\right) \\
& + \tau(p-1)\left(\Lambda_{u_0}(|x_\alpha - x_\gamma|) + t_j\Lambda_f(|x_\alpha - x_\gamma|)\right) \sum_{y_\beta \in \mathcal{G}_h} |\eta_\beta|^{p-2}\omega_\beta \\
& + \tau\Lambda_f(|x_\alpha - x_\gamma|) \\
= & \Lambda_{u_0}(|x_\alpha - x_\gamma|) + (t_j + \tau)\Lambda_f(|x_\alpha - x_\gamma|),
\end{aligned}$$

which concludes the proof. □

Remark 3.3 In particular, if both u_0 and f are Lipschitz functions with constants L_{u_0} and L_f respectively, the above result reads,

$$|U_\alpha^j - U_\gamma^j| \leq (L_{u_0} + t_j L_f)|x_\alpha - x_\gamma|.$$

We are now ready to state and prove the stability result: solutions with bounded data remain bounded (uniformly in the discretization parameters) for all times.

Proposition 3.4 *Under the assumptions of Proposition 3.2, we have that*

$$\sup_{y_\alpha \in \mathcal{G}_h} |U_\alpha^j| \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} + t_j\|f\|_{L^\infty(\mathbb{R}^d)}, \quad \text{for all} \quad j = 0, \ldots, N.$$

Proof By assumption ($\mathrm{A}_{u_0,f}$), we have that

$$\sup_{y_\alpha \in \mathcal{G}_h} |U_\alpha^0| \leq \sup_{y_\alpha \in \mathcal{G}_h} |u_0(x_\alpha)| \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}.$$

Assume by induction that

$$\sup_{y_\alpha \in \mathcal{G}_h} |U_\alpha^j| \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} + t_j\|f\|_{L^\infty(\mathbb{R}^d)}.$$

Direct computations lead to

$$\begin{aligned}
U_\alpha^{j+1} &= U_\alpha^j + \tau \sum_{y_\beta \in \mathcal{G}_h} |U_{\alpha+\beta}^j - U_\alpha^j|^{p-2}(U_{\alpha+\beta}^j - U_\alpha^j)\omega_\beta + \tau f_\alpha \\
&= U_\alpha^j \left(1 - \tau \sum_{y_\beta \in \mathcal{G}_h} |U_{\alpha+\beta}^j - U_\alpha^j|^{p-2}\omega_\beta\right) + \tau \sum_{y_\beta \in \mathcal{G}_h} |U_{\alpha+\beta}^j - U_\alpha^j|^{p-2} U_{\alpha+\beta}^j \omega_\beta + \tau f_\alpha.
\end{aligned}$$

By Proposition 3.2 we have that

$$|U^j_{\alpha+\beta} - U^j_\alpha|^{p-2} \leq (\Lambda_{u_0}(|y_\beta|) + t_j \Lambda_f(|y_\beta|))^{p-2},$$

which together with assumptions (A_ω) and (CFL) imply that

$$\tau \sum_{y_\beta \in \mathcal{G}_h} |U^j_{\alpha+\beta} - U^j_\alpha|^{p-2}\omega_\beta \leq \tau(\Lambda_{u_0}(r) + t_j\Lambda_f(r))^{p-2} \sum_{y_\beta \in \mathcal{G}_h} \omega_\beta \leq \frac{1}{p-1} \leq 1.$$

Direct computations plus the induction hypothesis allow us to conclude that

$$\begin{aligned}
|U^{j+1}_\alpha| \leq& \sup_{y_\alpha \in \mathcal{G}_h} |U^j_\alpha| \left(1 - \tau \sum_{y_\beta \in \mathcal{G}_h} |U^j_{\alpha+\beta} - U^j_\alpha|^{p-2}\omega_\beta\right)\\
&+ \tau \sup_{y_\alpha \in \mathcal{G}_h} |U^j_\alpha| \sum_{y_\beta \in \mathcal{G}_h} |U^j_{\alpha+\beta} - U^j_\alpha|^{p-2}\omega_\beta + \tau\|f\|_{L^\infty(\mathbb{R}^d)}\\
=& \sup_{y_\alpha \in \mathcal{G}_h} |U^j_\alpha| + \tau\|f\|_{L^\infty(\mathbb{R}^d)}\\
=& \|u_0\|_{L^\infty(\mathbb{R}^d)} + (t_j + \tau)\|f\|_{L^\infty(\mathbb{R}^d)},
\end{aligned}$$

which concludes the proof. □

3.3 Time equicontinuity for a discrete in time scheme

Now we extend the scheme from $\mathcal{G}_h$ to $\mathbb{R}^d$ by considering $U : \mathbb{R}^d \times \mathcal{T}_\tau$ defined by

$$\begin{cases} U^j(x) = U^{j-1}(x) + \tau\left(D^h_p U^{j-1}(x) + f(x)\right), & x \in \mathbb{R}^d,\ j = 1, \dots, N,\\ U^0(x) = u_0(x) & x \in \mathbb{R}^d. \end{cases} \tag{3.2}$$

Remark 3.5 Clearly, if we restrict the solution of (3.2) to $\mathcal{G}_h$, we recover the solution of (2.2).

Proposition 3.6 (Continuous dependence on the data) *Assume ($A_{u_0,f}$), (A_ω), $p \geq 2$, $r, h, \tau > 0$ and (CFL). Let $U, \widetilde{U}$ be the solutions of* (2.2) *corresponding to $u_0, \widetilde{u}_0$ and $f, \widetilde{f}$. For every $j = 0, \dots, N$, we have*

$$\|U^j - \widetilde{U}^j\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0 - \widetilde{u}_0\|_{L^\infty(\mathbb{R}^d)} + t_j\|f - \widetilde{f}\|_{L^\infty(\mathbb{R}^d)}.$$

Proof By assumption ($A_{u_0,f}$), we have that

$$\|U^0 - \widetilde{U}^0\|_{L^\infty(\mathbb{R}^d)} = \|u_0 - \widetilde{u}_0\|_{L^\infty(\mathbb{R}^d)}.$$

Assume by induction that

$$\|U^j - \widetilde{U}^j\|_{L^\infty(\mathbb{R}^d)} = \|u_0 - \widetilde{u}_0\|_{L^\infty(\mathbb{R}^d)} + t_j\|f - \widetilde{f}\|_{L^\infty(\mathbb{R}^d)}.$$

Similar computations as the ones in the proof of Proposition 3.2 yield

$$\begin{aligned}
U^{j+1}(x) - \widetilde{U}^{j+1}(x) =& (U^j(x) - \widetilde{U}^j(x))\left(1 - \tau(p-1)\sum_{y_\beta \in \mathcal{G}_h} |\eta_\beta|^{p-2}\omega_\beta\right) + \tau(p-1)\\
&\times \sum_{y_\beta \in \mathcal{G}_h} |\eta_\beta|^{p-2}(U^j(x+y_\beta) - \widetilde{U}^j(x+y_\beta))\omega_\beta + \tau(f(x) - \widetilde{f}(x)),
\end{aligned} \tag{3.3}$$

where $\eta_\beta \in \mathbb{R}$ is some number between $(U^j(x+y_\beta) - U^j(x))$ and $(\widetilde{U}^j(x+y_\beta) - \widetilde{U}^j(x))$. From here, the proof follows as in the proof of Proposition 3.2. □

Proposition 3.7 (Equicontinuity in time) *Assume* ($A_{u_0,f}$), (A_ω), $p \geq 2$, $r, h, \tau > 0$ *and* (CFL). *Let* U *be the solution of* (3.2). *Then*

$$\|U^{j+k} - U^j\|_{L^\infty(\mathbb{R}^d)} \leq \widetilde{K}(t_k)^{\frac{a}{2+(1-a)(p-2)}} + \|f\|_{L^\infty(\mathbb{R}^d)} t_k =: \overline{\Lambda}_{u_0,f}(t_k),$$

with

$$\tilde{K} = 4^{\frac{1+(1-a)(p-1)}{2+(1-a)(p-2)}} L_{u_0}^{\frac{p}{2+(1-a)(p-2)}} ((p-1)K_1^{p-2} K_2 M)^{\frac{a}{2+(1-a)(p-2)}}, \tag{3.4}$$

where M *comes from assumption* (A_ω), *and* K_1 *and* K_2 *are constants given in Sect.* 1 (*depending on a certain choice of mollifiers*).

Proof Consider a mollification of the initial data $u_{0,\delta} = u_0 * \rho_\delta$ where $\rho_\delta(x)$ is a standard mollifier (as defined in Appendix A). Let $(U_\delta)^j$ be the corresponding solution of (3.2) with $u_{0,\delta}$ as initial data. Then,

$$\|(U_\delta)^1 - (U_\delta)^0\|_{L^\infty(\mathbb{R}^d)} \leq \tau \|D_p^h u_{0,\delta}\|_{L^\infty(\mathbb{R}^d)} + \tau\|f\|_{L^\infty(\mathbb{R}^d)}.$$

Define $\widetilde{U}_\delta^j := U_\delta^{j+1}$ for all $j = 0, \ldots, N$. Clearly, $\widetilde{U}_\delta^j$ is the unique solution of (3.2) with initial data $\widetilde{U}_\delta^0 = U_\delta^1$ and right hand side f. By Proposition 3.6

$$\begin{aligned}\|U_\delta^{j+1} - U_\delta^j\|_{L^\infty(\mathbb{R}^d)} &= \|\widetilde{U}_\delta^j - U_\delta^j\|_{L^\infty(\mathbb{R}^d)} \leq \|\widetilde{U}_\delta^0 - U_\delta^0\|_{L^\infty(\mathbb{R}^d)} = \|U_\delta^1 - U_\delta^0\|_{L^\infty(\mathbb{R}^d)} \\ &\leq \tau\|D_p^h u_{0,\delta}\|_{L^\infty(\mathbb{R}^d)} + \tau\|f\|_{L^\infty(\mathbb{R}^d)}.\end{aligned}$$

A repeated use of the triangle inequality yields

$$\begin{aligned}\|U_\delta^{j+k} - U_\delta^j\|_{L^\infty(\mathbb{R}^d)} &\leq \sum_{i=0}^{k-1} \|U_\delta^{j+i+1} - U_\delta^{j+i}\|_{L^\infty(\mathbb{R}^d)} \\ &\leq (k\tau)\|D_p^h u_{0,\delta}\|_{L^\infty(\mathbb{R}^d)} + (k\tau)\|f\|_{L^\infty(\mathbb{R}^d)}.\end{aligned} \tag{3.5}$$

The symmetry of the weights ω_β together with Lemma B.1 implies

$$\begin{aligned}|D_p^h u_{0,\delta}(x)| &= \frac{1}{2}\left|\sum_{y_\beta \in \mathcal{G}_h} \big(J_p(u_{0,\delta}(x+y_\beta) - u_{0,\delta}(x)) - J_p(u_{0,\delta}(x) - u_{0,\delta}(x-y_\beta))\big)\,\omega_\beta\right| \\ &\leq \frac{p-1}{2} \sum_{y_\beta \in \mathcal{G}_h} \max\{|u_{0,\delta}(x+y_\beta) - u_{0,\delta}(x)|, |u_{0,\delta}(x) - u_{0,\delta}(x-y_\beta)|\}^{p-2} \\ &\quad \times \left|u_{0,\delta}(x+y_\beta) + u_{0,\delta}(x-y_\beta) - 2u_{0,\delta}(x)\right| \omega_\beta.\end{aligned} \tag{3.6}$$

Now note that, by the a-Hölder regularity of u_0 given by assumption ($A_{u_0,f}$), Lemma A.1 and Lemma A.2 imply

$$\begin{aligned}&|u_{0,\delta}(x \pm y_\beta) - u_{0,\delta}(x)| \leq K_1 L_{u_0} \delta^{a-1}|y_\beta|, \quad |u_{0,\delta}(x+y_\beta) + u_{0,\delta}(x-y_\beta) - 2u_{0,\delta}(x)| \\ &\quad \leq K_2 L_{u_0} \delta^{a-2}|y_\beta|^2,\end{aligned} \tag{3.7}$$

where K_1 and K_2 depend only on the mollifier ρ. Now note that, by (A_ω), we have

$$\sum_{y_\beta \in \mathcal{G}_h} |y_\beta|^p \omega_\beta \leq M. \tag{3.8}$$

Combining (3.5) and (3.8), we obtain

$$\|U_\delta^{j+k} - U_\delta^j\|_{L^\infty(\mathbb{R}^d)} \leq \frac{p-1}{2} t_k (K_1 L_{u_0}\delta^{a-1})^{p-2} K_2 L_{u_0}\delta^{a-2} \sum_{y_\beta\in\mathcal{G}_h} |y_\beta|^p \omega_\beta + t_k \|f\|_{L^\infty(\mathbb{R}^d)}$$
$$\leq \widehat{K}\delta^{(a-1)(p-2)+(a-2)} t_k + \|f\|_{L^\infty(\mathbb{R}^d)} t_k,$$

with $\widehat{K} = \frac{p-1}{2} K_1^{p-2} K_2 L_{u_0}^{p-1} M$. Using the triangle inequality, the above estimate and applying Proposition 3.6 several times we obtain

$$\|U^{j+k} - U^j\|_{L^\infty(\mathbb{R}^d)} \leq \|U^{j+k} - U_\delta^{j+k}\|_{L^\infty(\mathbb{R}^d)} + \|U_\delta^{j+k} - U_\delta^j\|_{L^\infty(\mathbb{R}^d)} + \|U^j - U_\delta^j\|_{L^\infty(\mathbb{R}^d)}$$
$$\leq 2\|u_0 - u_{0,\delta}\|_{L^\infty(\mathbb{R}^d)} + \widehat{K}\delta^{(a-1)(p-2)+(a-2)} t_k + \|f\|_{L^\infty(\mathbb{R}^d)} t_k$$
$$\leq 2L_{u_0}\delta^a + \widehat{K}\delta^{(a-1)(p-2)+(a-2)} t_k + \|f\|_{L^\infty(\mathbb{R}^d)} t_k.$$

By choosing $\delta = (\frac{\widehat{K}}{2L_{u_0}} t_k)^{\frac{1}{2+(1-a)(p-2)}}$ in the above estimate, we get the desired result

$$\|U^{j+k} - U^j\|_{L^\infty(\mathbb{R}^d)} \leq \tilde{K}(t_k)^{\frac{a}{2+(1-a)(p-2)}} + \|f\|_{L^\infty(\mathbb{R}^d)} t_k,$$

with

$$\tilde{K} = 4L_{u_0}\left(\frac{\widehat{K}}{2L_{u_0}}\right)^{\frac{a}{2+(1-a)(p-2)}}$$
$$= 4^{1-\frac{a}{2+(1-a)(p-2)}} L_{u_0}((p-1)K_1^{p-2} K_2 L_{u_0}^{p-2} M)^{\frac{a}{2+(1-a)(p-2)}}$$
$$= 4^{\frac{2+(1-a)(p-2)-a}{2+(1-a)(p-2)}} L_{u_0}^{\frac{p}{2+(1-a)(p-2)}} ((p-1)K_1^{p-2} K_2 M)^{\frac{a}{2+(1-a)(p-2)}}.$$

□

Remark 3.8 Actually, a close inspection of the previous proof reveals that for $u_0 \in C_b^2(\mathbb{R}^d)$ we can get

$$\|U^{j+k} - U^j\|_{L^\infty(\mathbb{R}^d)} \lesssim t_k.$$

3.4 Equiboundedness and equicontinuity estimates for a scheme in $\mathbb{R}^d \times [0, T]$

We now need to extend the numerical scheme in time in a continuous way. This is done by continuous interpolation, i.e.,

$$U(x,t) := \frac{t_{j+1}-t}{\tau} U^j(x) + \frac{t-t_j}{\tau} U^{j+1}(x) \quad \text{if} \quad t \in [t_j, t_{j+1}] \quad \text{for some} \quad j = 0, \ldots, N, \tag{3.9}$$

where U^j is the solution of (3.2).

Remark 3.9 It is standard to check that, for all $t \in [t_j, t_{j+1}]$, we have that the original scheme is preserved also outside the grid points, i.e.,

$$U(x,t) = U(x,t_j) + (t-t_j) D_p^h U(x,t_j) + (t-t_j) f(x). \tag{3.10}$$

We have the following result.

Proposition 3.10 (Stability and equicontinuity) *Assume* ($\mathrm{A}_{u_0,f}$), (A_ω), $p \geq 2$, $r, h, \tau > 0$ *and* (CFL). *Let* U *be the solution of* (3.9). *Then, we have:*

(a) (Equiboundedness) $\|U\|_{L^\infty(\mathbb{R}^N\times[0,T])} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} + T\|f\|_{L^\infty(\mathbb{R}^d)}$,
(b) (Equicontinuity) *For any $x, z \in \mathbb{R}^d$ and $t, \tilde{t} \in [0, T]$ we have that*

$$|U(x,t) - U(z,\tilde{t})| \leq \Lambda_{u_0}(|x-z|) + T\Lambda_f(|x-z|) + 3\overline{\Lambda}_{u_0,f}(|\tilde{t}-t|).$$

Proof Equiboundedness follows easily from a continuous in space version of Proposition 3.4, since

$$\begin{aligned}|U(x,t)| &\leq \frac{t_{j+1}-t}{\tau}\sup_{x\in\mathbb{R}^d}|U^j(x)| + \frac{t-t_j}{\tau}\sup_{x\in\mathbb{R}^d}|U^{j+1}(x)|\\ &\leq \frac{t_{j+1}-t}{\tau}\left(\|u_0\|_{L^\infty(\mathbb{R}^d)} + T\|f\|_{L^\infty(\mathbb{R}^d)}\right) + \frac{t-t_j}{\tau}\left(\|u_0\|_{L^\infty(\mathbb{R}^d)} + T\|f\|_{L^\infty(\mathbb{R}^d)}\right)\\ &\leq \|u_0\|_{L^\infty(\mathbb{R}^d)} + T\|f\|_{L^\infty(\mathbb{R}^d)}.\end{aligned}$$

Equicontinuity in space follows from the translation invariance of the scheme and Proposition 3.6:

$$|U(x+y,t) - U(x,t)| \leq \|u_0(\cdot+y) - u_0\|_{L^\infty(\mathbb{R}^d)} + T\|f(\cdot+y) - f\|_{L^\infty(\mathbb{R}^d)}.$$

To prove equicontinuity in time, we first consider $t, \tilde{t} \in [t_j, t_{j+1}]$ for some $j = 0, \ldots, N-1$. In this case we have

$$\begin{aligned}U(x,t) - U(x,\tilde{t}) &= \left(\frac{t_{j+1}-t}{\tau}U^j(x) + \frac{t-t_j}{\tau}U^{j+1}(x)\right)\\ &\quad - \left(\frac{t_{j+1}-\tilde{t}}{\tau}U^j(x) + \frac{\tilde{t}-t_j}{\tau}U^{j+1}(x)\right)\\ &= \frac{t-\tilde{t}}{\tau}\left(U^{j+1}(x) - U^j(x)\right).\end{aligned}$$

Then, from Proposition 3.7, we get

$$|U(x,t) - U(x,\tilde{t})| \leq |t-\tilde{t}|\frac{\overline{\Lambda}_{u_0,f}(\tau)}{\tau}$$

Note that the function $g(\tau) = \frac{\overline{\Lambda}_{u_0,f}(\tau)}{\tau}$ is decreasing. Thus, since $|t-\tilde{t}| \leq \tau$, we have $g(\tau) \leq g(|t-\tilde{t}|)$. It follows that

$$|U(x,t) - U(x,\tilde{t})| \leq \overline{\Lambda}_{u_0,f}(|t-\tilde{t}|).$$

Now consider $t \in [t_j, t_{j+1})$ and $\tilde{t} \in [t_{j+k}, t_{j+k+1})$ for $k \geq 1$. By the triangle inequality, the previous step and Proposition 3.7

$$\begin{aligned}&|U(x,t) - U(x,\tilde{t})|\\ &\quad\leq |U(x,t) - U(x,t_{j+1})| + |U(x,t_{j+k}) - U(x,\tilde{t})| + |U(x,t_{j+1}) - U(x,t_{j+k})|\\ &\quad\leq \overline{\Lambda}_{u_0,f}(|t_{j+1}-t|) + \overline{\Lambda}_{u_0,f}(|\tilde{t}-t_{j+k}|) + \overline{\Lambda}_{u_0,f}(|t_{j+k}-t_{j+1}|).\end{aligned}$$

Since $t \leq t_{j+1} \leq \tilde{t}$ and $t \leq t_{j+k} \leq \tilde{t}$, the above estimate yields

$$|U(x,t) - U(x,\tilde{t})| \leq 3\overline{\Lambda}_{u_0,f}(|\tilde{t}-t|).$$

Finally, we conclude space-time equicontinuity combining the above estimates to get

$$\begin{aligned}|U(x,t) - U(z,\tilde{t})| &\leq |U(x,t) - U(z,t)| + |U(z,t) - U(z,\tilde{t})|\\ &\leq \Lambda_{u_0}(|x-z|) + T\Lambda_f(|x-z|) + 3\overline{\Lambda}_{u_0,f}(|\tilde{t}-t|).\end{aligned}$$

□

By Arzelà-Ascoli, we obtain as a corollary that, up to a subsequence, the numerical solution converges locally uniformly to a limit.

Corollary 3.11 *Assume the hypotheses of Proposition 3.10. Let $\{U_h\}_{h>0}$ be a sequence of solutions of (3.9). Then, there exist a subsequence $\{U_{h_l}\}_{l=1}^{\infty}$ and a function $u \in C_b(\mathbb{R}^d \times [0,T])$ such that*

$$U_{h_l} \to u \quad \text{as} \quad l \to \infty \quad \text{locally uniformly in } \mathbb{R}^N \times [0,T].$$

4 Convergence of the numerical scheme

From Corollary 3.11, we have that the sequence of numerical solutions has a subsequence converging locally uniformly to some function v. We will now show that v is a viscosity solution of (1.1).

Theorem 4.1 *Let the assumptions of Corollary 3.11 hold. Then v is a viscosity solution of (1.1).*

Proof For notational simplicity, we avoid the subindex j and consider

$$U_h \to u \quad \text{as} \quad h \to 0 \quad \text{locally uniformly in } \mathbb{R}^N \times [0,T].$$

First of all, by the local uniform convergence,

$$u(x,0) = \lim_{h\to 0} U_h(x,0) = u_0(x),$$

locally uniformly. We will now show that u is a viscosity supersolution. The proof that u is a viscosity subsolution is similar.

Now let φ be a suitable test function for u at $(x^*, t^*) \in \mathbb{R}^d \times (0,T)$. We may assume that φ satisfies

(i) $\varphi(x^*,t^*) = u(x^*,t^*)$,
(ii) $u(x,t) > \varphi(x,t)$ for all $(x,t) \in B_R(x^*) \times (t^*-R, t^*] \setminus (x^*,t^*)$.

The local uniform convergence ensures (see Section 10.1.1 in [17]) that there exists a sequence $\{(x^h,t^h)\}_{h>0}$ such that

(i) $\varphi(x^h,t^h) - U_h(x^h,t^h) = \sup_{(x,t)\in B_R(x^h)\times(t^h-R,t^h]}\{\varphi(x,t) - U_h(x,t)\} =: M_h$,
(ii) $\varphi(x^h,t^h) - U_h(x^h,t^h) \geq \varphi(x,t) - U_h(x,t)$ for all $(x,t) \in B_R(x^h) \times (t^h - R, t^h] \setminus (x^h,t^h)$

and

$$(x^h,t^h) \to (x^*,t^*) \quad \text{as} \quad h \to 0.$$

Now consider $t_j \in \mathcal{T}_\tau$ such that $t^h \in [t_j, t_{j+1}]$ (note that the index j might depend on h, but this fact plays no role in the proof). By Remark 3.9,

$$U_h(x^h,t^h) = U_h(x^h,t_j) + (t^h - t_j) \sum_{y_\beta \in \mathcal{G}_h} J_p(U_h(x^h + y_\beta, t_j) - U_h(x^h,t_j))\omega_\beta$$
$$+ (t^h - t_j) f(x^h).$$

Define $\widetilde{U}_h = U_h + M_h$. It is clear that

$$\widetilde{U}_h(x^h, t^h) = \widetilde{U}_h(x^h, t_j) + (t^h - t_j) \sum_{y_\beta \in \mathcal{G}_h} J_p(\widetilde{U}_h(x^h + y_\beta, t_j) - \widetilde{U}_h(x^h, t_j))\omega_\beta$$
$$+ (t^h - t_j) f(x^h).$$

Clearly, $\widetilde{U}_h(x^h, t^h) = \varphi(x^h, t^h)$ and $\widetilde{U}_h \geq \varphi$, which implies that

$$\varphi(x^h, t^h) = \widetilde{U}_h(x^h, t_j) + (t^h - t_j) \sum_{y_\beta \in \mathcal{G}_h} J_p(\widetilde{U}_h(x^h + y_\beta, t_j) - \widetilde{U}_h(x^h, t_j))\omega_\beta$$
$$+ (t^h - t_j) f(x^h). \tag{4.1}$$

Now consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(\xi) = \xi + (t^h - t_j) \sum_{y_\beta \in \mathcal{G}_h} J_p(\widetilde{U}(x^h + y_\beta, t_j) - \xi)\omega_\beta$$

and note that

$$g'(\xi) = 1 - (t^h - t_j)(p - 1) \sum_{y_\beta \in \mathcal{G}_h} |\widetilde{U}(x^h + y_\beta, t_j) - \xi|^{p-2}\omega_\beta.$$

We will check now that $g'(\xi) \geq 0$ for any $\xi \in [\varphi(x^h, t_j), \widetilde{U}(x^h, t_j)]$. Indeed,

$$\begin{aligned}
|\widetilde{U}(x^h + y_\beta, t_j) - \xi| &\leq |\widetilde{U}(x^h + y_\beta, t_j) - \widetilde{U}(x^h, t_j)| + |\widetilde{U}(x^h, t_j) - \xi| \\
&\leq |U(x^h + y_\beta, t_j) - U(x^h, t_j)| + |\widetilde{U}(x^h, t_j) - \varphi(x^h, t_j)| \\
&\leq |U(x^h + y_\beta, t_j) - U(x^h, t_j)| + |U(x^h, t_j) - U(x^h, t^h)| + |\varphi(x^h, t^h) - \varphi(x^h, t_j)| \\
&\leq \Lambda_{u_0}(|y_\beta|) + T\Lambda_f(|y_\beta|) + 3\overline{\Lambda}_{u_0,f}(|t^h - t_j|) + |t^h - t_j| \|\partial_t \varphi\|_{L^\infty(B_R(x^h)\times[t^h-R,t^h])} \\
&\leq \Lambda_{u_0}(|y_\beta|) + T\Lambda_f(|y_\beta|) + 3\overline{\Lambda}_{u_0,f}(\tau) + \tau \|\partial_t \varphi\|_{L^\infty(B_R(x^h)\times[t^h-R,t^h])},
\end{aligned}$$

where we have used that $\widetilde{U}(x^h, t^h) = \varphi(x^h, t^h)$, Proposition 3.10 and the fact that $|t^h - t_j| \leq \tau$. By (CFL), and taking τ small enough, we have

$$\begin{aligned}
&3\overline{\Lambda}_{u_0,f}(\tau) + \tau \|\partial_t \varphi\|_{L^\infty(B_R(x^h)\times[t^h-R,t^h])} \\
&\qquad \leq 3\widetilde{K}\tau^{\frac{a}{2+(1-a)(p-2)}} + \left(3\|f\|_{L^\infty(\mathbb{R}^d)} + \|\partial_t \varphi\|_{L^\infty(B_R(x^h)\times[t^h-R,t^h])}\right)\tau \\
&\qquad \leq (3\widetilde{K} + 1)\tau^{\frac{a}{2+(1-a)(p-2)}} \\
&\qquad \leq (3\widetilde{K} + 1)r^a.
\end{aligned}$$

Thus,

$$\begin{aligned}
g'(\xi) &\geq 1 - (t^h - t_j)(p - 1) \sum_{y_\beta \in \mathcal{G}_h} |\Lambda_{u_0}(|y_\beta|) + T\Lambda_f(|y_\beta|) + (3\widetilde{K} + 1)r^a|^{p-2}\omega_\beta \\
&\geq 1 - \tau(p - 1)(L_{u_0} + TL_f + 3\widetilde{K} + 1)^{p-2} r^{a(p-2)} \sum_{y_\beta \in \mathcal{G}_h} \omega_\beta \\
&\geq 1 - \tau \frac{M(p - 1)(L_{u_0} + TL_f + 3\widetilde{K} + 1)^{p-2}}{r^{2+(1-a)(p-2)}} \\
&\geq 0,
\end{aligned}$$

where we have used (A_ω) and where the last inequality is due to the (CFL) condition. We can use this fact in (4.1) to get

$$\begin{aligned}\varphi(x^h, t^h) &= \widetilde{U}_h(x^h, t_j) + (t^h - t_j) \sum_{y_\beta \in \mathcal{G}_h} J_p(\widetilde{U}_h(x^h + y_\beta, t_j) - \widetilde{U}_h(x^h, t_j))\omega_\beta + (t^h - t_j) f(x^h) \\ &\geq \varphi(x^h, t_j) + (t^h - t_j) \sum_{y_\beta \in \mathcal{G}_h} J_p(\widetilde{U}_h(x^h + y_\beta, t_j) - \varphi(x^h, t_j))\omega_\beta + (t^h - t_j) f(x^h) \\ &\geq \varphi(x^h, t_j) + (t^h - t_j) \sum_{y_\beta \in \mathcal{G}_h} J_p(\varphi(x^h + y_\beta, t_j) - \varphi(x^h, t_j))\omega_\beta + (t^h - t_j) f(x^h).\end{aligned}$$

Consistency (A_c) yields

$$\partial_t \varphi(x^h, t^h) + o(\tau) \geq \Delta_p \varphi(x^h, t_j) + o_h(1) + f(x^h).$$

Passing to the limit as $h, \tau \to 0$, we get the desired result by the regularity of φ and the fact that $t^h, t_j \to t^*$ and $x^h \to x^*$ as $h \to 0$. □

We are now ready to prove convergence of the scheme.

Proof of Theorem 2.2 By Corollary 3.11 and Theorem 4.1, we know that, up to a subsequence, the sequence U_h converges to a viscosity solution of (1.1). Moreover, since viscosity solutions are unique (cf. Theorem 2.4), the whole sequence converges to the same limit.

5 Discretizations

In this section, we present two examples of discretizations and verify that the assumptions (A_c) and (A_ω) are satisfied. Moreover, we also give the precise form of corresponding CFL-condition.

5.1 Discretization in dimension $d = 1$

We consider the following finite difference discretization of Δ_p in dimension $d = 1$

$$D_p^h \phi(x) = \frac{J_p(\phi(x+h) - \phi(x)) + J_p(\phi(x-h) - \phi(x))}{h^p}.$$

A proof of consistency (A_c) can be found in Theorem 2.1 in [8]. Assumption (A_ω) is trivially true for $r = h$ since

$$\omega_1 = \omega_{-1} = \frac{1}{h^p} \quad \text{and} \quad \omega_\beta = 0 \quad \text{otherwise},$$

so that

$$\sum_{y_\beta \in \mathcal{G}_h} \omega_\beta = \frac{2}{h^p}.$$

5.2 Discretization in dimension $d > 1$

The following discretization was introduced in [9]:

$$D_p^h \phi(x) = \frac{h^d}{\mathcal{D}_{d,p}\, \omega_d\, r^{p+d}} \sum_{y_\beta \in B_r} J_p(\phi(x + y_\beta) - \phi(x)),$$

where ω_d denotes the measure of the unit ball in $\mathbb{R}^d$, the relation between r and h is given by

$$h = \begin{cases} o\big(r^{\frac{p}{p-1}}\big), & \text{if} \quad p \in (2, 3], \\ o\big(r^{\frac{3}{2}}\big), & \text{if} \quad p \in (3, \infty), \end{cases} \tag{5.1}$$

and $\mathcal{D}_{d,p} = \frac{d}{2(d+p)} \fint_{\partial B_1} |y_1|^p \, d\sigma(y)$. When $p \in \mathbb{N}$, a more explicit value of this constant is given in [9]. In general, the explicit value is given by

$$\mathcal{D}_{d,p} = \frac{d}{4\sqrt{\pi}} \cdot \frac{p-1}{d+p} \cdot \frac{\Gamma(\frac{d}{2})\Gamma(\frac{p-1}{2})}{\Gamma(\frac{d+p}{2})}.$$

A proof of consistency (A_c) can be found in Theorem 1.1 in [9]. Assumption (A_ω) trivially holds for $h = o(r^\alpha)$ for some $\alpha > 0$ according to (5.1) since

$$\omega_\beta = \omega_{-\beta} = \frac{h^d}{\mathcal{D}_{d,p}\, \omega_d \, r^{p+d}} \quad \text{if} \quad |h\beta| < r \quad \text{and} \quad \omega_\beta = 0 \quad \text{otherwise.}$$

To check (A_ω) we rely on the following estimate given in the proof of Theorem 1.1 in [9]:

$$\sum_{y_\beta \in B_r} h^d \le |B_{r+\sqrt{d}h}|.$$

In particular, taking for example $h \le r/\sqrt{d}$, we have

$$\sum_{y_\beta \in B_r} \omega_\beta = \frac{1}{\mathcal{D}_{d,p} r^p} \frac{|B_{r+\sqrt{d}h}|}{|B_r|} \le \frac{2^d}{\mathcal{D}_{d,p} r^p}.$$

6 Numerical experiments

We will perform the numerical tests comparing the numerical solution with the explicit Barenblatt solution of (1.1). For $p > 2$ this is given by

$$B(x,t) = K t^{-\alpha} \left(1 - \left(\frac{|x|}{t^\beta} \right)^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}},$$

where the constants are,

$$\alpha = \frac{d}{d(p-2)+p}, \quad \beta = \frac{1}{d(p-2)+p}, \quad \text{and} \quad K = \left(\frac{p-2}{p} \beta^{\frac{1}{p-1}} \right)^{\frac{p-1}{p-2}}.$$

6.1 Simulations in dimension $d = 1$

We consider the initial condition

$$u_0(x) = B(x,1) = K \left(1 - |x|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p-2}}$$

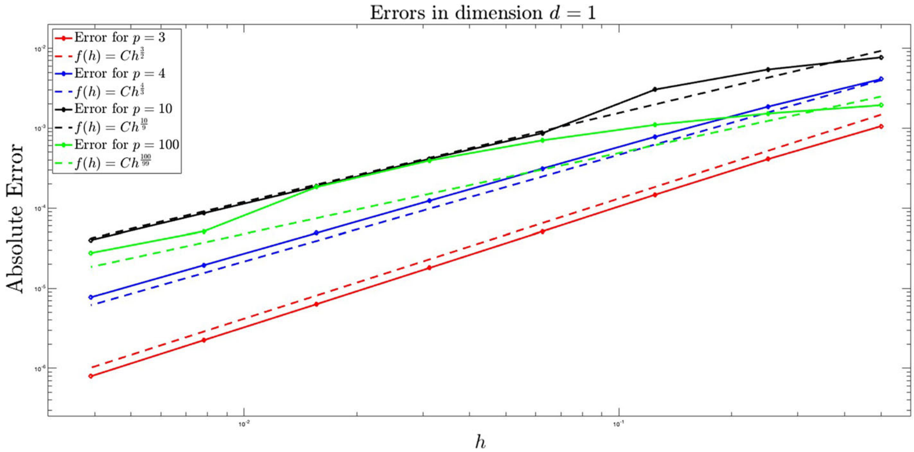

Fig. 1 Errors in dimension $d = 1$ for $p = 3, 4, 10, 100$

and $f = 0$. The corresponding solution of problem (1.1) is given by (see [24])

$$u(x,t) = B(x,t+1) = K(t+1)^{-\alpha}\left(1 - \left(\frac{|x|}{(t+1)^{\beta}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}.$$

Let us now comment on the CFL-condition (CFL). Clearly, u_0 is a Lipschitz function, and we can give an upper bound to its Lipschitz constant as follows

$$L_{u_0} = \sup_{x\in[-1,1]}\left|\frac{du_0}{dx}(x)\right| = \sup_{r\in[0,1]}\left\{K\frac{p}{p-2}\left(1-r^{\frac{p}{p-1}}\right)^{\frac{1}{p-2}} r^{\frac{1}{p-1}}\right\} \le K\frac{p}{p-2}.$$

Thus, for all $p > 2$, the CFL condition (CFL) can be take as $\tau \sim h^2$ (since $f = 0$ in this case). For completeness, we find the value of K in dimension $d = 1$. Note that

$$K = \left(\frac{p-2}{p}\frac{1}{(2(p-1))^{\frac{1}{p-1}}}\right)^{\frac{p-1}{p-2}} = \left(\frac{p-2}{p}\right)^{\frac{p-1}{p-2}}\frac{1}{(2(p-1))^{\frac{1}{p-2}}},$$

so that $L_{u_0} \le \left(\frac{p-2}{2p(p-1)}\right)^{\frac{1}{p-2}}$.

In Fig. 1, we show the numerical errors obtained. As it can be seen there, the errors seem to behave like $O(h^{p/(p-1)})$.

Acknowledgements Félix del Teso was supported by the Spanish Government through PGC2018-094522-B-I00, RYC2020-029589-I and PID2021-127105NB-I00 funded by the MICIN/AEI. He also wants to thank the Spanish Society of Applied Mathematics (SeMA) for their support on this work through the SeMA young Researcher Award 2021. Erik Lindgren was supported by the Swedish Research Council, 2017-03736. Part of this material is based upon work supported by the Swedish Research Council under grant no. 2016-06596 while the authors were in residence at Institut Mittag-Leffler in Djursholm, Sweden, during the research program "Geometric Aspects of Nonlinear Partial Differential Equations", fall of 2022.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Appendix A: Estimates for mollified Hölder continuous functions

Here we present some explicit estimates for mollifications needed in the proof of equicontinuity in Lemma 3.7. Let $\tau : [0, \infty) \to \mathbb{R}$ be a smooth function such that $\operatorname{supp} \tau \subset [0, 1]$. Define $\rho : \mathbb{R}^d \to \mathbb{R}$ given by $\rho(x) = \frac{M}{\omega_d}\tau(|x|)$ where ω_d is the measure of the unit sphere in dimension d and $M = M(d)$ is a constant defined by

$$M = \left(\int_0^1 \tau(r) r^{d-1}\,\mathrm{d}r\right)^{-1}.$$

In this way, we have that $\int_{B_1} \rho(x)\,dx = 1$. For $\delta > 0$ define also

$$\rho_\delta(x) = \frac{1}{\delta^d}\rho\left(\frac{x}{\delta}\right).$$

Then, for a function $f \in L^1_{\text{loc}}$ we define the mollification of f as

$$f_\delta(x) = (f * \rho_\delta)(x) = \int_{B_\delta} \rho_\delta(y) f(x-y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \rho_\delta(x-y) f(y)\,\mathrm{d}y.$$

The lemma below gives an estimate of the Lipschitz seminorm of f_δ when f is an α-Hölder continuous function for some $\alpha \in (0, 1]$.

Lemma A.1 *Let $\alpha \in (0, 1]$. Consider a function $f \in C^\alpha(\mathbb{R}^n)$ with $|f(x) - f(y)| \leq L|x-y|^\alpha$ for all $x, y \in \mathbb{R}^d$. Then, for all $x, y \in \mathbb{R}^d$, we have that*

$$|f_\delta(x) - f_\delta(y)| \leq K_1 L |x-y| \delta^{\alpha-1}, \quad \text{with} \quad K_1 = M\int_0^1 |\tau'(r)| r^{d-1}\,\mathrm{d}r.$$

Proof Since $\int_{\mathbb{R}^n} \nabla\rho_\delta(y)\,dy = \int_{\mathbb{R}^n} \nabla\rho(y)\,dy = 0$, it follows that

$$\begin{aligned} |\nabla f_\delta(x)| &= \left|\int_{B_\delta} \nabla\rho_\delta(y)\,(f(x-y) - f(x))\,\mathrm{d}y\right| \leq L\int_{B_\delta} |\nabla\rho_\delta(y)|\,|y|^\alpha\,\mathrm{d}y \\ &\leq \frac{M}{\omega_d} L\delta^\alpha \int_{B_\delta} \frac{1}{\delta^{d+1}}\left|\tau'\left(\frac{|y|}{\delta}\right)\right|\,\mathrm{d}y = ML\delta^{\alpha-1}\int_0^1 |\tau'(r)| r^{d-1}\,\mathrm{d}r. \end{aligned} \tag{A.1}$$

Thus,

$$|f_\delta(x) - f_\delta(y)| \leq \|\nabla f_\delta\|_{L^\infty(\mathbb{R}^d)} |x-y| = K_1 L |x-y| \delta^{\alpha-1}.$$

□

The lemma below gives an estimate of the second order central difference quotients of f_δ when f is an α-Hölder continuous function for some $\alpha \in (0, 1]$.

Lemma A.2 *Let $\alpha \in (0,1]$. Consider a function $f \in C^\alpha(\mathbb{R}^n)$ with $|f(x)-f(y)| \leq L|x-y|^\alpha$ for all $x, y \in \mathbb{R}^d$. Then, for all $x, y \in \mathbb{R}^d$, we have that*

$$|f_\delta(x+y) + f_\delta(x-y) - 2f_\delta(x)| \leq K_2 L|y|^2\delta^{\alpha-2},$$

with

$$K_2 = M\int_0^1 \left(\frac{|\tau'(r)|}{r} + |\tau''(r)|\right) r^{d-1}\,\mathrm{d}r.$$

Proof We note that we have the following formula for the second order derivatives of ρ_δ:

$$\partial_{ij}\rho_\delta(y) = \left(\frac{\delta_{ij}}{|y|} - \frac{y_i y_j}{|y|^3}\right)\frac{1}{\delta^{d+1}}\tau'\left(\frac{|y|}{\delta}\right) + \frac{y_i y_j}{\delta^{d+2}|y|^2}\tau''\left(\frac{|y|}{\delta}\right),$$

so that

$$\langle D^2\rho_\delta(y)\xi, \xi\rangle \leq \left(\frac{1}{\delta^{d+1}|y|}\left|\tau'\left(\frac{|y|}{\delta}\right)\right| + \frac{1}{\delta^{d+2}}\left|\tau''\left(\frac{|y|}{\delta}\right)\right|\right)|\xi|^2.$$

Similarly to the gradient, the Hessian also integrates to zero, that is,

$$\int_{\mathbb{R}^n} \partial_{ij}\rho_\delta(y)\,\mathrm{d}y = 0 \quad \text{for all} \quad i, j = 1, \ldots, d.$$

Indeed, when $i \neq j$, the result follows by antisymmetry in y. When $i = j$, we are integrating $\partial_{ii}\rho_\delta$, which yields zero since $\partial_i\rho_\delta = 0$ on $\mathbb{R}^d \setminus B_\delta$. As in the proof of the previous lemma, it follows that

$$\begin{aligned}
\|D^2 f_\delta\| &= \left\|\int_{B_\delta} D^2\rho_\delta(y)\,(f(x-y) - f(x))\,dy\right\| \leq L\int_{B_\delta} \|D^2\rho_\delta(y)\||y|^\alpha\,\mathrm{d}y \\
&\leq \frac{M}{\omega_d} L\delta^\alpha \int_{B_\delta}\left(\frac{1}{\delta^{d+1}|y|}\left|\tau'\left(\frac{|y|}{\delta}\right)\right| + \frac{1}{\delta^{d+2}}\left|\tau''\left(\frac{|y|}{\delta}\right)\right|\right)\mathrm{d}y \\
&= ML\delta^{\alpha-2}\int_0^1\left(\frac{|\tau'(r)|}{r} + |\tau''(r)|\right) r^{d-1}\,\mathrm{d}r,
\end{aligned} \tag{A.2}$$

where $\|\cdot\|$ denotes the operator norm. Now, by Taylor expansion

$$f_\delta(x \pm y) = f_\delta\phi(x) \pm \nabla f_\delta(x)\cdot y + \frac{1}{2}\sum_{\beta=1}\frac{\partial^{|\beta|} f_\delta}{\partial x_\beta}(z^\pm)y^\beta.$$

Thus,

$$\begin{aligned}
|f_\delta(x+y) + f_\delta(x-y) - 2f_\delta(x)| &\leq |y|^2\|D^2 f_\delta(z)\| \\
&\leq M\left(\int_0^1\left(\frac{|\tau'(r)|}{r} + |\tau''(r)|\right) r^{d-1}\,\mathrm{d}r\right) L\delta^{\alpha-2}|y|^2.
\end{aligned}$$

□

A.1. Explicit constants in dimensions one, two and three

Here we will compute explicit constants for the mollifier that is based on the choice

$$\tau(r) = e^{-\frac{1}{1-r^2}}\chi_{[0,1)}(r).$$

In one dimension: We have

$$M=\left(\int_0^1 \tau(r)\,\mathrm{d}r\right)^{-1} \le 4.51$$

and

$$K_1 = M\int_0^1 |\tau'(r)|\,\mathrm{d}r = M\int_0^1 (-\tau'(r))\,\mathrm{d}r = M\tau(0) = \frac{M}{e} \le 1.67.$$

Since

$$\int_0^1 \frac{|\tau'(r)|}{r} = 2\int_0^1 \frac{e^{-\frac{1}{1-r^2}}}{(1-r^2)^2}\,\mathrm{d}r \le 0.8,$$

and

$$\int_0^1 |\tau''(r)|dr = \int_0^1 \left| e^{-\frac{1}{1-r^2}} \frac{6r^4-2}{(1-r^2)^4}\right| \mathrm{d}r \le 1.6,$$

we conclude that

$$K_2 \le 2.4M \le 10.83.$$

In two dimensions: We have

$$M=\left(\int_0^1 \tau(r)r\,\mathrm{d}r\right)^{-1} \le 13.47$$

and

$$K_1 = M\int_0^1 |\tau'(r)|r\,\mathrm{d}r = M\int_0^1 (-\tau'(r)r)\,\mathrm{d}r = M\int_0^1 \tau(r)\,\mathrm{d}r \le 0.23M \le 3.13.$$

Since

$$\int_0^1 |\tau'(r)|\,\mathrm{d}r = \frac{1}{e}$$

and

$$\int_0^1 |\tau''(r)|r dr = \int_0^1 \left| e^{-\frac{1}{1-r^2}} \frac{6r^4-2}{(1-r^2)^4} r\right| \mathrm{d}r \le 1.04,$$

we conclude that

$$K_2 \le M(e^{-1}+1.04) \le 18.97.$$

In three dimensions: We have

$$M=\left(\int_0^1 \tau(r)r^2\,\mathrm{d}r\right)^{-1} \le 28.49$$

and

$$K_1 = M\int_0^1 |\tau'(r)|r^2\,\mathrm{d}r = M\int_0^1 (-\tau'(r)r^2)\,\mathrm{d}r = 2M\int_0^1 \tau(r)r\,\mathrm{d}r \le 2\times 0.08M \le 4.56.$$

Since

$$\int_0^1 |\tau'(r)|r\,\mathrm{d}r = \int_0^1 \tau(r)\,\mathrm{d}r \le 0.23$$

and

$$\int_0^1 |\tau''(r)|r^2 dr = \int_0^1 \left| e^{-\frac{1}{1-r^2}} \frac{6r^4-2}{(1-r^2)^4} r^2 \right| \mathrm{d}r \leq 0.79,$$

we conclude that

$$K_2 \leq M(0.23 + 0.79) \leq 29.06.$$

Appendix B: Pointwise inequalities

The following lemma follows from the Taylor expansion of the function $t \mapsto |t|^{p-2}t$.

Lemma B.1 *Let $p \geq 2$. Then*

$$\left| |a+b|^{p-2}(a+b) - |a|^{p-2}a \right| \leq (p-1)\max(|a|, |a+b|)^{p-2}|b|.$$

References

1. Andreianov, B.A., Gutnic, M., Wittbold, P.: Convergence of finite volume approximations for a nonlinear elliptic-parabolic problem: a “continuous” approach. SIAM J. Numer. Anal. **42**(1), 228–251 (2004)
2. Barrett, J.W., Liu, W.B.: Finite element approximation of the parabolic p-Laplacian. SIAM J. Numer. Anal. **31**(2), 413–428 (1994)
3. Bögelein, V.: Global gradient bounds for the parabolic p-Laplacian system. Proc. Lond. Math. Soc. **111**(3), 633–680 (2015)
4. Bucur, C., Squassina, M.: An asymptotic expansion for the fractional p-Laplacian and gradient dependent nonlocal operators. Commun. Contemp. Math. **24**(4), 2150021 (2022)
5. Codenotti, L., Lewicka, M., Manfredi, J.: Discrete approximations to the double-obstacle problem and optimal stopping of tug-of-war games. Trans. Am. Math. Soc. **369**(10), 7387–7403 (2017)
6. del Teso, F., Endal, J., Jakobsen, E.R.: Robust numerical methods for nonlocal (and local) equations of porous medium type. Part II: schemes and experiments. SIAM J. Numer. Anal. **56**(6), 3611–3647 (2018)
7. del Teso, F., Endal, J., Jakobsen, E.R.: Robust numerical methods for nonlocal (and local) equations of porous medium type. Part I: theory. SIAM J. Numer. Anal. **57**(5), 2266–2299 (2019)
8. del Teso, F., Lindgren, E.: A mean value formula for the variational p-Laplacian. NoDEA Nonlinear Differ. Equ. Appl. **28**, 27 (2021)
9. del Teso, F., Lindgren, E.: A finite difference method for the variational p-Laplacian. J. Sci. Comput. **90**, 67 (2022)
10. del Teso, F., Manfredi, J.J., Parviainen, M.: Convergence of dynamic programming principles for the p-Laplacian. Adv. Calc. Var. **15**(2), 191–212 (2022)
11. DiBenedetto, E.: Degenerate Parabolic Equations. Universitext, Springer, New York (1993)
12. DiBenedetto, E., Hoff, D.: An interface tracking algorithm for the porous medium equation. Trans. Am. Math. Soc. **284**(2), 463–500 (1984)
13. DiBenedetto, E., Gianazza, U., Vespri, V.: Harnack’s Inequality for Degenerate and Singular Parabolic Equations. Springer Monographs in Mathematics. Springer, New York (2012)
14. Diening, L., Ebmeyer, C., Ružička, M.: Optimal convergence for the implicit space-time discretization of parabolic systems with p-structure. SIAM J. Numer. Anal. **45**(2), 457–472 (2007)
15. Does, K.: An evolution equation involving the normalized p-Laplacian. Commun. Pure Appl. Anal. **10**(1), 361–396 (2011)
16. Emmrich, E., Šiška, D.: Full discretization of the porous medium/fast diffusion equation based on its very weak formulation. Commun. Math. Sci. **10**(4), 1055–1080 (2012)
17. Evans, L.C.: Partial Differential Equations. Graduate Studies in Mathematics, vol. 19. American Mathematical Society, Providence (1998)
18. Falcone, M., Finzi Vita, S., Giorgi, T., Smits, R.G.: A semi-Lagrangian scheme for the game p-Laplacian via p-averaging. Appl. Numer. Math. **73**, 63–80 (2013)
19. Ferreira, R., de Pablo, A., Pérez-Llanos, M.: Numerical blow-up for the p-Laplacian equation with a source. Comput. Methods Appl. Math. **5**(2), 137–154 (2005)

20. Iagar, R.G., Sánchez, A., Vázquez, J.L.: Radial equivalence for the two basic nonlinear degenerate diffusion equations. J. Math. Pures Appl. **89**(1), 1–24 (2008)
21. Ju, N.: Numerical analysis of parabolic p-Laplacian: approximation of trajectories. SIAM J. Numer. Anal. **37**(6), 1861–1884 (2000)
22. Juutinen, P., Kawohl, B.: On the evolution governed by the infinity Laplacian. Math. Ann. **335**(4), 819–851 (2006)
23. Juutinen, P., Lindqvist, P., Manfredi, J.J.: On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation. SIAM J. Math. Anal. **33**(3), 699–717 (2001)
24. Kamin, S., Vázquez, J.L.: Fundamental solutions and asymptotic behaviour for the p-Laplacian equation. Rev. Mat. Iberoam. **4**(2), 339–354 (1988)
25. Manfredi, J.J., Parviainen, M., Rossi, J.D.: Dynamic programming principle for tug-of-war games with noise. ESAIM Control Optim. Calc. Var. **18**(1), 81–90 (2012)
26. Monsaingeon, L.: An explicit finite-difference scheme for one-dimensional generalized porous medium equations: interface tracking and the hole filling problem. ESAIM Math. Model. Numer. Anal. **50**(4), 1011–1033 (2016)
27. Oberman, A.M.: A convergent difference scheme for the infinity Laplacian: construction of absolutely minimizing Lipschitz extensions. Math. Comput. **74**(251), 1217–1230 (2005)
28. Oberman, A.M.: Finite difference methods for the infinity Laplace and p-Laplace equations. J. Comput. Appl. Math. **254**, 65–80 (2013)
29. Vázquez, J.L.: Smoothing and Decay Estimates for Nonlinear Diffusion Equations. Oxford Lecture Series in Mathematics and its Applications, vol. 33. Oxford University Press, Oxford (2006)
30. Vázquez, J.L.: The Porous Medium Equation. Mathematical theory. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford (2007)