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Approximate optimality conditions and approximate duality theorems for nonlinear semi-infinite programming problems with uncertainty data

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Abstract
In this paper, we establish optimality conditions and duality theorems for a robust ε-quasi solution of a nonsmooth semi-infinite programming problem with data uncertainty in both the objective and constraints. Next, we provide an application to nonsmooth fractional semi-infinite optimization problem with data uncertainty in constraints. Finally, some examples are given to illustrate the obtained results.

Keywords Semi-infinite programming · Approximate optimality condition · Approximate duality · Clarke subdifferential · Robust optimization

Mathematics Subject Classification 49K99 · 65K10 · 90C34 · 90C26 · 90C46

1 Introduction

In recent years, the study of one among more a semi-infinite programming problem, which is an optimization problem on a feasible set described by an infinite number of inequality constraints, has occupied attention of researches. Many successful treatments of deterministic semi-infinite programming have been investigated from several different perspectives. We refer the readers to the papers [5,7,9,11,14,16,21,23,34,38,44,50,51], and the references therein. Semi-infinite programming problems could be applied in various fields such as in engineering design, mathematical physics, robotics, optimal control, transportation problems, fuzzy sets, cooperative games; see, for example [15,30].

Besides, robust optimization has emerged as a remarkable deterministic framework for studying optimization problems with uncertain data; see [1,2]. Many researchers have been attracted to work on the real-world application of robust optimization in engineering, business and management. Many interesting results have been published in [3,4,6,12,17,20,35,36,43,49,52] and the references therein.

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Furthermore, sometimes the exact solutions do not exist while the approximate ones do, even in the convex case; see [31,32] and other references therein. Therefore, the study of approximate solution is very significant from both the theoretical aspect and computational application. The results on optimality conditions and duality theorems for approximate solutions to multiobjective optimization problems were obtained in [8]. Approximate optimality theorems, approximate duality theorems and approximate saddle point theorems were established for the robust convex optimization problem [26,46]. In [13,47], authors studied optimality conditions, duality theorems and saddle point theorems for the approximate efficient solutions of nonsmooth robust multiobjective optimization problem. By using the Clarke subdifferential, Son et al. [41] obtained optimality conditions duality theorems and saddle point theorems for approximate solutions of nonconvex programming problem with an infinite number of constraints. Optimality conditions of approximate solutions for nonsmooth semi-infinite programming problem were given in [29]. In [25,40], authors investigated approximate optimality conditions, approximate duality theorems and approximate saddle point theorems of nonconvex multiobjective programming problem with an infinite number of constraints. Son et al. [42] established new necessary and sufficient optimality conditions for approximate solutions of a nonsmooth semi-infinite multiobjective optimization problem. By using the limiting/Mordukhovich subdifferential, Jiao et al. [22] established optimality conditions and duality theorems for approximate solution of semi-infinite programming problem. In [39], authors obtained necessary conditions for approximate solution of fractional semi-infinite multiobjective optimization problem. Besides, approximate optimality conditions and approximate duality theorems in robust convex semidefinite programming problems were given in [19]. In [27], authors studied optimality conditions and duality theorems for semi-infinite multiobjective optimization problems with data uncertainty in constraints. Recently, approximate optimality conditions and approximate duality theorems for semi-infinite convex optimization problem with data uncertainty in constraints have been obtained in [28]. By using the Clarke subdifferential, Khantree and Wangkeeree [24] have obtained approximate optimality conditions and approximate duality theorems for nonsmooth semi-infinite optimization problem with data uncertainty in constraints. More recently, approximate optimality conditions for semi-infinite programming problem with data uncertainty in both the objective and constraints have been given in [45]. By using the Clarke subdifferential, optimality conditions and duality theorems for approximate solutions of nonsmooth semi-infinite optimization problem with data uncertainty in both the objective and constraints have been obtained in [48]. As far as we know, the results on approximate optimality conditions as well as approximate duality theorems for semi-infinite programming problems with uncertainty in both the objective and constraints have been studied in few papers. For the considerations of references, we observe only references [45,48].

Inspired by the above observations, we provide some new results for approximate optimality conditions and approximate duality theorems for nonsmooth semi-infinite programming problem with data uncertainty in both the objective and constraints (USIP) via Clarke subdifferential. Next, an application to nonsmooth fractional semi-infinite programming problem is provided and some examples are also given to illustrate the obtained results.

The rest of the paper is organized as follows. Sections 1 and 2 present introduction, notations and preliminaries. In Sect. 3, we establish necessary and sufficient conditions for a robust ε-quasi solution to problem (USIP). In Sect. 4, we investigate approximate duality theorems for a Mond–Weir type dual problem with respect to the primal problem (USIP). In Sect. 5, we provide an application to nonsmooth fractional semi-infinite programming problem with data uncertainty in constraints. Finally, conclusions are given in Sect. 6.

2 Preliminaries

Throughout the paper we use the standard notation of variational analysis in [10,37]. In this paper, we write $\mathbb{R}^n$ instead of a finite real normed space (or Euclidean space of dimension $n \in \mathbb{N} := \{1, 2, \ldots\}$), and $\mathbb{R}^n$ for its topological dual, because $(\mathbb{R}^n)^* = \mathbb{R}^n$. The nonnegative (resp., nonpositive) orthant cone of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ (resp., $\mathbb{R}^n_-$). In any finite real normed space $\mathbb{R}^n$, a norm is always denoted by $||.||$ and the inner product is defined by $\langle ., . \rangle$. The symbol $\mathbb{B}$ stands for the closed unit ball of $\mathbb{R}^n$. The norm of an element ξ of $\mathbb{R}^n$, denote by $||\xi||$, is given by

$$||\xi|| := \sup\{\langle \xi, d\rangle \mid d \in \mathbb{R}^n, ||d|| \leq 1\}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a given real valued function. We say that f is locally Lipschitz, if for any $x \in \mathbb{R}^n$, there exist a positive constant L and an open neighbourhood $\mathcal{N}(x)$ of x, such that for any $x_1, x_2 \in \mathcal{N}(x)$,

$$|f(x_1) - f(x_2)| \leq L||x_1 - x_2||.$$

For any $d \in \mathbb{R}^n$, the usual one-side directional derivative of f at $x \in \mathbb{R}^n$ is defined as follows:

$$f'(x; d) := \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}.$$

The Clarke generalized directional derivative of f at $x \in \mathbb{R}^n$ is defined as follows:

$$f^C(x; d) := \limsup_{y \to x, t \downarrow 0} \frac{f(y + td) - f(y)}{t}.$$

The Clarke subdifferential of f at $x \in \mathbb{R}^n$ is defined as follows:

$$\partial^C f(x) := \{\xi \in \mathbb{R}^n \mid f^C(x; d) \geq \langle \xi, d\rangle, \forall d \in \mathbb{R}^n\}.$$

Definition 1 [10] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The function f is said to be quasi-differentiable or regular at $x \in \mathbb{R}^n$ (in the sense of Clarke), if $f'(x; d)$ exists and equals to $f^C(x; d)$, for any $d \in \mathbb{R}^n$.

Definition 2 [10] Let $\Omega \subset \mathbb{R}^n$ be a nonempty subset and $x \in \Omega$. The Clarke normal cone to Ω at x is defined by

$$N^C(x; \Omega) := \{\xi \in \mathbb{R}^n \mid \langle \xi, d\rangle \leq 0, \forall v \in T^C(x; \Omega)\},$$

where $T^C(x, \Omega)$ denotes the tangent cone of Ω and

$$T^C(x; \Omega) := \{v \in \mathbb{R}^n \mid \forall t_n \downarrow 0, \forall x_n \to x, \exists v_n \to v, \text{ with } x_n + t_n v_n \in \Omega, \forall n \in \mathbb{N}\},$$

which is equivalent to $T^C(x; \Omega) := \{v \in \mathbb{R}^n \mid d^C_\Omega(x; v) = 0\}$, where d_Ω denotes the distance function to Ω.

Definition 3 [37] Let $\Omega \subset \mathbb{R}^n$. Ω is said to be a convex set if for all $x, y \in \Omega$ and $\lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \in \Omega$.

Definition 4 [37] Let $\Omega \subset \mathbb{R}^n$ be a nonempty subset. A function $f : \Omega \to \mathbb{R}$ is said to be

(i) convex if for all $x, y \in \Omega$ and all $\lambda \in [0, 1]$, then $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$;
(ii) concave if $-f$ is convex.

Remark 1 Suppose that $\Omega \subset \mathbb{R}^n$ is a nonempty closed convex subset. Then, $N^C(x;\Omega)$ coincides with the cone normal in the sense of convex analysis and

$$N^C(x;\Omega) := \{\xi \in \mathbb{R}^n \mid \langle \xi, y-x\rangle \le 0, \forall y \in \Omega\}.$$

The following important properties of the Clarke subdifferential will be used later in this paper.

Lemma 1 [10] *Suppose that $\Omega \subset \mathbb{R}^n$ is a nonempty subset and $x \in \Omega$. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz near x and attains a minimum over Ω at x. Then,*

$$0 \in \partial^C f(x) + N^C(x;\Omega).$$

Lemma 2 [10] *Suppose that $f_k : \mathbb{R}^n \to \mathbb{R}$, $k = 1, 2, \ldots, m$ are locally Lipschitz functions. Then, for any $x \in \mathbb{R}^n$,*

$$\partial^C(f_1 + f_2 + \cdots + f_m)(x) \subset \partial^C f_1(x) + \partial^C f_2(x) + \cdots + \partial^C f_m(x).$$

Lemma 3 [10] *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz functions near x, and suppose that $g(x) \neq 0$. Then $\dfrac{f}{g}$ is locally Lipschitz near x, and one has*

$$\partial^C\left(\frac{f}{g}\right)(x) \subset \frac{g(x)\partial^C f(x) - f(x)\partial^C g(x)}{g^2(x)}.$$

In this paper, we are interested in the study of a semi-infinite programming problem with inequality constraints having the following form

$$\text{(SIP)} \quad \begin{array}{l} \min\limits_{x\in\Omega} f(x), \\ \text{s.t. } g_t(x) \le 0, \quad \forall t \in T, \end{array}$$

where $\Omega \subset \mathbb{R}^n$ is a nonempty closed (not necessarily convex) set, T is a nonempty infinite index set and $f, g_t : \mathbb{R}^n \to \mathbb{R}, t \in T$ are locally Lipschitz functions. This problem (SIP) with data uncertainty can be captured by

$$\text{(USIP)} \quad \begin{array}{l} \min\limits_{x\in\Omega} f(x,u), \\ \text{s.t. } g_t(x, v_t) \le 0, \quad \forall t \in T, \end{array}$$

where the uncertain parameters u and $v_t, t \in T$ belong to convex compact sets $\mathcal{U} \subset \mathbb{R}^m$ and $\mathcal{V}_t \subset \mathbb{R}^q, t \in T$, respectively. $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, t \in T$ are locally Lipschitz functions.

The uncertainty set-valued mapping $\mathcal{V} : T \rightrightarrows \mathbb{R}^q$ is defined as $\mathcal{V}(t) := \mathcal{V}_t$ for all $t \in T$. The notation $v \in \mathcal{V}$ means that v is a selection of $\mathcal{V}$, i.e., $v : T \to \mathbb{R}^q$ and $v_t \in \mathcal{V}_t$ for all $t \in T$. So, the uncertainty set is the graph of $\mathcal{V}$, that is, $\text{gph}\mathcal{V} := \{(t, v_t) \mid v_t \in \mathcal{V}_t, t \in T\}$.

The robust counterpart of problem (USIP) is

$$\text{(RUSIP)} \quad \begin{array}{l} \min\limits_{x\in\Omega} \left\{\max\limits_{u\in\mathcal{U}} f(x,u)\right\}, \\ \text{s.t. } g_t(x, v_t) \le 0, \quad \forall v_t \in \mathcal{V}_t, \forall t \in T. \end{array}$$

Let $\mathbb{R}^{(T)}$ be the linear space given below

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t\in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ except for finitely many } \lambda_t \neq 0\}.$$

Let $\mathbb{R}_+^{(T)}$ be the positive cone in $\mathbb{R}^{(T)}$ defined by

$$\mathbb{R}_+^{(T)} := \{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0 \text{ for all } t \in T\}.$$

With $\lambda \in \mathbb{R}^{(T)}$, its supporting set, $T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}$, is a finite subset of T. $\{z_t\} \subset Z, t \in T, Z$ being a real linear space, we understand that

$$\sum_{t\in T} \lambda_t z_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t z_t, & \text{if } \; T(\lambda) \neq \emptyset, \\ 0, & \text{if } \; T(\lambda) = \emptyset. \end{cases}$$

For $g_t, t \in T$,

$$\sum_{t\in T} \lambda_t g_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t g_t, & \text{if } \; T(\lambda) \neq \emptyset, \\ 0, & \text{if } \; T(\lambda) = \emptyset. \end{cases}$$

3 Robust approximate optimality conditions

In this section, we establish the necessary and sufficient optimality conditions for approximate solution of problem (USIP).

Definition 5 The robust feasible set F of problem (USIP) is defined by

$$F := \{x \in \Omega \mid g_t(x, v_t) \leq 0, \forall v_t \in \mathcal{V}_t, \forall t \in T\},$$

where $\Omega \subset \mathbb{R}^n$ is a nonempty closed (not necessarily convex) set.

Definition 6 [48] Let $\varepsilon \in \mathbb{R}_+\backslash\{0\}$. A point $\bar{x} \in F$ is said to be a robust ε-quasi solution of problem (USIP), if it is an ε-quasi solution of problem (RUSIP), i.e, for any $x \in F$,

$$\max_{u\in\mathcal{U}} f(\bar{x}, u) \leq \max_{u\in\mathcal{U}} f(x, u) + \sqrt{\varepsilon}||x - \bar{x}||.$$

The following constraint qualification is an extension of Definition 3.2 in [48].

Definition 7 Let $\bar{x} \in F$. We say that the following robust constraint qualification (RCQ) is satisfied at $\bar{x} \in F$ if

$$N^C(\bar{x}; F) \subseteq \bigcup_{\substack{\lambda\in A(\bar{x}) \\ v_t\in\mathcal{V}_t}} \left[\sum_{t\in T} \lambda_t \partial_x^C g_t(\bar{x}, v_t)\right] + N^C(\bar{x}; \Omega),$$

where $A(\bar{x}) := \{\lambda \in \mathbb{R}_+^{(T)} \mid \lambda_t g_t(\bar{x}, v_t) = 0, \forall v_t \in \mathcal{V}_t, \forall t \in T\}$ is set of active constraint multipliers at $\bar{x} \in \Omega \subset \mathbb{R}^n$.

If $\mathcal{V}_t, t \in T$ is singleton, the qualification condition (RCQ) becomes the qualification condition (CQ) for problem (SIP). The qualification conditions (CQ) have been introduced and used in [29] and the references therein.

In what follows, the uncertain objective function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ of problem (USIP) is assumed to satisfy the hypotheses (see [48]).

(H1) The function $u \in \mathcal{U} \mapsto f(x, u)$ is upper semicontinuous for each $x \in \mathbb{R}^n$;

(H2) The function $x \in \mathbb{R}^n \mapsto f(x, u)$ is locally Lipschitz and regular for each $u \in \mathcal{U}$;

(H3) $\partial_x^C f(x, u)$ is upper semicontinuous in $(x, u) \in \mathbb{R}^n \times \mathcal{U}$, where $\partial_x^C f(x, u)$ denotes the Clarke subdifferential of f with respect to x.

Lemma 4 [48] *Suppose that the hypotheses* (H1)-(H3) *are fulfilled. Further, suppose that* $\mathcal{U}$ *is convex and compact and* $f(x, .)$ *is concave on* $\mathcal{U}$*, for any* $x \in \Omega$*. Then,*

$$\partial^C \left(\max_{u\in\mathcal{U}} f(., u) \right)(x) = \bigcup_{u\in\mathcal{U}(x)} \partial_x^C f(x, u),$$

where $\mathcal{U}(x) := \{\bar{u} \in \mathcal{U} \mid f(x, \bar{u}) = \max_{u\in\mathcal{U}} f(x, u)\}$.

Theorem 1 *Let* $\varepsilon \in \mathbb{R}_+\backslash\{0\}$ *and let* $\bar{x} \in F$ *be a robust* ε*-quasi solution of problem* (USIP)*. Suppose that* $f(\bar{x}, .)$ *is concave on* $\mathcal{U}$*, and that qualification condition* (RCQ) *at* $\bar{x} \in F$ *holds. Then, there exist* $\bar{u} \in \mathcal{U}$*,* $(\bar{\lambda}_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ *and* $\bar{v}_t \in \mathcal{V}_t, t \in T$*, such that*

$$0 \in \partial_x^C f(\bar{x}, \bar{u}) + \sum_{t\in} \bar{\lambda}_t \partial_x^C g_t(\bar{x}, \bar{v}_t) + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B}, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0, \tag{1}$$

and

$$f(\bar{x}, \bar{u}) = \max_{u\in\mathcal{U}} f(\bar{x}, u). \tag{2}$$

Proof Suppose that $\bar{x} \in F$ is a robust ε-quasi solution of problem (USIP). Then, for any $x \in F$,

$$\max_{u\in\mathcal{U}} f(\bar{x}, u) \leq \max_{u\in\mathcal{U}} f(x, u) + \sqrt{\varepsilon}||x - \bar{x}||. \tag{3}$$

For any $x \in \mathbb{R}^n$, we set $\Phi(x) := \max_{u\in\mathcal{U}} f(x, u) + \sqrt{\varepsilon}||x - \bar{x}||$. From (3), it implies that Φ is locally Lipschitz at $\bar{x}$ and $\bar{x}$ is a minimizer of the following problem $\min_{x\in F} \Phi(x)$. By Lemma 1, we have

$$0 \in \partial^C \Phi(\bar{x}) + N^C(\bar{x}; F). \tag{4}$$

Because, one has $\partial^C||. - \bar{x}||(\bar{x}) = \mathbb{B}$. So, from the Lemma 2, we obtain

$$0 \in \partial^C \left(\max_{u\in\mathcal{U}} f(., u) \right)(\bar{x}) + N^C(\bar{x}; F) + \sqrt{\varepsilon}\mathbb{B}. \tag{5}$$

From qualification condition (RCQ), we deduce that (5) is equivalent to

$$0 \in \partial^C \left(\max_{u\in\mathcal{U}} f(., u) \right)(\bar{x}) + \bigcup_{\substack{\lambda\in A(\bar{x})\\ v_t\in\mathcal{V}_t}} \left[\sum_{t\in T} \lambda_t \partial_x^C g_t(\bar{x}, v_t) \right] + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B},$$

where $A(\bar{x}) := \{\lambda \in \mathbb{R}_+^{(T)} \mid \lambda_t g_t(\bar{x}, v_t) = 0, \forall v_t \in \mathcal{V}_t, \forall t \in T\}$. Therefore, there exist $(\bar{\lambda}_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ and $\bar{v}_t \in \mathcal{V}_t, t \in T$, such that

$$0 \in \partial^C \left(\max_{u\in\mathcal{U}} f(., u) \right)(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t \partial_x^C g_t(\bar{x}, \bar{v}_t) + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B}, \tag{6}$$

and $\bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0$. Furthermore, from Lemma 4, we have

$$\partial^C \left(\max_{u\in\mathcal{U}} f(., u) \right)(\bar{x}) = \bigcup_{\bar{u}\in\mathcal{U}(\bar{x})} \partial_x^C f(\bar{x}, \bar{u}), \tag{7}$$

where $\mathcal{U}(\bar{x}) := \{\bar{u} \in \mathcal{U} \mid f(\bar{x}, \bar{u}) = \max_{u\in\mathcal{U}} f(\bar{x}, u)\}$. Thus, it follows from (6) and (7) that there exist $\bar{u} \in \mathcal{U}$, $(\bar{\lambda}_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ and $\bar{v}_t \in \mathcal{V}_t, t \in T$ such that (1) and (2) hold. □

Remark 2 Theorem 1 improves Theorem 3.1 in [22], Theorem 3.1 in [29], Theorem 4.1 in [41], and Theorem 3.1 in [48].

Now, let us provide an example illustrating Theorem 1.

Example 1 We consider problem (USIP) with $\varepsilon = \frac{1}{4}$, $\Omega = (-\infty, 0] \subset \mathbb{R}$, $x \in \mathbb{R}$, $u \in \mathcal{U} = [0, 1]$, $t \in T = [0, 1]$ and $v_t \in \mathcal{V}_t = [2-t, 2+t]$ for any $t \in T$. Take the functions $f(x, u) = |x| + u - 1$ and $g_t(x, v_t) = -v_t x^2$. We can see that the robust feasible set $F = (-\infty, 0]$. Therefore, $\bar{x} = 0$ is a robust ε-quasi solution of problem (USIP). Indeed, one has

$$\max_{u \in \mathcal{U}} f(x, u) + \sqrt{\varepsilon}||x - \bar{x}|| = |x| + \frac{1}{2}|x| \geq 0 = \max_{u \in \mathcal{U}} f(\bar{x}, u), \quad \forall x \in F.$$

Note that the qualification condition (RCQ) is satisfied at $\bar{x} = 0 \in F$. Indeed, we have $N^C(\bar{x}; \Omega) = N^C(\bar{x}; (-\infty, 0]) = [0, +\infty)$, $\partial_x^C g_t(\bar{x}, v_t) = \{0\}$ for any $v_t \in \mathcal{V}_t$, $t \in T$ and

$$\bigcup_{\substack{\lambda \in A(\bar{x}) \\ v_t \in \mathcal{V}_t}} \left[\sum_{t \in T} \lambda_t \partial_x^C g_t(\bar{x}, v_t) \right] + N^C(\bar{x}; \Omega) = [0, +\infty).$$

Besides, one has $N^C(\bar{x}; F) = N^C(\bar{x}; (-\infty, 0]) = [0, +\infty)$. Therefore, the qualification condition (RCQ) holds at $\bar{x} = 0$. Take $\varepsilon = \frac{1}{4}$, $\bar{u} = 1$, $\bar{x} = 0$ and $\mathbb{B} = [-1, 1]$. It is easy to see that $\partial_x^C f(\bar{x}, \bar{u}) = [-1, 1]$, $\partial_x^C g_t(\bar{x}, v_t) = \{0\}$, $\forall v_t \in \mathcal{V}_t$, $t \in T$ and

$$\begin{aligned} 0 \in \left[-\frac{3}{2}, +\infty\right) &= [-1, 1] + [0, +\infty) + \left[-\frac{1}{2}, \frac{1}{2}\right] \\ &= \partial_x^C f(\bar{x}, \bar{u}) + \sum_{t \in T} \lambda_t \partial_x^C g_t(\bar{x}, v_t) + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B}, \end{aligned}$$

for any $(\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ and $v_t \in \mathcal{V}_t$, $t \in T$, $\lambda_t g_t(\bar{x}, \bar{v}_t) = 0$, and $f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u) = 0$. Therefore, Theorem 1 is satisfied.

Now, we will introduce a concept of a robust ε-approximate (KKT) condition for problem (USIP).

Definition 8 Let $\varepsilon \in \mathbb{R}_+ \backslash \{0\}$. A point $\bar{x} \in F$ is said to satisfy the robust ε-approximate (KKT) condition with respect to problem (USIP) if there exist $(\bar{\lambda}_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, which $(\bar{\lambda}_t)_{t \in T}$ are not all zero and $\bar{u} \in \mathcal{U}$, $\bar{v}_t \in \mathcal{V}_t$, $t \in T$, such that

$$0 \in \partial_x^C f(\bar{x}, \bar{u}) + \sum_{t \in T} \bar{\lambda}_t \partial_x^C g_t(\bar{x}, \bar{v}_t) + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B}, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0,$$

and $f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u)$.

Motivated by the definition of generalized convexity due to [13], we will introduce a concept of ε-quasi generalized convexity as follows:

Definition 9 Let $g_T := (g_t)_{t \in T}$ and $\varepsilon \in \mathbb{R}_+ \backslash \{0\}$. We say that (f, g_T) is ε-quasi generalized convex on Ω at $\bar{x} \in \Omega$, if for any $x \in \Omega$, $x_0 \in \partial_x^C f(\bar{x}, u)$, $u \in \mathcal{U}$ and $x_t \in \partial_x^C g_t(\bar{x}, v_t)$, $v_t \in \mathcal{V}_t$, $t \in T$, there exists $w \in T^C(\bar{x}; \Omega)$ such that

$$\begin{aligned} &\langle x_0, w \rangle + \sqrt{\varepsilon}||x - \bar{x}|| \geq 0 \Rightarrow f(x, u) + \sqrt{\varepsilon}||x - \bar{x}|| \geq f(\bar{x}, u), \\ &g_t(x, v_t) \leq g_t(\bar{x}, v_t) \Rightarrow \langle x_t, w \rangle \leq 0, \quad \forall t \in T, \end{aligned}$$

and

$$\langle b, w \rangle \leq ||x - \bar{x}||, \quad \forall b \in \mathbb{B}.$$

Remark 3 (i) According to Remark 3.9 in [13], if (f, g_T) is generalized convex (Definition 4.2 in [48]) at $\bar{x}$, then for $\varepsilon \in \mathbb{R}_+\backslash\{0\}$, (f, g_T) is ε-quasi generalized convex at $\bar{x}$.
(ii) Furthermore, by a similar argument as in [13, Example 3.10], we can prove that the class of ε-quasi generalized convex functions is properly larger the one of generalized convex functions (Definition 4.2 in [48]).

Now, we will provide an example to illustrate Definition 9.

Example 2 Let $x \in \mathbb{R}$, $\Omega = [0, +\infty) \subset \mathbb{R}$, $t \in T = [0, 1]$, $u \in \mathcal{U} = [1, 2]$ and $v_t \in \mathcal{V}_t = [2 - t, t + 2]$ for any $t \in T$, $\mathbb{B} = [-1, 1]$. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{V}_t \to \mathbb{R}$ be defined by

$$f(x) = u(|x| + x^3) \quad \text{and} \quad g_t(x, v_t) = -v_t x^2.$$

Let us consider $\bar{x} = 0, \bar{u} = 1$, we have $\partial^C f(\bar{x}, \bar{u}) = [-1, 1]$ and $\partial^C_x g(\bar{x}, v_t) = \{0\}$, $T^C(\bar{x}; \Omega) = T^C(\bar{x}; [0, +\infty)) = [0, +\infty)$. Now, consider $x_0 = 0 \in \partial^C f(\bar{x}, \bar{u})$, $x_t \in \partial^C_x g(\bar{x}, v_t)$. For any $x \in \Omega$, by taking $w = x \in \Omega = [0, +\infty) = T^C(\bar{x}; \Omega)$, $\varepsilon = \dfrac{1}{4}$, it follows that

$$\begin{aligned}
&\langle x_0, w\rangle + \sqrt{\varepsilon}||x - \bar{x}|| = \frac{1}{2}|x| \geq 0, \quad \forall w \in T^C(\bar{x}; \Omega)\\
&\quad \Rightarrow f(x, u) + \sqrt{\varepsilon}||x - \bar{x}|| = u(|x| + x^3) + \frac{1}{2}|x| \geq 0 = f(\bar{x}, u), \quad \forall u \in \mathcal{U},\\
&g_t(x, v_t) = -v_t x^2 \leq 0 = g_t(\bar{x}, v_t), \quad \forall v_t \in \mathcal{V}_t, t \in T\\
&\quad \Rightarrow \langle x_t, w\rangle = 0 \leq 0, \quad \forall w \in T^C(\bar{x}; \Omega),
\end{aligned}$$

and

$$||x - \bar{x}|| = |x| \geq x \geq bx = \langle b, w\rangle, \quad \forall b \in \mathbb{B} = [-1, 1].$$

This shows that (f, g_T) is ε-quasi generalized convex on Ω at $\bar{x} \in \Omega$.

Next, we will propose a type sufficient optimality condition for a robust ε-quasi solution of problem (USIP) in the following theorem.

Theorem 2 *Let $\varepsilon \in \mathbb{R}_+\backslash\{0\}$ and $g_T := (g_t)_{t\in T}$. Suppose that $\bar{x} \in F$ satisfies the robust ε-approximate* (KKT) *condition with respect to problem* (USIP)*. If (f, g_T) is ε-quasi generalized convex on Ω at $\bar{x} \in \Omega$, then $\bar{x}$ is a robust ε-quasi solution of problem* (USIP).

Proof Since $\bar{x} \in F$ satisfies the robust ε-approximate (KKT) condition with respect to problem (USIP), there exist $(\bar{\lambda}_t)_{t\in T} \in \mathbb{R}^{(T)}_+$, which $(\bar{\lambda}_t)_{t\in T}$ are not all zero and $x_0 \in \partial^C_x f(\bar{x}, \bar{u})$, $\bar{u} \in \mathcal{U}$, $x_t \in \partial^C_x g(\bar{x}, \bar{v}_t)$, $\bar{v}_t \in \mathcal{V}_t$, $t \in T$ and $b \in \mathbb{B}$ such that $\bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0$ and

$$-\left(x_0 + \sum_{t\in T} \bar{\lambda}_t x_t + \sqrt{\varepsilon} b\right) \in N^C(\bar{x}; \Omega) \tag{8}$$

and

$$f(\bar{x}, \bar{u}) = \max_{u\in\mathcal{U}} f(\bar{x}, u). \tag{9}$$

Suppose on contrary that $\bar{x}$ is not a robust ε-quasi solution of problem (USIP). It then follows that there exists $x \in F$ such that

$$\max_{u\in\mathcal{U}} f(x, u) + \sqrt{\varepsilon}||x - \bar{x}|| < \max_{u\in\mathcal{U}} f(\bar{x}, u). \tag{10}$$

Note further that

$$\max_{u\in\mathcal{U}} f(x,u) \geq f(x,\bar{u}), \quad \bar{u} \in \mathcal{U}. \tag{11}$$

From (9), (10), and (11), we deduce that

$$f(x,\bar{u}) + \sqrt{\varepsilon}||x-\bar{x}|| < f(\bar{x},\bar{u}). \tag{12}$$

On the other hand, if $t \in T(\bar{\lambda})$, then $g_t(\bar{x},\bar{v}_t) = 0$. Note that for any $x \in F$, $g_t(x,\bar{v}_t) \leq 0$ for any $t \in T$. It follows that

$$g_t(x,\bar{v}_t) \leq 0 = g_t(\bar{x},\bar{v}_t), \tag{13}$$

for any $x \in F$ and $t \in T(\bar{\lambda})$. By the ε-quasi generalized convexity of (f, g_T) on Ω at $\bar{x} \in \Omega$ and (12), (13), for such x, there exist $x_0 \in \partial_x^C f(\bar{x},\bar{u})$, $\bar{u} \in \mathcal{U}$, $x_t \in \partial_x^C g_t(\bar{x},\bar{v}_t)$, $\bar{v}_t \in \mathcal{V}_t$, $t \in T$ and $w \in T^C(\bar{x};\Omega)$ such that

$$\langle x_0, w\rangle + \sqrt{\varepsilon}||x-\bar{x}|| < 0, \tag{14}$$

$$\langle x_t, w\rangle \leq 0, \tag{15}$$

$$\langle b, w\rangle \leq ||x-\bar{x}||, \quad \forall b \in \mathbb{B}. \tag{16}$$

On the other hand, by (15), we conclude that

$$\sum_{t\in T} \bar{\lambda}_t \langle x_t, w\rangle \leq 0. \tag{17}$$

From (14) and (17), we can assert that

$$\langle x_0, w\rangle + \sqrt{\varepsilon}||x-\bar{x}|| + \sum_{t\in T} \bar{\lambda}_t \langle x_t, w\rangle < 0. \tag{18}$$

Combining (16) and (18), we imply that

$$\langle x_0, w\rangle + \sqrt{\varepsilon}\langle b, w\rangle + \sum_{t\in T} \bar{\lambda}_t \langle x_t, w\rangle < 0. \tag{19}$$

On the other hand, since $w \in T^C(\bar{x};\Omega)$ and (8), we obtain

$$\langle x_0, w\rangle + \sqrt{\varepsilon}\langle b, w\rangle + \sum_{t\in T} \bar{\lambda}_t \langle x_t, w\rangle \geq 0,$$

which contradicts to (19). Therefore, $\bar{x}$ is a robust ε-quasi solution of problem (USIP). □

Remark 4 Theorem 2 improves Theorem 3.2 in [22] and Theorem 3.4 in [24].

Finally in this section, we will provide an example to show the importance of the ε-quasi generalized convexity of (f, g_T) in Theorem 2.

Example 3 Let $x \in \mathbb{R}$, $\Omega = (-\infty, 0] \subset \mathbb{R}$, $u \in \mathcal{U} = [1,2]$, $t \in T = [0,1]$ and $v_t \in \mathcal{V}_t = [2-t, 2+t]$ for any $t \in T$. Consider the functions

$$f(x,u) = \begin{cases} ux^2\cos\frac{1}{x}, & \text{if } x \neq 0,\\ 0, & \text{if } x = 0,\end{cases}$$

and

$$g_t(x,v_t) = tx^2 + v_t x.$$

Then, $F = [-1, 0]$, $N^C(\bar{x}; \Omega) = N^C(\bar{x}; (-\infty, 0]) = [0, +\infty)$, and $T^C(\bar{x}; \Omega) = T^C(\bar{x}; (-\infty, 0]) = (-\infty, 0]$. By selecting $\bar{x} = 0$, $\bar{u} = 1$ and $\bar{v}_t = 2 - t$, one has

$$\partial_x^C f(\bar{x}, \bar{u}) = [-1, 1] \quad \text{and} \quad \partial_x^C g_t(\bar{x}, \bar{v}_t) = \{2 - t\}.$$

Now, take an arbitrarily ε such that $0 < \varepsilon \leq \frac{1}{4\pi^2}$. Then, it implies that $\bar{x} \in F$ satisfies the robust ε-approximate (KKT) condition. Indeed, let us select $\varepsilon = \frac{1}{4\pi^2}$, $\bar{x} = 0$, $\bar{u} = 1$, $\bar{\lambda}_t = 0$, $\bar{v}_t = 2 - t$ and $\mathbb{B} = [-1, 1]$. Then,

$$\begin{aligned} 0 \in \left[-1 - \frac{1}{2\pi}, +\infty\right) &= [-1, 1] + [0, +\infty) + \left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right] \\ &= \partial_x^C f(\bar{x}, \bar{u}) + \sum_{t \in T} \bar{\lambda}_t \partial_x^C g_t(\bar{x}, \bar{v}_t) + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B}, \end{aligned}$$

$\bar{\lambda}_t g(\bar{x}, \bar{v}_t) = 0$ and $\max_{u \in \mathcal{U}} f(x, u) = f(\bar{x}, \bar{u}) = 0$. However, $\bar{x} = 0$ is not a robust ε-quasi solution of problem (USIP). In order to see this, let us take $x = -\frac{1}{\pi} \in F = [-1, 0]$ and $\bar{x} = 0$, $\varepsilon = \frac{1}{4\pi^2}$. Then,

$$\max_{u \in \mathcal{U}} f(x, u) + \sqrt{\varepsilon}||x - \bar{x}|| = -\frac{1}{\pi^2} + \frac{1}{2\pi}.\frac{1}{\pi} = -\frac{1}{2\pi^2} < 0 = \max_{u \in \mathcal{U}} f(\bar{x}, u).$$

The reason is that (f, g_T) is not ε-quasi generalized convex at $\bar{x} = 0$. Indeed, take $x = -\frac{1}{\pi} \in F = [-1, 0]$, $\varepsilon = \frac{1}{4\pi^2}$, $\bar{x} = 0$, $\bar{u} = 1$ and $x_0 = 0 \in \partial_x^C f(\bar{x}, \bar{u}) = [-1, 1]$. Clearly,

$$\langle x_0, w \rangle + \sqrt{\varepsilon}||x - \bar{x}|| = \frac{1}{2\pi}.\frac{1}{\pi} = \frac{1}{2\pi^2} > 0, \quad \forall w \in T^C(\bar{x}; \Omega) = (-\infty, 0].$$

However,

$$f(x, \bar{u}) + \sqrt{\varepsilon}||x - \bar{x}|| = -\frac{1}{\pi^2} + \frac{1}{2\pi}.\frac{1}{\pi} = -\frac{1}{2\pi^2} < 0 = f(\bar{x}, \bar{u}).$$

4 Mond–Weir duality for approximate solution

In this section, we address a Mond–Weir type dual problem (MUSID) with respect to the primal problem (USIP).

Let $x \in \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is a nonempty closed (not necessarily convex) set, $\lambda \in \mathbb{R}_+^{(T)}$, and $u \in \mathcal{U}$, $v_t \in \mathcal{V}_t$, $t \in T$. Now, we introduce the Lagrangian function L with respect to problem (USIP) as follows:

$$L(x, \lambda, u, v_t) := f(x, u).$$

Let $\varepsilon \in \mathbb{R}_+ \backslash \{0\}$. We consider the dual problem (MUSID) of problem (USIP) as follows:

$$\text{(MUSID)} \quad \begin{cases} \max \; L(y, \lambda, u, v_t) \\ \text{s.t.} \quad 0 \in \partial_x^C f(y, u) + \sum_{t \in T} \lambda_t \partial_x^C g_t(y, v_t) + N^C(y; \Omega) + \sqrt{\varepsilon}\mathbb{B}, \\ \qquad \sum_{t \in T} \lambda_t g_t(y, v_t) \geq 0, \\ \qquad y \in \Omega, \lambda \in \mathbb{R}_+^{(T)}, u \in \mathcal{U}, v_t \in \mathcal{V}_t, t \in T. \end{cases}$$

The robust feasible set of problem (MUSID) is defined by

$$F_{\text{MUSID}} := \Bigg\{ (y, \lambda, u, v_t) \in \Omega \times \mathbb{R}^{(T)}_{+} \times \mathcal{U} \times \mathcal{V}_t) \mid 0 \in \partial^C_x f(y, u) + \sum_{t \in T} \lambda_t \partial^C_x g_t(y, v_t) + N^C(y; \Omega) + \sqrt{\varepsilon}\mathbb{B}, \sum_{t \in T} \lambda_t g_t(y, v_t) \geq 0 \Bigg\}.$$

Now, we will introduce the following definition of a robust ε-quasi solution for problem (MUSID).

Definition 10 Let $\varepsilon \in \mathbb{R}_+ \backslash \{0\}$. We say that $(\bar{y}, \bar{\lambda}, \bar{u}, \bar{v}_t) \in F_{\text{MUSID}}$ is a robust ε-quasi solution of problem (MUSID) if for any $(y, \lambda, u, v_t) \in F_{\text{MUSID}}$,

$$L(\bar{y}, \bar{\lambda}, \bar{u}, \bar{v}_t) + \sqrt{\varepsilon}||\bar{y} - y|| \geq L(y, \lambda, u, v_t).$$

The following theorem describes duality relations between the primal problem (USIP) and the dual problem (MUSID).

Theorem 3 *Let $\varepsilon \in \mathbb{R}_+ \backslash \{0\}$ and $g_T := (g_t)_{t \in T}$. Suppose that $\bar{x}$ is an ε-quasi solution of problem* (USIP)*, the qualification condition* (RCQ) *is satisfied at $\bar{x}$ and $f(\bar{x}, .)$ is concave on $\mathcal{U}$. Then there exists $(\bar{\lambda}, \bar{u}, \bar{v}_t) \in \mathbb{R}^{(T)}_{+} \times \mathcal{U} \times \mathcal{V}_t$ such that $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t) \in F_{\text{MUSID}}$ and* $\max_{u \in \mathcal{U}} f(\bar{x}, u) = L(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t) = f(\bar{x}, \bar{u})$*. Besides, if (f, g_T) is ε-quasi generalized convex on Ω at any $y \in \Omega$, then $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t)$ is a robust ε-quasi solution of problem* (MUSID).

Proof According to Theorem 1, there exist $\bar{u} \in \mathcal{U}$, $(\bar{\lambda}_t)_{t \in T} \in \mathbb{R}^{(T)}_{+}$ and $\bar{v}_t \in \mathcal{V}_t$, $t \in T$, such that

$$0 \in \partial^C_x f(\bar{x}, \bar{u}) + \sum_{t \in} \bar{\lambda}_t \partial^C_x g_t(\bar{x}, \bar{v}_t) + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B}, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0, \tag{20}$$

and

$$f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u). \tag{21}$$

Therefore, $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t) \in F_{\text{MUSID}}$. From (21), we have

$$\max_{u \in \mathcal{U}} f(\bar{x}, u) = f(\bar{x}, \bar{u}) = L(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t). \tag{22}$$

Now, we prove that if (f, g_T) is ε-quasi generalized convex on Ω at any $y \in \Omega$, then $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t)$ is a robust ε-quasi solution of problem (MUSID). Suppose on contrary that $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t)$ is not a robust ε-quasi solution of problem (MUSID). Then there exists $(y, \lambda, u, v_t) \in F_{\text{MUSID}}$ such that

$$L(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t) + \sqrt{\varepsilon}||\bar{x} - y|| < L(y, \lambda, u, v_t), \tag{23}$$

where $L(y, \lambda, u, v_t) = f(y, u)$. From (22) and (23), we can assert that

$$f(y, u) > f(\bar{x}, \bar{u}) + \sqrt{\varepsilon}||\bar{x} - y||. \tag{24}$$

Since $(y, \lambda, u, v_t) \in F_{\text{MUSID}}$, there exist $\lambda \in \mathbb{R}^{(T)}_{+}$, $x_0 \in \partial^C_x f(y, u)$, $u \in \mathcal{U}$, $x_t \in \partial^C_x g(y, v_t)$, $v_t \in \mathcal{V}_t$, $t \in T$ and $b \in \mathbb{B}$ such that

$$-\left(x_0 + \sum_{t \in T} \lambda_t x_t + \sqrt{\varepsilon} b \right) \in N^C(y; \Omega), \tag{25}$$

$$\sum_{t \in T} \lambda_t g_t(y, v_t) \geq 0. \tag{26}$$

From (26) and $(\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)}$, it follows that $g_t(y, v_t) \geq 0$ if $t \in T(\lambda)$. Note that for $\bar{x} \in F$, we have $g_t(\bar{x}, v_t) \leq 0$ for any $t \in T$. It deduces that

$$g_t(\bar{x}, v_t) \leq 0 \leq g_t(y, v_t), \tag{27}$$

for any $\bar{x} \in F$ and $t \in T(\lambda)$. By the ε-quasi generalized convexity of (f, g_T) on Ω at $y \in \Omega$ and (24), (27), for such $\bar{x}$, there exist $x_0 \in \partial_x^C f(y, u)$, $u \in \mathcal{U}$, $x_t \in \partial_x^C g_t(y, v_t)$, $v_t \in \mathcal{V}_t$, $t \in T$ and $w \in T^C(y; \Omega)$ such that

$$\langle x_0, w\rangle + \sqrt{\varepsilon}||\bar{x} - y|| < 0, \tag{28}$$

$$\langle x_t, w\rangle \leq 0, \tag{29}$$

$$\langle b, w\rangle \leq ||\bar{x} - y||, \quad \forall b \in \mathbb{B}. \tag{30}$$

Thus, we deduce from (29) that

$$\sum_{t\in T} \lambda_t \langle x_t, w\rangle \leq 0. \tag{31}$$

From (28) and (31), we can assert that

$$\langle x_0, w\rangle + \sqrt{\varepsilon}||\bar{x} - y|| + \sum_{t\in T} \lambda_t \langle x_t, w\rangle < 0. \tag{32}$$

Combining (30) and (32), we imply that

$$\langle x_0, w\rangle + \sqrt{\varepsilon}\,\langle b, w\rangle + \sum_{t\in T} \lambda_t \langle x_t, w\rangle < 0. \tag{33}$$

On the other hand, since $w \in T^C(y; \Omega)$ and (25), we obtain

$$\langle x_0, w\rangle + \sqrt{\varepsilon}\,\langle b, w\rangle + \sum_{t\in T} \lambda_t \langle x_t, w\rangle \geq 0,$$

which contradicts to (33). Therefore, $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t)$ is a robust ε-quasi solution of problem (MUSID). □

Remark 5 Theorem 3 improves Theorem 4.2 in [22] and Theorem 4.6 in [24].

The next example asserts the importance of the qualification condition (RCQ) imposed in Theorem 3. More precisely, if $\bar{x}$ is a robust ε-quasi solution of problem (USIP) at which the qualification condition (RCQ) is not satisfied, then we may not find out a triplet $(\bar{\lambda}, \bar{u}, \bar{v}_t) \in \mathbb{R}_+^{(T)} \times \mathcal{U} \times \mathcal{V}_t$ such that $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t)$ belongs to the robust feasible set F_{MUSID} of the dual problem (MUSID).

Example 4 We consider problem (USIP) with $\Omega = (-\infty, 0] \subset \mathbb{R}$ and $x \in \mathbb{R}$, $u \in \mathcal{U} = [0, 1]$, $t \in T = [0, 1]$ and $v_t \in \mathcal{V}_t = [2 - t, 2 + t]$ for any $t \in T$. Take the functions $f(x, u) = x + u - 1$ and $g_t(x, v_t) = v_t x^2$. We can see that the robust feasible set $F = \{0\}$. Now, take $\bar{x} = 0$, $\varepsilon = \dfrac{1}{4}$. Then, it is easy to prove that $\bar{x} = 0$ is a robust ε-quasi solution of problem (USIP). Indeed, one has

$$\max_{u\in\mathcal{U}} f(x, u) + \sqrt{\varepsilon}||x - \bar{x}|| = x + \frac{1}{2}|x| \geq 0 = \max_{u\in\mathcal{U}} f(\bar{x}, u), \quad \forall x \in F.$$

Next, consider the dual problem (MUSID) with respect to problem (USIP). Take $\varepsilon = \dfrac{1}{4}$, $\bar{u} = 1$ and $\mathbb{B} = [-1, 1]$. It is easy to see that $\partial_x^C f(\bar{x}, \bar{u}) = \{1\}$, $\partial_x^C g_t(\bar{x}, v_t) = \{0\}$, $\forall v_t \in \mathcal{V}_t$, $t \in$

T, and

$$0 \notin \left[\frac{1}{2}, \frac{3}{2}\right] = \{1\} + \left[-\frac{1}{2}, \frac{1}{2}\right] = \partial_x^C f(\bar{x}, \bar{u}) + \sum_{t\in T} \lambda_t \partial_x^C g_t(\bar{x}, v_t) + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B},$$

for any $(\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ and $\forall v_t \in \mathcal{V}_t, t \in T$. It follows easily that $(\bar{x}, \lambda, \bar{u}, v_t) \notin F_{\text{MUSID}}$ for any $\lambda \in \mathbb{R}_+^{(T)}$ and $\forall v_t \in \mathcal{V}_t, t \in T$. The reason is that the qualification condition (RCQ) is not satisfied at $\bar{x} = 0 \in F$. Indeed, we have $N^C(\bar{x}; \Omega) = N^C(\bar{x}; (-\infty, 0]) = [0, +\infty)$, $\partial_x^C g_t(\bar{x}, v_t) = \{0\}$, for any $v_t \in \mathcal{V}_t, t \in T$ and

$$\bigcup_{\substack{\lambda\in A(\bar{x})\\ v_t\in\mathcal{V}_t}} \left[\sum_{t\in T} \lambda_t \partial_x^C g_t(\bar{x}, v_t)\right] + N^C(\bar{x}; \Omega) = [0, +\infty).$$

Besides, one has $N^C(\bar{x}; F) = \mathbb{R}$. Therefore, the qualification condition (RCQ) is not satisfied at $\bar{x} = 0$. Hence, Theorem 3 is not valid.

Now, we present an example to show the importance of the ε-quasi generalized convexity of (f, g_T) in Theorem 3.

Example 5 We consider problem (USIP) with $\varepsilon = \dfrac{1}{25}$, $\Omega = [-1, 0] \subset \mathbb{R}, x \in \mathbb{R}, u \in \mathcal{U} = [0, 1], t \in T = [0, 1]$ and $v_t \in \mathcal{V}_t = [2 - t, 2 + t]$ for any $t \in T$. Take the functions $f(x, u) = x^3 + u$ and $g_t(x, v_t) = -v_t x^2$. We can see that the robust feasible set $F = [-1, 0]$. Therefore, $\bar{x} = -1$ is a robust ε-quasi solution of problem (USIP). Indeed, one has

$$\max_{u\in\mathcal{U}} f(x, u) + \sqrt{\varepsilon}||x - \bar{x}|| = x^3 + 1 + \frac{1}{5}|x + 1| \geq 0 = \max_{u\in\mathcal{U}} f(\bar{x}, u), \quad \forall x \in F.$$

Note that the qualification condition (RCQ) is satisfied at $\bar{x} = -1 \in F$. Indeed, we have $N^C(\bar{x}; \Omega) = N^C(\bar{x}; [-1, 0]) = (-\infty, 0]$, $\partial_x^C g_t(\bar{x}, v_t) = \{0\}$, for any $v_t \in \mathcal{V}_t, t \in T$ and

$$\bigcup_{\substack{\lambda\in A(\bar{x})\\ v_t\in\mathcal{V}_t}} \left[\sum_{t\in T} \lambda_t \partial_x^C g_t(\bar{x}, v_t)\right] + N^C(\bar{x}; \Omega) = (-\infty, 0].$$

Besides, one has $N^C(\bar{x}; F) = N^C(\bar{x}; [-1, 0]) = (-\infty, 0]$. Therefore, the qualification condition (RCQ) holds at $\bar{x} = -1$. Now, consider the dual problem (MUSID) with respect to problem (USIP). Take $\varepsilon = \dfrac{1}{25}$, $\bar{\lambda} = 0, \bar{u} = 0, \bar{x} = -1, \bar{v}_t = 2 - t$ and $\mathbb{B} = [-1, 1]$. It is easy to see that $\partial_x^C f(\bar{x}, \bar{u}) = \{3\}$, $\partial_x^C g_t(\bar{x}, \bar{v}_t) = \{2(2 - t)\}$ and

$$\begin{aligned} 0 \in \left(-\infty, \frac{16}{5}\right] &= \{3\} + (-\infty, 0] + \left[-\frac{1}{5}, \frac{1}{5}\right] \\ &= \partial_x^C f(\bar{x}, \bar{u}) + \sum_{t\in T} \bar{\lambda}_t \partial_x^C g_t(\bar{x}, \bar{v}_t) + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B} \end{aligned}$$

and $\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0$. It follows easily that

$$(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t) \in F_{\text{MUSID}} \text{ and } \max_{u\in\mathcal{U}} f(\bar{x}, u) = L(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t) = f(\bar{x}, \bar{u}) = 0,$$

where F_{MUSID} is the robust feasible set of problem (MUSID). However, $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t)$ is not a robust ε-quasi solution of problem (MUSID). In order to see this, let us take $\varepsilon =$

$\frac{1}{25}$, $(y, \lambda, u, v_t) = (0, 1, 1, 2+t) \in \Omega \times \mathbb{R}_+^{(T)} \times \mathcal{U} \times \mathcal{V}_t$. Clearly,

$$L(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t) + \sqrt{\varepsilon}||\bar{x} - y|| = \frac{1}{5} < 1 = L(y, \lambda, u, v_t).$$

The reason is that (f, g_T) is not the ε-quasi generalized convex on Ω at $y = 0 \in \Omega$. Indeed, by choosing $\varepsilon = \frac{1}{25}$, $z = -\frac{1}{2} \in \Omega$, $u = 1$ and $x_0 \in \partial_x^C f(y, u) = \{0\}$, $T^C(y; \Omega) = (-\infty, 0]$, we have

$$\langle x_0, w\rangle + \sqrt{\varepsilon}||z - y|| = \frac{1}{10} > 0, \quad \forall w \in T^C(y; \Omega) = (-\infty, 0].$$

However,

$$f(z, u) + \sqrt{\varepsilon}||z - y|| = -\frac{1}{8} + 1 + \frac{1}{10} = \frac{39}{40} < 1 = f(y, u).$$

Theorem 4 *Let $\varepsilon \in \mathbb{R}_+\backslash\{0\}$ and $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t) \in F_{\text{MUSID}}$ such that $\max_{u\in\mathcal{U}} f(\bar{x}, u) = L(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t) = f(\bar{x}, \bar{u})$. If $\bar{x} \in F$ and (f, g_T) is ε-quasi generalized convex on Ω at $\bar{x} \in \Omega$, then $\bar{x}$ is a robust ε-quasi solution of problem* (USIP).

Proof Since $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}_t) \in F_{\text{MUSID}}$, there exist $(\bar{\lambda}_t)_{t\in T} \in \mathbb{R}_+^{(T)}$, $x_0 \in \partial_x^C f(\bar{x}, \bar{u})$, $\bar{u} \in \mathcal{U}$, $x_t \in \partial_x^C g(\bar{x}, \bar{v}_t)$, $\bar{v}_t \in \mathcal{V}_t$, $t \in T$ and $b \in \mathbb{B}$ such that

$$-\left(x_0 + \sum_{t\in T} \bar{\lambda}_t x_t + \sqrt{\varepsilon} b\right) \in N^C(\bar{x}; \Omega), \tag{34}$$

$$\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) \geq 0. \tag{35}$$

Suppose on contrary that $\bar{x}$ is not a robust ε-quasi solution of problem (USIP). It then follows that there exists $x \in F$ such that

$$\max_{u\in\mathcal{U}} f(x, u) + \sqrt{\varepsilon}||x - \bar{x}|| < \max_{u\in\mathcal{U}} f(\bar{x}, u). \tag{36}$$

Note further that

$$\max_{u\in\mathcal{U}} f(x, u) \geq f(x, \bar{u}), \quad \bar{u} \in \mathcal{U}. \tag{37}$$

From (36), (37), and $f(\bar{x}, \bar{u}) = \max_{u\in\mathcal{U}} f(\bar{x}, u)$, we deduce that

$$f(x, \bar{u}) + \sqrt{\varepsilon}||x - \bar{x}|| < f(\bar{x}, \bar{u}). \tag{38}$$

By (35) and $(\bar{\lambda}_t)_{t\in T} \in \mathbb{R}_+^{(T)}$, it follows that $g_t(\bar{x}, \bar{v}_t) \geq 0$ if $t \in T(\bar{\lambda})$. Note that for $x \in F$, we have $g_t(x, \bar{v}_t) \leq 0$ for any $t \in T$. We deduce that

$$g_t(x, \bar{v}_t) \leq 0 \leq g_t(\bar{x}, \bar{v}_t), \tag{39}$$

for any $x \in F$ and $t \in T(\bar{\lambda})$. By the ε-quasi generalized convexity of (f, g_T) on Ω at $\bar{x} \in \Omega$ and (38), (39), for such x, there exist $x_0 \in \partial_x^C f(\bar{x}, \bar{u})$, $\bar{u} \in \mathcal{U}$, $x_t \in \partial_x^C g_t(\bar{x}, \bar{v}_t)$, $\bar{v}_t \in \mathcal{V}_t$, $t \in T$ and $w \in T^C(\bar{x}; \Omega)$ such that

$$\langle x_0, w\rangle + \sqrt{\varepsilon}||x - \bar{x}|| < 0, \tag{40}$$

$$\langle x_t, w\rangle \leq 0, \tag{41}$$

$$\langle b, w\rangle \leq ||x - \bar{x}||, \quad \forall b \in \mathbb{B}. \tag{42}$$

Then, from (41), we follow that

$$\sum_{t \in T} \bar{\lambda}_t \langle x_t, w \rangle \leq 0. \tag{43}$$

From (40) and (43), we can assert that

$$\langle x_0, w \rangle + \sqrt{\varepsilon}||x - \bar{x}|| + \sum_{t \in T} \bar{\lambda}_t \langle x_t, w \rangle < 0. \tag{44}$$

Combining (42) and (44), we imply that

$$\langle x_0, w \rangle + \sqrt{\varepsilon} \langle b, w \rangle + \sum_{t \in T} \bar{\lambda}_t \langle x_t, w \rangle < 0. \tag{45}$$

On the other hand, since $w \in T^C(\bar{x}; \Omega)$ and (34), we obtain

$$\langle x_0, w \rangle + \sqrt{\varepsilon} \langle b, w \rangle + \sum_{t \in T} \bar{\lambda}_t \langle x_t, w \rangle \geq 0,$$

which contradicts to (45). Therefore, $\bar{x}$ is a robust ε-quasi solution of problem (USIP). □

Remark 6 Theorem 4 improves Theorem 4.3 in [22].

5 Application to fractional semi-infinite programming problem

In this section, we consider nonsmooth fractional semi-infinite programming with uncertainty data in constraints:

$$\text{(UFSIP)} \quad \begin{array}{l} \min\limits_{x \in \Omega} \quad f(x) := \dfrac{p(x)}{q(x)}, \\ \text{s.t. } g_t(x, v_t) \leq 0, \quad \forall t \in T, \end{array}$$

where $\Omega \subset \mathbb{R}^n$ is a nonempty closed (not necessarily convex) set, T is a nonempty infinite index set, the functions $p, q : \mathbb{R}^n \to \mathbb{R}$, and $g_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $t \in T$ are locally Lipschitz functions, and for each $t \in T$, $v_t \in \mathbb{R}^q$ is an uncertain parameter, which belongs to some convex compact set $\mathcal{V}_t \subset \mathbb{R}^q$.

The uncertainty set-valued mapping $\mathcal{V} : T \rightrightarrows \mathbb{R}^q$ is defined as $\mathcal{V}(t) := \mathcal{V}_t$ for all $t \in T$. The notation $v \in \mathcal{V}$ means that v is a selection of $\mathcal{V}$, i.e., $v : T \to \mathbb{R}^q$ and $v_t \in \mathcal{V}_t$ for all $t \in T$. So, the uncertainty set is the graph of $\mathcal{V}$, that is, $\text{gph}\mathcal{V} := \{(t, v_t) \mid v_t \in \mathcal{V}_t, t \in T\}$.

In what follows, for the sake of convenience, we further assume that $q(x) > 0$ for all $x \in \Omega$, and that $p(\bar{x}) \leq 0$ for the reference point $\bar{x} \in \Omega$.

Definition 11 The robust feasible set of problem (UFSIP) is defined by

$$F := \{x \in \Omega \mid g_t(x, v_t) \leq 0, \forall v_t \in \mathcal{V}_t, \forall t \in T\}.$$

Definition 12 Let $\varepsilon \in \mathbb{R}_+ \backslash \{0\}$ and $f := \dfrac{p}{q}$. A point $\bar{x} \in F$ is said to be a robust ε-quasi solution to problem (UFSIP) if

$$f(x) + \sqrt{\varepsilon}||x - \bar{x}|| \geq f(\bar{x}), \quad \forall x \in F.$$

Theorem 5 *Let $\varepsilon \in \mathbb{R}_+\backslash\{0\}$. Suppose that $\bar{x} \in F$ is a robust ε-quasi solution of problem* (UFSIP). *Suppose that the qualification condition* (RCQ) *at $\bar{x}$ holds. Then, there exist $(\bar{\lambda}_t)_{t\in T} \in \mathbb{R}_+^{(T)}$, $\bar{v}_t \in \mathcal{V}_t$, $t \in T$, such that*

$$0 \in \frac{q(\bar{x})\partial^C p(\bar{x}) - p(\bar{x})\partial^C q(\bar{x})}{q^2(\bar{x})} + \sum_{t\in T} \bar{\lambda}_t \partial_x^C g_t(\bar{x}, \bar{v}_t) + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B}, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0. \tag{46}$$

Proof Suppose that $\bar{x} \in F$ is a robust ε-quasi solution of problem (UFSIP), it follows that $\bar{x}$ is a robust ε-quasi solution of problem (USIP) with $f := \frac{p}{q}$. By applying Theorem 1, there exist $(\bar{\lambda}_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ and $\bar{v}_t \in \mathcal{V}_t$, $t \in T$, such that

$$0 \in \partial^C f(\bar{x}) + \sum_{t\in} \bar{\lambda}_t \partial_x^C g_t(\bar{x}, \bar{v}_t) + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B}, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0. \tag{47}$$

Thanks to Lemma 3, one has

$$\partial^C f(\bar{x}) = \partial^C \left(\frac{p}{q}\right)(\bar{x}) \subset \frac{q(\bar{x})\partial^C p(\bar{x}) - p(\bar{x})\partial^C q(\bar{x})}{q^2(\bar{x})}. \tag{48}$$

Combining (47) with (48), we can assert that

$$0 \in \frac{q(\bar{x})\partial^C p(\bar{x}) - p(\bar{x})\partial^C q(\bar{x})}{q^2(\bar{x})} + \sum_{t\in T} \bar{\lambda}_t \partial_x^C g_t(\bar{x}, \bar{v}_t) + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B}, \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0.$$

The proof of the theorem is complete. □

The following simple example shows that the qualification condition (RCQ) is essential in Theorem 5.

Example 6 We consider problem (UFSIP) with $\Omega = (-\infty, 0] \subset \mathbb{R}$, $x \in \mathbb{R}$, $t \in T = [0, 1]$ and $v_t \in \mathcal{V}_t = [2 - t, 2 + t]$ for any $t \in T$. Take the functions $f(x) = \frac{p(x)}{q(x)}$, where $p(x) = x$, $q(x) = x^2 + 1$ and $g_t(x, v_t) = v_t x^2$. We can see that the robust feasible set $F = \{0\}$. Now, take $\bar{x} = 0$, $\varepsilon = \frac{1}{4}$. Then, it is easy to show that $\bar{x} = 0$ is a robust ε-quasi solution of problem (UFSIP). Indeed, one has

$$f(x) + \sqrt{\varepsilon}||x - \bar{x}|| = \frac{x}{x^2 + 1} + \frac{1}{2}|x| \geq 0 = f(\bar{x}), \quad \forall x \in F.$$

Since $N^C(\bar{x}; \Omega) = N^C(\bar{x}; (-\infty, 0]) = [0, +\infty)$, $\partial_x^C g_t(\bar{x}, v_t) = \{0\}$ at $\bar{x} = 0$ for any $v_t \in \mathcal{V}_t$, $t \in T$, one has

$$\bigcup_{\substack{\lambda\in A(\bar{x}) \\ v_t\in\mathcal{V}_t}} \left[\sum_{t\in T} \lambda_t \partial_x^C g_t(\bar{x}, v_t)\right] + N^C(\bar{x}; \Omega) = [0, +\infty).$$

Moreover, $N^C(\bar{x}; F) = \mathbb{R}$. Therefore, the qualification condition (RCQ) is not satisfied at $\bar{x} = 0$. On the other hand, take $\varepsilon = \frac{1}{4}$ and $\mathbb{B} = [-1, 1]$. It is easy to see that $\partial^C p(\bar{x}) =$

$\{1\}$, $\partial^C q(\bar{x}) = \{0\}$ and

$$0 \notin \left[\frac{1}{2}, +\infty\right) = \{1\} + [0, +\infty) + \left[-\frac{1}{2}, \frac{1}{2}\right]$$
$$= \frac{q(\bar{x})\partial^C p(\bar{x}) - p(\bar{x})\partial^C q(\bar{x})}{q^2(\bar{x})} + \sum_{t\in T} \lambda_t \partial_x^C g_t(\bar{x}, v_t) + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B},$$

for any $(\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ and $v_t \in \mathcal{V}_t$, $t \in T$. Hence, Theorem 5 is not valid.

The following simple example proves that, in general, a feasible point may satisfy the qualification condition (RCQ), but if this point is not a robust ε-quasi solution of problem (UFSIP), then (46) does not hold.

Example 7 Let $x \in \mathbb{R}$, $t \in T = [0, 1]$ and $v_t \in \mathcal{V}_t = [2 - t, 2 + t]$ for any $t \in T$. Consider the functions $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x) = x$, $q(x) = x^2 + 1$ and $g_t(x, v_t) := -v_t x^2$. We consider problem (UFSIP) with $\Omega := (-\infty, 0]$. By simple computation, one has $F = (-\infty, 0]$, $N^C(\bar{x}; \Omega) = N^C(\bar{x}; (-\infty, 0]) = [0, +\infty)$ and $\partial_x^C g_t(\bar{x}, v_t) = \{0\}, \forall v_t \in \mathcal{V}_t, t \in T$. Therefore, we have

$$\bigcup_{\substack{\lambda\in A(\bar{x})\\ v_t\in\mathcal{V}_t}} \left[\sum_{t\in T} \lambda_t \partial_x^C g_t(\bar{x}, v_t)\right] + N^C(\bar{x}; \Omega) = [0, +\infty).$$

Moreover, we have $N^C(\bar{x}; F) = N^C(\bar{x}; (-\infty, 0]) = [0, +\infty)$. Clearly, the qualification condition (RCQ) holds at $\bar{x}$. On the other hand, take $\bar{x} = 0$, $\varepsilon = \dfrac{1}{4}$ and $\mathbb{B} = [-1, 1]$. It is easy to see that $\partial^C p(\bar{x}) = \{1\}$, $\partial^C q(\bar{x}) = \{0\}$ and

$$0 \notin \left[\frac{1}{2}, +\infty\right) = \{1\} + [0, +\infty) + \left[-\frac{1}{2}, \frac{1}{2}\right]$$
$$= \frac{q(\bar{x})\partial^C p(\bar{x}) - p(\bar{x})\partial^C q(\bar{x})}{q^2(\bar{x})} + \sum_{t\in T} \lambda_t \partial_x^C g_t(\bar{x}, v_t) + N^C(\bar{x}; \Omega) + \sqrt{\varepsilon}\mathbb{B},$$

for any $(\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ and $v_t \in \mathcal{V}_t$, $t \in T$. Hence, condition (46) is not true. The reason is that $\bar{x} = 0$ is not a robust ε-quasi solution of problem (UFSIP). Indeed, we can choose $\varepsilon = \dfrac{1}{4}$, $x = -\dfrac{1}{5} \in F = (-\infty, 0]$. Clearly,

$$f(x) + \sqrt{\varepsilon}||x - \bar{x}|| = \frac{x}{x^2 + 1} + \frac{1}{2}|x| = -\frac{6}{65} < 0 = f(\bar{x}).$$

6 Conclusion

In this paper, we studied the optimality conditions for a robust ε-quasi solution of problem (USIP). Next, we established approximate duality theorems in term of Mond–Weir type which is formulated in approximate form. Finally, an application to fractional semi-infinite programming problem was provided. The results obtained in this paper improve the corresponding results reported in recent literature.

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