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Well-posedness and energy decay for some thermoelastic systems of Timoshenko type with Kelvin–Voigt damping

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Abstract
This paper aims to study the well-posedness and the stability of two thermoelastic systems. The derivation of the first system is based on a classical coupling between the mechanical equations of Timoshenko and the thermal effects which are based on the conductivity of Fourier's law. Whereas, the second system is derivable through a thermal coupling on the shear force. Furthermore, the damping of Kelvin–Voigt type is simultaneously presented in both the shear stress and the bending moment for the two systems.

Keywords Timoshenko system · Kelvin–Voigt damping · Viscoelsticity · Thermoelastic system · Well-posedness · Energy decay · Contraction semigroup

Mathematics Subject Classification 35B37 · 35L55 · 93D15 · 74D05

1 Introduction

The study of Kelvin–Voigt materials with viscoelastic structures has been the subject of study for many researchers, in this regard, we can refer to the basic works [8,16] through which it was explained that the damping structure of these materials depends on the combination of elasticity and viscosity. In another context, by considering the mechanical models under presence of an effective thermal conductivity, e.g., see papers [4,5] and the references therein, we note that the thermal effects at the level of the elastic structure may plays an important role in stabilizing the previous materials. In addition, looking at the papers [2,18], the authors used

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different types of dissipation, but did not present the Kelvin–Voigt type. Therefore, the present work will address this gap by studying the coupled hyperbolic-parabolic system to achieve stability results regarding the dissipation efficiency of Kelvin–Voigt in both thermoelastic type. More precisely, in this paper, on the one hand, we study Timoshenko system with thermal effects that are effective on the bending moment, the system is given as follows

$$
\begin{aligned}
&\rho_1 u_{tt} = k(u_x+\varphi)_x + \gamma_1 (u_x+\varphi)_{xt}, && \text{in } (0,1)\times\mathbb{R}_+,\\
&\rho_2 \varphi_{tt} = b\varphi_{xx} + \gamma_2\varphi_{xxt} - k(u_x+\varphi) - \gamma_1(u_x+\varphi)_t - \beta\theta_x, && \text{in } (0,1)\times\mathbb{R}_+,\\
&\rho_3\theta_t = \gamma_3\theta_{xx} - \beta\varphi_{xt}, && \text{in } (0,1)\times\mathbb{R}_+,
\end{aligned} \tag{1.1}
$$

subject to the Dirichlet boundary conditions for u and θ

$$u(0,t) = u(1,t) = \theta(0,t) = \theta(1,t) = 0, \quad t \geq 0, \tag{1.2}$$

and in addition to Neumann boundary condition for φ

$$\varphi_x(0,t) = \varphi_x(1,t) = 0, \quad t \geq 0. \tag{1.3}$$

On the other hand, we will study the Timoshenko system with thermal effects acting on shear force, the system is given as follows

$$
\begin{aligned}
&\rho_1 u_{tt} = k(u_x+\varphi)_x + \gamma_1 (u_x+\varphi)_{xt} - \beta\theta_x, && \text{in } (0,1)\times\mathbb{R}_+,\\
&\rho_2 \varphi_{tt} = b\varphi_{xx} + \gamma_2\varphi_{xxt} - k(u_x+\varphi) - \gamma_1(u_x+\varphi)_t + \beta\theta, && \text{in } (0,1)\times\mathbb{R}_+,\\
&\rho_3\theta_t = \gamma_3\theta_{xx} - \beta(u_x+\varphi)_t && \text{in } (0,1)\times\mathbb{R}_+,
\end{aligned} \tag{1.4}
$$

subject to the Dirichlet boundary conditions for φ and θ

$$\varphi(0,t) = \varphi(1,t) = \theta(0,t) = \theta(1,t) = 0, \quad t \geq 0, \tag{1.5}$$

in addition to Neumann boundary condition for u

$$u_x(0,t) = u_x(1,t) = 0, \quad t \geq 0. \tag{1.6}$$

The unknown functions

$$(u,\varphi,\theta) \;: (x,t) \in \mathcal{I}\times[0,\infty) \longmapsto \mathbb{R}, \quad \text{with} \quad \mathcal{I} = (0,1),$$

represents the transverse displacement, the angle of rotation and the relative temperature of the beam respectively. $\rho_1, \rho_2, \rho_3, k, b, \gamma_3, \gamma_1$ and γ_2 are strictly positive fixed constants, β is coupling coefficient.

The both systems (1.1) and (1.4) are complemented with the following initial conditions

$$
\begin{aligned}
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \varphi(x,0) = \varphi_0(x), \quad x \in \mathcal{I},\\
&\varphi_t(x,0) = \varphi_1(x), \quad \theta(x,0) = \theta_0(x), \quad x \in \mathcal{I}.
\end{aligned} \tag{1.7}
$$

Models derivation

We note that, we can derivate our systems by considering the following evolution equations of thermoelastic Timoshenko model, for the model representation see [20].

$$
\begin{aligned}
&\rho_1 u_{tt} - T_x = 0, && \text{in } \mathcal{I}\times\mathbb{R}_+,\\
&\rho_2\varphi_{tt} - H_x + T = 0, && \text{in } \mathcal{I}\times\mathbb{R}_+,
\end{aligned} \tag{1.8}
$$

where t is the time, x is the distance along the center line of the beam structure, u is the transverse displacement, and φ is the rotation of the neutral axis due to bending. Here, $\rho_1 = \rho A$ and $\rho_2 = \rho I$, where ρ is the density, A is the cross-sectional area, and I is the second moment of area of the cross-sectional area. When the stress–strain constitutive law is of Kelvin–Voigt type [11], we can adopt the following two coupling:

First coupling that produce system (1.1)

The Timoshenko system with thermoelastic dissipation, effective in the bending moment equation is given as follows

$$\begin{aligned} T &= k(u_x + \varphi) + \gamma_1 (u_x + \varphi)_t, \\ H &= b\varphi_x + \gamma_2 \varphi_{xt} + \beta\theta. \end{aligned} \tag{1.9}$$

Second coupling that produce system (1.4)

The Thermoelastic Timoshenko system acting on shear force is given as follows

$$\begin{aligned} T &= k(u_x + \varphi) + \gamma_1 (u_x + \varphi)_t + \beta\theta, \\ H &= b\varphi_x + \gamma_2 \varphi_{xt}. \end{aligned} \tag{1.10}$$

In systems (1.9) and (1.10) the temperature θ following the Fourier law [7] is given as follows

$$\begin{aligned} \rho_3 \theta_t &= -q_x - \beta\varphi_{xt}, && \text{in } \mathcal{I} \times \mathbb{R}_+, \\ \rho_3 \theta_t &= -q_x - \beta(u_x + \varphi)_t, && \text{in } \mathcal{I} \times \mathbb{R}_+, \end{aligned} \tag{1.11}$$

where the flux is obtained by

$$q = -\gamma_3 \theta_x, \quad \text{in } \mathcal{I} \times \mathbb{R}_+. \tag{1.12}$$

Finally, by substituting system (1.9) in the system (1.8) and using equation $(1.11)_1$, we obtain system (1.1) . Similarly, by substituting system (1.10) in the system (1.8) and using equation $(1.11)_2$, we obtain system (1.4) .

Earlier results

Malacarne and Rivera [11] studied the Timoshenko system [20] with the viscoelastic dissipative mechanism of Kelvin–Voigt type. Consequently, they found a new results related to the solution behavior for system

$$\begin{aligned} \rho_1 \phi_{tt} &= k(\phi_x + \psi)_x + \gamma_1 (\phi_x + \psi)_{tx}, && \text{in } \ ]0, L[\times]0, \infty[, \\ \rho_2 \psi_{tt} &= b\psi_{xx} + \gamma_2 \psi_{xxt} + k(\phi_x + \psi) + \gamma_1 (\phi_x + \psi)_t, && \text{in } \ ]0, L[\times]0, \infty[. \end{aligned} \tag{1.13}$$

To be more precise, the authors proved that the semigroup is analytical if and only if the viscoelastic damping is present at the same time in the shear stress and in the bending moment, i.e., $\gamma_1, \gamma_2 > 0$. Otherwise, the corresponding semigroup is not exponentially stable even for equal wave speeds. The final result showed that the polynomial rate of decay is $t^{-1/2}$. In addition, Liu and Zhang [9] studied the stability and regularity of solution for the system (1.13).

In [19], Tian and Zhang considered a Timoshenko system with local Kelvin–Voigt damping modeled by the following system

$$\begin{aligned} &\rho_1 \omega_{tt} - [\kappa (\omega_x + \phi) + D_1 (\omega_{xt} + \phi_t)]_x = 0, && \text{in } \ (0, L) \times \mathbb{R}_+, \\ &\rho_2 \phi_{tt} - (\mu\phi_x + D_2 \phi_{xt})_x + \kappa (\omega_x + \phi) + D_1 (\omega_{xt} + \phi_t) = 0, && \text{in } \ (0, L) \times \mathbb{R}_+, \end{aligned} \tag{1.14}$$

where

$$D_1(\cdot), D_2(\cdot) : [0, L] \longrightarrow \mathbb{R}_+ \cup \{0\}, \tag{1.15}$$

are continuous functions. They proved by using method based on frequency analysis that the system energy decays exponentially or polynomially where the decay rate depends on the properties of the material coefficient functions (1.15), for more detail, see references [1,6,12,14–16].

In the same context but in case of a porous-elastic system with Kelvin–Voigt damping, Almeida and Ramos [17] proposed the following system

$$\begin{aligned} &\rho u_{tt} - (\mu u_x + \gamma_1 u_{tx})_x - b\phi_x = 0, \qquad \text{in} \quad (0, L) \times (0, \infty), \\ &J\phi_{tt} - (\delta\phi_x + \gamma_2\phi_{tx})_x + bu_x + \xi\phi = 0, \quad \text{in} \quad (0, L) \times (0, \infty), \end{aligned} \tag{1.16}$$

where, the variables u and ϕ represent the displacement of a solid elastic material and the volume fraction. On the one hand, they showed that the semigroup associated with the system (1.16) under both Dirichlet–Dirichlet and Dirichlet–Neumann boundary conditions is founded analytic and consequently exponentially stable. On the other hand, they proved that the system (1.16) with Dirichlet–Neumann boundary conditions has lack of exponential decay for the case

$$\gamma_1 > 0, \ \gamma_2 = 0 \quad \text{or} \quad \gamma_1 = 0, \ \gamma_2 > 0.$$

Moreover, they proved the same result in the case of Timoshenko's model, i.e., if

$$\mu = \xi = b.$$

In view of the previous works in which there is a total absence of thermal effects in the systems, we decided to study the Timoshenko system in two different states imposed by the thermal effects. Hence, in this work we have proved some results about the well-posedness of solutions and we showed the energy decay of the studied problems.

Remark 1.1

- As in [3], the choice of the spaces of zero-mean functions for the variable u, φ and its derivative is consistent. Indeed, noting

$$\chi_1(t) = \int_{\mathcal{I}} \varphi(x, t)\, dx \quad \text{and} \quad \chi_2(t) = \int_{\mathcal{I}} u(x, t)\, dx.$$

Integrating the previous equations on $\mathcal{I}$, we obtain respectively the following two differential equations

$$\rho_2 \ddot{\chi}_1(t) - \gamma_1 \dot{\chi}_1(t) - k\chi_1(t) = 0 \quad \text{and} \quad \rho_1 \ddot{\chi}_2(t) = 0. \tag{1.17}$$

Hence, if

$$\chi_1(0) = \dot{\chi}_1(0) = 0 \quad \text{and} \quad \chi_2(0) = \dot{\chi}_2(0) = 0. \tag{1.18}$$

Then, it follows that

$$\chi_1(t) \equiv 0 \quad \text{and} \quad \chi_2(t) \equiv 0.$$

Thus, the use of Poincaré's inequality for the function φ in system (1.1) and the function u in system (1.4) is justified.

- By virtue of Cauchy–Schwarz and Poincaré's inequalities, we have the following inequality

$$\|\Sigma\|^2_{L^2(\mathcal{I})} \leq \|\varphi_t\|^2_{L^2(\mathcal{I})}, \quad \forall \varphi \in L^2(\mathcal{I}),$$

  where

$$\Sigma(x,t) = \int_0^x \varphi_t(y,t)\, dy.$$

- By virtue of Poincaré's inequality, we have the following estimate

$$\begin{aligned}\|u_x\|^2_{L^2(\mathcal{I})} &= \int_{\mathcal{I}} [(u_x+\varphi)-\varphi]^2 \; dx \\ &\leq 2\|u_x+\varphi\|^2_{L^2(\mathcal{I})} + 2\|\varphi_x\|^2_{L^2(\mathcal{I})}, \quad \forall u \in L^2(\mathcal{I}).\end{aligned}$$

It is worth noting that this observations will play a major role in the attainment of the results of this paper.

Paper plan

The paper respects the following plan. Firstly, in Sect. 2, we will prove the well-posedness of the problems. After that, in Sect. 3, we will show the exponential stability results of both systems (1.1) and (1.4). Finally, in Sect. 4, we will give some open problem for the interested readers.

2 Well-posedness

This section will be concerned with the existence and uniqueness of global solutions based on the classical Lumer–Phillips Theorem, see e.g., the books [10,13].

Theorem 2.1 *Let $\mathcal{A}_i$ be a densely defined linear operators on a Hilbert spaces $\mathcal{H}_i$, for $(i = 1, 2)$. Then $\mathcal{A}_i$ is the infinitesimal generator of a contraction semigroup $s(t)$ if and only if*

- *$\mathcal{A}_i$ are dissipative; and*
- *$0 \in \varrho(\mathcal{A}_i)$.*

In order to define the operators $\mathcal{A}_i$, we introduce the new variables $v^i = u^i_t$, $\phi^i = \varphi^i_t$. Now, we consider the following Hilbert phase spaces

$$\begin{aligned}\mathcal{H}_1 &= H^1_0(\mathcal{I}) \times L^2(\mathcal{I}) \times H^1_*(\mathcal{I}) \times L^2_*(\mathcal{I}) \times L^2(\mathcal{I}),\\ \mathcal{H}_2 &= H^1_*(\mathcal{I}) \times L^2_*(\mathcal{I}) \times H^1_0(\mathcal{I}) \times L^2(\mathcal{I}) \times L^2(\mathcal{I}).\end{aligned}$$

We note that the closed subspace of $L^2(\mathcal{I})$ is defined by

$$L^2_*(\mathcal{I}) = \left\{\vartheta \in L^2(\mathcal{I}) : \int_{\mathcal{I}} \vartheta \; dx = 0\right\}, \tag{2.1}$$

and the following Sobolev spaces is defined by

$$H^2_*(\mathcal{I}) = \{\vartheta \in H^2(\mathcal{I}) : \vartheta_x(0) = \vartheta_x(1) = 0\}, \quad H^1_*(\mathcal{I}) = L^2_*(\mathcal{I}) \cap H^1(\mathcal{I}). \tag{2.2}$$

The corresponding norm in $\mathcal{H}_i$ is given by

$$\|U^i\|^2_{\mathcal{H}_i} = \rho_1\|v^i\|^2_{L^2(\mathcal{I})} + \rho_2\|\phi^i\|^2_{L^2(\mathcal{I})} + \rho_3\|\theta^i\|^2_{L^2(\mathcal{I})} + k\|u^i_x+\varphi^i\|^2_{L^2(\mathcal{I})} + b\|\varphi^i_x\|^2_{L^2(\mathcal{I})}. \tag{2.3}$$

where $U^i = (u^i, v^i, \varphi^i, \phi^i, \theta^i) \in \mathcal{H}_i$ and $U^i_* = (u^i_*, v^i_*, \varphi^i_*, \phi^i_*, \theta^i_*) \in \mathcal{H}_i$.

The inner product is given by

$$\langle U^i, U^i_* \rangle_{\mathcal{H}_i} = \int_{\mathcal{I}} \left[\rho_1 v^i v^i_* + \rho_2 \phi^i \phi^i_* + \rho_3 \theta^i \theta^i_* + k(u^i_x + \varphi^i)(u^i_{*x} + \varphi^i_*) + b\varphi^i_x \varphi^i_{*x} \right] dx. \tag{2.4}$$

Let us introduce the operators $\mathcal{A}_i : \mathcal{D}(\mathcal{A}_i) \subset \mathcal{H}_i \longrightarrow \mathcal{H}_i$ as follows

$$\mathcal{A}_1 = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \frac{k}{\rho_1}\partial_x^2(\cdot) & \frac{\gamma_1}{\rho_1}\partial_x^2(\cdot) & \frac{k}{\rho_1}\partial_x(\cdot) & \frac{\gamma_1}{\rho_1}\partial_x(\cdot) & -\frac{\beta}{\rho_1}\partial_x(\cdot) \\ 0 & 0 & 0 & I & 0 \\ -\frac{k}{\rho_2}\partial_x(\cdot) & -\frac{\gamma_1}{\rho_2}\partial_x(\cdot) & \frac{b}{\rho_2}\partial_x^2(\cdot) - \frac{k}{\rho_2} & -\frac{\gamma_2}{\rho_2}\partial_x^2(\cdot) - \frac{\gamma_1}{\rho_2} & \frac{\beta}{\rho_2} \\ 0 & -\frac{\beta}{\rho_3}\partial_x(\cdot) & 0 & -\frac{\beta}{\rho_3} & \frac{\gamma_3}{\rho_3}\partial_x^2(\cdot) \end{pmatrix}, \tag{2.5}$$

and

$$\mathcal{A}_2 = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \frac{k}{\rho_1}\partial_x^2(\cdot) & \frac{\gamma_1}{\rho_1}\partial_x^2(\cdot) & \frac{k}{\rho_1}\partial_x(\cdot) & \frac{\gamma_1}{\rho_1}\partial_x(\cdot) & 0 \\ 0 & 0 & 0 & I & 0 \\ -\frac{k}{\rho_2}\partial_x(\cdot) & -\frac{\gamma_1}{\rho_2}\partial_x(\cdot) & \frac{b}{\rho_2}\partial_x^2(\cdot) - \frac{k}{\rho_2} & -\frac{\gamma_2}{\rho_2}\partial_x^2(\cdot) - \frac{\gamma_1}{\rho_2} & -\frac{\beta}{\rho_2}\partial_x(\cdot) \\ 0 & 0 & 0 & -\frac{\beta}{\rho_3}\partial_x(\cdot) & \frac{\gamma_3}{\rho_3}\partial_x^2(\cdot) \end{pmatrix}, \tag{2.6}$$

with the domains $\mathcal{D}(\mathcal{A}_i)$ defined as follows

$$\mathcal{D}(\mathcal{A}_1) = \left\{ U^1 \in \mathcal{H}_1 \middle| \begin{array}{r} u^1 \in H_0^1(\mathcal{I}) \cap H^2(\mathcal{I}) \\ v^1 \in H_0^1(\mathcal{I}) \\ \varphi^1 \in H^2(\mathcal{I}) \cap H_*^1(\mathcal{I}) \\ \phi^1 \in H_*^1(\mathcal{I}) \\ \theta^1 \in H^2(\mathcal{I}) \end{array} \right\}, \tag{2.7}$$

and

$$\mathcal{D}(\mathcal{A}_2) = \left\{ U^2 \in \mathcal{H}_2 \middle| \begin{array}{r} u^2 \in H_*^1(\mathcal{I}) \cap H_*^2(\mathcal{I}) \\ v^2 \in H_*^1(\mathcal{I}) \\ \varphi^2 \in H^2(\mathcal{I}) \cap H_0^1(\mathcal{I}) \\ \phi^2 \in H_0^1(\mathcal{I}) \\ \theta^2 \in H^2(\mathcal{I}) \end{array} \right\}. \tag{2.8}$$

Then, the problems (1.1)–(1.7) is equivalent to the following evolution Cauchy problems

$$\begin{aligned} &\frac{d}{dt}U^i(t) = \mathcal{A}_i U^i(t), \quad t > 0, \\ &U^i(0) = U^i_0 = \left(u^i_0, u^i_1, \varphi^i_0, \varphi^i_1, \theta^i_0\right)^T. \end{aligned}$$

Proposition 2.2 *The operators $\mathcal{A}_i$ are the infinitesimal generators of a C_0-semigroup of contractions $s(t)$ over the space $\mathcal{H}_i$.*

For the proof of the previous Proposition we need to prove the followingLemmas

Lemma 2.3 *The linear operator $\mathcal{A}_i$ for $i = 1, 2$ is a dissipative operator.*

Proof In fact, we observe that if $U^i \in \mathcal{D}(\mathcal{A}_i)$, by using the inner product (2.4) and the operator $\mathcal{A}_i$ defined in (2.5) and (2.6), we can get

$$\begin{aligned}\langle \mathcal{A}_i U^i, U^i\rangle_{\mathcal{H}_i} &= -\gamma_1\|v_x^i+\phi^i\|^2_{L^2(\mathcal{I})} - \gamma_2\|\phi_x^i\|^2_{L^2(\mathcal{I})} - \gamma_3\|\theta_x^i\|^2_{L^2(\mathcal{I})}\\ &\le 0.\end{aligned} \tag{2.9}$$

Then, the proof of Lemma (2.3) is finished. □

Lemma 2.4 *Let the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ defined by (2.5) and (2.6) respectively. Then, we have*

$$0 \in \varrho(\mathcal{A}_i).$$

Proof For any $F^* = (f_1^*, f_2^*, f_3^*, f_4^*, f_5^*)^T \in \mathcal{H}_1$, $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}_2$, we want to find $U^i = (u^i, v^i, \varphi^i, \phi^i, \theta^i) \in \mathcal{D}(\mathcal{A}_i)$ such that

$$\begin{aligned}\mathcal{A}_1 U^1 &= F^*,\\ \mathcal{A}_2 U^2 &= F.\end{aligned}$$

In terms of the components, we get

$$\begin{cases} v^1 = f_1^*,\\ ku^1_{xx} + \gamma_1 v^1_{xx} + k\varphi^1_x + \gamma_1\phi^1_x = \rho_1 f_2^*,\\ \phi^1 = f_3^*,\\ b\varphi^1_{xx} - ku^1_x - \gamma_1 v^1_x - k\varphi^1 + \gamma_2^1\phi_{xx} - \gamma_1\phi^1 - \beta\theta^1_x = \rho_2 f_4^*,\\ \gamma_3\theta^1_{xx} - \beta\phi^1_x = \rho_3 f_5^*, \end{cases} \tag{2.10}$$

and

$$\begin{cases} v^2 = f_1,\\ ku^2_{xx} + \gamma_1 v^2_{xx} + k\varphi^2_x + \gamma_1\phi^2_x - \beta\theta^2_x = \rho_1 f_2,\\ \phi^2 = f_3,\\ b\varphi^2_{xx} - ku^2_x - \gamma_1 v^2_x - k\varphi^2 + \gamma_2\phi^2_{xx} - \gamma_1\phi^2 + \beta\theta^2 = \rho_2 f_4,\\ \gamma_3\theta^2_{xx} - \beta v^2_x - \beta\phi^2 = \rho_3 f_5. \end{cases} \tag{2.11}$$

By using the equations $(2.10)_1$ and $(2.10)_3$, we can deduce that

$$v^1 \in H_0^1(\mathcal{I}), \phi^1 \in H_*^1(\mathcal{I}). \tag{2.12}$$

Also, by using the equations $(2.11)_1$ and $(2.11)_3$, we can deduce that

$$v^2 \in H_*^1(\mathcal{I}), \phi^2 \in H_0^1(\mathcal{I}). \tag{2.13}$$

Now, by using Eqs. $(2.10)_5$ and $(2.10)_5$, we can write

$$\gamma_3\theta^1_{xx} = \vartheta_3^* \in L^2(\mathcal{I}), \tag{2.14}$$

and

$$\gamma_3 \theta^2_{xx} = \vartheta_3 \in L^2(\mathcal{I}), \tag{2.15}$$

where

$$\vartheta_3^* = \rho_3 f_5^* + \beta (f_3^*)_x,$$
$$\vartheta_3 = \rho_3 f_5 + \beta (f_1)_x + \beta f_3.$$

We conclude that there exists a unique function $\theta^i \in H_0^1(\mathcal{I})$ for $i = 1, 2$. Then, the remaining point is to prove that there exist u^i and φ^i satisfying the following systems

$$\begin{cases} k u^1_{xx} + k\varphi^1_x = \vartheta_1^*, \\ b\varphi^1_{xx} - k u^1_x - k\varphi^1 = \vartheta_2^*, \end{cases} \tag{2.16}$$

and

$$\begin{cases} k u^2_{xx} + k\varphi^2_x = \vartheta_1, \\ b\varphi^2_{xx} - k u^2_x - k\varphi^2 = \vartheta_2, \end{cases} \tag{2.17}$$

where

$$\begin{cases} \vartheta_1^* = \rho_1 f_2^* - \gamma_1 (f_1^*)_{xx} - \gamma_1 (f_3^*)_x, \\ \vartheta_2^* = \rho_2 f_4^* - \gamma_2 (f_3^*)_{xx} + \gamma_1 (f_1^*)_x - \gamma_1 f_3^* + \beta \theta^1_x, \\ \vartheta_1 = \rho_1 f_2 - \gamma_1 (f_3)_x - \gamma_1 (f_1)_{xx} + \beta \theta^2_x, \\ \vartheta_2 = \rho_2 f_4 - \beta \theta^2 + \gamma_1 f_3 - \gamma_2 (f_3)_{xx} + \gamma_1 (f_1)_x. \end{cases} \tag{2.18}$$

Introducing the spaces

$$\mathcal{X}_1 = H^1_*(\mathcal{I}) \cap H^1_0(\mathcal{I}), \quad \text{and} \quad \mathcal{X}_2 = H^1_0(\mathcal{I}) \cap H^1_*(\mathcal{I}).$$

Denote the bilinear forms for the systems (2.16) and (2.17) as follows

$$\mathcal{B}_i(U^i, \tilde{U}^i) = \int_{\mathcal{I}} -k u^i_x \tilde{u}^i_x + k\varphi^i_x \tilde{u}^i_x - b\varphi^i_x \tilde{\varphi}^i_x - k u^i_x \tilde{\varphi}^i_x - k\varphi^i \tilde{\varphi}^i \, dx, \text{ for } i = 1, 2.$$

The linear forms is defined as follows

$$\mathcal{L}_1(V^1) = \int_{\mathcal{I}} \vartheta_1^* \tilde{u}^1 + \vartheta_2^* \tilde{\varphi}^1 \, dx, \quad \forall V^1 = (\tilde{u}^1, \tilde{\varphi}^1) \in \mathcal{X}_1,$$
$$\mathcal{L}_2(V^2) = \int_{\mathcal{I}} \vartheta_1 \tilde{u}^2 + \vartheta_2 \tilde{\varphi}^2 \, dx, \quad \forall V^2 = (\tilde{u}^2, \tilde{\varphi}^2) \in \mathcal{X}_2.$$

We conclude that the bilinear forms $\mathcal{B}_i(\cdot,\cdot)$ are coercive and continuous over the Hilbert space $\mathcal{X}_i$. Also the linear forms $\mathcal{L}_i(\cdot)$ are continuous over the Hilbert space $\mathcal{X}_i$. Therefore, according to the Lax-Milgram conditions, we deduce that there exists a unique solution to the following variational formula

$$\mathcal{B}_i(U^i, V^i) = \mathcal{L}_i(V^i) \in \mathcal{X}_i, \quad \text{for} \quad i = 1, 2. \tag{2.19}$$

Thus, the proof of Lemma (2.4) is completed. □

Proof of Proposition (2.2) Based on Lemmas (2.3) and (2.4), the operators $\mathcal{A}_i$ are m-dissipative. Then, using the Lumer–Phillips Theorem (2.1), we conclude that the operators $\mathcal{A}_i$ are infinitesimal generators of a C_0-semigroup of contractions. Then, the proof of Proposition (2.2) is completed. □

Now, we present our main result as follows

Theorem 2.5 *Let $U_0^i \in \mathcal{D}(\mathcal{A}_i)$, Problems (1.1)–(1.7) have a unique classical solution*

$$U^i \in C\left(\mathbb{R}_+, \mathcal{D}(\mathcal{A}_i)\right) \cap C^1\left(\mathbb{R}_+, \mathcal{H}_i\right)$$

Moreover, if $U_0^i \in \mathcal{H}_i$, then there exists unique mild solution of the problems (1.1)–(1.7)

$$U^i \in C\left(\mathbb{R}_+, \mathcal{H}_i\right).$$

Proof Firstly, it is clear that $\mathcal{D}(\mathcal{A}_i)$ dense in $\mathcal{H}_i$. Then, under Proposition (2.2), the proof of Theorem (2.5) is finished. □

3 Energy decay

In this section, we introduce our exponential decay result for the problems (1.1)–(1.7).

Remark 3.1 Throughout this section, c is used to denote a generic positive constant that will be changed from one inequality to another.

First, we define the energy of the problems (1.1)–(1.7) as follows

$$E(t) := \frac{1}{2}\left[\rho_1\|u_t\|^2_{L^2(\mathcal{I})} + \rho_2\|\varphi_t\|^2_{L^2(\mathcal{I})} + \rho_3\|\theta\|^2_{L^2(\mathcal{I})} + b\|\varphi_x\|^2_{L^2(\mathcal{I})} + k\|u_x + \varphi\|^2_{L^2(\mathcal{I})}\right]. \tag{3.1}$$

Lemma 3.2 *Let (u, φ, θ) solution to the problems* (1.1)–(1.7). *Then, the energy functional (3.1) satisfy the following equality*

$$\begin{aligned} \frac{d}{dt}E(t) &= -\gamma_3\|\theta_x\|^2_{L^2(\mathcal{I})} - \gamma_2\|\varphi_{xt}\|^2_{L^2(\mathcal{I})} - \gamma_1\|(u_x + \varphi)_t\|^2_{L^2(\mathcal{I})} \\ &\le 0. \end{aligned} \tag{3.2}$$

Proof Multiplying the equations of system (1.1) respectively by (u, φ) and θ and integrating over $\mathcal{I}$, by using integration by parts and the boundary conditions (1.2)–(1.3). By adding the result, we get the equality (3.2). The same procedure for the problem (1.4)–(1.7) gives the result. □

Exponential stability of problem (1.1)–(1.3).
Defining the primitives

$$\begin{aligned} &\bullet\ \Theta(x,t) := \int_0^x u_t(y,t)\,dy. \\ &\bullet\ \Sigma(x,t) := \int_0^x \varphi_t(y,t)\,dy. \\ &\bullet\ \Pi(x,t) := \int_0^x (u_x + \varphi)(y,t)\,dy. \\ &\bullet\ \Upsilon(x,t) := \int_0^x \theta(y,t)\,dy. \end{aligned} \tag{3.3}$$

We introduce the following functionals

$$
\begin{aligned}
&\bullet\ \Phi_1(t) := \rho_1\rho_2 \left\langle u_t(t), \Sigma(t)\right\rangle_{L^2(\mathcal{I})}.\\
&\bullet\ \Psi_1(t) := \rho_2 \left\langle \varphi(t), \varphi_t(t)\right\rangle_{L^2(\mathcal{I})}.\\
&\bullet\ \Lambda_1(t) := -\rho_1 \left\langle (u_x+\varphi)(t), \Theta(t)\right\rangle_{L^2(\mathcal{I})}.
\end{aligned}
\tag{3.4}
$$

Let N a large positive number, we define the Lyapunov functional $\mathcal{F}_1$ as follows

$$
\mathcal{F}_1(t) := NE(t) + N_0\Phi_1(t) + N_1\Psi_1(t) + N_2\Lambda_1(t). \tag{3.5}
$$

Lemma 3.3 *For N large enough, there exist two positive constants α_1 and α_2 such that*

$$
\alpha_1 E(t) \le \mathcal{F}_1(t) \le \alpha_2 E(t), \forall t \ge 0. \tag{3.6}
$$

Proof Defining the functional $\tilde{\mathcal{F}}_1$ by

$$
\tilde{\mathcal{F}}_1(t) = N_0\Phi_1(t) + N_1\Psi_1(t) + N_2\Lambda_1(t). \tag{3.7}
$$

Then, by using Young, Poincaré's inequalities and the previous functionals (3.4), we obtain

$$
|\tilde{\mathcal{F}}_1(t)| \le \widehat{N}\int_{\mathcal{I}} \left(\rho_1 u_t^2 + \rho_2\varphi_t^2 + \rho_3\theta^2 + k(u_x+\varphi)^2 + b\varphi_x^2\right)\, dx.
$$

Consequently, we get

$$
|\mathcal{F}_1(t) - NE(t)| \le \widehat{N}E(t),
$$

where

$$
\widehat{N} > 0.
$$

That is

$$
(N-\widehat{N})E(t) \le \mathcal{F}_1(t) \le (N+\widehat{N})E(t).
$$

By choosing N large enough, the inequality (3.6) follows. □

Lemma 3.4 *The functionals Φ_1, Ψ_1 and Λ_1 satisfy the following differential inequalities*

$$
\begin{aligned}
&\bullet\ \frac{d}{dt}\Phi_1(t) + \frac{\gamma_1\rho_1}{2}\|u_t\|^2_{L^2(\mathcal{I})} \le c\left(\|u_x+\varphi\|^2_{L^2(\mathcal{I})} + \|\varphi_x\|^2_{L^2(\mathcal{I})} + \|\varphi_{xt}\|^2_{L^2(\mathcal{I})}\right).\\
&\bullet\ \frac{d}{dt}\Psi_1(t) + \frac{b}{2}\|\varphi_x\|^2_{L^2(\mathcal{I})} \le c\left(\|\varphi_{xt}\|^2_{L^2(\mathcal{I})} + \|u_x+\varphi\|^2_{L^2(\mathcal{I})}\right.\\
&\qquad\qquad \left. + \|(u_x+\varphi)_t\|^2_{L^2(\mathcal{I})} + \|\theta_x\|^2_{L^2(\mathcal{I})}\right).\\
&\bullet\ \frac{d}{dt}\Lambda_1(t) + \frac{k}{2}\|u_x+\varphi\|^2_{L^2(\mathcal{I})} \le \varepsilon\|u_t\|^2_{L^2(\mathcal{I})} + \frac{c}{\varepsilon}\|(u_x+\varphi)_t\|^2_{L^2(\mathcal{I})},
\end{aligned}
\tag{3.8}
$$

where $\varepsilon > 0$.

Proof

- Taking the derivative of $(3.4)_1$, by using the first and the second equations in (1.1) and the boundary conditions (1.2)–(1.3), yields

$$\begin{aligned}&\frac{d}{dt}\Phi_1(t)+\gamma\rho_1\|u_t\|^2_{L^2(\mathcal{I})}\\&\quad=-\rho_2 k\left\langle(u_x+\varphi)(t),\varphi_t(t)\right\rangle_{L^2(\mathcal{I})}-\rho_2\gamma_1\left\langle(u_x+\varphi)_t(t),\varphi_t(t)\right\rangle_{L^2(\mathcal{I})}\\&\qquad+\rho_1\left\langle u_t(t),b\varphi_x(t)+\gamma_2\varphi_{xt}(t)-k\Pi(t)-\gamma_1\Sigma(t)-\beta\theta(t)\right\rangle_{L^2(\mathcal{I})}.\end{aligned}\tag{3.9}$$

By using Young's inequality and Remark (1.1), we obtain

$$\begin{aligned}-k\rho_1\left\langle u_t(t),\Pi(t)\right\rangle_{L^2(\mathcal{I})}&=-k\rho_1\left\langle u_t(t),u(t)+\int_0^x\varphi(y,t)dy\right\rangle_{L^2(\mathcal{I})}\\&\le\frac{3\gamma_1\rho_1}{10}\|u_t\|^2_{L^2(\mathcal{I})}+c\left(\|u_x+\varphi\|^2_{L^2(\mathcal{I})}+\|\varphi_x\|^2_{L^2(\mathcal{I})}\right).\end{aligned}$$

Now, applying Young and Poincaré's inequalities to estimate all remaining terms in equality (3.9). Then, we get the desired inequality $(3.8)_1$.

- Taking the derivative of $(3.4)_2$, by using the second equations in (1.1) and the boundary conditions (1.2)–(1.3), we get

$$\begin{aligned}\frac{d}{dt}\Psi_1(t)+b\|\varphi_x\|^2_{L^2(\mathcal{I})}=&-\left\langle\varphi(t),k(u_x+\varphi)(t)+\gamma_1(u_x+\varphi)_t(t)\right\rangle_{L^2(\mathcal{I})}\\&+\left\langle\varphi_x(t),\beta\theta(t)+\gamma_2\varphi_{xt}(t)\right\rangle_{L^2(\mathcal{I})}.\end{aligned}$$

  Applying Young and Poincaré's inequalities, it appear directly inequality $(3.8)_2$.
- Taking the derivative of $(3.4)_3$, by using the first equations in (1.1), we obtain

$$\frac{d}{dt}\Lambda_1(t)+k\|u_x+\varphi\|^2_{L^2(\mathcal{I})}=-\left\langle(u_x+\varphi)_t(t),\rho_1\Theta(t)+\gamma_1(u_x+\varphi)(t)\right\rangle_{L^2(\mathcal{I})}.$$

  Applying Young and Poincaré's inequalities, we obtain the inequality $(3.8)_3$. □

The main result of stability for the system (1.1) is given by the following Theorem.

Theorem 3.5 *Let $(u,\varphi,\theta)\in\mathcal{H}_1$ solution to the system (1.1) with boundary conditions (1.2)–(1.3) and initial condition (1.7). Then, the energy functional (3.1) satisfies*

$$E(t)\le\lambda_1 e^{-\lambda_2 t},\ \forall t\ge 0,\tag{3.10}$$

for λ_1,λ_2 positive constants.

Proof By differentiating equality (3.5) and by using estimates (3.8), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t)\le&-\varpi_1\|u_t\|^2_{L^2(\mathcal{I})}-\varpi_2\|\varphi_x\|^2_{L^2(\mathcal{I})}-\varpi_3\|u_x+\varphi\|^2_{L^2(\mathcal{I})}-\varpi_4\|\varphi_{xt}\|^2_{L^2(\mathcal{I})}\\&-\varpi_5\|\theta_x\|^2_{L^2(\mathcal{I})}-\varpi_6\|(u_x+\varphi)_t\|^2_{L^2(\mathcal{I})},\end{aligned}\tag{3.11}$$

where

$$\begin{aligned}&\varpi_1=N_0\frac{\gamma\rho_1}{2}-N_2\varepsilon,\\&\varpi_2=N_1\frac{b}{2}-N_0c,\quad\varpi_3=N_2\frac{k}{2}-c(N_0+N_1),\\&\varpi_4=N\gamma_2-c(N_0+N_1),\\&\varpi_5=N\gamma_3-N_1c,\\&\varpi_6=N\gamma_1-N_1c-N_2\frac{c}{\varepsilon}.\end{aligned}\tag{3.12}$$

Now, all the terms on the right-hand side of (3.11) become negative if we select our parameters carefully.

First, let us pick $\varepsilon = \frac{\gamma_1 \rho_1}{2N_2}$. Then, we choose N_1 large enough such that

$$N_0 \frac{\gamma_1 \rho_1}{2} - N_2 \varepsilon > 0.$$

Hence, we get

$$N_0 > 1.$$

We choose N_1 such that

$$\frac{N_1 b}{2} - N_0 c > 0,$$

we can choose N_2 such that

$$\frac{N_2 k}{2} - c N_0 - c N_1 > 0.$$

The constants N_0, N_1 and N_2 check the following inequality

$$N_2 \gg N_1 \gg N_0. \tag{3.13}$$

Finally, we choose N large enough such that

$$\begin{aligned} &N\gamma_2 - c(N_0 + N_1) > 0, \\ &N\gamma_3 - N_1 c > 0, \\ &N\gamma_1 - N_1 c - N_2 \tfrac{c}{\varepsilon} > 0. \end{aligned} \tag{3.14}$$

Then, ϖ_i, for $i = 1, \ldots, 6$ are all negative constants.

At this point, there exists a constant ϑ, and, further, $\mathcal{F}_1(t) \sim E(t)$. Then, inequality (3.11) takes the following form

$$\frac{d}{dt}\mathcal{F}_1(t) \leq -\vartheta_1 E(t), \forall t \geq 0. \tag{3.15}$$

Recalling by (3.6), the fact that $\mathcal{F}_1 \sim E$, so we get

$$\frac{d}{dt}\mathcal{F}_1(t) \leq -\zeta_1 \mathcal{F}_1(t), \forall t \geq 0,$$

for some positive constant ζ_1.

Integrating the last inequality over $(0, t)$, we arrive at

$$\mathcal{F}_1(t) \leq \mathcal{F}_1(0) e^{-\zeta_1 t}, \forall t \geq 0.$$

By using the other side of the equivalence relation (3.6). Then, the proof of Theorem (3.5) is finished. □

Exponential stability of problem (1.4)–(1.6)

Let $\tilde{N}$ a large positive number, we define the Lyapunov functional $\mathcal{F}_2$ as follows

$$\mathcal{F}_2(t) := \tilde{N} E(t) + N_3 \Phi_2(t) + N_4 \Psi_2(t) + N_5 \Lambda_2(t), \tag{3.16}$$

where

$$\begin{aligned} \Phi_2(t) &:= \rho_1 \rho_3 \left\langle \Upsilon(t), u_t(t) \right\rangle_{L^2(\mathcal{I})}. \\ \Psi_2(t) &:= \rho_2 \left\langle \varphi(t), \varphi_t(t) \right\rangle_{L^2(\mathcal{I})}. \\ \Lambda_2(t) &:= -\rho_1 \left\langle \Theta(t), (u_x + \varphi)(t) \right\rangle_{L^2(\mathcal{I})}. \end{aligned} \tag{3.17}$$

Lemma 3.6 *For $\tilde{N}$ large enough. Then, there exist two positive constants α_3 and α_4 such that*

$$\alpha_3 E(t) \leq \mathcal{F}_2(t) \leq \alpha_4 E(t), \quad \forall t \geq 0. \tag{3.18}$$

Proof For the proof, we use the same procedure that used for the proof of Lemma (3.3). □

Lemma 3.7 *The functionals Φ_2, Ψ_2 and Λ_2 satisfy the following differential inequalities*

$$\bullet \ \frac{d}{dt}\Phi_2(t) + \frac{\beta\rho_1}{2}\|u_t\|^2_{L^2(\mathcal{I})} \leq \varepsilon_1\|u_x+\varphi\|^2_{L^2(\mathcal{I})} + c\left(\|\theta_x\|^2_{L^2(\mathcal{I})} + \|\varphi_{xt}\|^2_{L^2(\mathcal{I})}\right) + \frac{c}{\varepsilon_1}\|(u_x+\varphi)_t\|^2_{L^2(\mathcal{I})}.$$

$$\bullet \ \frac{d}{dt}\Psi_2(t) + \frac{b}{2}\|\varphi_x\|^2_{L^2(\mathcal{I})} \leq c\left(\|u_x+\varphi\|^2_{L^2(\mathcal{I})} + \|(u_x+\varphi)_t\|^2_{L^2(\mathcal{I})} + \|\theta_x\|^2_{L^2(\mathcal{I})} + \|\varphi_{xt}\|^2_{L^2(\mathcal{I})}\right).$$

$$\bullet \ \frac{d}{dt}\Lambda_2(t) + \frac{k}{2}\|u_x+\varphi\|^2_{L^2(\mathcal{I})} \leq \varepsilon_2\|u_t\|^2_{L^2(\mathcal{I})} + c\|\theta\|^2_{L^2(\mathcal{I})} + \frac{c}{\varepsilon_2}\|(u_x+\varphi)_t\|^2_{L^2(\mathcal{I})}, \tag{3.19}$$

where $\varepsilon_1, \varepsilon_2 > 0$.

Proof • Taking the derivative of $(3.17)_1$, by using both the first and the third equation of system (1.4) together with boundary conditions (1.5)–(1.6), yields

$$\frac{d}{dt}\Phi_2(t) + \beta\rho_1\|u_t\|^2_{L^2(\mathcal{I})} = \rho_3\beta\|\theta\|^2_{L^2(\mathcal{I})} - \rho_3\langle\theta(t), k(u_x+\varphi)(t) + \gamma_1(u_x+\varphi)_t(t)\rangle_{L^2(\mathcal{I})} + \rho_1\langle u_t(t), \gamma_3\theta_x(t) - \beta\Sigma(t)\rangle_{L^2(\mathcal{I})}. \tag{3.20}$$

Now, for the remaining terms in equality (3.20), we use Young, Poincaré and Cauchy-Schwarz's inequalities. Hence, the inequality $(3.19)_1$ appears directly.

- Taking the derivative of $(3.17)_2$, by using the second equation in system (1.4), we get

$$\begin{aligned}&\frac{d}{dt}\Psi_2(t) + b\|\varphi_x\|^2_{L^2(\mathcal{I})}\\ &\quad= \rho_2\|\varphi_t\|^2_{L^2(\mathcal{I})} - \gamma_2\langle\varphi_x(t), \varphi_{xt}(t)\rangle_{L^2(\mathcal{I})} - k\langle\varphi(t), (u_x+\varphi)(t)\rangle_{L^2(\mathcal{I})}\\ &\qquad - \gamma_1\langle\varphi(t), (u_x+\varphi)_t(t)\rangle_{L^2(\mathcal{I})} + \beta\langle\varphi(t), \theta(t)\rangle_{L^2(\mathcal{I})}.\end{aligned} \tag{3.21}$$

By Young and Poincaré's inequalities, we obtain the inequality $(3.19)_2$.

- Taking the derivative of $(3.17)_3$, by using the first equation in system (1.4), we obtain

$$\begin{aligned}&\frac{d}{dt}\Lambda_2(t) + k\|u_x+\varphi\|^2_{L^2(\mathcal{I})}\\ &\quad= -\gamma_1\langle(u_x+\varphi)_t(t), (u_x+\varphi)(t)\rangle_{L^2(\mathcal{I})} + \beta\langle\theta(t), (u_x+\varphi)(t)\rangle_{L^2(\mathcal{I})}\\ &\qquad - \langle\Theta(t), (u_x+\varphi)_t(t)\rangle_{L^2(\mathcal{I})}.\end{aligned} \tag{3.22}$$

By Young and Poincaré's inequalities, we obtain the inequality $(3.19)_3$. □

The main result of stability for the system (1.4) is given by the following Theorem.

Theorem 3.8 *Let $(u, \varphi, \theta) \in \mathcal{H}_2$ solution to the system (1.4) with boundary conditions (1.5)–(1.6) and initial condition (1.7). Then, the energy functional (3.1) satisfies*

$$E(t) \leq \lambda_3 e^{-\lambda_4 t}, \quad \forall t \geq 0, \tag{3.23}$$

for λ_3, λ_4 *positive constants.*

Proof Let $\tilde{N} > 0$, we define the Lyapunov functional $\mathcal{F}_2$ as follows

$$\mathcal{F}_2(t) = \tilde{N}E(t) + N_3\Phi_2(t) + N_4\Psi_2(t) + N_5\Lambda_2(t). \tag{3.24}$$

By differentiating inequality (3.24) and by using the inequalities (3.19), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2(t) \leq & -\tilde{\varpi}_1\|u_t\|^2_{L^2(\mathcal{I})} - \tilde{\varpi}_2\|\varphi_x\|^2_{L^2(\mathcal{I})} - \tilde{\varpi}_3\|u_x+\varphi\|^2_{L^2(\mathcal{I})} - \tilde{\varpi}_4\|\varphi_{xt}\|^2_{L^2(\mathcal{I})} \\ & -\tilde{\varpi}_5\|\theta_x\|^2_{L^2(\mathcal{I})} - \tilde{\varpi}_6\|(u_x+\varphi)_t\|^2_{L^2(\mathcal{I})},\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}\tilde{\varpi}_1 &= N_3\frac{\beta\rho_1}{2} - N_5\varepsilon_2,\\ \tilde{\varpi}_2 &= N_4\frac{b}{2},\\ \tilde{\varpi}_3 &= N_5\frac{k}{2} - N_3\varepsilon_1 - N_4 c,\\ \tilde{\varpi}_4 &= \tilde{N}\gamma_2 - c(N_3+N_4),\\ \tilde{\varpi}_5 &= \tilde{N}\gamma_3 - c(N_3+N_4+N_5),\\ \tilde{\varpi}_6 &= \tilde{N}\gamma_1 - N_3\frac{c}{\varepsilon_1} - N_4 c - N_5\frac{c}{\varepsilon_2}.\end{aligned} \tag{3.26}$$

We choose the constants N_3, N_4 and N_5 such that

$$N_5 \gg N_4 \quad \text{and} \quad N_3, N_4 > 0. \tag{3.27}$$

Then, We choose $\varepsilon_1, \varepsilon_2$ small enough such that

$$\begin{aligned}\varepsilon_2 &< \frac{N_3\beta\rho_1}{2N_5},\\ \varepsilon_1 &< \frac{N_5 k}{2N_3} - \frac{N_4 c}{N_3}.\end{aligned} \tag{3.28}$$

Next, we can choose $\tilde{N}$ large enough such that

$$\begin{aligned}&\tilde{N}\gamma_1 - N_3\frac{c}{\varepsilon_1} - N_4 c - N_5\frac{c}{\varepsilon_2} > 0,\\ &\tilde{N}\gamma_3 - c(N_3+N_4+N_5) > 0,\\ &\tilde{N}\gamma_2 - c(N_3+N_4) > 0.\end{aligned} \tag{3.29}$$

Hence, the constants $\tilde{\varpi}_i$ for$i = 1, \ldots, 6$ are all negative; at this point, there exists a constant ϑ_2, and, further, $\mathcal{F}_2(t) \sim E(t)$, so inequality (3.25) takes the following form

$$\frac{d}{dt}\mathcal{F}_2(t) \leq -\vartheta_2 E(t), \forall t \geq 0. \tag{3.30}$$

Now, by using the fact that $\mathcal{F}_2 \sim E$ we get

$$\frac{d}{dt}\mathcal{F}_2(t) \leq -\zeta_2\mathcal{F}_2(t), \forall t \geq 0, \tag{3.31}$$

for some positive constant ζ_2. Integrating the last inequality over $(0, t)$, we finally arrive at

$$\mathcal{F}_2(t) \leq \mathcal{F}_2(0)e^{-\zeta_2 t}, \forall t \geq 0. \tag{3.32}$$

By using the other side of the equivalence relation (3.6). Then, the proof of Theorem (3.8) is finished. □

4 Some open problems

In this section, we will give some open problems for the interested reader.

- Viscoelasticity more Cattaneo's law acting on bending moment:

$$\begin{aligned}
&\rho_1 u_{tt} - k(u_x+\varphi)_x - \gamma_1(u_x+\varphi)_{xt} = 0, \quad \text{in} \quad (0,L)\times(0,\infty),\\
&\rho_2\varphi_{tt} - b\varphi_{xx} - \gamma_2\varphi_{xxt} + k(u_x+\varphi) + \gamma_1(u_x+\varphi)_t + \delta\theta_x = 0, \quad \text{in} \quad (0,L)\times(0,\infty),\\
&\rho_3\theta_t + q_x + \delta\varphi_{xt} = 0, \quad \text{in} \quad (0,L)\times(0,\infty),\\
&\tau q_t + \beta q + \theta_x = 0, \quad \text{in} \quad (0,L)\times(0,\infty).
\end{aligned}$$

- Viscoelasticity more Cattaneo's law acting on shear force:

$$\begin{aligned}
&\rho_1 u_{tt} - k(u_x+\varphi)_x - \gamma_1(u_x+\varphi)_{xt} + \delta\theta_x = 0, \quad \text{in} \quad (0,L)\times(0,\infty),\\
&\rho_2\varphi_{tt} - b\varphi_{xx} - \gamma_2\varphi_{xxt} + k(u_x+\varphi) + \gamma_1(u_x+\varphi)_t - \delta\theta = 0, \quad \text{in} \quad (0,L)\times(0,\infty),\\
&\rho_3\theta_t + q_x + \delta(u_x+\varphi)_t = 0, \quad \text{in} \quad (0,L)\times(0,\infty),\\
&\tau q_t + \beta q + \theta_x = 0, \quad \text{in} \quad (0,L)\times(0,\infty).
\end{aligned}$$

- Viscoelasticity more Cattaneo's law acting on bending moment for truncated version:

$$\begin{aligned}
&\rho_1 u_{tt} - k(u_x+\varphi)_x - \gamma_1(u_x+\varphi)_{xt} = 0, \quad \text{in} \quad (0,L)\times(0,\infty),\\
&-\rho_2\varphi_{xxt} - b\varphi_{xx} + k(u_x+\varphi) + \gamma_1(u_x+\varphi)_t + \delta\theta_x = 0, \quad \text{in} \quad (0,L)\times(0,\infty),\\
&\rho_3\theta_t + q_x + \delta\varphi_{xt} = 0, \quad \text{in} \quad (0,L)\times(0,\infty),\\
&\tau q_t + \beta q + \theta_x = 0, \quad \text{in} \quad (0,L)\times(0,\infty).
\end{aligned}$$

- Viscoelasticity more Cattaneo's law acting on shear force for truncated version:

$$\begin{aligned}
&\rho_1 u_{tt} - k(u_x+\varphi)_x - \gamma_1(u_x+\varphi)_{xt} + \delta\theta_x = 0, \quad \text{in} \quad (0,L)\times(0,\infty),\\
&-\rho_2\varphi_{xxt} - b\varphi_{xx} + k(u_x+\varphi) + \gamma_1(u_x+\varphi)_t - \delta\theta = 0, \quad \text{in} \quad (0,L)\times(0,\infty),\\
&\rho_3\theta_t + q_x + \delta(u_x+\varphi)_t = 0, \quad \text{in} \quad (0,L)\times(0,\infty),\\
&\tau q_t + \beta q + \theta_x = 0, \quad \text{in} \quad (0,L)\times(0,\infty).
\end{aligned}$$

It will be interesting to extend our results to the previous systems.

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this manuscript. The authors declare that they have no competing interests.

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