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Existence of renomalized solution for nonlinear elliptic boundary value problem without Δ_2 -condition

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Abstract
In this paper we will prove in Musielak–Orlicz spaces, the existence of renomalized solution for nonlinear elliptic equations of Leray-Lions type, in the case where the Musielak–Orlicz function φ doesn't satisfy the Δ_2 condition while the right hand side f belongs to $W^{-1}E_{\psi}(\Omega)$.

Keywords Musielak–Orlicz–Sobolev spaces · Elliptic equation · Renormalized solutions · Truncations

Mathematics Subject Classification 35J25 · 35J60 · 46E30

1 Introduction and basic assumptions

This work deals with existence of solutions for strongly nonlinear boundary value problem whose model is:

$$\begin{cases} A(u) - \operatorname{div} \Phi(u) + g(x,u,\nabla u) = f \quad in \quad \Omega \\ \qquad\qquad u \equiv 0, \quad on \quad \partial\Omega \end{cases} \tag{1.1}$$

where Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 2$, $A(u) = -\operatorname{div} a(x,u,\nabla u)$ be a Leray-Lions operator defined from the space $W_0^1 L_{\varphi}(\Omega)$ into its dual $W^{-1}L_{\psi}(\Omega)$, and $\Phi \in \mathrm{C}^0\left(\mathrm{R}, \mathrm{R}^N\right)$. where a is a function satisfying the following conditions :

$$a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N \text{is a Carathéodory function.} \tag{1.2}$$

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There exist two Musielak–Orlicz functions φ and γ such that $\gamma \prec\prec \varphi$, a positive function $d(\cdot) \in E_{\psi}(\Omega)$ and positive constants k_1, k_2 and k_3 such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$

$$|a(x,s,\xi)| \leq k_1 \left(d(x) + \psi_x^{-1}\gamma(x,k_2|s|)\right) + \psi_x^{-1}\varphi(x,k_3|\xi|); \tag{1.3}$$

$$\left(a(x,s,\xi) - a\left(x,s,\xi'\right)\right)\left(\xi - \xi'\right) > 0; \tag{1.4}$$

$$a(x,s,\xi).\xi \geq \alpha\varphi(x,|\xi|). \tag{1.5}$$

Furthermore, let $g(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$, satisfying the following conditions

$$|g(x,s,\xi)| \leq c(x) + b(|s|)\varphi(x,|\xi|); \tag{1.6}$$

$$g(x,s,\xi)s \geq 0; \tag{1.7}$$

where $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous positive function which belongs to $L^1\left(\mathbb{R}^+\right)$ and $c(\cdot) \in L^1(\Omega)$ The right-hand side of (1.1) and $\Phi : \mathbb{R} \to \mathbb{R}^N$ are assumed to satisfy

$$f \in W^{-1}E_{\psi}(\Omega); \tag{1.8}$$

$$\Phi \in \mathcal{C}^0\left(\mathbb{R}, \mathbb{R}^N\right). \tag{1.9}$$

Note that no growth hypothesis is assumed on the function Φ, which implies that the term $-\text{div}\ \Phi(u)$ may be meaningless, even as a distribution.

Several researches deals with the existence solutions of elliptic and parabolic problems under various assumptions and in different contexts (see [1–10,13–20,24–28,35,37,39,40] for more details), indeed we can't recite all examples; we will just choose some of them, So we mention that:

the problem (1.1) was treated by Boccardo (see [23]) in the case $g \equiv 0$ and for p such that $2 - 1/N < p < N$ where he proved the existence and regularity of an entropy solution u that is $u \in W_0^{1,q}(\Omega)$, $q < \tilde{p} = \frac{(p-1)N}{N-1}$, $T_k(u) \in W_0^{1,p}(\Omega)$, $\forall k > 0$. The same problem have been studied by Diperna and lions in [26] where they introduced the idea of renormalized solutions.

In the framework of variable exponent Sobolev spaces in [12] have proved the existence result of solutions for the problem 1.1 without sign condition involving nonstandard growth.

In the setting of Musielak spaces and in variational case, the existence of a weak solution for the problem (1.1) was treated by Ahmed Oubeid, Benkirane and Sidi El Vally in [11] where div $\Phi \equiv 0$.

Our purpose in this paper is to show the existence of renormalized solutions for problem (1.1) in Musielak Orlicz spaces in the case where the Musielak–Orlicz function φ doesn't satisfy the Δ_2 condition,while the right-hand side belongs to $W^{-1}E_{\psi}(\Omega)$, $\Phi \in \mathcal{C}^0\left(\mathbb{R}, \mathbb{R}^N\right)$. and a nonlinearity $g(x,s,\xi)$ having natural growth with respect to the gradient.

The paper is organized as follows: In Sect. 2 , we give some preliminaries and background. Section 3 is devoted to some technical lemmas which can be used to our result. In the final Sect. 4, we state our main result and give the prove of an existence solution.

2 Some preliminaries and background

Here we give some definitions and properties that concern Musielak–Orlicz spaces (see [34]).

Let Ω be an open subset of $\mathbb{R}^n$, a Musielak–Orlicz function φ is a real-valued function defined in $\Omega \times \mathbb{R}_+$ such that

(a) $\varphi(x,t)$ is an N-function i.e. convex, nondecreasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0$ and

$$\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t}=0,\qquad \lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t}\quad=0$$

(b) $\varphi(x,t)$ is a measurable function for all $t\geq 0$.
Now, let $\varphi_x(t)=\varphi(x,t)$ and let φ_x^{-1} be the non-negative reciprocal function with respect to t, i.e the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t))=\varphi\left(x,\varphi_x^{-1}(t)\right)=t$$

The Musielak–Orlicz function φ is said to satisfy the Δ_2 -condition if for some $k>0$, and a non negative function h, integrable in Ω, we have

$$\varphi(x,2t)\leq k\varphi(x,t)+h(x)\text{ for all } x\in\Omega \text{ and } t\geq 0. \tag{2.1}$$

When (2.1) holds only for $t\geq t_0>0$, then φ is said to satisfy the Δ_2 -condition near infinity. Let φ and γ be two Musielak–Orlicz functions, we say that φ dominate γ and we write $\gamma\prec\varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x\in\Omega$

$\gamma(x,t)\leq\varphi(x,ct)$ for all $t\geq t_0$, (resp. for all $t\geq 0$ i.e. $t_0=0$) We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma\prec\prec\varphi$ if for every positive constant c we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0,\quad \left(\text{resp. }\lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0\right)$$

Remark 2.1 (see [29]) If $\gamma\prec\varphi$ near infinity such that γ is locally integrable on Ω, then $\forall c>0$ there exists a nonnegative integrable function h such that

$$\gamma(x,t)\leq\varphi(x,ct)+h(x),\ \text{ for all } t\geq 0 \text{ and for a. e. } x\in\Omega.$$

For a Musielak–Orlicz function φ and a measurable function $u:\Omega\longrightarrow\mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u)=\int_\Omega\varphi(x,|u(x)|)dx$$

The set $K_\varphi(\Omega)=\left\{u:\Omega\longrightarrow\mathbb{R}\text{ measurable }/\rho_{\varphi,\Omega}(u)<\infty\right\}$ is called the Musielak–Orlicz class (or generalized Orlicz class). The Musielak–Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega)=\left\{u:\Omega\longrightarrow\mathbb{R}\text{ measurable }/\rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right)<\infty,\ \text{for some }\lambda>0\right\}$$

For a Musielak–Orlicz function φ we put: $\psi(x,s)=\sup_{t>0}\{st-\varphi(x,t)\}$, ψ is the Musielak–Orlicz function complementary to φ (or conjugate of φ) in the sens of Young with respect to the variable s In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega}=\inf\left\{\lambda>0/\int_\Omega\varphi\left(x,\frac{|u(x)|}{\lambda}\right)dx\leq 1\right\}$$

which is called the Luxemburg norm and the so-called Orlicz norm by:

$$\||u|\|_{\varphi,\Omega}=\sup_{\|v\|_\psi\leq 1}\int_\Omega|u(x)v(x)|dx$$

where ψ is the Musielak Orlicz function complementary to φ. These two norms are equivalent (see [34])

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_\varphi(\Omega)$, It is a separable space (see [34], Theorem 7.10) .

We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \left\{u \in L_\varphi(\Omega) : \forall|\alpha| \le m,\, D^\alpha u \in L_\varphi(\Omega)\right\}$$

and

$$W^m E_\varphi(\Omega) = \left\{u \in E_\varphi(\Omega) : \forall|\alpha| \le m,\, D^\alpha u \in E_\varphi(\Omega)\right\}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + \ldots + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.

Let

$$\bar{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\le m} \rho_{\varphi,\Omega}\left(D^\alpha u\right) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\left\{\lambda > 0 : \bar{\rho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \le 1\right\}$$

for $u \in W^m L_\varphi(\Omega)$. These functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\left(W^m L_\varphi(\Omega), \|\,\|^m_{\varphi,\Omega}\right)$ is a Banach space if φ satisfies the following condition (see [34]):

$$\text{there exist a constant } c_0 > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \ge c_0 \tag{2.2}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\le m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closed.

The space $W_0^m L_\varphi(\Omega)$ is defined as the $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. and the space $W_0^m E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W_0^m L_\varphi(\Omega)$ be the $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$ The following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \left\{ f \in D'(\Omega);\, f = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \right\}$$

and

$$W^{-m} E_\psi(\Omega) = \left\{ f \in D'(\Omega);\, f = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \right\}$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \bar{\rho}_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0$$

For φ and her complementary function ψ, the following inequality is called the Young inequality (see [34]):

$$ts \leq \varphi(x,t) + \psi(x,s), \quad \forall t, s \geq 0, x \in \Omega \tag{2.3}$$

This inequality implies that

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1 \tag{2.4}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} > 1 \tag{2.5}$$

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} \leq 1 \tag{2.6}$$

For two complementary Musielak Orlicz functions φ and ψ, let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, then we have the Holder inequality (see [34]):

$$\left| \int_\Omega u(x)v(x)dx \right| \leq \|u\|_{\varphi,\Omega} \||v|\|_{\psi,\Omega} \tag{2.7}$$

We will use the following technical lemmas.

3 Some technical lemmas

Lemma 3.1 [19] *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak–Orlicz functions which satisfy the following conditions:*

(i) *There exist a constant $c > 0$ such that $\inf_{x\in\Omega}\varphi(x,1) \geq c$.*
(ii) *There exist a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$ we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{\log\left(\frac{1}{|x-y|}\right)}\right)}, \quad \forall t \geq 1 \tag{3.1}$$

(iii)

$$\textit{If } D \subset \Omega \textit{ is a bounded measurable set, then } \int_D \varphi(x,1)dx < \infty \tag{3.2}$$

(iv) *There exist a constant $C > 0$ such that $\psi(x,1) \leq C$ a.e in Ω.*

Under this assumptions, $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\bar{\Omega})$ is dense in $W^1L_\varphi(\Omega)$ the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L_\psi(\Omega)$ on an element u of $W_0^1L_\varphi(\Omega)$ is well defined. It will be denoted by $< S, u >$.

Lemma 3.2 [36] *Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak–Orlicz function and let $u \in W_0^1L_\varphi(\Omega)$. Then $F(u) \in W_0^1L_\varphi(\Omega)$ Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e\ in \{x \in \Omega : u(x) \in D\} \\ 0 & a.e\ in\ \{x \in \Omega : u(x) \notin D\} \end{cases}$$

Lemma 3.3 [29] (Poincare's inequality) *Let φ a Musielak Orlicz function which satisfies the assumptions of lemma 3.1, suppose that $\varphi(x,t)$ decreases with respect of one of coordinate of x Then, that exists a constant $c > 0$ depends only of Ω such that*

$$\int_\Omega \varphi(x, |u(x)|)dx \leq \int_\Omega \varphi(x, c|\nabla u(x)|)dx, \quad \forall u \in W_0^1 L_\varphi(\Omega)$$

Lemma 3.4 [19] *Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then, there exists a sequence $(u_n) \subset \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_\varphi(\Omega)$$

Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.

Lemma 3.5 *Let (f_n), $f \in L^1(\Omega)$ such that*

(i) $f_n \geq 0$ *a.e in* Ω
(ii) $f_n \longrightarrow f$ *a.e in* Ω
(iii) $\int_\Omega f_n(x)dx \longrightarrow \int_\Omega f(x)dx$ *then* $f_n \longrightarrow f$ *strongly in* $L^1(\Omega)$

Lemma 3.6 [20] *If a sequence $g_n \in L_\varphi(\Omega)$ converges in measure to a measurable function g and if g_n remains bounded in $L_\varphi(\Omega)$, then $g \in L_\varphi(\Omega)$ and $g_n \rightharpoonup g$ for $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$*

Lemma 3.7 (Jensen inequality) [38] *Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ a convex function and $g : \Omega \longrightarrow \mathbb{R}$ is function measurable, then*

$$\varphi\left(\int_\Omega g d\mu\right) \leq \int_\Omega \varphi \circ g d\mu.$$

Lemma 3.8 (The Nemytskii Operator) [29] *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak Orlicz functions. Let $f : \Omega \times \mathbb{R}^p \longrightarrow \mathbb{R}^q$ be a Carathodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:*

$$|f(x,s)| \leq c(x) + k_1 \psi_x^{-1} \varphi\,(x, k_2|s|)$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$ Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is continuous from

$$\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right)^p = \prod\left\{u \in L_M(\Omega) : d\,(u, E_M(\Omega)) < \frac{1}{k_2}\right\}$$

into $\left(L_\psi(\Omega)\right)^q$ for the modular convergence.
Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right)^p$ to $\left(E_\gamma(\Omega)\right)^q$

Lemma 3.9 *Let Ω be a bounded open subset of R^N with the segment property. If $u \in (W_0^1 L_\varphi(\Omega))^N$ then $\int_\Omega div\, u\, dx = 0$.*

Proof of lemma 3.9 The proof of this lemma is based on [[30], Lemma 3.2]

4 Main result

We consider the following boundary value problem

$$(\mathcal{P})\begin{cases} A(u) - \operatorname{div}\Phi(u) + g(.,u,\nabla u) = f \in W^{-1}E_\psi(\Omega), & \text{in } \Omega \\ u \equiv 0, & \text{on } \partial\Omega \end{cases}$$

Let us define

$$\mathcal{T}_0^{1,\varphi}(\Omega) = \left\{u \text{ measurable} \quad \text{such that} \quad T_k(u) \in W_0^1L_\varphi(\Omega), \forall k > 0\right\}.$$

As in [21] we define the following notion of renormalized solution, which gives a meaning to a possible solution of $(\mathcal{P})$

Definition 4.1 Assume that (1.2)–(1.4), (1.6) hold true. A function u is a renormalized solution of the problem $(\mathcal{P})$ if

$$\begin{cases} u \in \mathcal{T}_0^{1,\varphi}(\Omega), g(.,u,\nabla u) \in L^1(\Omega), g(.,u,\nabla u)u \in L^1(\Omega) \\ \qquad \displaystyle\int_\Omega a(x,u,\nabla u)h(u)\nabla v dx + \int_\Omega a(x,u,\nabla u)h'(u)\nabla u v dx \\ \qquad + \displaystyle\int_\Omega \Phi(u)h(u)\nabla v dx + \int_\Omega \Phi(u)h'(u)\nabla u v dx \\ \qquad + \displaystyle\int_\Omega g(x,u,\nabla u)h(u)v dx = \int_\Omega f h(u) v dx \end{cases} \tag{4.1}$$

for all $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support in $\mathbb{R}$, and for all $v \in W_0^1L_\varphi(\Omega) \cap L^\infty(\Omega)$.

The weaker problem (4.1) is obtained by using the test function $h(u)v$ where $h \in W^{1,\infty}(\mathbb{R})$. and $v \in W_0^1L_\varphi(\Omega) \cap L^\infty(\Omega)$ in $(\mathcal{P})$.

Remark 1 Let us note that in (4.1) every term is meaningful in the distributional sense.

Theorem 4.1 *Under assumptions* (1.2)–(1.4),(1.6) *there exists at least a renormalized solution u in the sense of definition 4.1 of problem* $(\mathcal{P})$.

Let us introduce the truncate operator. For a given constant $k > 0$, we define the function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} s \text{ if } |s| \le k, \\ k\frac{s}{|s|} \text{ if } |s| > k. \end{cases}$$

4.1 Proof of Theorem 4.1

4.1.1 Approximate problem and a priori estimate

We use an idea contained in [37] (Theorem 1.1), based on the approximation of the original problem and a priori estimate. For $n \in \mathbb{N}$, let $(f_n)_n$ be a sequence in $W^{-1}E_\psi(\Omega) \cap L^1(\Omega)$ such that $f_n \longrightarrow f$ in $L^1(\Omega)$ with $\|f_n\|_1 \le \|f\|_1$, $\phi_n(s) = \phi(T_n(s))$ and $g_n(x,s,\xi) = T_n(g(x,s,\xi))$. The following approximate problem

$$(P_n)\begin{cases} -\operatorname{div}(a(\cdot,u_n,\nabla u_n)) + g_n(\cdot,u_n,\nabla u_n) = f_n + \operatorname{div}(\Phi_n(u_n)) & \text{in } D'(\Omega) \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution u_n in $W_0^1 L_\varphi(\Omega)$.

Now Choosing u_n as a function test in problem (P_n), we have

$$\int_\Omega a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n dx+\int_\Omega \Phi_n\left(u_n\right) \cdot \nabla u_n dx+\int_\Omega g_n\left(x, u_n, \nabla u_n\right) u_n dx=\left\langle f, u_n\right\rangle \tag{4.2}$$

By posing

$$\widetilde{\Phi}_n(t)=\int_0^t \Phi_n(\tau) d \tau$$

we obtain

$$\bar{\Phi}_n(0)=0.$$

As each component of $\bar{\Phi}_n$ is uniformly Lipschitizian, and according to [[32], Lemma 2], it follows that the function $\bar{\Phi}_n\left(u_n\right)$ belongs to $\left(W_0^1 L_\varphi(\Omega)\right)^N$.

therefore by using Lemma 3.9

$$\int_\Omega \Phi_n\left(u_n\right) \cdot \nabla u_n dx=\int_\Omega \operatorname{div}\left(\widetilde{\Phi}_n\left(u_n\right)\right) dx=0$$

According to (1.7) and using Young's inequality, we have

$$\left|\int_\Omega a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n dx\right| \leq C_1+\frac{\alpha}{2} \int_\Omega \varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right) dx. \tag{4.3}$$

which together with (1.5) gives

$$\int_\Omega \varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right) dx \leq C_2 \tag{4.4}$$

Poincare inequality (see Lemma3.3) implies that

$$\int_\Omega \varphi\left(x, \frac{\left|T_k\left(u_n\right)\right|}{c}\right) dx \leq \int_\Omega \varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right) dx \leq c_2 k \tag{4.5}$$

On the other hand we have

$$\int_\Omega g_n\left(x, u_n, \nabla u_n\right) u_n dx \leq C_3 \tag{4.6}$$

so it follows that $\left(T_k\left(u_n\right)\right)_n$ and $\left(\nabla T_k\left(u_n\right)\right)_n$ are bounded in $L_\varphi(\Omega)$, Thus

$$\left(T_k\left(u_n\right)\right)_n \; is \text{ bounded in } W_0^1 L_\varphi(\Omega),$$

there exists some $v_k \in W_0^1 L_\varphi(\Omega)$ such that

$$\begin{cases} T_k\left(u_n\right) \rightharpoonup v_k & \text{weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma\left(\Pi L_\varphi, \Pi E_\psi\right) \\ T_k\left(u_n\right) \longrightarrow v_k & \text{strongly in } E_\psi(\Omega). \end{cases} \tag{4.7}$$

Now one suppose that exists a function φ satisfies $\lim_{t \rightarrow \infty} \frac{\varphi(t)}{t}=\infty$ and $\varphi(t) \leq \operatorname{ess\,inf}_{x \in \Omega} \varphi(x, t)$ Let $k>0$ large enough, by using (4.5) we have

$$\begin{aligned} \varphi(k) \operatorname{meas}\left\{\left|u_n\right|>k\right\} & =\int_{\left\{\left|u_n\right|>k\right\}} \varphi\left(\left|T_k\left(u_n\right)\right|\right) dx \\ & \leq \int_{\left\{\left|u_n\right|>k\right\}} \varphi\left(x,\left|T_k\left(u_n\right)\right|\right) dx \leq \int_\Omega \varphi\left(x,\left|T_k\left(u_n\right)\right|\right) dx \\ & \leq c_3 k \end{aligned}$$

Hence

$$\text{meas}\{|u_n| > k\} \le \frac{c_3 k}{\varphi(k)} \longrightarrow 0 \text{ as } k \longrightarrow \infty$$

For every $\lambda > 0$, we have

$$\begin{aligned} \text{meas}\{|u_n - u_m| > \lambda\} \le \lambda\} &\le \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ &+ \text{meas}\{|T_k(u_n) - T_k(u_m)| > \lambda\} \end{aligned} \tag{4.8}$$

then, by using (4.5) one suppose that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω, Let $\varepsilon > 0$, then by (4.8) there exists some $k = k(\varepsilon) > 0$ such that

$$\text{meas}\{|u_n - u_m| > \lambda\} < \varepsilon, \quad \text{for all } n, m \ge h_0(k(\varepsilon), \lambda)$$

which means that $(u_n)_n$ is a Cauchy sequence in measure in Ω, thus converge almost every where to u.

Consequently

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma\left(\Pi L_\varphi, \Pi E_\psi\right) \\ u_n \longrightarrow u \text{ strongly in } E_\psi(\Omega). \end{cases} \tag{4.9}$$

4.1.2

In this step we shall show the boundedness of $(a(\cdot, T_k(u_n), \nabla T_k(u_n)))_n$ in $\left(L_\psi(\Omega)\right)^N$

Let $\vartheta \in E_\varphi(\Omega)^N$ such that $\|\vartheta\|_{\varphi,\Omega} \le 1$, the hypothesis (1.4) gives We have

$$\int_\Omega \left[a(x, T_k(u_n), \nabla T_k(u_n)) - a\left(x, T_k(u_n), \frac{\vartheta}{k_3}\right)\right]\left[\nabla T_k(u_n) - \frac{\vartheta}{k_3}\right] dx > 0$$

This implies that

$$\begin{aligned} &\int_\Omega \frac{1}{k_3} a(x, T_k(u_n), \nabla T_k(u_n))\, \vartheta\, dx \\ &\le \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\, \nabla T_k(u_n)\, dx \\ &\quad - \int_\Omega a\left(x, T_k(u_n), \frac{\vartheta}{k_3}\right)\left(\nabla T_k(u_n) - \frac{\vartheta}{k_3}\right) dx \\ &\le c_2 k - \int_\Omega a\left(x, T_k(u_n), \frac{\vartheta}{k_3}\right) \nabla T_k(u_n)\, dx \\ &\quad + \frac{1}{k_3}\int_\Omega a\left(x, T_k(u_n), \frac{\vartheta}{k_3}\right) \vartheta\, dx \end{aligned}$$

By using Young's inequality in the last two terms of the last side and (4.5) we get

$$
\begin{aligned}
\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \vartheta d x \leq & c_2 k k_3 \\
& +3 k_1\left(1+k_3\right) \int_\Omega \psi\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right|}{3 k_1}\right) d x \\
& +3 k_1 k_3 \int_\Omega \varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right) d x+3 k_1 \int_\Omega \varphi(x,|\vartheta|) d x \\
\leq & c_2 k k_3+3 k_1 k_3 c_2 k+3 k_1 \\
& +3 k_1\left(1+k_3\right) \int_\Omega \psi\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right|}{3 k_1}\right) d x
\end{aligned}
$$

Now, by using (1.3) and the convexity of ψ we get

$$
\psi\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right|}{3 k_1}\right) \leq \frac{1}{3}\left(\psi(x, d(x))+\gamma\left(x, k_2\left|T_k\left(u_n\right)\right|\right)+\varphi(x,|\vartheta|)\right) .
$$

Thanks to **"Remark 2.1"** there exists $h \in L^1(\Omega)$ such that

$$
\gamma\left(x, k_2\left|T_k\left(u_n\right)\right|\right) \leq \gamma\left(x, k_2 k\right) \leq \varphi(x, 1)+h(x);
$$

then by integrating over Ω we deduce that

$$
\begin{aligned}
\int_\Omega \psi\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{v}{k_3}\right)\right|}{3 k_1}\right) d x \leq & \frac{1}{3}\left(\int_\Omega \psi(x, d(x)) d x+\int_\Omega h(x) d x\right. \\
& \left.+\int_\Omega \varphi(x, 1) d x+\int_\Omega \varphi(x,|\vartheta|) d x\right) \leq c_k,
\end{aligned}
$$

where c_k is a constant depending on k. So,

$$
\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \vartheta d x \leq c_k, \quad \forall \vartheta \in\left(E_{\varphi}(\Omega)\right)^N \quad \text{with } \|\vartheta\|_{\varphi, \Omega}=1
$$

and thus $\left\|a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\right\|_{\psi, \Omega} \leq c_k$, which implies that,

$$
\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\right)_n \text{ is bounded in } L_\psi(\Omega)^N . \tag{4.10}
$$

4.1.3

Let us show that :

$$
\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\left(m \leq\left|u_n\right| \leq m+1\right|} a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n d x=0
$$

Defining

$$
\theta_m(r)=T_{m+1}(r)-T_m(r) \qquad \text{For any } m \geq 1,
$$

in view of [[32], Lemma2] one get $\theta_m\left(u_n\right) \in W_0^1 L_{\varphi}(\Omega)$.

Now let us taking $\theta_m(u_n)$ as a test function in (P_n) we have

$$\int_\Omega a(x, u_n, \nabla u_n) \nabla\theta_m(u_n)\,dx + \int_\Omega \Phi_n(u_n) \nabla\theta_m(u_n)\,dx + \int_\Omega g_n(x, u_n, \nabla u_n)\,\theta_m(u_n)\,dx = \int_\Omega f_n\theta_m(u_n)\,dx$$

Consider,

$$\phi(t) = \Phi_n(t)\chi_{\{s\in\mathbb{R}, m\le|s|\le m+1\}}(t)$$
$$\tilde{\phi}(t) = \int_0^t \phi(\tau)d\tau$$

hence $\tilde{\phi}(u_n) \in \left(W_0^1L_\varphi(\Omega)\right)^N$ (by Lemma3.2). We obtain, by Lemma 3.9,

$$\int_\Omega \Phi_n(u_n)\nabla\theta_m(u_n)\,dx = \int_\Omega \Phi_n(u_n)\,\chi_{\{s\in\mathbb{R}, m\le|s|\le m+1\}}(u_n)\nabla u_n dx = \int_\Omega \phi(u_n)\nabla u_n dx = \int_\Omega \operatorname{div}\left(\tilde{\phi}(u_n)\right)dx = 0$$

Using the sign condition (1.7) we have $g_n(x, u_n, \nabla u_n)\,\theta_m(u_n) \ge 0$ a.e. in Ω, and knowing that $\nabla\theta_m(u_n) = \nabla u_n\chi_{\{m\le|u_n|\le m+1\}}$ a.e. in Ω, we get

$$\int_{\{m\le|u_n|\le m+1\}} a(x, u_n, \nabla u_n)\nabla u_n dx \le \langle f, \theta_m(u_n)\rangle\,.$$

It is not difficult to see that

$$\|\nabla\theta_m(u_n)\|_{\varphi,\Omega} \le \|\nabla u_n\|_{\varphi,\Omega}\,.$$

then in view of (4.4) and (4.9) it follows that

$$\theta_m(u_n) \rightharpoonup \theta_m(u) \quad \text{weakly in } W_0^1L_\varphi(\Omega) \quad \text{for } \sigma\left(\Pi L_\varphi(\Omega), \Pi E_\varphi(\Omega)\right)$$

Therefore, we get

$$\lim_{n\to\infty}\int_{\{m\le|u_u|\le m+1\}} a(x, u_n, \nabla u_n)\cdot\nabla u_n dx \le \langle f, \theta_m(u)\rangle$$

as $\theta_m(u)\rightharpoonup 0$ weakly in $W_0^1L_\varphi(\Omega,)$ for $\sigma\left(\Pi L_\varphi(\Omega), \Pi E_\varphi(\Omega)\right)$ one obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{(m\le||n_n|\le m+1|} a(x, u_n, \nabla u_n)\cdot\nabla u_n dx \le \lim_{m\to\infty}\langle f, \theta_m(u)\rangle = 0$$

By (1.5), we get

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{(m\le|u_k|\le m+1|} a(x, u_n, \nabla u_n)\cdot\nabla u_n dx = 0 \tag{4.11}$$

4.1.4

In this subsubsection we pose $\phi(s) = se^{\lambda s^2}$ where $\lambda = \left(\frac{b(k)}{2\alpha}\right)^2$. it is easy to get,

$$\text{for all } s\in\mathbb{R}, \qquad \phi'(s) - \frac{b(k)}{\alpha}|\phi(s)| \ge \frac{1}{2} \tag{4.12}$$

For $m \geq k$, definning

$$\psi_m(s) = \begin{cases} 1 & \text{if } |s| \leq m \\ m+1-|s| & \text{if } m \leq |s| \leq m+1 \\ 0 & \text{if } |s| \geq m+1 \end{cases}$$

Let $\left\{v_j\right\}_j \subset D(\Omega)$ be a sequence such that $v_j \rightarrow u$ in $W_0^1 L_{\varphi}(\Omega)$ for the modular convergence and a e. in Ω. And let us define the functions

$$\theta_n^j = T_k\left(u_n\right) - T_k\left(v_j\right), \theta^j = T_k(u) - T_k\left(v_j\right) \text{ and } z_{n,m}^j = \phi\left(\theta_n^j\right) \psi_m\left(u_n\right).$$

Using $z_{n,m}^j \in W_0^1 L_{\varphi}(\Omega)$ as a test function in (P_n) we get

$$\begin{aligned} & \int_\Omega a\left(x, u_n, \nabla u_n\right) \cdot \nabla z_{n,m}^j dx \\ & + \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n\left(u_n\right) \cdot \nabla u_n \psi_m'\left(u_n\right) \phi\left(T_k\left(u_n\right) - T_k\left(v_j\right)\right) dx \\ & + \int_\Omega \Phi_n\left(u_n\right) \cdot \nabla \phi\left(T_k\left(u_n\right) - T_k\left(v_j\right)\right) \psi_m\left(u_n\right) dx \\ & + \int_\Omega g_n\left(x, u_n, \nabla u_n\right) z_{n,m}^j dx = \int_\Omega f z_{n,m}^j dx \end{aligned} \tag{4.13}$$

From now on, we denote by $\epsilon_i(n, j), i = 0, 1, 2, \ldots,$ various sequences of real numbers which tend to zero as n and $j \rightarrow \infty$, i.e.,

$$\lim_{j \rightarrow +\infty} \lim_{n \rightarrow +\infty} \epsilon_i(n, j) = 0$$

by using (4.7) one has $z_{n,m}^j \rightarrow \phi\left(\theta^j\right) \psi_m(u)$ weakly in $L^\infty(\Omega)$ for $\sigma^*\left(L^\infty, L^1\right)$ as $n \rightarrow \infty$ which give

$$\lim_{n \rightarrow \infty} \int_\Omega f z_{n,m}^j dx = \int_\Omega f \phi\left(\theta^j\right) \psi_m(u) dx$$

and $\phi\left(\theta^j\right) \rightarrow 0$ weakly in $L^\infty(\Omega)$ for $\sigma\left(L^\infty, L^1\right)$ as $j \rightarrow \infty$, we have

$$\lim_{j \rightarrow \infty} \int_\Omega f \phi\left(\theta^j\right) \psi_m(u) dx = 0$$

Therefore, by denoting

$$\int_\Omega f z_{n,m}^j dx = \epsilon_0(n, j),$$

the divergence lemma implies that

$$\int_{\{m \leq |u_n| \leq m+1\}} \Phi_n\left(u_n\right) \cdot \nabla u_n \psi_m'\left(u_n\right) \phi\left(T_k\left(u_n\right) - T_k\left(v_j\right)\right) dx = 0.$$

The third term in the left-hand side of (4.13) can be written as follows

$$\begin{aligned} & \int_\Omega \Phi_n\left(u_n\right) \cdot \nabla \phi\left(T_k\left(u_n\right) - T_k\left(v_j\right)\right) \psi_m\left(u_n\right) dx \\ & = \int_\Omega \Phi_n\left(u_n\right) \cdot \nabla T_k\left(u_n\right) \phi'\left(\theta_n^j\right) \psi_m\left(u_n\right) dx \\ & \quad - \int_\Omega \Phi_n\left(u_n\right) \cdot \nabla T_k\left(v_j\right) \phi'\left(\theta_n^j\right) \psi_m\left(u_n\right) dx \end{aligned}$$

Applying the divergence lemma we have,

$$\int_\Omega \Phi_n(u_n)\cdot\nabla T_k(u_n)\,\phi'\left(\theta_n^j\right)\psi_m(u_n)\,dx=0.$$

By (4.7) one obtain

$$\Phi_n(u_n)\,\phi'\left(\theta_n^j\right)\psi_m(u_n)\to\Phi(u)\phi'\left(\theta^j\right)\psi_m(u)\text{ a.e. in }\Omega\quad as\quad n\to+\infty$$

now, we can verify that

$$\left\|\Phi_n(u_n)\,\phi'\left(\theta_n^j\right)\psi_m(u_n)\right\|_{\varphi,\Omega}\le\psi\left(x,c_m\phi'(2k)\right)|\Omega|+1$$

with $c_m=\max_{|t|\le m+1}\Phi(t)$.
Thanks to [[33], Theorem 14.6], we have

$$\lim_{n\to+\infty}\int_\Omega \Phi_n(u_n)\cdot\nabla T_k\left(v_j\right)\phi'\left(\theta_n^j\right)\psi_m(u_n)\,dx=\int_\Omega \Phi(u)\cdot\nabla T_k\left(v_j\right)\phi'\left(\theta^j\right)\psi_m(u)dx$$

Using the modular convergence of the sequence $\left\{v_j\right\}_j$, it follows that

$$\lim_{j\to+\infty}\lim_{n\to+\infty}\int_\Omega \Phi_n(u_n)\cdot\nabla T_k\left(v_j\right)\phi'\left(\theta_n^j\right)\psi_m(u_n)\,dx=\int_\Omega \Phi(u)\cdot\nabla T_k(u)\psi_m(u)dx$$

Then, thanks to Lemma 3.9 we obtain

$$\int_\Omega \Phi(u)\cdot\nabla T_k(u)\psi_m(u)dx=0$$

Therefore,we denote

$$\int_\Omega \Phi_n(u_n)\cdot\nabla\phi\left(T_k(u_n)-T_k\left(v_j\right)\right)\psi_m(u_n)\,dx=\epsilon_1(n,j).$$

since $g_n(x,u_n,\nabla u_n)\,z_{nm}^j\ge0$ on the set $\{|u_n|>k\}$ and $\psi_m(u_n)=1$ on the set $\{|u_n|\le k\}$, by according to 4.13 we get

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla z_{nm}^j dx+\int_{\{|u_n|\le k|} g_n(x,u_n,\nabla u_n)\,\phi\left(\theta_n^j\right)dx\le\epsilon_2(n,j)\tag{4.14}$$

For the first term of the left-hand side of (4.14) we can write

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla z_{n,m}^j dx&=\int_\Omega a(x,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)\\&\quad-\nabla T_k\left(v_j\right))\,\phi'\left(\theta_n^j\right)\psi_m(u_n)\,dx\\&\quad+\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\phi\left(\theta_n^j\right)\psi_m'(u_n)\,dx\\&=\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\left(\nabla T_k(u_n)-\nabla T_k\left(v_j\right)\right)\phi'\left(\theta_n^j\right)dx\\&\quad-\int_{||u_n|>k|} a(x,u_n,\nabla u_n)\cdot\nabla T_k\left(v_j\right)\phi'\left(\theta_n^j\right)\psi_m(u_n)\,dx\\&\quad+\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\phi\left(\theta_n^j\right)\psi_m'(u_n)\,dx\end{aligned}$$

therefore

$$\int_{\Omega} a\left(x, u_n, \nabla u_n\right) \cdot \nabla z_{nm}^{j} dx$$
$$= \int_{\Omega} \left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right)\right)$$
$$\left(\nabla T_k\left(u_n\right) - \nabla T_k\left(v_j\right) \chi_j^s\right) \phi'\left(\theta_n^j\right) dx$$
$$+ \int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right) \cdot \left(\nabla T_k\left(u_n\right) - \nabla T_k\left(v_j\right) \chi_j'\right) \phi'\left(\theta_n^j\right) dx$$
$$- \int_{\Omega \setminus \Omega_j^*} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(v_j\right) \phi'\left(\theta_n^j\right) dx$$
$$- \int_{[|u_u|>k]} a\left(x, u_n, \nabla u_n\right) \cdot \nabla T_k\left(v_j\right) \phi'\left(\theta_n^j\right) \psi_m\left(u_n\right) dx$$
$$+ \int_{\Omega} a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n \phi\left(\theta_n^j\right) \psi_m'\left(u_n\right) dx \tag{4.15}$$

let us define $x_j^s, s > 0$, and the characteristic function of the subset $\Omega_j^s = \left\{x \in \Omega : \left|\nabla T_k\left(v_j\right)\right| \leq s\right\}$.

By fixing m and s, we will pass to the limit in n and in j in the second, third, fourth and fifth term on the right hand side of (4.15) .

For the second term, we have

$$\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right) \cdot \left(\nabla T_k\left(u_n\right) - \nabla T_k\left(v_j\right) \chi_j^s\right) \phi'\left(\theta_n^j\right) dx$$
$$\to \int_{\Omega} (a\left(x, T_k(u), \nabla T_k\left(v_j\right) \chi_j^s\right) \cdot \left(\nabla T_k(u) - \nabla T_k\left(v_j\right) \chi_j^s\right) \phi'\left(\theta^j\right)) dx \quad \text{as } n \to +\infty$$

thinks to 3.8, one has

$$a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) x_j^s\right) \phi'\left(\theta_n^j\right) \to$$
$$a\left(x, T_k(u), \nabla T_k\left(v_j\right) \chi_j^s\right) \phi'\left(\theta^j\right) \text{ strongly in } \left(E_{\varphi}(\Omega)\right)^N \text{ as n } \to \infty$$

and by (4.4)

$$\nabla T_k\left(u_n\right) \rightharpoonup \nabla T_k(u) \quad \text{weakly in } \left(L_{\varphi}(\Omega)\right)^N$$

Let us define χ^s the characteristic function of the subset

$$\Omega^s = \{x \in \Omega : |\nabla T_k(u)| \leq s\}$$

As $\nabla T_k\left(v_j\right) \chi_j^s \to \nabla T_k(u) \chi^s$ strongly in $\left(E_{\varphi}(\Omega)\right)^N$ as $j \to \infty$, we get

$$\int_{\Omega} a\left(x, T_k(u), \nabla T_k\left(v_j\right) \chi_j^s\right) \cdot \left(\nabla T_k(u) - \nabla T_k\left(v_j\right) \chi_j^s\right) \phi'\left(\theta^j\right) dx \to 0 \quad \text{as } j \to \infty$$

thus,

$$\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right) \cdot \left(\nabla T_k\left(u_n\right) - \nabla T_k\left(v_j\right) \chi_j^s\right) \phi'\left(\theta_n^j\right)) dx = \epsilon_3(n, j) \tag{4.16}$$

For third term estimation of (4.15) . It's it is clear that by (1.5) one can verify that $a(x, s, 0) = 0$ for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$.

Thus, from (4.10) we have that

$$(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right))_n \text{ is bounded in } \left(L_{\varphi}(\Omega)\right)^N \text{ for all } k \geq 0.$$

Therefore, there exist a subsequence still indexed by n and a function l_k in $\left(L_{\varphi}(\Omega)\right)^N$ such that

$$a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \rightharpoonup l_k \text{ weakly in } \left(L_{\varphi}(\Omega)\right)^N \text{ for } \sigma\left(\Pi L_{\psi}, \Pi E_{\varphi}\right).$$

Then, by using the fact that $\nabla T_k\left(v_j\right) \chi_{\Omega \backslash \Omega_j^s} \in \left(E_{\varphi}(\Omega)\right)^N$, we get

$$\int_{\Omega | \Omega_j^s} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(v_j\right) \phi'\left(\theta_n^j\right) dx$$
$$\rightarrow \int_{\Omega \backslash \Omega_j^s} l_k \cdot \nabla T_k\left(v_j\right) \phi'\left(\theta^j\right) dx \quad \text{as } n \rightarrow \infty.$$

The modular convergence of $\left\{v_j\right\}$ give

$$-\int_{\Omega \backslash \Omega_j} l_k \cdot \nabla T_k\left(v_j\right) \phi'\left(\theta^j\right) dx \rightarrow -\int_{\Omega | \Omega^s} l_k \cdot \nabla T_k(u) dx \quad \text{as } j \rightarrow \infty$$

Consequently

$$-\int_{\Omega \backslash \Omega_j^{*=s}} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(v_j\right) \phi'\left(\theta_n^j\right) dx = -\int_{\Omega \backslash \Omega^s} l_k \cdot \nabla T_k(u) dx + \epsilon_4(n, j) \tag{4.17}$$

For the fourth term, we remark that $\psi_m\left(u_n\right) = 0$ on the subset $\{|u_n| \geq m+1\}$, then we obtain

$$-\int_{\{|u_n|>k\}} a\left(x, u_n, \nabla u_n\right) \cdot \nabla T_k\left(v_j\right) \phi'\left(\theta_n^j\right) \psi_m\left(u_n\right) dx$$
$$= -\int_{\{|u_n|>k\}} a\left(x, T_{m+1}\left(u_n\right), \nabla T_{m+1}\left(u_n\right)\right) \cdot \nabla T_k\left(v_j\right) \phi'\left(\theta_n^j\right) \psi_m\left(u_n\right) dx$$

By using the same procedure as above we have

$$-\int_{\{|u_n|>k\}} a\left(x, T_{m+1}\left(u_n\right), \nabla T_{m+1}\left(u_n\right)\right) \cdot \nabla T_k\left(v_j\right) \phi'\left(\theta_n^j\right) \psi_m\left(u_n\right) dx$$
$$= -\int_{\{|u|>k\}} l_{m+1} \cdot \nabla T_k(u) \psi_m(u) dx + \epsilon_5(n, j)$$

By observing that $\nabla T_k(u) = 0$ on the subset $\{|u| > k\}$, we can write

$$-\int_{\{|u_n|>k\}} a\left(x, u_n, \nabla u_n\right) \cdot \nabla T_k\left(v_j\right) \phi'\left(\theta_n^j\right) \psi_m\left(u_n\right) dx = \epsilon_5(n, j) \tag{4.18}$$

For the last term of (4.15) we obtain

$$\begin{aligned}&\left|\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n\phi\left(\theta_n^j\right)\psi_m'\left(u_n\right)dx\right|\\&=\left|\int_{\{m\le|u_k|\le m+1\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n\phi\left(\theta_n^j\right)\psi_m'\left(u_n\right)dx\right|\\&\le\phi(2k)\int_{\{m\le|u_k|\le m+1\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n dx\end{aligned}$$

By taking $T_1\left(u_n-T_m\left(u_n\right)\right)\in W_0^1L_\varphi(\Omega)$ as test in (P_n) one has

$$\begin{aligned}&\int_{\{m\le|u_k|\le m+1\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n dx+\int_{\{m\le|u_n|\le m+1\}}\Phi_n\left(u_n\right)\cdot\nabla u_n dx\\&+\int_{\{|u_x|\ge m\}} g_n\left(x,u_n,\nabla u_n\right)T_1\left(u_n-T_m\left(u_n\right)\right)dx=\left\langle f,T_1\left(u_n-T_m\left(u_n\right)\right)\right\rangle.\end{aligned}$$

by according to Lemma 3.9, we get

$$\int_{\{m\le|u_k|\le m+1\}}\Phi_n\left(u_n\right)\cdot\nabla u_n dx=0$$

Since $g_n\left(x,u_n,\nabla u_n\right)T_1\left(u_n-T_m\left(u_n\right)\right)\ge 0$ on the subset $\{|u_n|\ge m\}$, we have

$$\int_{\{m\le|u_k|\le m+1\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n dx\le\left\langle f,T_1\left(u_n-T_m\left(u_n\right)\right)\right\rangle$$

By observing f as $f=-\operatorname{div}F$, where $F\in\left(E_\varphi(\Omega)\right)^N$, and applying Young's inequality, we get

$$\int_{\{m\le|u_k|\le m+1\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n dx\le\alpha\int_{\{m\le|u_n|\le m+1\}}\psi\left(x,\frac{2}{\alpha}|F|\right)dx$$

which implies that

$$\left|\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n\phi\left(\theta_n^j\right)\psi_m'\left(u_n\right)dx\right|\le\alpha\phi(2k)\int_{\{m\le|u_k|\le m+1\}}\psi\left(x,\frac{2}{\alpha}|F|\right)dx \tag{4.19}$$

thinks to (4.15), (4.17), (4.18) and 4.19 we get

$$\begin{aligned}&\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\nabla z_{n,m}^j dx\\&\ge\int_\Omega\left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)x_j^s\right)\right)\left(\nabla T_k\left(u_n\right)\right.\\&\left.-\nabla T_k\left(v_j\right)x_j^s\right)\phi'\left(\theta_n^j\right)dx &(4.20)\\&-\alpha\phi(2k)\int_{\{m\le|u_k|\le m+1\}}\psi\left(x,\frac{2}{\alpha}|F|\right)dx\\&-\int_{\Omega\setminus\Omega^s} l_k\cdot\nabla T_k(u)dx+e_6(n,j) &(4.21)\end{aligned}$$

Now, we turn to the second term on the left-hand side of (4.15) and by using the hypothesis (1.6) one has

$$\begin{aligned}
&\left|\int_{\{|u_n|\leq k\}} g_n\left(x,u_n,\nabla u_n\right)\phi\left(\theta_n^j\right)dx\right| \\
&\quad=\left|\int_{\{|u_x|\leq k\}} g_n\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\phi\left(\theta_n^j\right)dx\right| \\
&\quad\leq b(k)\int_\Omega \varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right)\left|\phi\left(\theta_n^j\right)\right|dx+b(k)\int_\Omega c(x)\left|\phi\left(\theta_n^j\right)\right|dx \\
&\quad\leq \frac{b(k)}{\alpha}\int_\Omega a_n\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(u_n\right)\left|\phi\left(\theta_n^j\right)\right|dx+\epsilon_7(n,j)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\int_{|u_k|\leq k} g_n\left(x,u_n,\nabla u_n\right)\phi\left(\theta_n^j\right)dx\right| \\
&\leq \frac{b(k)}{\alpha}\int_\Omega\left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)\left(\nabla T_k\left(u_n\right)\right. \\
&\left.-\nabla T_k\left(v_j\right)\chi_j^s\right)\left|\phi\left(\theta_n^j\right)\right|dx \\
&+\frac{b(k)}{\alpha}\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\left|\phi\left(\theta_n^j\right)\right|dx \\
&+\frac{b(k)}{\alpha}\int_\Omega a_n\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(v_j\right)\chi_j^s\left|\phi\left(\theta_n^j\right)\right|dx+\epsilon_7(n,j)
\end{aligned}\tag{4.22}$$

Using the same procedure as above we get

$$\frac{b(k)}{\alpha}\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\left|\phi\left(\theta_n^j\right)\right|dx=\epsilon_8(n,j)$$

and

$$\frac{b(k)}{\alpha}\int_\Omega a_n\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(v_j\right)\chi_j^s\left|\phi\left(\theta_n^j\right)\right|dx=\epsilon_9(n,j).$$

thus, we obtain

$$\begin{aligned}
&\left|\int_{\{|u_n|\leq k\}} g_n\left(x,u_n,\nabla u_n\right)\phi\left(\theta_n^j\right)dx\right|\leq\frac{b(k)}{\alpha}\int\left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\right. \\
&\left.-a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right) \\
&\times\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\left|\phi\left(\theta_n^j\right)\right|dx+\epsilon_{10}(n,j)
\end{aligned}\tag{4.23}$$

By combining (4.14),(4.20) and (4.23) we have

$$\begin{aligned}
&\int_\Omega\left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right) \\
&\quad\times\left(\phi'\left(\theta_n^j\right)-\frac{b(k)}{\alpha}\left|\phi\left(\theta_n^j\right)\right|\right)dx \\
&\leq\int_{\Omega\backslash\Omega^s} l_k\cdot\nabla T_k(u)dx+\alpha\phi(2k)\psi\left(x,\frac{2}{\alpha}|F|\right)dx+\epsilon_{11}(n,j)
\end{aligned}$$

thinks to (4.12), we get

$$
\begin{aligned}
&\int_{\Omega}\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right)\right) \\
&\quad\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right) \rho(x) d x \\
&\leq 2 \int_{\Omega \backslash \Omega^s} l_k \cdot \nabla T_k(u) d x+2 \alpha \phi(2 k) \\
&\quad\int_{\{m \leq|u_n| \leq m+1\}} \psi\left(x, \frac{2}{\alpha}|F|\right) d x+\epsilon_{11}(n, j)
\end{aligned}
\tag{4.24}
$$

On the other hand, we have

$$
\begin{aligned}
&\int_{\Omega}\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k(u) \chi^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k(u) \chi^s\right) d x \\
&=\int_{\Omega}\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right)\right) \\
&\quad\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right) d x \\
&\quad+\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot\left(\nabla T_k\left(v_j\right) \chi_j^s-\nabla T_k(u) \chi^s\right) d x \\
&\quad-\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k(u) \chi^s\right) \cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k(u) \chi^s\right) d x \\
&\quad+\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right) \cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right) d x
\end{aligned}
$$

We will passe to the limit in n and then in j in the last three terms of the right-hand side of the above equality.

using the same procedure as is done in (4.15) and (4.22), we get

$$
\begin{aligned}
&\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot\left(\nabla T_k\left(v_j\right) \chi_j^s-\nabla T_k(u) \chi^s\right) d x=\epsilon_{12}(n, j) \\
&\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k(u) \chi^s\right) \cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k(u) \chi^s\right) d x=\epsilon_{13}(n, j) \\
&\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right) \cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right) d x=\epsilon_{14}(n, j)
\end{aligned}
\tag{4.25}
$$

Therefore,

$$
\begin{aligned}
&\int_{\Omega}\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k(u) \chi^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k(u) \chi^s\right) d x \\
&=\int_{\Omega}\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right)\right) \\
&\quad\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right) d x+\epsilon_{15}(n, j)
\end{aligned}
\tag{4.26}
$$

Let $r \leq s$. Thinks to (1.4) , (4.24) and (4.26) we have

$$
\begin{aligned}
0 &\leq \int_{\Omega^r} \left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(x, T_k\left(u_n\right), \nabla T_k(u)\right)\right)\left(\nabla T_k\left(u_n\right) - \nabla T_k(u)\right) dx \\
&\leq \int_{\Omega^s} \left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(x, T_k\left(u_n\right), \nabla T_k(u)\right)\right)\left(\nabla T_k\left(u_n\right) - \nabla T_k(u)\right) dx \\
&= \int_{\Omega^s} \left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(x, T_k\left(u_n\right), \nabla T_k(u)\chi^s\right)\right)\left(\nabla T_k\left(u_n\right) - \nabla T_k(u)\chi^s\right) dx \\
&\leq \int_{\Omega} \left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(x, T_k\left(u_n\right), \nabla T_k(u)\chi^s\right)\right)\left(\nabla T_k\left(u_n\right) - \nabla T_k(u)\chi^s\right) dx \\
&= \int_{\Omega} \left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right)\chi_j^s\right)\right) \\
&\quad \left(\nabla T_k\left(u_n\right) - \nabla T_k\left(v_j\right)\chi_j^s\right) dx + \epsilon_{15}(n, j)
\end{aligned}
$$

by passing to the limit in n and then in j one has,

$$
\begin{aligned}
0 &\leq \limsup_{n\to\infty} \int \left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(x, T_k\left(u_n\right), \nabla T_k(u)\right)\right)\left(\nabla T_k\left(u_n\right) - \nabla T_k(u)\right) dx \\
&\leq 2\int_{\Omega\setminus\Omega^s} l_k \cdot \nabla T_k(u) dx + 2\alpha\phi(2k) \int_{\{m\leq|u|\leq m+1\}} \psi\left(x, \frac{2}{\alpha}|F|\right) dx.
\end{aligned}
$$

Let $s \to +\infty$ and $m \to +\infty$, using the fact that $l_k \cdot \nabla T_k(u) \in L^1(\Omega), |F| \in \left(E_\varphi(\Omega)\right)^N$, $|\Omega\setminus\Omega^s| \to 0$ and $|\{m \leq |u| \leq m+1\}| \to 0$, we obtain

$$
\begin{aligned}
&\int_{\Omega^r} \left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(x, T_k\left(u_n\right), \nabla T_k(u)\right)\right) \\
&\quad \cdot \left(\nabla T_k\left(u_n\right) - \nabla T_k(u)\right) dx \to 0 \quad \text{as } n \to \infty
\end{aligned}
$$

Thinks to [31] there exists a subsequence of $\{u_n\}$ still indexed by n such that

$$
\nabla u_n \to \nabla u \quad \text{a.e.in } \Omega \tag{4.27}
$$

Thus, by taking account that (4.7),(4.9) and (4.10) we can apply [[33], Theorem 14.6] to obtain $a(x, u, \nabla u) \in \left(L_\varphi(\Omega)\right)^N$ and

$$
a\left(x, u_n, \nabla u_n\right) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } \left(L_\varphi(\Omega)\right)^N \text{ for } \sigma\left(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega)\right). \tag{4.28}
$$

4.1.5

Now,we shall prove that

$$
\text{for every } k > 0, \quad T_k\left(u_n\right) \to T_k(u) \quad \text{in } W_0^1 L_\varphi(\Omega) \text{ for the modular convegence}
$$

.

From inequality (4.24), we obtain

$$\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(u_n\right) dx \leq \int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(v_j\right) \chi_j^s dx$$
$$+\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) x_j^s\right) \cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right) dx$$
$$+2 \alpha \phi(2k) \quad \int_{\{m \leq |u_k| \leq m+1\}} \psi\left(x, \frac{2}{\alpha}|F|\right) dx$$
$$+2 \int_{\Omega | \Omega^s} a\left(x, T_k(u), \nabla T_k(u)\right) \cdot \nabla T_k(u) dx+\epsilon_{11}(n, j).$$

thinks to (4.25), we obtain

$$\begin{aligned}
&\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(u_n\right) dx \\
&\leq \int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(v_j\right) \chi_j^s dx+2 \alpha \phi(2k) \\
&\quad \int_{\{m \leq |u_n| \leq m+1\}} \psi\left(x, \frac{2}{\alpha}|F|\right) dx \\
&\quad +2 \int_{\Omega | \Omega^s} a\left(x, T_k(u), \nabla T_k(u)\right) \cdot \nabla T_k(u) dx+\epsilon_{17}(n, j).
\end{aligned}$$

the passage to the limitto the limit in n on both sides of this inequality and using (4.28) implies that

$$\begin{aligned}
&\limsup_{n \rightarrow \infty} \int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(u_n\right) dx \\
&\leq \int_{\Omega} a\left(x, T_k(u), \nabla T_k(u)\right) \cdot \nabla T_k\left(v_j\right) \chi_j^s dx+2 \alpha \phi(2k) \\
&\quad \int_{\{m \leq |u| \leq m+1\}} \psi\left(x, \frac{2}{\alpha}|F|\right) dx \\
&\quad +2 \int_{\Omega \backslash \Omega^s} a\left(x, T_k(u), \nabla T_k(u)\right) \cdot \nabla T_k(u) dx .
\end{aligned}$$

and by passing to the limit in j we obtain

$$\begin{aligned}
&\limsup_{n \rightarrow \infty} \int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(u_n\right) dx \\
&\leq \int_{\Omega} a\left(x, T_k(u), \nabla T_k(u)\right) \cdot \nabla T_k(u) \chi^s dx+2 \alpha \phi(2k) \\
&\quad \int_{\{m \leq |u| \leq m+1\}} \psi\left(x, \frac{2}{\alpha}|F|\right) dx \\
&\quad +2 \int_{\Omega \backslash \Omega^s} a\left(x, T_k(u), \nabla T_k(u)\right) \cdot \nabla T_k(u) dx .
\end{aligned}$$

Let s and $m \rightarrow \infty$, we get

$$\limsup_{n \rightarrow \infty} \int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(u_n\right) dx \leq \int_{\Omega} a\left(x, T_k(u), \nabla T_k(u)\right) \cdot \nabla T_k(u) dx$$

Now, thinks to (1.5),(4.4),(4.27) and applying Fatou's lemma, we get

$$\int_\Omega a\left(x, T_k(u), \nabla T_k(u)\right)\cdot \nabla T_k(u)dx \leq \liminf_{n\to\infty}\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\cdot \nabla T_k\left(u_n\right)dx$$

thus,

$$\lim_{n\to\infty}\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\cdot \nabla T_k\left(u_n\right)dx = \int_\Omega a\left(x, T_k(u), \nabla T_k(u)\right)\cdot \nabla T_k(u)dx$$

In view of Lemma 3.5, we deduce that for every $k > 0$

$$a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\cdot \nabla T_k\left(u_n\right) \to a\left(x, T_k(u), \nabla T_k(u)\right)\cdot \nabla T_k(u) \quad \text{strongly in } L^1(\Omega) \tag{4.29}$$

by vertu of hypothesis (1.5) and using the convexity of φ we get

$$\begin{aligned}&\varphi\left(x, \frac{\left|\nabla T_k\left(u_n\right) - \nabla T_k(u)\right|}{2}\right)\\ &\leq \frac{1}{2\alpha}a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\cdot \nabla T_k\left(u_n\right) + \frac{1}{2\alpha}a\left(x, T_k(u), \nabla T_k(u)\right)\cdot \nabla T_k(u)\end{aligned}$$

By applying Vitali's theorem we obtain

$$\lim_{|\varepsilon|\to 0}\sup_n\int_E \varphi\left(x, \frac{\left|\nabla T_k\left(u_n\right) - \nabla T_k(u)\right|}{2}\right)dx = 0$$

Consequently, for every $k > 0$

$$T_k\left(u_n\right) \to T_k(u) \quad \text{in } W_0^1 L_\varphi(\Omega).$$

for the modular convegence.

4.1.6

We shall show that

$$g_n\left(x, u_n, \nabla u_n\right) \to g(x, u, \nabla u) \quad in \quad L^1(\Omega) \tag{4.30}$$

From (4.9) and (4.27) we have

$$g_n\left(x, u_n, \nabla u_n\right) \to g(x, u, \nabla u) \quad \text{a.e. in } \Omega \tag{4.31}$$

Let E be a measurable subset of Ω and let $m > 0$. by taking account of (1.5) and (1.6) we obtain

$$\begin{aligned}\int_E \left|g_n\left(x, u_n, \nabla u_n\right)\right| dx &= \int_{E\cap[|u_k|\leq m\}} \left|g_n\left(x, u_n, \nabla u_n\right)\right| dx\\ &\quad + \int_{E\cap\{|u_n|>m\}} \left|g_n\left(x, u_n, \nabla u_n\right)\right| dx\\ &\leq b(m)\int_E c(x)dx + b(m)\int_E a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right)\cdot \nabla T_m\left(u_n\right)dx\\ &\quad + \frac{1}{m}\int_\Omega g_n\left(x, u_n, \nabla u_n\right)u_n dx\end{aligned}$$

By (1.7) and (4.6) it follows that

$$\lim_{m\to\infty}\frac{1}{m}\int_\Omega g_n\left(x, u_n, \nabla u_n\right)u_n dx = 0$$

By using (4.29) the sequence

$$\{a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right) \cdot \nabla T_m\left(u_n\right)\}_n \text{ is equi-integrable },$$

Consequently

$$\lim_{|E|\to 0} \sup_n \int_E a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right) \cdot \nabla T_m\left(u_n\right) dx = 0$$

This proves that $g_n\left(x, u_n, \nabla u_n\right)$ is equi-integrable.

Therefore, Vitali's theorem allows us to get

$$g(x, u, \nabla u) \in L^1(\Omega),$$

and

$$g_n\left(x, u_n, \nabla u_n\right) \to g(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega). \tag{4.32}$$

4.1.7

In this subsubsection we prove that

$$\lim_{m\to\infty} \int_{\{m\le|u|\le m+1\}} a(x, u, \nabla u) \cdot \nabla u dx = 0 \tag{4.33}$$

for any $m \geq 0$ we have

$$\begin{aligned}
\int_{\{m\le|u_n|\le m+1\}} a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n dx &= \int_\Omega a\left(x, u_n, \nabla u_n\right) \cdot \left(\nabla T_{m+1}\left(u_n\right) - \nabla T_m\left(u_n\right)\right) dx \\
&= \int_\Omega a\left(x, T_{m+1}\left(u_n\right), \nabla T_{m+1}\left(u_n\right)\right) \cdot \nabla T_{m+1}\left(u_n\right) dx \\
&\quad - \int_\Omega a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right) \cdot \nabla T_m\left(u_n\right) dx
\end{aligned}$$

Thinks to (4.29) and passing to the limit as $n \to +\infty$ for fixed $m \geq 0$

$$\begin{aligned}
\lim_{n\to\infty} &\int_{\{m\le|u_n|\le m+1\}} a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n dx \\
&= \int_\Omega a\left(x, T_{m+1}(u), \nabla T_{m+1}(u)\right) \cdot \nabla T_{m+1}(u) dx - \int_\Omega a\left(x, T_m(u), \nabla T_m(u)\right) \nabla T_m(u) dx \\
&= \int_\Omega a(x, u, \nabla u) \cdot \left(\nabla T_{m+1}(u) - \nabla T_m(u)\right) dx \\
&= \int_{\{m\le|u|\le m+1\}} a(x, u, \nabla u) \cdot \nabla u dx.
\end{aligned}$$

according to (4.11), we can pass to the limit as $m \to +\infty$ in order to have (4.33).

4.1.8

Finally, in this step thanks to (4.29) and Lemma 3.5, one has

$$a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n \to a(x, u, \nabla u) \cdot \nabla u \quad \text{strongly in } L^1(\Omega) \tag{4.34}$$

Let $h \in \mathbb{C}_c^1(\mathbb{R})$ and $\varrho \in \mathcal{D}(\Omega)$. we choose $h(u_n)\varrho$ as a test function in (P_n) we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n h'(u_n)\varrho dx + \int_\Omega \rho(x)a(x,u_n,\nabla u_n)\cdot\nabla\varrho h(u_n)\,dx$$
$$+\int_\Omega \Phi_n(u_n)\cdot\nabla(h(u_n)\varrho)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)h(u_n)\varrho dx = \langle f, h(u_n)\varrho\rangle. \tag{4.35}$$

Now, we can pass to the limit as $n\to+\infty$ in each term of equality (4.35). since h and h' have a compact support on $\mathbb{R}$, there exists a real number $v>0$, such that supp $h\subset[-v,v]$ and supp $h'\subset[-v,v]$. For $n>v$, we can have

$$\Phi_n(t)h(t)=\Phi(T_v(t))h(t) \quad\text{and}\quad \Phi_n(t)h'(t)=\Phi(T_v(t))h'(t)$$

Moreover,

$$\text{the functions } \Phi h \text{ and } \Phi h' \text{ belong to } \left(\mathcal{C}^0(\mathbb{R})\cap L^\infty(\mathbb{R})\right)^N.$$

Now we can see that the sequence $\{h(u_n)\varrho\}_n$ is bounded in $W_0^1L_\varphi(\Omega)$.
Indeed, let $c^{'}>0$ be a positive constant such that $\|h(u_n)\nabla\varrho\|_\infty\le c^{'}$ and $\left\|h'(u_n)\varrho\right\|_\infty\le c^{'}$. Thinks to (3.2) we obtain

$$\begin{aligned}\int_\Omega\varphi\left(x,\frac{|\nabla(h(u_n)\varrho)|}{2c^{'}}\right)dx &\le \int_\Omega\varphi\left(x,\frac{|h(u_n)\nabla\varrho|+\left|h'(u_n)\varrho\|\nabla u_n\right|}{2c^{'}}\right)dx\\ &\le \frac12\int_\Omega\varphi(x,1)dx+\frac12\int_\Omega\varphi(x,|\nabla T_M(u_n)|)\,dx\\ &\le c\end{aligned}$$

Which jointly with (4.9) it follows that

$$h(u_n)\varrho\to h(u)\varrho \text{ weakly in } W_0^1L_\varphi(\Omega) \text{ for for } \sigma\left(\Pi L_\varphi,\Pi E_\psi\right). \tag{4.36}$$

Which give

$$\langle f,h(u_n)\varphi\rangle\to\langle f,h(u)\varphi\rangle.$$

Let E be a measurable subset of Ω. we pose $c_v=\max_{|t|\le v}\Phi(t)$. And denoting by $\|v|_{\varphi,\Omega}$ the Orlicz norm of a function $v\in L_\varphi(\Omega)$. We thinking to the strengthened $H\ddot{o}lder$ inequality with both Orlicz and Luxemburg norms, we have

$$\begin{aligned}\|\Phi(T_v(u_n))\chi_E\|_{\psi,\Omega} &= \sup_{\|v\|_{\varphi,\Omega}\le1}\left|\int_E\Phi(T_v(u_n))\,v dx\right|\\ &\le c_v\sup_{\|v\|_{\varphi,\Omega}\le1}\|\chi_E\|_{\psi,\Omega}\|v\|_{\varphi,\Omega}\\ &\le c_v|E|\varphi^{-1}\left(x,\frac{1}{|E|}\right)\end{aligned}$$

Consequently,

$$\lim_{|E|\to0}\sup_n\|\Phi(T_v(u_n))\chi_E\|_{(\psi,\Omega)}=0$$

Then, in view of (4.9) and by applying [[33], Lemma 11.2], we get

$$\Phi(T_v(u_n))\to\Phi(T_v(u)) \quad \text{strongly in } \left(E_\psi(\Omega)\right)^N$$

which together with (4.36) allow us to pass to the limit in the third term of (4.35) to obtain

$$\int_\Omega \Phi\left(T_v\left(u_n\right)\right) \cdot \nabla\left(h\left(u_n\right) \varrho\right) dx \rightarrow \int_\Omega \Phi\left(T_v(u)\right) \cdot \nabla(h(u) \varrho) dx$$

Observing that

$$\left|a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n h'\left(u_n\right) \varrho\right| \leq c' a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n$$

Therefore, thinks to (4.34) and applying Vitali's theorem, we get

$$\int_\Omega a\left(x, u_n, \nabla u_n\right) \cdot \nabla u_n h'\left(u_n\right) \varrho dx \rightarrow \int_\Omega a(x, u, \nabla u) \cdot \nabla u h'(u) \varrho dx$$

Concerning the second term of (4.35) using the same procedure as above we obtain

$$h\left(u_n\right) \nabla \varrho \rightarrow h(u) \nabla \varrho \quad \text{strongly in } \left(E_\varphi(\Omega)\right)^N$$

which jointly with (4.28) implies that

$$\int_\Omega a\left(x, u_n, \nabla u_n\right) \cdot \nabla \varrho h\left(u_n\right) dx \rightarrow \int_\Omega a(x, u, \nabla u) \cdot \nabla \varrho h(u) dx$$

Remark that $h\left(u_n\right) \varrho \rightarrow h(u) \varrho$ weakly in $L^{\infty}(\Omega)$ for $\sigma^*\left(L^{\infty}, L^1\right)$ with (4.30) we can pass to the limit in the Fourth term of (4.35) in order to have

$$\int_\Omega g_n\left(x, u_n, \nabla u_n\right) h\left(u_n\right) \varrho dx \rightarrow \int_\Omega g(x, u, \nabla u) h(u) \varrho dx$$

Finally, we can pass to the limit in each term of (4.35) so as to obtain

$$\begin{aligned} & \int_\Omega a(x, u, \nabla u) \cdot\left[\nabla \varphi h(u)+h'(u) \varrho \nabla u\right] dx+\int \Phi(u) h'(u) \varrho \cdot \nabla u dx \\ & +\int_\Omega \Phi(u) h(u) \cdot \nabla \varrho dx+\int_\Omega g(x, u, \nabla u) h(u) \varrho dx=\langle f, h(u) \varrho\rangle \end{aligned}$$

for all $h \in \mathcal{C}_c^1(\mathbb{R})$ and for all $\varrho \in \mathcal{D}(\Omega)$. Thus, as well (1.7),(4.6) and (4.31) we apply Fatou's lemma to get $g(x, u, \nabla u) u \in L^1(\Omega)$.

Consequently, thinks to (4.9),(4.28),(4.32),(4.33), the function u is a renormalized solution of problem $(\mathcal{P})$.

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