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Polynomial-exponential stability and blow-up solutions to a nonlinear damped viscoelastic Petrovsky equation

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Abstract
This work is concerned with the initial boundary value problem for a nonlinear viscoelastic Petrovsky equation

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau - \Delta u_t - \Delta u_{tt} + u_t|u_t|^{m-1} = u|u|^{p-1}.$$

We prove that the solution energy has polynomial rate of decay, even if the kernel g decays exponentially provided $m > 1$ while decay rates is exponentially in the case of weak damping. The unbounded properties of solutions in two cases $m = 1$ and $p > m \geq 1$ have been also investigated. For the first case, we prove the blow-up of solutions with different ranges of initial energy. For the second case, we prove blow-up of solutions under some restrictions on g when the initial energy is negative or non negative at less than potential well depth.

Keywords Stability · Blow-up · Viscoelastic Petrovsky equation

Mathematics Subject Classification 35L35 · 35B35 · 35B40

1 Introduction

In this paper, we investigate the problem

$$\begin{aligned} &u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau - \Delta u_t \\ &\quad -\Delta u_{tt} + u_t|u_t|^{m-1} = u|u|^{p-1}, \quad x \in \Omega, t \geq 0, \qquad (1.1) \\ &u(x,t) = \partial_\nu u(x,t) = 0, \qquad x \in \partial\Omega, \quad t \geq 0, \qquad (1.2) \\ &u(x,0) = u_0, \quad u_t(x,0) = u_1, \qquad x \in \Omega, \qquad (1.3) \end{aligned}$$

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where Ω is a bounded domain in R^n, $n \geq 1$, with a smooth boundary $\partial\Omega$, $m \geq 1$, $p > 1$, ν is the unit outer normal on $\partial\Omega$ and g is a non-negative function that represents the kernel of memory term.

In [3], Cavalvcanti et al. studied the global existence result and the uniform exponential decay of energy for the following equation:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma \Delta u_t = 0. \tag{1.4}$$

In the case $\gamma = 0$, Messaoudi and Tatar [15] showed that the solution goes to zero with an exponential or polynomial rate. Using the potential well method, the same authors [16] obtained global existence and an exponential decay result in the presence of a nonlinear source term. Moreover, for sufficiently large values of the initial data and for a suitable relation between p and the relaxation function, they proved an unboundedness result. In [22], Wu proved the general decay of solutions for the nonlinear Eq. (1.4) in the presence of nonlinear damping and source terms in the case $\gamma = 0$. Recently in [23], the author established same result when the nonlinear damping term is replaced by a weak damping term. In this regard, without nonlinear source term, we may recall the work by Han and Wang [5] in which the authors obtained a general decay of solutions. For more related studies in connecting with the existence, finite time blow-up and asymptotic properties of solutions for nonlinear wave equations we refer the reader to [6,13,14,17,18,24,26,28] and references therein.

In [4], Guesmia considered the equation

$$u_{tt} + \Delta^2 u + h(u_t) = f(u), \qquad x \in \Omega, \ \ t > 0, \tag{1.5}$$

with boundary and initial conditions of Dirichlet type who established global existence, uniqueness and decay results under suitable growth conditions on h by exploiting the semi-group approach. Later, based on fixed point theorem, Messaoudi [12] studied (1.5) for $h(u_t) = u_t|u_t|^{m-2}$, $f(u) = u|u|^{p-2}$ and proved an existence result when $m \geq p$ with an arbitrary initial data and an unboundedness result if $m < p$ and the initial energy is negative. Then, Wu and Tsai [25] showed that the solution decays algebraically without the relation between m and p while it blows up in finite time if $p > m$ and the initial energy is nonnegative. In [2], Amroun and Benaissa obtained the global solvability of (1.5) subject to the same boundary and initial conditions as (1.2), (1.3) where $f(u) = bu|u|^{p-2}$ and h satisfies

$$c_1|s| \leq |h(s)| \leq c_2|s|^r, \qquad |s| \geq 1, \ \ c_1, c_2 > 0,$$

under some appropriate restrictions on p and r. In the presence of the strong damping, Li et al. [9] considered the following Petrovsky equation:

$$u_{tt} + \Delta^2 u - \Delta u_t + u_t|u_t|^{m-1} = u|u|^{p-1}, \qquad x \in \Omega, \ \ t \geq 0,$$

with the boundary and initial conditions (1.2) and (1.3). The authors obtained the global existence and uniform decay of solutions if the initial data are in some stable set without any interaction between the damping mechanism $u_t|u_t|^{m-1}$ and the source term $u|u|^{p-1}$. Moreover, they established the blow up properties of local solution in the case $p > m$ where the initial energy is less than the potential well depth.

In the study of plates, Rivera et al. [19] considered the following viscoelastic equation

$$u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau = 0.$$

They proved that the first and second order energy, associated with the solutions, decay exponentially provided the kernel of the memory also decays exponentially. On the other hand, the authors in [20] considered the equation

$$u_{tt} + Au - (g * A^{\alpha}u)(t) = 0,$$

where A is a positive self-adjoint operator with domain $D(A)$ in a Hilbert space H. They showed that the dissipation given by the memory effect is not strong enough to produce exponential stability, when $0 \leq \alpha < 1$, while such dissipation is capable to produce polynomial decay even if the kernel g decays exponentially. Recently, the authors in [11] considered (1.1)–(1.3) and established some asymptotic behavior and blow up results for solutions with positive initial energy. Very recently, Li and Gao [8] considered the Petrovsky equation

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(t,s)ds + |u_t|^{m-2}u_t = |u|^{p-2}u,$$

and obtained blow up results in both nonlinear and linear damping cases. We may also recall the recent related works in [21] and [27].

In the present work, our study will be devoted to the problem (1.1)–(1.3). We show that, under suitable assumptions on the function g, the solution is global provided that the initial data are small enough. We also show that the solution energy decays exponentially for the linear damping case ($m = 1$). We prove that the energy also has polynomially rate of decay, even if the kernel g decays exponentially, provided $m > 1$. To this end, we use the inequality (Lemma 2.4) given by Komornik [7]. We investigate the unbounded properties of solutions in two cases: $m = 1$ and $p > m \geq 1$. For the first case, we prove the blow-up of solutions with different ranges of initial energy. Estimates of the lifespan of solutions are also given. For the second case, we prove blow-up of solutions under some restrictions on g when the initial energy is negative or nonnegative at less than potential well depth.

The plan of this paper is as follows. In Sect. 2, we introduce some notations, lemmas and our main results. In Sect. 3, we present the global existence result, Lemma 3.2, and decay rates of the energy, Theorem 2.8. Unboundedness results, Theorems 2.9 and 2.10 , are given in Sect. 4.

2 Preliminaries and main results

To prove our main results, we shall give some lemmas, assumptions and notations.

Lemma 2.1 (Sobolev–Poincarè inequality [1]) *Let q be a number with $2 \leq q < \infty$ ($n = 1, 2, 3, 4$) or $2 \leq q \leq \frac{2n}{n-4}$ ($n \geq 5$), then for $u \in H_0^2(\Omega)$ there is a constant $C_* = C(\Omega, q)$ such that*

$$\|u\|_q \leq C_* \|\Delta u\|_2.$$

For nonlinear terms and the relaxation function we assume that

(G1) m and p satisfy

$$1 < p < \infty \quad (n = 1, 2, 3, 4) \qquad \text{or} \qquad 1 < p \leq \frac{n}{n-4} \quad (n \geq 5), \tag{2.1}$$

$$1 < m < \infty \quad (n = 1, 2, 3, 4) \qquad \text{or} \qquad 1 < m \leq \frac{n+4}{n-4} \quad (n \geq 5). \tag{2.2}$$

(G2) $g \in C^1(\mathbb{R}^+) \cup L^1(\mathbb{R}^+)$ such that $g \geq 0$; $g' \leq 0$ and

$$1 - \int_0^{\infty} g(s)ds = l > 0, \qquad g(0) > 0. \tag{2.3}$$

(G3) The kernel g decays exponentially to zero, as $t \to \infty$, namely

$$g'(t) + k_0 g(t) \leq 0, \qquad \forall t \in (0, +\infty), \quad \text{for some} \quad k_0 > 0.$$

Let us define C^1-functionals $I, J, E : \mathbb{R}^+ \to \mathbb{R}$ by

$$I[u](t) = I(t) = \left(1 - \int_0^t g(\tau)d\tau\right) \|\Delta u\|_2^2 + (g \circ \Delta u)(t) - \|u\|_{p+1}^{p+1}, \tag{2.4}$$

$$J[u](t) = J(t) = \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right) \|\Delta u\|_2^2 + \frac{1}{2}(g \circ \Delta u)(t) - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{2.5}$$

$$E[u](t) = E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + J[u](t), \tag{2.6}$$

where $u = u(x, t)$ is arbitrary solution of the problem (1.1)–(1.3) and

$$(g \circ v)(t) = \int_0^t g(t - \tau) \int_{\Omega} |v(t) - v(\tau)|^2 dx d\tau.$$

Lemma 2.2 *$E(t)$ is a non-increasing function for $t \geq 0$ and*

$$E'(t) = -\frac{1}{2}g(t)\|\Delta u\|_2^2 - \|\nabla u_t\|_2^2 + \frac{1}{2}(g' \circ \Delta u)(t) - \|u_t\|_{m+1}^{m+1}. \tag{2.7}$$

Proof Multiplying (1.1) by u_t, integrating over Ω and using the boundary conditions, we get

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}\left(\|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|\Delta u\|_2^2\right) - \frac{1}{p+1}\|u\|_{p+1}^{p+1}\right\} \\ &\qquad - \int_{\Omega} \Delta u_t(t) \int_0^t g(t-\tau)\Delta u(\tau) d\tau dx = -\|\nabla u_t\|_2^2 - \|u_t\|_{m+1}^{m+1}. \end{aligned} \tag{2.8}$$

For the last term in the left hand side of (2.8) we have

$$\begin{aligned} &\int_{\Omega} \Delta u_t(t) \int_0^t g(t-\tau)\Delta u(\tau) d\tau dx \\ &= \int_0^t g(t-\tau) \int_{\Omega} \Delta u_t(t)\Big(\Delta u(\tau) - \Delta u(t)\Big) dx d\tau + \frac{1}{2}\int_0^t g(\tau)d\tau \frac{d}{dt}\Big(\|\Delta u(t)\|_2^2\Big) \\ &= -\frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\Big(\|\Delta u(\tau) - \Delta u(t)\|_2^2\Big) - \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 \\ &\quad + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(\tau)d\tau \|\Delta u(t)\|_2^2\right) \\ &= -\frac{1}{2}\frac{d}{dt}(g \circ \Delta u)(t) + \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 \\ &\quad + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(\tau)d\tau \|\Delta u(t)\|_2^2\right), \end{aligned}$$

which combining with (2.8) and using (2.6) we obtain (2.7). □

Define

$$d = \inf_{u \in H_0^2(\Omega)\backslash\{0\}} \sup_{\lambda \geq 0} J(\lambda u).$$

Lemma 2.3 *We have*

$$0 < d_1 \leq d \leq d_2(u) = \sup_{\lambda \geq 0} J(\lambda u),$$

where

$$d_1 = \frac{p-1}{2(p+1)} \left(\frac{l}{C_*^2} \right)^{\frac{p+1}{p-1}},$$

and

$$d_2(u) = \frac{p-1}{2(p+1)} \left(\frac{(1 - \int_0^t g(\tau)d\tau)\|\Delta u\|_2^2 + (g \circ \Delta u)(t)}{\|u\|_{p+1}^2} \right)^{\frac{p+1}{p-1}}.$$

Proof See Lemma 5 in [11]. □

Lemma 2.4 (Komornik [7]) *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function and assume that there are two constants $r \geq 0$ and $C > 0$ such that*

$$\int_t^\infty \varphi^{r+1}(s)ds \leq C^{-1}\varphi^r(0)\varphi(t), \qquad \forall t \geq 0,$$

then, we have for each $t \geq 0$,

$$\begin{cases} \varphi(t) \leq \varphi(0)\exp(1 - Ct), & r = 0, \\ \varphi(t) \leq \varphi(0)\left(\frac{1+rCt}{1+r}\right)^{\frac{-1}{r}}, & r > 0. \end{cases}$$

Lemma 2.5 (Li and Tsai [10]) *Let $\delta > 0$ and $B(t) \in C^2(0, \infty)$ be a nonnegative function satisfying*

$$B''(t) - 4(\delta + 1)B'(t) + 4(\delta + 1)B(t) \geq 0. \tag{2.9}$$

If

$$B'(0) > r_2 B(0) + K_0, \tag{2.10}$$

with $r_2 = 2(\delta + 1) - 2\sqrt{(\delta + 1)\delta}$, then $B'(t) > K_0$ for $t > 0$, where K_0 is a constant.

Lemma 2.6 (Li and Tsai [10]) *If $M(t)$ is a non-increasing function on $[t_0, \infty)$, $t_0 \geq 0$, and satisfies the differential inequality*

$$M'(t)^2 \geq \alpha + \beta M(t)^{2+\frac{1}{\delta}}, \qquad t \geq t_0,$$

where $\alpha > 0$, $\beta \in \mathbb{R}$, then there exists a finite time T^* such that

$$\lim_{t \longrightarrow T^{*-}} M(t) = 0,$$

and the upper bound of T^* is estimated, respectively by the following cases

(1) *If* $\beta < 0$, *then*

$$T^* \leq t_0 + \frac{1}{\sqrt{-\beta}} \ln \frac{\sqrt{-\alpha/\beta}}{\sqrt{-\alpha/\beta} - M(t_0)}.$$

(2) *If* $\beta = 0$, *then*

$$T^* \leq t_0 + \frac{M(t_0)}{M'(t_0)}.$$

(3) *If* $\beta > 0$, *then*

$$T^* \leq \frac{M(t_0)}{\sqrt{\alpha}} \quad or \quad T^* \leq t_0 + 2^{(3\delta+1)/2\delta} \frac{\delta c}{\sqrt{\alpha}} \left[1 - (1 + cM(t_0))^{-1/2\delta}\right],$$

where $c = (\alpha/\beta)^{2+\frac{1}{\delta}}$.

We state a local existence theorem that can be established by combining the arguments of [2,12,28].

Theorem 2.7 *Suppose that* (2.1), (2.2) *hold and* $u_0 \in H_0^2(\Omega)$, $u_1 \in H_0^1(\Omega)$. *Then there exists a unique weak solution* $u(t)$ *such that*

$$u \in C\left([0,T]; H_0^2(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right),$$
$$u_t \in L^2\left([0,T]; H_0^1(\Omega)\right) \cap L^{m+1}(\Omega \times (0,T)),$$

for some positive constant T.

Now we are in a position to state our main results.

Theorem 2.8 *Suppose that* $(G1) - (G3)$ *hold. Let* $(u_0, u_1) \in H_0^2(\Omega) \times H_0^1(\Omega)$ *be given which satisfies*

$$I(u_0) > 0, \qquad E(0) < d_1.$$

Then there exists a positive constant C *such that the global solution of* (1.1)–(1.3) *satisfies,* $\forall t \geq 0$,

$$\begin{cases} E(t) \leq E(0)e^{1-Ct}, & if \;\; m = 1, \\ E(t) \leq E(0)\left(\frac{2+(m-1)Ct}{m+1}\right)^{-2/(m-1)}, & if \;\; m > 1. \end{cases}$$

Theorem 2.9 *Suppose that* $m = 1$ *and* (2.1), $(G2)$ *hold and*

$$a_1 = (p-1) - \big(p - 1 + 1/(p+1)\big) \int_0^\infty g(\tau) d\tau > 0. \tag{2.11}$$

Assume that $(u_0, u_1) \in H_0^2(\Omega) \times H_0^1(\Omega)$ *and that either one of the following conditions is satisfied*

(1) $E(0) < 0$,
(2) $E(0) = 0$ *and* $\int_\Omega \big(u_0 u_1 + \nabla u_0 . \nabla u_1\big) dx > 0$,
(3) $0 < E(0) < \frac{a_1 d_1}{l(p-1)}$ *and* $I(u_0) < 0$,

(4) $\frac{a_1 d_1}{l(p-1)} \leq E(0) < A$ *such that*

$$A = \min\left\{ \frac{\left(\int_\Omega (u_0 u_1 + \nabla u_0 . \nabla u_1) dx\right)^2}{2(T_1+1)\left(\|u_0\|_2^2 + \|\nabla u_0\|_2^2\right)}, \right.$$
$$\left. \frac{p+3}{p+1}\left[\frac{1}{2}\left(1+\sqrt{\frac{p-1}{p+3}}\right)\left(\int_\Omega \left(u_0 u_1 + \nabla u_0 . \nabla u_1\right) dx\right) - \left(\|u_0\|_2^2 + \|\nabla u_0\|_2^2\right)\right]\right\},$$

then the solution $u(t)$ *blows up at finite time* T^* *in the sense of*

$$\lim_{t \nearrow T^{*-}} \|\Delta u(t)\|_2^2 = +\infty.$$

Theorem 2.10 *Suppose that* $(G1)$, $(G2)$ *hold and* $p > m \geq 1$. *Assume that* $(u_0, u_1) \in H_0^2(\Omega) \times H_0^1(\Omega)$ *satisfy* $I(u_0) < 0$.

(i) *If* $E(0) < \beta d_1$ $(\beta < 1)$ *and* g *satisfies*

$$\int_0^\infty g(\tau) d\tau < \frac{(p-1)(1-\beta)}{(p-1)(1-\beta) + 1/[(p+1)-(p-1)\beta]}, \tag{2.12}$$

then the solution of (1.1)–(1.3) *blows up in finite time.*

(ii) *Suppose that there exists* $2 < \theta < p+1$ *such that*

$$a_2 = \left(\theta/2 - 1\right) - \left(\theta/2 - 1 + 1/(2\theta)\right) \int_0^\infty g(\tau) d\tau > 0, \tag{2.13}$$

and $E(0) < \left(\frac{2a_2}{p-1}\right) d_1$ *(one can verify that* $\frac{2a_2}{p-1} < 1$*). Then the solution of* (1.1)–(1.3) *blows up in finite time.*

3 Global existence and energy decay

Lemma 3.1 *Suppose that* (2.1) *and* $(G2)$ *hold. Assume that* $(u_0, u_1) \in H_0^2(\Omega) \times H_0^1(\Omega)$. *If* $I(u_0) > 0$ *and* $E(0) < d_1$ *then* $I(u(t)) > 0$ *for all* $t \geq 0$.

Proof Since $I(u_0) > 0$, then by continuity, there exists $T_* \leq T$ such that $I(u(t)) \geq 0$ for all $t \in [0, T_*]$. From (2.4), (2.6), (2.7) and the fact that $1 - \int_0^t g(\tau) d\tau > 1 - \int_0^\infty g(\tau) d\tau$, for all $t \in [0, T_*]$ we have

$$\begin{aligned} J(t) &= \frac{p-1}{2(p+1)}\left\{\left(1 - \int_0^t g(\tau) d\tau\right)\|\Delta u\|_2^2 + (g \circ \Delta u)(t)\right\} + \frac{1}{p+1} I(t) \\ &\geq \frac{p-1}{2(p+1)}\left\{\left(1 - \int_0^t g(\tau) d\tau\right)\|\Delta u\|_2^2 + (g \circ \Delta u)(t)\right\} \\ &\geq \frac{l(p-1)}{2(p+1)}\|\Delta u\|_2^2. \end{aligned} \tag{3.1}$$

Using (2.6), (3.1) and lemma 2.2 we obtain

$$\|\Delta u\|_2^2 \leq \frac{2(p+1)}{l(p-1)} J(t) \leq \frac{2(p+1)}{l(p-1)} E(t) \leq \frac{2(p+1)}{l(p-1)} E(0), \tag{3.2}$$

for all $t \in [0, T_*]$. Then, by lemma 2.1 and (3.2) we have

$$\|u\|_{p+1}^{p+1} \le C_*^{p+1}\|\Delta u\|_2^{p+1} \le C_*^{p+1}\left[\frac{2(p+1)}{l(p-1)}E(0)\right]^{\frac{p-1}{2}}\|\Delta u\|_2^2 < l\|\Delta u\|_2^2 \le \left(1-\int_0^t g(\tau)d\tau\right)\|\Delta u\|_2^2,$$

which implies $I(u(t)) > 0$ for all $t \in [0, T_*]$. By repeating this procedure, T_* can be extended to T. □

Lemma 3.2 *Suppose that* $(G1)$ *and* (2.3) *hold. Under the assumptions of Lemma 3.1 the solution of* (1.1)–(1.3) *is global and bounded in time.*

Proof We use (2.4)–(2.6) and lemma 2.2 to get

$$\begin{aligned}E(0) \ge E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + J(t)\\ &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{p+1}I(t)\\ &\quad + \frac{p-1}{2(p+1)}\left\{\left(1-\int_0^t g(\tau)d\tau\right)\|\Delta u\|_2^2 + (g\circ\Delta u)(t)\right\}.\end{aligned}$$

By the 3.1, $I(t) \ge 0$. Using the assumption $(G1)$ we deduce

$$\|u_t(t)\|_2^2 + \|\nabla u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 + (g\circ\Delta u)(t) \le KE(t) \le KE(0), \qquad \forall t \ge 0, \tag{3.3}$$

where $K = \frac{2(p+1)}{l(p-1)}$. This shows that $\|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|\Delta u\|_2^2$ is uniformly bounded and independent of t. Therefore the solution of (1.1)–(1.3) is bounded and global. □

Proof of Theorem 2.8 Multiplying by $E^r(t)u(t)$, with $r = \frac{m-1}{2}$, on both sides of (1.1) and integrating over $\Omega \times [t_1, t_2]$ we obtain

$$\begin{aligned}0 &= \int_{t_1}^{t_2} E^r(t)\int_\Omega uu_{tt}dxdt + \int_{t_1}^{t_2} E^r(t)\int_\Omega \nabla u(t).\nabla u_{tt}(t)dxdt\\ &\quad - \int_{t_1}^{t_2} E^r(t)\int_\Omega \Delta u(t)\int_0^t g(t-\tau)\Delta u(\tau)d\tau dxdt\\ &\quad + \int_{t_1}^{t_2} E^r(t)\|\Delta u(t)\|_2^2 dt + \int_{t_1}^{t_2} E^r(t)\int_\Omega \nabla u(t).\nabla u_t(t)dxdt\\ &\quad + \int_{t_1}^{t_2} E^r(t)\int_\Omega uu_t|u_t|^{m-1}dxdt - \int_{t_1}^{t_2} E^r(t)\|u(t)\|_{p+1}^{p+1}dt.\end{aligned} \tag{3.4}$$

For the first term in the right hand side of (3.4) we have

$$\begin{aligned}\int_{t_1}^{t_2} E^r(t)\int_\Omega uu_{tt}dxdt &= \int_\Omega E(t)^r uu_t dx\Big|_{t_1}^{t_2} - r\int_{t_1}^{t_2}\int_\Omega E(t)^{r-1}E'(t)uu_t dxdt\\ &\quad - \int_{t_1}^{t_2} E^r(t)\|u_t(t)\|_2^2 dt.\end{aligned} \tag{3.5}$$

Similarly, for the second term we obtain

$$\begin{aligned}\int_{t_1}^{t_2} E^r(t)\int_\Omega \nabla u(t).\nabla u_{tt}(t)dxdt &= \int_\Omega E^r(t)\nabla u(t).\nabla u_t(t)dx\Big|_{t_1}^{t_2}\\ &- r\int_{t_1}^{t_2}\int_\Omega E^{r-1}(t)E'(t)\nabla u(t).\nabla u_t(t)dxdt - \int_{t_1}^{t_2} E^r(t)\|\nabla u_t(t)\|_2^2 dt.\end{aligned} \tag{3.6}$$

Using (3.5), (3.6) and relation

$$\begin{aligned}&\frac{2}{p+1}\int_{t_1}^{t_2} E^r(t)\|u\|_{p+1}^{p+1}dt\\ &= \int_{t_1}^{t_2} E^r(t)\Big(\|u_t\|_2^2 + \|\nabla u_t\|_2^2\Big)dt + \int_{t_1}^{t_2} E^r(t)\left(1-\int_0^t g(\tau)d\tau\right)\|\Delta u(t)\|_2^2 dt\\ &+ \int_{t_1}^{t_2} E^r(t)(g\circ\Delta u)(t)dt - 2\int_{t_1}^{t_2} E^{r+1}(t)dt,\end{aligned}$$

Eq. (3.4) can be written in the form

$$\begin{aligned}&2\int_{t_1}^{t_2} E^{r+1}(t)dt - \left(\frac{p-1}{p+1}\right)\int_{t_1}^{t_2} E^r(t)\|u(t)\|_{p+1}^{p+1}dt\\ &= -\int_\Omega E^r(t)\Big(uu_t + \nabla u.\nabla u_t\Big)dx\Big|_{t_1}^{t_2} + 2\int_{t_1}^{t_2} E^r(t)\Big(\|u_t(t)\|_2^2 + \|\nabla u_t(t)\|_2^2\Big)dt\\ &+ r\int_{t_1}^{t_2}\int_\Omega E^{r-1}(t)E'(t)\Big(uu_t + \nabla u.\nabla u_t\Big)dxdt\\ &- \int_{t_1}^{t_2} E^r(t)\int_\Omega \nabla u(t).\nabla u_t(t)dxdt - \int_{t_1}^{t_2} E^r(t)\int_\Omega uu_t|u_t|^{m-1}dxdt\\ &+ \int_{t_1}^{t_2} E^r(t)\int_\Omega \Delta u(t)\int_0^t g(t-\tau)\Delta u(\tau)d\tau dxdt - \int_{t_1}^{t_2} E^r(t)\|\Delta u(t)\|_2^2 dt\\ &+ \int_{t_1}^{t_2} E^r(t)\Big(1-\int_0^t g(\tau)d\tau\Big)\|\Delta u(t)\|_2^2 dt + \int_{t_1}^{t_2} E^r(t)(g\circ\Delta u)(t)dt.\end{aligned} \tag{3.7}$$

We now estimate the terms in the right hand side of (3.7). For the first term, we use Young's inequality, lemma 2.1, (3.3) and lemma 2.2 to obtain

$$\begin{aligned}\left|-\int_\Omega E^r(t)uu_t dx\Big|_{t_1}^{t_2}\right| &\le \sum_{i=1}^{2}\left|E^r(t)\Big(\frac{C_*^2}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_2^2\Big)\right|_{t=t_i}\\ &\le \sum_{i=1}^{2}\left|\Big(\frac{p+1}{p-1}\Big)\Big(\frac{C_*^2+1}{l}\Big)E^{r+1}(t)\right|_{t=t_i} \le 2\Big(\frac{p+1}{p-1}\Big)\Big(\frac{C_*^2+1}{l}\Big)E^{r+1}(t_1).\end{aligned} \tag{3.8}$$

Similarly, by the same way and using the Poincaré inequality we get

$$\begin{aligned}\left|-\int_\Omega E^r(t)\nabla u.\nabla u_t dx\Big|_{t_1}^{t_2}\right| &\le \sum_{i=1}^{2}\left|E^r(t)\Big(\frac{\rho}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|\nabla u_t(t)\|_2^2\Big)\right|_{t=t_i}\\ &\le \sum_{i=1}^{2}\left|\Big(\frac{p+1}{p-1}\Big)\Big(\frac{\rho+1}{l}\Big)E^{r+1}(t)\right|_{t=t_i} \le 2\Big(\frac{p+1}{p-1}\Big)\Big(\frac{\rho+1}{l}\Big)E^{r+1}(t_1),\end{aligned} \tag{3.9}$$

where ρ denotes the Poincaré constant. For the second term, in the right hand side of (3.7), for any $\varepsilon > 0$, we have

$$2\int_{t_1}^{t_2}\int_{\Omega} E^r(t)|u_t(t)|^2 dxdt \le 2\varepsilon|\Omega|\int_{t_1}^{t_2} E^{r+1}(t)dt + 2c(\varepsilon)\int_{t_1}^{t_2} \|u_t(t)\|_{2(r+1)}^{2(r+1)}dt. \tag{3.10}$$

By (2.7) we get

$$\int_{t_1}^{t_2} \|u_t(t)\|_{2(r+1)}^{2(r+1)}dt = \int_{t_1}^{t_2} \|u_t(t)\|_{m+1}^{m+1}dt \le -\int_{t_1}^{t_2} E'(t)dt \le E(t_1). \tag{3.11}$$

Using (3.10) and (3.11) we obtain

$$2\int_{t_1}^{t_2}\int_{\Omega} E^r(t)|u_t(t)|^2 dxdt \le 2\varepsilon|\Omega|\int_{t_1}^{t_2} E^{r+1}(t)dt + 2c(\varepsilon)E(t_1). \tag{3.12}$$

We use again (2.7) to find

$$2\int_{t_1}^{t_2}\int_{\Omega} E^r(t)|\nabla u_t(t)|^2 dxdt \le -2\int_{t_1}^{t_2} E^r(t)E'(t)dt \le \frac{2}{r+1}E^{r+1}(t_1). \tag{3.13}$$

For the third term in the right hand side of (3.7) we use Young's inequality, Poincaré inequality, lemma 2.1 and (3.3) to find

$$\begin{aligned}
&\left| r\int_{t_1}^{t_2}\int_{\Omega} E^{r-1}(t)E'(t)\Big(uu_t + \nabla u.\nabla u_t\Big)dxdt\right| \\
&\le -r\int_{t_1}^{t_2} E^{r-1}(t)E'(t)\left(\frac{C_*^2}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_2^2 + \frac{\rho}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|\nabla u_t(t)\|_2^2\right)dt \\
&\le -\frac{rK}{2}\big(C_*^2+\rho+2\big)\int_{t_1}^{t_2} E^{r-1}(t)E'(t)E(t)dt \le \frac{rK}{2(r+1)}\big(C_*^2+\rho+2\big)E^{r+1}(t_1).
\end{aligned} \tag{3.14}$$

By (2.7) and (3.3) we estimate the fourth term in the form

$$\begin{aligned}
\left|-\int_{t_1}^{t_2} E^r(t)\int_{\Omega}\nabla u(t).\nabla u_t(t)dxdt\right| &\le \int_{t_1}^{t_2} E^r(t)\Big(\frac{\varepsilon\rho}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2\varepsilon}\|\nabla u_t(t)\|_2^2\Big)dt \\
&\le \varepsilon\Big(\frac{K\rho}{2}\Big)\int_{t_1}^{t_2} E^{r+1}(t)dt - \frac{1}{2\varepsilon}\int_{t_1}^{t_2} E^r(t)E'(t)dt \\
&\le \varepsilon\Big(\frac{K\rho}{2}\Big)\int_{t_1}^{t_2} E^{r+1}(t)dt + \frac{1}{2\varepsilon(r+1)}E^{r+1}(t_1).
\end{aligned} \tag{3.15}$$

For the fifth term we have

$$\begin{aligned}
&\left|-\int_{t_1}^{t_2} E^r(t)\int_{\Omega} uu_t|u_t|^{m-1}dxdt\right| \le \int_{t_1}^{t_2} E^r(t)\Big(\varepsilon\|u(t)\|_{m+1}^{m+1} + c(\varepsilon)\|u_t(t)\|_{m+1}^{m+1}\Big)dt \\
&\le \varepsilon C_*^{m+1}\int_{t_1}^{t_2} E^r(t)\|\Delta u(t)\|_2^{m+1}dt - c(\varepsilon)\int_{t_1}^{t_2} E^r(t)E'(t)dt \\
&\le \varepsilon\Big(C_*^{m+1}K\big(KE(0)\big)^{\frac{m-1}{2}}\Big)\int_{t_1}^{t_2} E^{r+1}(t)dt + \frac{c(\varepsilon)}{r+1}E^{r+1}(t_1).
\end{aligned} \tag{3.16}$$

By the use of Young's inequality, for the sixth term in the right hand side of (3.7), we have

$$\begin{aligned}
&\left|\int_{t_1}^{t_2} E^r(t)\int_\Omega \Delta u(t)\int_0^t g(t-\tau)\Delta u(\tau)d\tau dxdt\right|\\
&\le \int_{t_1}^{t_2} E^r(t)\int_\Omega\left(\int_0^t g(t-\tau)|\Delta u(\tau)-\Delta u(t)||\Delta u(t)|d\tau\right)dxdt\\
&\quad+\int_{t_1}^{t_2} E^r(t)\int_0^t g(\tau)d\tau\|\Delta u(t)\|_2^2dt\\
&\le(\delta+1)\int_{t_1}^{t_2} E^r(t)\int_0^t g(\tau)d\tau\|\Delta u(t)\|_2^2dt+\frac{1}{4\delta}\int_{t_1}^{t_2} E^r(t)(g\circ\Delta u)(t)dt.
\end{aligned}\tag{3.17}$$

Using the estimates (3.8), (3.9) and (3.12)–(3.17), from the Eq. (3.7), we obtain

$$\begin{aligned}
&2\int_{t_1}^{t_2} E^{r+1}(t)dt-\left(\frac{p-1}{p+1}\right)\int_{t_1}^{t_2} E^r(t)\|u(t)\|_{p+1}^{p+1}dt\\
&\le 2c(\varepsilon)E(t_1)+M_1E^{r+1}(t_1)+\varepsilon M_2\int_{t_1}^{t_2} E^{r+1}(t)dt\\
&\quad+\delta\int_{t_1}^{t_2} E^r(t)\int_0^t g(\tau)d\tau\|\Delta u(t)\|_2^2dt+\left(\frac{1}{4\delta}+1\right)\int_{t_1}^{t_2} E^r(t)(g\circ\Delta u)(t)dt,
\end{aligned}\tag{3.18}$$

where

$$\begin{aligned}
M_1&=\left(\frac{2(p+1)}{l(p-1)}+\frac{rK}{2(r+1)}\right)(C_*^2+\rho+2)+\frac{1}{r+1}\left(2+\frac{1}{2\varepsilon}+c(\varepsilon)\right),\\
M_2&=2|\Omega|+\frac{K\rho}{2}+C_*^{m+1}K\big(KE(0)\big)^{\frac{m-1}{2}}.
\end{aligned}$$

For the last two terms in the right hand side of (3.18) we use $(G3)$, (2.7) and (3.3) to get

$$\begin{aligned}
&\delta\int_{t_1}^{t_2} E^r(t)\int_0^t g(\tau)d\tau\|\Delta u(t)\|_2^2dt+\left(\frac{1}{4\delta}+1\right)\int_{t_1}^{t_2} E^r(t)(g\circ\Delta u)(t)dt\\
&\le\delta\left(\frac{2(p+1)}{p-1}\right)\left(\frac{1-l}{l}\right)\int_{t_1}^{t_2} E^{r+1}(t)dt-\left(\frac{1}{4\delta k_0}+\frac{1}{k_0}\right)\int_{t_1}^{t_2} E^r(t)(g'\circ\Delta u)(t)dt\\
&\le\delta\left(\frac{2(p+1)}{p-1}\right)\left(\frac{1-l}{l}\right)\int_{t_1}^{t_2} E^{r+1}(t)dt+\left(\frac{1}{2\delta k_0}+\frac{2}{k_0}\right)\frac{1}{r+1}E^{r+1}(t_1).
\end{aligned}\tag{3.19}$$

On the other hand, by the use of lemma 2.1 and (3.3), we have

$$\left(\frac{p-1}{p+1}\right)\int_{t_1}^{t_2} E^r(t)\|u(t)\|_{p+1}^{p+1}dt\le\frac{2}{l}C_*^{p+1}\left(\frac{2(p+1)}{l(p-1)}E(0)\right)^{\frac{p-1}{2}}\int_{t_1}^{t_2} E^{r+1}(t)dt,$$

which implies

$$\begin{aligned}
&2\int_{t_1}^{t_2} E^{r+1}(t)dt-\left(\frac{p-1}{p+1}\right)\int_{t_1}^{t_2} E^r(t)\|u(t)\|_{p+1}^{p+1}dt\\
&\ge 2\left[1-\left(\frac{E(0)}{d_1}\right)^{\frac{p-1}{2}}\right]\int_{t_1}^{t_2} E^{r+1}(t)dt.
\end{aligned}\tag{3.20}$$

Since $E(0)<d_1$, then

$$1-\left(\frac{E(0)}{d_1}\right)^{\frac{p-1}{2}}>0.$$

By (3.19) and (3.20) and choosing ε and δ small enough such that

$$1-\left(\frac{E(0)}{d_1}\right)^{\frac{p-1}{2}} - \delta\left(\frac{p+1}{p-1}\right)\left(\frac{1-l}{l}\right) - \varepsilon\frac{M_2}{2} > 0,$$

the estimate (3.18) takes the form

$$\int_{t_1}^{t_2} E^{r+1}(t)dt \le \gamma^{-1}\left(2c(\varepsilon)E^{-r}(0) + \widehat{M}_1\right)E^r(0)E(t_1) \le C^{-1}E^r(0)E(t_1), \quad (3.21)$$

where γ and C are some positive constants and $\widehat{M}_1 = M_1 + \left(\frac{1}{2k_0} + \frac{2}{k_0}\right)\frac{1}{r+1}$. An application of lemma 2.4 completes the proof. □

4 Blow up

In this section, we investigate unboundedness results for the solutions of (1.1)-(1.3). First, we give the following lemma which will be used frequently throughout this section.

Lemma 4.1 *Suppose that* $(G1)$, $(G2)$ *hold and* $(u_0, u_1) \in H_0^2(\Omega) \times H_0^1(\Omega)$. *Assume further that* $E(0) < d_1$ *and* $I(0) < 0$. *Then*

$$I(t) < 0, \qquad \forall t \in [0, T), \quad (4.1)$$

and

$$d_1 < \frac{p-1}{2(p+1)}\left[\left(1 - \int_0^t g(\tau)d\tau\right)\|\Delta u(t)\|_2^2 + (g \circ \Delta u)(t)\right] < \frac{p-1}{2(p+1)}\|u(t)\|_{p+1}^{p+1}, \quad (4.2)$$

for all $t \in [0, T)$.

Proof See Lemma 6 in [11]. □

Remark 4.2 Under the assumptions of lemma 4.1 and using lemma 2.1 it is easy to see

$$\|\Delta u\|_2^2 > \left(\frac{p+1}{p-1}\right)\left(\frac{2d_1}{l}\right).$$

4.1 Blow-up with different ranges of initial energy: the case $m = 1$

Let us to define

$$a(t) = \int_\Omega \left(u^2 + |\nabla u|^2\right)dx + \int_0^t \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)dt. \quad (4.3)$$

Lemma 4.3 *Suppose that* (2.1), $(G2)$ *and* (2.11) *hold. Then*

$$\begin{aligned} &a''(t) - (p+3)\left(\|u_t\|_2^2 + \|\nabla u_t\|_2^2\right) \\ &\quad \ge -2(p+1)E(0) + 2(p+1)\int_0^t \left(\|u_t\|_2^2 + \|\nabla u_t\|_2^2\right)dt. \end{aligned} \quad (4.4)$$

Proof From (4.3) we have

$$a'(t) = 2\int_\Omega \big(uu_t + \nabla u.\nabla u_t\big)dx + \|u\|_2^2 + \|\nabla u\|_2^2,$$

and

$$\begin{aligned} a''(t) &= 2\big(\|u_t\|_2^2 + \|\nabla u_t\|_2^2\big) - 2\Big(1 - \int_0^t g(\tau)d\tau\Big)\|\Delta u\|_2^2 \\ &\quad - 2\int_\Omega \Delta u(t)\int_0^t g(t-\tau)\big(\Delta u(t) - \Delta u(\tau)\big)d\tau dx + 2\|u\|_{p+1}^{p+1}. \end{aligned} \tag{4.5}$$

By using Young's inequality, for $\eta > 0$, we obtain

$$\int_\Omega \Delta u(t)\int_0^t g(t-\tau)\big(\Delta u(t) - \Delta u(\tau)\big)d\tau dx \le \eta\|\Delta u\|_2^2\int_0^t g(\tau)d\tau + \frac{1}{4\eta}(g\circ\Delta u)(t). \tag{4.6}$$

Then by (2.8), (4.5) and (4.6) we have

$$\begin{aligned} a''(t) - (p+3)\big(\|u_t\|_2^2 + \|\nabla u_t\|_2^2\big) &\ge -2(p+1)E(0) \\ &+ 2(p+1)\int_0^t \big(\|u_t\|_2^2 + \|\nabla u_t\|_2^2\big)dt + \Big(p+1-\frac{1}{2\eta}\Big)(g\circ\Delta u)(t) \\ &+ \Big[(p-1) - \big(p-1+2\eta\big)\int_0^t g(\tau)d\tau\Big]\|\Delta u\|_2^2. \end{aligned} \tag{4.7}$$

Letting $\eta = \frac{1}{2(p+1)}$ and using (2.11) we obtain (4.4). □

We now consider different cases on the sign of the initial energy:

(1) If $E(0) < 0$ then from (4.4), we have

$$a'(t) \ge a'(0) - 2(p+1)E(0)t, \quad t \ge 0.$$

Thus, we get $a'(t) > \|u_0\|_2^2 + \|\nabla u_0\|_2^2$ for $t > t^*$ where

$$t^* = \max\left\{\frac{a'(0) - \big(\|u_0\|_2^2 + \|\nabla u_0\|_2^2\big)}{2(p+1)E(0)}, 0\right\}. \tag{4.8}$$

(2) If $E(0) = 0$, then $a''(t) \ge 0$ for $t \ge 0$. Furthermore, if $a'(0) > \|u_0\|_2^2 + \|\nabla u_0\|_2^2$, then $a'(t) > \|u_0\|_2^2 + \|\nabla u_0\|_2^2$ for $t \ge 0$.

(3) If $0 < E(0) < \frac{a_1 d_1}{l(p-1)}$, and $I(u_0) < 0$, we have

$$\begin{aligned} a''(t) - (p+3)\big(\|u_t\|_2^2 + \|\nabla u_t\|_2^2\big) &\ge -2(p+1)E(0) \\ &+ 2(p+1)\int_0^t \big(\|u_t\|_2^2 + \|\nabla u_t\|_2^2\big)dt + a_1\|\Delta u\|_2^2. \end{aligned} \tag{4.9}$$

By (4.9) and remark 4.2 we have

$$a''(t) \ge -2(p+1)E(0) + a_1\|\Delta u\|_2^2 \ge 2(p+1)\left(\frac{a_1 d_1}{l(p-1)} - E(0)\right) > 0. \tag{4.10}$$

Then we obtain $a'(t) > \|u_0\|_2^2 + \|\nabla u_0\|_2^2$ for $t > t^*$ where

$$t^* = \max\left\{\frac{\|u_0\|_2^2 + \|\nabla u_0\|_2^2 - a'(0)}{2(p+1)\left(\frac{a_1 d_1}{l(p-1)} - E(0)\right)}, 0\right\}. \tag{4.11}$$

(4) For the case that $E(0) \geq \frac{a_1 d_1}{l(p-1)}$, we first note that

$$\|u(t)\|_2^2 - \|u_0\|_2^2 = 2\int_0^t \int_\Omega u(t) u_t(t) dx dt, \tag{4.12}$$

and

$$\|\nabla u(t)\|_2^2 - \|\nabla u_0\|_2^2 = 2\int_0^t \int_\Omega \nabla u(t).\nabla u_t(t) dx dt. \tag{4.13}$$

By using Hölder's inequality and Young's inequality, we have from (4.12) and (4.13)

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 + \int_0^t \|u(t)\|_2^2 dt + \int_0^t \|u_t(t)\|_2^2 dt, \tag{4.14}$$

$$\|\nabla u(t)\|_2^2 \leq \|\nabla u_0\|_2^2 + \int_0^t \|\nabla u(t)\|_2^2 dt + \int_0^t \|\nabla u_t(t)\|_2^2 dt. \tag{4.15}$$

By Hölder's inequality and Young's inequality, from (4.14) and (4.15), we get

$$\begin{aligned} a'(t) \leq\ & a(t) + \|u_0\|_2^2 + \|\nabla u_0\|_2^2 \\ & + \int_\Omega \left(u_t^2 + |\nabla u_t|^2\right) dx + \int_0^t \left(\|u_t\|_2^2 + \|\nabla u_t\|_2^2\right) dt. \end{aligned} \tag{4.16}$$

Hence by (4.4) and (4.16) we obtain

$$a''(t) - (p+3)a'(t) + (p+3)a(t) + a \geq 0,$$

where

$$a = (p+3)\left(\|u_0\|_2^2 + \|\nabla u_0\|_2^2\right) + 2(p+1)E(0).$$

Let

$$B(t) = a(t) + \frac{a}{p+3}, \qquad t > 0.$$

Then $B(t)$ satisfies (2.9) with $\delta = \frac{p-1}{4}$. Condition (2.10) with $K_0 = \|u_0\|_2^2 + \|\nabla u_0\|_2^2$ is equivalent to

$$a'(0) > \frac{p+3}{2}\left(1 - \sqrt{\frac{p-1}{p+3}}\right)\left(a(0) + \frac{a}{p+3}\right) + \|u_0\|_2^2 + \|\nabla u_0\|_2^2,$$

which means

$$E(0) < \frac{p+3}{p+1}\left[\frac{1}{2}\left(1 + \sqrt{\frac{p-1}{p+3}}\right)\left(\int_\Omega \left(u_0 u_1 + \nabla u_0.\nabla u_1\right) dx\right) - \left(\|u_0\|_2^2 + \|\nabla u_0\|_2^2\right)\right].$$

Then by lemma 2.5 we find $a'(t) \geq \|u_0\|_2^2 + \|\nabla u_0\|_2^2$.

Consequently, we have proved the following lemma:

Lemma 4.4 *Under the assumptions of Theorem 2.9, we have*

$$a'(t) \geq \|u_0\|_2^2 + \|\nabla u_0\|_2^2, \quad \text{for} \quad t > t_0,$$

where $t_0 = t^*$ is given by (4.8) and (4.11) in cases (1) and (3) and $t^* = 0$ in cases (2) and (4).

Proof of Theorem 2.9 Let

$$M(t) = \left[a(t) + (T_1 - t)(\|u_0\|_2^2 + \|\nabla u_0\|_2^2)\right]^{-\delta} \quad \text{for} \quad t \in [0, T_1], \tag{4.17}$$

where $\delta = (p-1)/4$ and $T_1 > 0$ is a certain constant which will be specified later. We have

$$\begin{aligned} M'(t) &= -\delta \left[a(t) + (T_1 - t)(\|u_0\|_2^2 + \|\nabla u_0\|_2^2)\right]^{-\delta-1} \left[a'(t) - \left(\|u_0\|_2^2 + \|\nabla u_0\|_2^2\right)\right] \\ &= -\delta M^{1+\frac{1}{\delta}}(t) \left[a'(t) - \left(\|u_0\|_2^2 + \|\nabla u_0\|_2^2\right)\right]. \end{aligned}$$

and

$$M''(t) = -\delta M^{1+\frac{2}{\delta}}(t) V(t), \tag{4.18}$$

where

$$\begin{aligned} V(t) = a''(t) &\left[a(t) + (T_1 - t)(\|u_0\|_2^2 + \|\nabla u_0\|_2^2)\right] \\ &- (\delta + 1) \left[a'(t) - \left(\|u_0\|_2^2 + \|\nabla u_0\|_2^2\right)\right]^2. \end{aligned} \tag{4.19}$$

For simplicity of calculation, we denote

$$\begin{aligned} P_u &= \int_\Omega u^2 dx, \quad Q_u = \int_0^t \|u\|_2^2 dt, \quad R_u = \int_\Omega u_t^2 dx, \quad S_u = \int_0^t \|u_t\|_2^2 dt, \\ P'_u &= \int_\Omega |\nabla u|^2 dx, \quad Q'_u = \int_0^t \|\nabla u\|_2^2 dt, \quad R'_u = \int_\Omega |\nabla u_t|^2 dx, \quad S'_u = \int_0^t \|\nabla u_t\|_2^2 dt. \end{aligned}$$

By (4.12), (4.13) and Hölder's inequality we have

$$\begin{aligned} a'(t) &= 2 \int_\Omega \left(u u_t + \nabla u . \nabla u_t\right) dx + 2 \int_0^t \int_\Omega \left(u u_t + \nabla u . \nabla u_t\right) dx dt + \|u_0\|_2^2 + \|\nabla u_0\|_2^2 \\ &\leq 2 \left(\sqrt{R_u P_u} + \sqrt{Q_u S_u} + \sqrt{R'_u P'_u} + \sqrt{Q'_u S'_u}\right) + \|u_0\|_2^2 + \|\nabla u_0\|_2^2. \end{aligned} \tag{4.20}$$

If case (1) or (2) holds, It follows from (4.4) that

$$a''(t) \geq (-8\delta - 4) E(0) + 4(\delta + 1) \left(R_u + S_u + R'_u + S'_u\right). \tag{4.21}$$

Then, using (4.19)–(4.21), we get

$$\begin{aligned} V(t) \geq &\left[(-8\delta - 4) E(0) + 4(\delta + 1)(R_u + S_u + R'_u + S'_u)\right] M^{-\frac{1}{\delta}}(t) \\ &- 4(\delta + 1) \left(\sqrt{R_u P_u} + \sqrt{Q_u S_u} + \sqrt{R'_u P'_u} + \sqrt{Q'_u S'_u}\right)^2. \end{aligned}$$

By (4.17) we find

$$\begin{aligned} V(t) \geq &(-8\delta - 4) E(0) M^{-\frac{1}{\delta}}(t) \\ &+ 4(\delta + 1)\left[(R_u + S_u + R'_u + S'_u)(T_1 - t)(\|u_0\|_2^2 + \|\nabla u_0\|_2^2) + K(t)\right], \end{aligned}$$

where

$$K(t) = (R_u + S_u + R'_u + S'_u)(P_u + Q_u + P'_u + Q'_u) - \left(\sqrt{R_u P_u} + \sqrt{Q_u S_u} + \sqrt{R'_u P'_u} + \sqrt{Q'_u S'_u}\right)^2.$$

By Schwarz inequality and $K(t)$ being nonnegative, we have

$$V(t) \geq (-8\delta - 4)E(0)M^{-\frac{1}{\delta}}(t), \qquad \forall t : t_0 < t < T_1. \tag{4.22}$$

Therefore, by (4.18) and (4.22), we get

$$M''(t) \leq \delta(8\delta + 4)E(0)M^{1+\frac{1}{\delta}}(t), \qquad \forall t : t_0 < t < T_1. \tag{4.23}$$

Note that by lemma 4.4, $M'(t) < 0$ for $t \geq t_0$. Multiplying (4.23) by $M'(t)$ and integrating it from t_0 to t, we have

$$M'(t)^2 \geq \alpha + \beta M^{2+1/\delta}(t) \qquad \forall t : t_0 < t < T_1, \tag{4.24}$$

where

$$\alpha = \left(\frac{p-1}{2}\right)^2 M^{\frac{2p+6}{p-1}}(t_0)\left[\left(\int_\Omega (u_0 u_1 + \nabla u_0.\nabla u_1)dx\right)^2 - 2E(0)M^{\frac{-4}{p-1}}(t_0)\right] > 0,$$
$$\beta = \frac{1}{2}(p-1)^2 E(0). \tag{4.25}$$

In the case (3), from (4.4) and (4.10) we obtain

$$a''(t) \geq (8\delta + 4)c_1 + 4(\delta + 1)\left(R_u + S_u + R'_u + S'_u\right),$$

where $c_1 = \frac{a_1 d_1}{l(p-1)} - E(0)$. Following similar procedure in case (1), we find

$$M''(t) \leq -\delta(8\delta + 4)c_1 - M^{1+\frac{1}{\delta}}(t) \qquad \forall t : t_0 < t < T_1,$$
$$M'(t)^2 \geq \alpha + \beta M^{2+1/\delta}(t) \qquad \forall t : t_0 < t < T_1,$$

where

$$\alpha = \left(\frac{p-1}{2}\right)^2 M^{\frac{2p+6}{p-1}}(t_0)\left[\left(\int_\Omega (u_0 u_1 + \nabla u_0.\nabla u_1)dx\right)^2 + 2c_1 M^{\frac{-4}{p-1}}(t_0)\right] > 0,$$
$$\beta = -\frac{c_1}{2}(p-1)^2. \tag{4.26}$$

For the case (4), by the steps of case (1), we obtain (4.24) with $\alpha, \beta > 0$ in (4.25) if

$$E(0) < \frac{\left(\int_\Omega (u_0 u_1 + \nabla u_0.\nabla u_1)dx\right)^2}{2(T_1 + 1)(\|u_0\|_2^2 + \|\nabla u_0\|_2^2)}.$$

Then by lemma 2.6, there exists a finite time T^* such that $\lim_{t \nearrow T^{*-}} M(t) = 0$. This means that $\lim_{t \nearrow T^{*-}} \left(\|u\|_2^2 + \|\nabla u\|_2^2\right) = +\infty$. Using lemma 2.1 and Poincaré inequality we obtain $\|\Delta u(t)\|_2^2 \to +\infty$ as $t \to T^{*-}$. This completes the proof.

Remark 4.5 By lemma 2.6, the upper bounds of T^* can be estimated respectively according to the sign of $E(0)$. In the case (1)

$$T^* \leq t_0 - \frac{M(t_0)}{M'(t_0)}.$$

Furthermore, if $M(t_0) < \min\{1, \sqrt{-\alpha/\beta}\}$, we have

$$T^* \leq t_0 + \frac{1}{\sqrt{-\beta}} \ln \frac{\sqrt{-\alpha/\beta}}{\sqrt{-\alpha/\beta} - M(t_0)},$$

where α and β are defined in (4.25). In case (2),

$$T^* \leq t_0 + \frac{M(t_0)}{\sqrt{\alpha}},$$

where α is defined in (4.25). In cases (3) and (4),

$$T^* \leq \frac{M(t_0)}{\sqrt{\alpha}}, \quad \text{or} \quad T^* \leq t_0 + 2^{\frac{3p+1}{2(p-1)}} \left(\frac{\alpha^2}{\beta^2}\right)^{\frac{p+1}{p-1}} \frac{(p-1)}{4\sqrt{\alpha}} \left[1 - \left(1 + cM(t_0)\right)^{-\frac{2}{p-1}}\right].$$

Moreover, in case (3), α and β are defined in (4.26) and in case (4), α and β are defined in (4.25).

Remark 4.6 We note that T_1 is feasible provided that $T_1 \geq T^*$. However, the choice of T_1 in (4.17) is possible under some conditions. When $E(0) < 0$, from (4.23) it is clear that $M''(t) < 0$. Therefore, $\frac{d}{dt}\left[\frac{M(t)}{M'(t)}\right] = 1 - \left[M(t)M''(t)/(M'(t))^2\right] \geq 1$ and so $\frac{M(t)}{M'(t)}$ is increasing. Since, by lemma 4.4 $M'(0) \leq 0$, in the case $\int_\Omega (u_0 u_1 + \nabla u_0 . \nabla u_1) dx > 0$ we take $T_1 \geq -\frac{M(0)}{M'(0)}$ which means

$$T_1 \geq \frac{\|u_0\|_2^2 + \|\nabla u_0\|_2^2}{\int_\Omega (u_0 u_1 + \nabla u_0 . \nabla u_1) dx}.$$

On the other hand, an integrating of $\frac{d}{dt}\left[\frac{M(t)}{M'(t)}\right]$ over $(0, T^*)$ gives us $T^* \leq -\frac{M(0)}{M'(0)}$. Then we get $T^* \leq T_1$. For the case $\int_\Omega (u_0 u_1 + \nabla u_0 . \nabla u_1) dx \leq 0$ by (4.8) we have $t^* = \frac{a'(0) - \left(\|u_0\|_2^2 + \|\nabla u_0\|_2^2\right)}{2(p+1)E(0)}$ and we choose $T_1 \geq t^* - \frac{M(t^*)}{M'(t^*)}$. If $E(0) = 0$, then the condition $a'(0) > \|u_0\|_2^2 + \|\nabla u_0\|_2^2$ implies that $\int_\Omega (u_0 u_1 + \nabla u_0 . \nabla u_1) dx > 0$ and so we can choose again $T_1 \geq -\frac{M(0)}{M'(0)}$. If $0 < E(0) < \min\{k_1, k_2\}$ where

$$k_1 = \frac{\delta + 1}{r_2(2\delta + 1)} \left[a'(0) - 2r_2 a(0) - (\|u_0\|_2^2 + \|\nabla u_0\|_2^2)\right],$$

$$k_2 = \frac{\left(\int_\Omega (u_0 u_1 + \nabla u_0 . \nabla u_1) dx\right)^2}{2(T_1 + 1)\left(\|u_0\|_2^2 + \|\nabla u_0\|_2^2\right)},$$

then we choose T_1 such that $\hat{k}_1 \leq T_1 \leq \hat{k}_2$ where

$$\hat{k}_1 = \frac{\left(\int_\Omega (u_0 u_1 + \nabla u_0 . \nabla u_1) dx\right)^2}{2k_1\left(\|u_0\|_2^2 + \|\nabla u_0\|_2^2\right)} - 1,$$

$$\hat{k}_2 = \frac{\left(\int_\Omega (u_0 u_1 + \nabla u_0 . \nabla u_1) dx\right)^2}{2E(0)\left(\|u_0\|_2^2 + \|\nabla u_0\|_2^2\right)} - 1.$$

4.2 Blow-up with initial energy less than potential well depth: the case $p > m \geq 1$

In this section we prove an unboundedness result, Theorem 2.10, for certain solutions of (1.1)–(1.3) with non-positive initial energy as well as positive initial energy.

Proof of Theorem 2.10 (i) See Theorem 4 in [11].
(ii) On the contrary, under the conditions in Theorem 2.10, suppose that the existence time of solution $u(t)$ can be extended to the whole interval $[0, \infty)$. Let

$$\phi(t) = \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2. \tag{4.27}$$

Twice differentiating of (4.27), it follows from (1.1) that

$$\frac{1}{2}\phi''(t) = \|u_t\|_2^2 + \|\nabla u_t\|_2^2 - \Big(1 - \int_0^t g(\tau)d\tau\Big)\|\Delta u\|_2^2 - \int_\Omega \nabla u.\nabla u_t dx + \|u\|_{p+1}^{p+1}$$
$$- \int_\Omega \Delta u(t) \int_0^t g(t-\tau)(\Delta u(t) - \Delta u(\tau))d\tau dx - \int_\Omega u u_t |u_t|^{m-1} dx.$$

By the Young's inequality, for $\eta > 0$, we have

$$\Big|\int_\Omega \Delta u(t) \int_0^t g(t-\tau)\big(\Delta u(\tau) - \Delta u(t)\big)d\tau dx\Big| \le \eta(g \circ \Delta u)(t) + \frac{1}{4\eta}\int_0^t g(\tau)d\tau\|\Delta u\|_2^2. \tag{4.28}$$

Using (4.28) and (2.4), for $\eta > 0$, we have

$$\begin{aligned} \frac{1}{2}\phi''(t) \ge \|u_t\|_2^2 + \|\nabla u_t\|_2^2 - \frac{1}{4\eta}\int_0^t g(\tau)d\tau\ \|\Delta u\|_2^2 - \int_\Omega \nabla u.\nabla u_t dx \\ + (1-\eta)(g\circ\Delta u)(t) - \int_\Omega u u_t|u_t|^{m-1}dx - I(u(t)). \end{aligned} \tag{4.29}$$

For the sixth term in the right hand side of (4.29) we use the Hölder's inequality to obtain

$$\Big|\int_\Omega u u_t|u_t|^{m-1}dx\Big| \le \|u(t)\|_{m+1}\|u_t(t)\|_{m+1}^m. \tag{4.30}$$

Taking $2 < m+1 < p+1$ into account and using the standard interpolation inequality we have

$$\|u(t)\|_{m+1} \le \|u(t)\|_2^k \|u(t)\|_{p+1}^{1-k}, \tag{4.31}$$

where $\frac{k}{2} + \frac{1-k}{p+1} = \frac{1}{m+1}$ which gives $k = \frac{2(p-m)}{(m+1)(p-1)} > 0$. Using lemma 4.1 we know that

$$\|\Delta u(t)\|_2^2 \le \alpha_1\|u(t)\|_{p+1}^{p+1}, \tag{4.32}$$

for some $\alpha_1 > 0$. Then, by (4.32) and lemma 2.1, we deduce

$$\|u(t)\|_2^2 \le \alpha_2\|u(t)\|_{p+1}^{p+1}, \tag{4.33}$$

where α_2 is a positive constant. Therefore, from (4.30),(4.31), (4.33) and Young's inequality, we have, for all $\delta > 0$,

$$\begin{aligned} \Big|\int_\Omega u u_t|u_t|^{m-1}dx\Big| &\le \alpha_3\|u(t)\|_{p-1}^{1-k-(p+1)/(m+1)-(p+1)k/2}\|u(t)\|_{p+1}^{(p+1)/(m+1)}\|u_t(t)\|_{m+1}^m \\ &\le \alpha_3\left\{\left(\frac{\delta^{m+1}}{m+1}\right)\|u(t)\|_{p+1}^{p+1} + \frac{m}{m+1}\delta^{-(m+1)/m}\|u_t(t)\|_{m+1}^{m+1}\right\}, \end{aligned} \tag{4.34}$$

for some $\alpha_3 > 0$. Hence, by the inequality (4.34), for any $\varepsilon > 0$, we can write

$$\Big|\int_\Omega u u_t|u_t|^{m-1}dx\Big| \le \varepsilon\|u(t)\|_{p+1}^{p+1} + c(\varepsilon)\|u_t(t)\|_{m+1}^{m+1}, \tag{4.35}$$

and by the use of Young's inequality, we get

$$\left|\int_{\Omega} \nabla u.\nabla u_t dx\right| \leq \varepsilon \|\nabla u\|_2^2 + c(\varepsilon)\|\nabla u_t\|_2^2. \tag{4.36}$$

By lemma 4.1 we know that

$$\begin{aligned} I(u(t)) &\leq I(u(t)) + \theta\big(E(0) - E(t)\big) \\ &\leq -(\theta/2)\big(\|u_t\|_2^2 + \|\nabla u_t\|_2^2\big) + (1-\theta/2)\,(g \circ \Delta u)(t) \\ &\quad + \big(\theta/(p+1) - 1\big)\|u\|_{p+1}^{p+1} + (1-\theta/2)\Big(1 - \int_0^t g(\tau)d\tau\Big)\|\Delta u\|_2^2 + \theta E(0). \end{aligned} \tag{4.37}$$

Thus, by (4.29), and (4.35)-(4.37), we arrive at

$$\begin{aligned} &\frac{1}{2}\phi''(t) + c(\varepsilon)\Big(\|\nabla u_t\|_2^2 + \|u_t\|_{m+1}^{m+1}\Big) \\ &\geq \left(1 + \frac{\theta}{2}\right)\big(\|u_t\|_2^2 + \|\nabla u_t\|_2^2\big) + \left(\frac{\theta}{2} - \eta\right)(g \circ \Delta u)(t) \\ &\quad + \left[\left(\frac{\theta}{2} - 1\right) - \left(\frac{\theta}{2} - 1 + \frac{1}{4\eta}\right)\int_0^t g(\tau)d\tau\right]\|\Delta u\|_2^2 \\ &\quad + \left(1 - \frac{\theta}{p+1} - \varepsilon\right)\|u\|_{p+1}^{p+1} - \varepsilon\|\nabla u\|_2^2 - \Big(\frac{2a_2\theta}{p-1}\Big)d_1. \end{aligned} \tag{4.38}$$

Letting $\eta = \theta/2$, in (4.38) and using (2.13) we have

$$\begin{aligned} \frac{1}{2}\phi''(t) + c(\varepsilon)\Big(\|\nabla u_t\|_2^2 + \|u_t\|_{m+1}^{m+1}\Big) &\geq a_2 l\|\Delta u\|_2^2 + \left(1 - \frac{\theta}{p+1} - \varepsilon\right)\|u\|_{p+1}^{p+1} \\ &\quad - \varepsilon\|\nabla u\|_2^2 - \Big(\frac{2a_2\theta}{p-1}\Big)d_1. \end{aligned} \tag{4.39}$$

In (4.39), for $u \in H_0^2(\Omega)$, we know that $\|u\|_{H^2(\Omega)} < \infty$ which implies that $\|\nabla u\|_2^2 < c_0$ is bounded. Recalling remark 4.2, for sufficiently small ε, from (4.39) we obtain

$$\begin{aligned} &\frac{1}{2}\phi''(t) + c(\varepsilon)\Big(\|\nabla u_t\|_2^2 + \|u_t\|_{m+1}^{m+1}\Big) \\ &\geq \left(1 - \frac{\theta}{p+1} - \varepsilon\right)\|u\|_{p+1}^{p+1} + \left(\frac{2a_2}{p-1}\right)(p+1)d_1 - c_0\varepsilon - \Big(\frac{2a_2\theta}{p-1}\Big)d_1 \\ &> \left(1 - \frac{\theta}{p+1} - \varepsilon\right) l\|\Delta u\|_2^2 + \left(\frac{2a_2}{p-1}\right)(p+1)d_1 - c_0\varepsilon - \Big(\frac{2a_2\theta}{p-1}\Big)d_1 \\ &> \left\{1 - \frac{\theta}{p+1} - \left(1 + \frac{c_0(p-1)}{2(p+1)d_1}\right)\varepsilon\right\}\frac{2(p+1)}{p-1}d_1 =: A > 0. \end{aligned} \tag{4.40}$$

By integrating (4.40) from 0 to t, we get

$$\frac{1}{2}\phi'(t) + c(\varepsilon)\int_0^t \Big(\|\nabla u_t(\tau)\|_2^2 + \|u_t(\tau)\|_{m+1}^{m+1}\Big)d\tau > At + \frac{1}{2}\phi'(0). \tag{4.41}$$

From (2.7), we know that

$$\int_0^t \Big(\|\nabla u_t(\tau)\|_2^2 + \|u_t(\tau)\|_{m+1}^{m+1}\Big)d\tau \leq E(0) - E(t) < d_1, \quad \forall t \geq 0. \tag{4.42}$$

Then, (4.41) gives us

$$\frac{1}{2}\phi'(t) > At + \frac{1}{2}\phi'(0) - c(\varepsilon)d_1, \quad \forall t \geq 0. \tag{4.43}$$

Integrating (4.43) over $(0, t)$, we have

$$\phi(t) > At^2 + \big(\phi'(0) - 2c(\varepsilon)d_1\big)t + \phi(0), \quad \forall t \geq 0. \tag{4.44}$$

On the other hand, we estimate $\phi(t)$ in the following form. By the Hölder's inequality we have

$$|u(t)|^2 = \left|u_0 + \int_0^t u_t(\tau)d\tau\right|^2 \leq 2|u_0|^2 + 2t\int_0^t |u_t(\tau)|^2 d\tau.$$

Therefore, by Poincaré inequality, we get

$$\begin{aligned}\|u(t)\|_2^2 &\leq 2\|u_0\|_2^2 + 2t\int_0^t \|u_t(t)\|_2^2 dt \\ &\leq 2\|u_0\|_2^2 + 2\rho t\int_0^t \|\nabla u_t(t)\|_2^2 dt < 2\|u_0\|_2^2 + 2\rho d_1 t,\end{aligned} \tag{4.45}$$

where ρ denotes the Poincaré constant. Similarly,

$$\|\nabla u(t)\|_2^2 \leq 2\|\nabla u_0\|_2^2 + 2t\int_0^t \|\nabla u_t(t)\|_2^2 dt < 2\|\nabla u_0\|_2^2 + 2d_1 t. \tag{4.46}$$

By (4.45) and (4.46) we obtain

$$\phi(t) < 2\phi(0) + (1+\rho)2d_1 t. \tag{4.47}$$

The inequality (4.44) says that the function ϕ grows at least as a quadratic function while the inequality (4.47) shows that ϕ is a function at most of linear form. This is a contradiction and so the proof of Theorem 2.10 is completed. □

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