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Homogenization of a nonlinear parabolic problem corresponding to a Leray–Lions monotone operator with right-hand side measure

Mohammed Abdellaoui[1] · **Elhoussine Azroul**[1]

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Abstract

In this paper we deal with asymptotic behaviour of renormalized solutions u_n to the nonlinear parabolic problems whose model is

$$\begin{cases} (u_n)_t - \operatorname{div}(a_n(t, x, \nabla u_n)) = \mu_n & \text{in } Q = (0, T) \times \Omega, \\ u_n(t, x) = 0 & \text{on } (0, T) \times \partial\Omega, \\ u_n(0, x) = u_0^n & \text{in } \Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 1$, $T > 0$ and $u_0^n \in C_0^\infty(\Omega)$ that approaches u_0 in $L^1(\Omega)$. Moreover $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of Radon measures with bounded variation in Q which converges to μ in the narrow topology of measures. The main result states that, under the assumption of G-convergence of the operators $A_n(v) = -\operatorname{div}(a_n(t, x, \nabla v_n))$, defined for $v_n \in L^p(0, T; W_0^{1,p}(\Omega))$ for $p > 1$, to the operator $A_0(v) = -\operatorname{div}(a_0(t, x, \nabla v))$ and up to subsequences, (u_n) converges a.e. in Q to the renormalized solution u of the problem

$$\begin{cases} u_t - \operatorname{div}(a_0(t, x, \nabla u)) = \mu & \text{in } Q = (0, T) \times \Omega, \\ u(t, x) = 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0, x) = u_0 & \text{in } \Omega. \end{cases}$$

The proposed renormalized formulation differs from the usual one by the fact that truncated function $T_k(u_n)$ (which depend on the solutions) are used in place of the solutions u_n. We prove existence of such a limit-solution and we discuss its main properties in connection with G-convergence, we finally show the relationship between the new approach and the previous ones and we extend this result using capacitary estimates and auxiliary test functions.

Keywords Nonlinear parabolic problems · Homogenization (G-convergence) · Measure data

✉ Mohammed Abdellaoui
mohammed.abdellaoui3@usmba.ac.ma

Elhoussine Azroul
elhoussine.azroul@usmba.ac.ma

[1] Laboratory LAMA, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, University of Fez, B.P. 1796, Atlas Fez, Morocco

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Contents

1 Introduction

We are interested in the asymptotic behaviour of the solutions u_n to the (homogeneous) parabolic problems

$$\begin{cases} (u_n)_t - \operatorname{div}(a_n(t,x,\nabla u_n)) = \mu_n & \text{in } (0,T)\times\Omega, \\ u_n(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_n(0,x) = u_0^n & \text{in } \Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $(u_0^n)_{n\in\mathbb{N}}$ is a sequence of smooth functions that approaches u_0 in $L^1(\Omega)$, $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of Radon measures with bounded total variation on $Q = (0,T)\times\Omega$ which converges to μ in the narrow topology of measures, and $(a_n(t,x,\zeta))_{n\in\mathbb{N}}$ are Carathéodory functions such that the corresponding operators $A_n : L^p(0,T,W_0^{1,p}(\Omega)) \mapsto L^{p'}(0,T;W^{-1,p'}(\Omega))$, defined by $A_n(v_n) = -\operatorname{div}(a_n(t,x,\nabla v_n))$, turn out to be monotone, continuous and coercive between the Sobolev space $L^p(0,T;W_0^{1,p}(\Omega))$, $p > 1$, and its dual space $L^{p'}(0,T,W^{-1,p'}(\Omega))$, $\frac{1}{p}+\frac{1}{p'} = 1$. Under these assumptions, for every $F_n \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$ there exists a unique variational solution v_n to the problem

$$\begin{cases} (v_n)_t - \operatorname{div}(a_n(t,x,\nabla v_n)) = F_n \text{ in } Q = (0,T)\times\Omega, \\ v_n(0,x) = u_0^n \text{ in } \Omega, \quad v_n(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases}$$

that is a unique function $v_n \in W \cap C(0,T;L^2(\Omega))$ in the weak sense

$$\begin{cases} -\displaystyle\int_\Omega u_0^n w(0)\mathrm{d}x - \int_0^T \langle w_t, v_n\rangle \mathrm{d}t + \int_Q a(t,x,\nabla v_n)\cdot\nabla w \,\mathrm{d}x\mathrm{d}t = \int_0^T \langle F_n, w\rangle, \ \forall w\in W \\ \text{with } w(T)=0,\ W = \{u\in L^p(0,T;V),\ u_t\in L^{p'}(0,T;V')\} \text{ and } V = W_0^{1,p}(\Omega)\cap L^2(\Omega), \end{cases} \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual duality pairing between the spaces $W_0^{1,p}(\Omega)\cap L^2(\Omega)$ and $(W^{-1,p'}(\Omega)\cap L^2(\Omega))'$ (see [46] for the case $p \geq 2$ and [44] for $1 < p < 2$). Let emphasize that variational solutions are solutions to problems with $L^{p'}(Q)$-data. This kind of problems has been largely studied in different context, but here the obtained results are related to the theory of homogenization, which is the study of the asymptotic behaviour for solutions of (1.1) corresponding to $a_n(t, x, \zeta) = a(nt, nx, \zeta)$, $n \in \mathbb{N}$, with suitable periodicity assumptions on $a(\cdot, \cdot, \zeta)$. If there exists $A_0(v) = -\mathrm{div}(a_0(t, x, \nabla v)) : L^p(0, T; W_0^{1,p}(\Omega)) \mapsto L^{p'}(0, T; W^{-1,p'}(\Omega))$ such that for every $F \in L^{p'}(Q)$, the solutions v_n of (1.1) converge weakly in $L^p(0, T; W_0^{1,p}(\Omega))$ to the variational solution v of the problem

$$\begin{cases} v_t - \mathrm{div}(a_0(t, x, \nabla v)) = F \text{ in } Q = (0, T) \times \Omega, \\ v(0, x) = u_0 \text{ in } \Omega, \quad v(t, x) = 0 \text{ on } (0, T) \times \partial\Omega, \end{cases}$$

and the momenta $a_n(t, x, \nabla v_n)$ converge to $a_0(t, x, \nabla v)$ weakly in $(L^{p'}(Q))^N$, the sequence A_n is said to be *G-converging* (or *H-converging*) to A_0. Hence the sequence of operators (A_n) G-converges to A_0 if the asymptotic behaviour of solutions corresponding to A_n is described by the problem corresponding to A_0. In the case of the theory of Homogenization: the operator A_0 represents the macroscopic model associated to the microscopic structures described, at different scale, by A_n. Recall that the notion of G-convergence was introduced in Spagnolo [73] for parabolic operators (the name comes from the fact that G-convergence is defined in terms of Green's operators), the extension to the elliptic case is defined in [72], especially to the second order symmetric linear elliptic operators in [71] (See also [55,68,69] for problems with lower order terms and [50,70,78] for non-symmetric case where the last two authors use the name H-convergence, H stands for Homogenization). For the case of elliptic operators with arbitrary order we refer to [54,79], for the parabolic case we refer to Colombini and Spagnolo [19,20] and Spagnolo [74], while the arbitrary order case is studied in [80] and for a class of non-divergence parabolic operators in [81]. The properties of G-convergence for some classes of quasi-linear elliptic operators with linear principal part are studied in [5,8,10,58] and also [40] for the study of conditions under which the weak convergence of coefficients implies the G-convergence of the corresponding operators. The case of quasi-linear monotone operators in divergence form was studied by Tartar (unpublished notes, 1981), by Pankov [56], Del Vecchio [28], and Francu [32] under some equi-continuity assumptions, by Chiado Piat et al. [18] and Defranceschi [27] without any continuity conditions. The relationship between G-convergence of quasi-linear elliptic monotone operators and Γ-convergence of the corresponding convex functionals is studied in [26], the case of degenerate monotone operators in divergence form is considered in [25]. A discrete notion of G-convergence for finite difference equations can be found in [41], we refer to [39,42,43] for the case of quasi-linear parabolic operators and [21,75] for hyperbolic case (see also [14,22,38,57] for more references). As a consequence of stability properties proved in the context of Dirichlet problems with elliptic operators and measure data [23,48,49], we drive such new results for parabolic problems with measures which depend on time and the extension to more general spaces seems to be always possible. We consider, as starting point, a sequence (A_n) of operators in divergence form G-converging to an operator A_0 of the same form. Hence A_0 is a good model for the asymptotic behaviour of the variational solutions of (1.1). The main point is to study the possibility to describe the asymptotic behaviour of solutions corresponding to the operators A_n in the case where μ lies in the space of measures, some partial results can be found in [67] when μ lies in $L^1(Q)$ and in [1,61] in the case of "uniform" convergence of $a_n(t, x, \zeta)$ (see for instance [62] for nonexistence results (concentration phenomena) and [2] for blowing-up problems). In particular if $0 < \alpha < \beta$ and $\mathcal{M}(\alpha, \beta)$ is the set of all matrices $A(t, x) \in (L^\infty(Q))^{N \times N}$ such that

$$A(t, x)\, \zeta \cdot \zeta \geq \alpha|\zeta|^2, \quad A^{-1}(t, x)\, \zeta \cdot \zeta \geq \beta^{-1}|\zeta|^2 \quad \forall \zeta \in \mathbb{R}^N,$$

and if $(A_n) \in \mathcal{M}(\alpha, \beta)$ G-converges to $A_0 \in \mathcal{M}(\alpha, \beta)$, then the above assumptions implies that the adjoint matrices $(\overline{A}_n)$ G-converges to the adjoint $\overline{A}_0$ of A_0. Letting μ be a fixed measure with bounded variation in Q, we consider the solutions u_n of problems

$$\begin{cases} (u_n)_t - \operatorname{div}(A_n(t,x)\nabla u_n) = \mu_n \text{ in } Q, \\ u_n(0,x) = u_0^n \text{ in } \Omega, \quad u_n(t,x) = 0 \text{ on } (0,T) \times \partial\Omega. \end{cases} \tag{1.2}$$

Regardless of the assumptions on A_n, compactness results on the solutions of problems (1.2) plays a crucial role in the existence and uniqueness of solutions (u_n) (see [9,66,76]), u_n is a solution of (1.2) if $u_n \in L^q(0,T; W_0^{1,q}(\Omega)) \cap L^\infty(0,T; L^1(\Omega))$ for every $q < p - \frac{N}{N+1}$ and, for every $f_n \in L^\infty(Q)$, u_n satisfies

$$\int_\Omega u_0^n v_n(0)\mathrm{d}x + \int_Q u_n f_n \mathrm{d}x\mathrm{d}t = \int_Q v_n \mathrm{d}\mu_n, \tag{1.3}$$

where v_n is the variational solution to

$$\begin{cases} (v_n)_t - \operatorname{div}(\overline{A}_n(t,x)\nabla v_n) = f_n \text{ in } Q = (0,T) \times \Omega, \\ v_n(0,x) = u_0^n \text{ in } \Omega, \quad v_n(t,x) = 0 \text{ on } (0,T) \times \partial\Omega. \end{cases}$$

On the other hand, solution u_n of (1.2) is unique and that there exists $C > 0$, depending only on N, α and β, such that $\|u_n\|_{L^q(0,T;W_0^{1,q}(\Omega))} \leq C$ (see [60,66]). Precisely, we can extract a subsequence (u_n) which converges weakly in $L^q(0,T; W_0^{1,q}(\Omega))$ to a function u. In particular, (u_n) converges to u strongly in $L^1(Q)$, and then

$$\lim_{n\to+\infty} \int_Q u_n f_n \mathrm{d}x\mathrm{d}t = \int_Q u f \mathrm{d}x\mathrm{d}t, \tag{1.4}$$

for every $f \in L^\infty(Q)$. As a consequence of G-convergence hypothesis on the operators, the solutions v_n converge weakly in $L^2(0,T; H_0^1(\Omega))$ to the solution v of problem

$$\begin{cases} v_t - \operatorname{div}(\overline{A}_0(t,x)\nabla v) = f \text{ in } Q = (0,T) \times \Omega, \\ v(0,x) = u_0 \text{ in } \Omega, \quad v(t,x) = 0 \text{ on } (0,T) \times \partial\Omega. \end{cases}$$

As a consequence of classical regularity results, (v_n) turns out to be a sequence of equi-Hölder continuous functions and hence the sequence converges to v uniformly in Q. We then have

$$\lim_{n\to\infty} \int_Q v_n \mathrm{d}\mu_n = \int_Q v \mathrm{d}\mu,$$

which, together with (1.3)–(1.4), implies that u is solution of problem

$$\begin{cases} u_t - \operatorname{div}(\overline{A}_0(t,x)\nabla u) = f \text{ in } Q = (0,T) \times \Omega, \\ u(0,x) = u_0 \text{ in } \Omega, \quad u(t,x) = 0 \text{ on } (0,T) \times \partial\Omega, \end{cases}$$

and the whole sequence (u_n) converges to u. Losely speaking, the method of G-convergence of linear operators with measures allows to describe a similar model valid for variational solutions. It is worth noting that, in this paper, such a construction can be achieved (at least with the same technique) as far as nonlinear parabolic operators are concerned. Note that in this case we miss two important properties: first, the fact that the formulation in terms of duality (1.3) translates the problem of the passage to the limit for solutions corresponding to measure data in a problem of convergence of solutions corresponding to regular data (solved by the assumption of G-convergence) (see [60,76]), Second, the property that in the linear case the solution corresponding to measure data is unique, while for nonlinear equations the uniqueness of the solution for general measure data is still an open problem [61,63], we would like to emphasize that due to the first difficulty we are led to choose to set our results in the framework of the so-called renormalized solution given in [61] in the context of parabolic problems

$$\begin{cases} (u_n)_t - \operatorname{div}(a_n(t,x,\nabla u_n)) = \mu_n \text{ in } Q = (0,T)\times\Omega, \\ u_n(0,x) = u_0^n \text{ in } \Omega, \quad u_n(t,x) = 0 \text{ on } (0,T)\times\partial\Omega. \end{cases} \tag{1.5}$$

Let us recall that the main property of renormalized solutions u_n of (1.5) is that all truncations $T_k(u_n)$ are variational solutions of the boundary value problems (adapted to its truncations) defined by

$$\begin{cases} (T_k(u_n))_t - \operatorname{div}(a_n(t,x,\nabla T_k(u_n))) = \mu_{k,n} \text{ in } Q = (0,T)\times\Omega, \\ T_k(u_n)(0,x) = T_k(u_0^n) \text{ in } \Omega, \quad T_k(u_n)(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{1.6}$$

where $\mu_{k,n}$ are suitable regular measures (more precisely $\mu_{k,n}$ does not charge sets of zero parabolic p-capacity) which converges to μ as k goes to $+\infty$. Our results are actually in the same spirit as those in [48] which concern the elliptic equation

$$-\operatorname{div}(a_n(x,\nabla u_n)) = \mu_n, \text{ in } \Omega, \quad u_n = 0 \text{ on } \partial\Omega.$$

The paper is planned in the following way, in Sect. 2 we will precise the notion of capacity and some basic properties of measures and we will state the main result using assumptions on Leray–Lions operators and the definitions of renormalized solutions, whose proof, which is rather technical, is left to Sect. 3, where the strategy is to pass to the limit as $n \to +\infty$ in (1.6) for every $k > 0$ fixed, instead of trying to pass to the limit in the original problems. This approach has the advantage of attacking the problem from a variational point of view. Nevertheless, the passage to the limit in (1.6) cannot be performed directly using the hypothesis of G-convergence of the operators, due to the presence of varying right-hand sides, that converge only in a very weak sense. First we prove some a priori estimates on the elements u_n, $T_k(u_n)$, $a_n(t,x,\nabla u_n)$, $a_n(t,x,\nabla T_k(u_n))$ and $\mu_{k,n}$ in order to obtain a limit equation where information about operators and data are lost (Sect. 4). As a consequence, we reconstruct the datum in Sect. 5, and following the approach of Minty's Lemma, we reconstruct the operator in Sect. 6. Because of the lack of uniqueness result, we obtain that for every fixed μ and for every choice of a sequence of renormalized solutions to problem (1.5) it is possible to extract a subsequence (possibly depending on μ) which converges to a renormalized solution of the problem corresponding to the G-limit A_0 and with datum μ.

2 Preliminaries and general results

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, $N \geq 2$, and let p and p' be two real numbers with $1 < p \leq N$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let us fix some notations. Henceforward, we will consider, respectively, $|\zeta|$ and $\zeta \cdot \zeta'$ the Euclidean norm of a vector $\zeta \in \mathbb{R}^N$ and the scalar product between ζ and $\zeta' \in \mathbb{R}^N$. For formally, a certain property holds almost everywhere (or a.e.) if it holds for all cases except for a certain subset which is very small. Frequently it will be convenient to describe situations that hold except on sets of zero measure. So by convention, a property is said to hold μ-almost everywhere (μ-a.e.) if the set of points on which it doesn't hold has μ-measure zero, similarly, this notations can be used in the case of convergence. Moreover, in what follows, ω will indicate any quantity that vanishes as the parameters in its argument go to their (obvious, if not explicitly stressed) limit point with the same order in which they appear.

2.1 Capacity

For every Borel set $B \subseteq Q$, its p-capacity $\text{cap}_p(B, Q)$ with respect to Q is defined by (see [37,65])

$$\text{cap}_p(B, Q) = \inf\{\|u\|_W\},$$

where the infimum is taken over all the functions $u \in W$ such that $u \geq 1$ a.e. in a neighborhood of B. We say that a property $\mathcal{P}(t, x)$ holds cap_p-quasi everywhere if $\mathcal{P}(t, x)$ holds for every (t, x) outside a subset of Q of zero p-capacity. A function u defined on Q is said to be cap_p-quasi continuous if for every $\epsilon > 0$ there exists $B_\epsilon \subseteq Q$ with $\text{cap}_p(B_\epsilon, Q) < \epsilon$ such that the restriction of u to $Q \backslash B$ is continuous. It is well known that every function in W_2 defined by

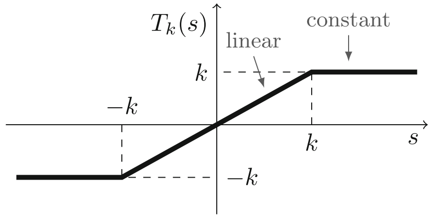
$$W_2 = \left\{u \in L^2(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q);\ u_t \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)\right\},$$

has a unique cap_p-quasi continuous representative $\tilde{u}$, whose values are defined (and finite) cap_p-quasi everywhere in Q. In what follows we always identify a function $u \in W_2$ with its cap_p-quasi continuous representative $\tilde{u}$. A set $E \subseteq Q$ is said to be cap_p-quasi open if for every $\epsilon > 0$ there exists an open set U_ϵ such that $E \subseteq U_\epsilon \subseteq Q$ and $\text{cap}_p(U \backslash E, Q) \leq \epsilon$. It can be easily seen that, if u is a cap_p-quasi continuous function, then for every $k \in \mathbb{R}$ the sets $\{u > k\} = \{(t, x) \in Q : u(t, x) > k\}$ and $\{u < k\} = \{(t, x) \in Q : u(t, x) < k\}$ are cap_p-quasi open. The characteristic function of a cap_p-quasi open set can be approximated by a monotonic sequence of functions in the energy space W_2, as stated in the following lemma (see [31, Theorem 2.11, Lemma 2.20]).

Lemma 2.1 *For every cap_p-quasi open set $E \subseteq Q$, there exists an increasing sequence (w_n) of nonnegative functions in W which converges to χ_E cap_p-quasi everywhere in Q.*

2.2 Truncations

For every $k > 0$, we define the truncation function $T_k : \mathbb{R} \mapsto \mathbb{R}$ (see Fig. 1) by $T_k(s) :$ $\max(-k, \min(k, s))$. Let us consider the space $\mathcal{T}_0^{1,p}(Q)$ of all functions $u : Q \to \overline{\mathbb{R}}$ which are measurable and finite a.e. in Q, and such that $T_k(u)$ belongs to $L^p(0, T; W_0^{1,p}(\Omega))$ for every $k > 0$. It is easy to see that every function $u \in \mathcal{T}_0^{1,p}(Q)$ has a cap_p-quasi continuous



$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k & \text{if } s > k, \\ -k & \text{if } s < -k. \end{cases} \tag{2.1}$$

Fig. 1 The function $T_k(s)$

representative, that will always be identified with u. Moreover, for every $u \in \mathcal{T}_0^{1,p}(Q)$, there exists a measurable function $v : Q \mapsto \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{\{|u|\leq k\}}$ a.e. in Q, for every $k > 0$, and v is unique up to almost everywhere equivalence [4, Lemma 2.1]. Hence it is possible to define a generalized gradient ∇u of $u \in \mathcal{T}_0^{1,p}(Q)$, setting $\nabla u = v$. If $u \in L^1(0, T; W^{1,1}(\Omega))$, the gradient coincides with the usual one, while for $u \in L^1(0, T; L^1_{\text{loc}}(\Omega))$, it may differ from the distributional gradient of u.

2.3 Measures

Let us denote with $\mathcal{M}_b(Q)$ the set of all Radon measures on Q with bounded total variation and $C_b(Q)$ the space of all bounded, continuous functions on Q, so that $\int_Q \varphi \mathrm{d}\mu$ is defined for $\varphi \in C_b(Q)$ and $\mu \in \mathcal{M}_b(Q)$ where the positive, the negative and the total variation parts of a measure μ in $\mathcal{M}_b(Q)$ are denoted by μ^+, μ^- and $|\mu|$, respectively. We recall that for a measure μ in $\mathcal{M}_b(Q)$ and a Borel set $E \subseteq Q$, the measure $\mu \llcorner E$ is defined by $(\mu \llcorner E)(B) = \mu(E \cap B)$ for any Borel set $B \subseteq Q$. Analogous, we define $\mathcal{M}_0(Q)$ as the set of all measures μ in $\mathcal{M}_b(Q)$ with bounded variation over Q that does not charge the sets of zero parabolic p-capacity that is if $\mu \in \mathcal{M}_0(Q)$ then $\mu(B) = 0$ for every Borel set $B \subseteq Q$ such that $\text{cap}_p(B, Q) = 0$, while $\mathcal{M}_s(Q)$ will be the set of all measures μ in $\mathcal{M}_b(Q)$ for which there exists a Borel set $E \subset Q$, with $\text{cap}_p(E, Q) = 0$, such that $\mu = \mu \llcorner E$.

Remark 2.2 A measure $\mu_0 \in \mathcal{M}_0(Q)$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\mu_0(B) < \epsilon$ for every Borel set $B \subseteq Q$ with $\text{cap}_p(B, Q) < \delta$.

Proposition 2.3 *If $\mu_0 \in \mathcal{M}_0(Q)$ and if v is a function in W_2. Then v is measurable with respect to μ_0. If v further belongs to $L^\infty(Q)$, then v belongs to $L^\infty(Q, \mu_0)$ and $\|v\|_{L^\infty(Q,\mu_0)} = \|v\|_{L^\infty(Q)}$.*

Proof See [59, corollary 4.9]. □

Thanks to this result and to the dominated convergence theorem, we derive the following limit

Corollary 2.4 *If $\mu_0 \in \mathcal{M}_0(Q)$ and $(v_n) \in W_2 \cap L^\infty(Q)$, bounded in $L^\infty(Q)$, which converges to a function v cap_p-quasi everywhere. Then (v_n) converges to v μ_0-almost everywhere, and*

$$\lim_{n\to 0} \int_Q v_n \mathrm{d}\mu_0 = \int_Q v \mathrm{d}\mu_0.$$

Remark 2.5 Let (ρ_n) be a sequence of $L^1(Q)$-functions converging to ρ weakly in $L^1(Q)$. Let (Θ_n) be a sequence of functions belonging to $L^\infty(Q)$, bounded in the same space, and converging a.e. in Q to a function Θ. Then, according to the Egorov theorem, we have

$$\lim_{n\to 0} \int_Q \Theta_n \rho_n \mathrm{d}x\mathrm{d}t = \int_Q \Theta \rho \mathrm{d}x\mathrm{d}t,$$

this result will be often used in what follows.

On the other hand, if (v_n) is a sequence of functions in W_2 which converges weakly to v, then for every $\mu_0 \in \mathcal{M}_0(Q)$ the sequence (v_n) converges to v in μ_0-measure (see [16]). Before passing to the convergence results, let us state an interesting result about the decomposition of measures in $\mathcal{M}_0(Q)$.

Lemma 2.6 *If $\mu_0 \in \mathcal{M}_0(Q)$ then, there exist a decomposition (f, G, g) of μ_0 such that $\mu_0 = f - \text{div}(G) + g_t$ where $f \in L^1(Q)$, $G \in (L^{p'}(Q))^N$ and $g \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^2(\Omega)$. Moreover*

$$\int_Q v \mathrm{d}\mu_0 = \int_Q v f \,\mathrm{d}x\mathrm{d}t + \int_0^T G \cdot \nabla v \,\mathrm{d}x\mathrm{d}t + \int_0^T \langle v_t, g \rangle \mathrm{d}t$$

for every $v \in W \cap L^\infty(Q)$.

Proof See [31, Theorem 2.1]. □

The standard argument of Lemma 2.6 plays a key role in the proof of the following convergence result.

Lemma 2.7 *If $\mu_0 \in \mathcal{M}_0(Q)$ and (v_n) is a sequence of functions in $W_2 \cap L^\infty(Q)$ which converges to a function $v \in W_2 \cap L^\infty(Q)$ weakly in W_2 . Assume that $\|v_n\|_{L^\infty(Q)} \leq C$ for every $n \in \mathbb{N}$. Then (v_n) converges to v strongly in $L^2(Q, \mu_0)$.*

Let us state a general decomposition result of measures in $\mu \in \mathcal{M}_b(Q)$ and used several times in the next.

Theorem 2.8 *Let $\mu \in \mathcal{M}_b(Q)$, then there exists a unique decomposition (μ_0, μ_s) such that $\mu = \mu_0 + \mu_s$, $\mu_0 \in \mathcal{M}_0(Q)$ and $\mu_s \in \mathcal{M}_s(Q)$.*

Proof See [33, Lemma 2.1]. □

Recall that a sequence (μ_n) of measures in $\mathcal{M}_b(Q)$ converges to a measure μ in $\mathcal{M}_b(Q)$ in the narrow topology of measures if

$$\lim_{n \to +\infty} \int_Q \varphi \mathrm{d}\mu_n = \int_Q \varphi \mathrm{d}\mu \tag{2.2}$$

for every $\varphi \in C_b(Q)$. If (2.2) holds for all continuous functions φ with compact support in Q (i.e. $\varphi \in C_c(Q)$), then it coincides with the usual weak-* convergence in $\mathcal{M}_b(Q)$.

Remark 2.9 Recall also that a sequence of nonnegative measures (μ_n) converges to μ in the narrow topology if and only if it converges to μ in the weak-* topology, and the masses $(\mu_n(Q))$ converges to $\mu(Q)$. Then, for nonnegative measures, the narrow convergence is equivalent to the convergence in (2.2) for every $\varphi \in C^\infty(\overline{Q})$.

2.4 The operator

A function $a : Q \times \mathbb{R}^N \to \mathbb{R}^N$ is said to be a Carathéodory function if $a(\cdot, \cdot, \zeta)$ is measurable on Q for every $\zeta \in \mathbb{R}^N$ and $a(t, x, \cdot)$ is continuous on $\mathbb{R}^N$ for almost every (t, x) in Q. Fixed two constants $c_0, c_1 > 0$ and a nonnegative function $b_0 \in L^s(Q)$, $s > \frac{N}{p}$, we say that $a : Q \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies hypothesis $H(c_0, c_1, b_0)$ if for almost every $(t, x) \in Q$ the following assumptions hold

$$a(t, x, \zeta) \cdot \zeta \geq c_0 |\zeta|^p \quad \forall \zeta \in \mathbb{R}^N, \tag{2.3}$$

$$|a(t, x, \zeta)| \leq b_0(t, x) + c_1 |\zeta|^{p-1} \quad \forall \zeta \in \mathbb{R}^N, \tag{2.4}$$

$$(a(t, s, \zeta) - a(t, x, \zeta')) \cdot (\zeta - \zeta') > 0 \quad \forall \zeta, \zeta' \in \mathbb{R}^N, \ \zeta \neq \zeta'. \tag{2.5}$$

Remark 2.10 We observe that if $a(t, x, \zeta)$ is a Carathéodory function satisfying (2.3), then $a(t, x, 0) = 0$ for a.e. (t, x) in Q.

Thanks to the assumptions (2.3)–(2.5), the differential operator $u \mapsto -\mathrm{div}(a(t,x,\nabla u))$ turns out to be a coercive and monotone operator acting from the space $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual space $L^{p'}(0,T;W^{-1,p'}(\Omega))$. It is well known, by the standard theory of monotone operators (see for instance [45,46]), if $F \in L^{p'}(Q)$, then there exists a unique variational solution v of the problem

$$\begin{cases} v_t - \mathrm{div}(a(t,x,\nabla v)) = f \text{ in } Q = (0,T)\times\Omega, \\ v(0,x) = u_0 \text{ in } \Omega, \quad v(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{2.6}$$

in the sense that v belongs to $W \cap C(0,T;L^2(\Omega))$ and

$$-\int_\Omega u_0\varphi(0)\mathrm{d}x - \int_0^T \langle \varphi_t, v\rangle \mathrm{d}t + \int_Q a(t,x,\nabla v)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t = \int_0^T \langle F,\varphi\rangle_{W^{-1,p'},W_0^{1,p}}\mathrm{d}t, \tag{2.7}$$

for all $\varphi \in W$ such that $\varphi(T) = 0$, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

Remark 2.11 We also recall that, since $\partial\Omega$ is smooth and a satisfies assumptions $H(c_0,c_1,b_0)$ with $b_0 \in L^s(Q)$ with $s > \frac{N}{p}$, for every $F = -\mathrm{div}(G_0)$, $G_0 \in (L^\infty(Q))^N$ the solution v of (2.6) is a Hölder continuous function. Moreover, there exists $C > 0$, depending only on p, c_0, c_1, b_0 and $\mathcal{L}^N(Q)$, such that $\|v\|_{C^{0,\alpha}(Q)} \le C\|G_0\|_{(L^\infty(Q))^N}$ (see [35,36,45]).

Let us define a $\mathcal{M}_0$-version of Minty's Lemma (an elliptic version of this Lemma can be found in [11, Lemma 7] and different type of Minty's Lemma are established in [13] for parabolic problems with p-growth). Our Minty type Lemma can be proved as the elliptic case and reads as follows

Lemma 2.12 *Let a be Carathéodory function satisfying $H(c_0,c_1,b_0)$, let γ_0 be measure in $\mathcal{M}_0(Q)$ and v be a function in $\mathcal{T}_0^{1,p}(Q)$. Then v is such that*

$$\begin{aligned} &-\int_0^T \langle T_k(v)_t, T_k(v)-w\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\mathrm{d}t + \int_Q a(t,x,\nabla T_k(v))\cdot\nabla(T_k(v)-w)\mathrm{d}x\mathrm{d}t \\ &= \int_Q (T_k(v)-w)d\gamma_0 \end{aligned} \tag{2.8}$$

for every $w \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q) \cap C(0,T;L^1(\Omega))$ if and only if v satisfies

$$-\int_0^T \langle w_t, T_k(v)-w\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\mathrm{d}t + \int_Q a(t,x,w)\cdot\nabla(T_k(v)-w)\mathrm{d}x\mathrm{d}t \le \int_Q (T_k(v)-w)d\gamma_0 \tag{2.9}$$

for every $w \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q) \cap C(0,T;L^1(\Omega))$ with $w_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$.

In order to better specify the definition of G-convergence (also called Homogenization, Fig. 2), recall that the present ideas of this paper differs from the several papers cited above. In fact, in other works, especially parabolic problems, the G-convergence is related to finite energy solutions. But here, this concept fits with the renormalized framework, it deals with the homogenization of the renormalized formulation in the case where the sequence of momenta (a_n) considered, bounded, in $(L^{p'}(Q))^N$ and satisfies (2.3)–(2.5) for some fixed $\alpha, \beta > 0$. It consists in proving that if $u_0^n \in L^1(\Omega)$, (a_n) G-converges to a_0 (see the definition of this convergence in Definition 2.13 bellow) and that the sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ tightly

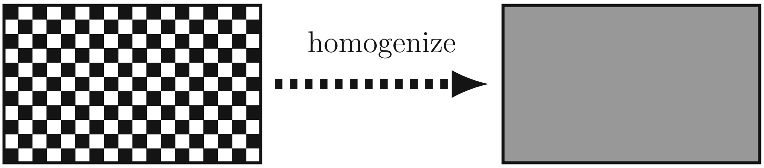


Fig. 2 Homogenization theory

converges to μ (i.e., in the narrow topology of measures), a subsequence of the sequence of the solutions of the renormalized equation relative to a_n converges to a solution of the renormalized equation relative to a_0, this is in order to see that the notion of renormalized solution is thus robust in the sense that it is stable under the G-convergence of the momenta, which is the "weakest possible" convergence for the corresponding operators. It is also worth noticing that the proof that we will present in Sect. 6 to prove this homogenization result of renormalized solutions is closed to the proof used in [48] (see also [52]) to obtain the asymptotic behaviour of renormalized solutions and to illustrate the robustness of the method in elliptic case. The robustness of both the notion of renormalized solution and of the method of proof (using truncate functions) is emphasized by the stability result of renormalized solutions with respect to variations of the right-hand side which given in the following classical result: Consider a sequence of weak solutions u_n relative to some operators (a_n) and to a sequence of right-hand sides F_n which converges weakly in $L^{p'}(0, T; W^{-1,p'}(\Omega))$ to F (see 2.10) and under a special assumptions of equi-integrability of F_n, a subsequence of the sequences u_n is proved to converge weakly in $L^p(0, T; W_0^{1,p}(\Omega))$ to a solution of the weak equation relative to the right-hand side F. The above main result was generalized by Malusa and Orsina [48] under Leray–Lions and G-convergence assumptions in the case of singular measures, this argument, however, quite simply extends to measures converging weak-* was observed in [47] in order to prove the asymptotic behavior of Stampacchia solutions and under weak-* convergence of the data (fixed measures) in the theory of "cheap" control. Moreover, the asymptotic behaviour (G-convergence, $H-$convergence, numerical approximation) of the solutions of Dirichlet problems in $L^1(\Omega)$ can be found in [6,7,12,17,24, 53], All these homogenization elliptic problems are considered Dirichlet setting, for which the solutions is zero in the boundary. In a different setting, let us mention [15] where renormalized solutions and homogenization are mixed and the more recent work [34] where the authors study, with the help of renormalized solutions, the homogenization of a linear elliptic problem with L^1-data, Neumann boundary conditions and highly oscillating boundary. In addition, Ben Cheikh Ali in [3] studied the homogenization of a renormalized solution in perforated domains with a Neumann boundary condition on the boundary of the holes and the authors in [30] considered the homogenization of a class of quasilinear elliptic problems in a periodically perforated domain with L^1-data and nonlinear Robin conditions on the boundary of the holes. Observe that variational solutions corresponding to the data $f \in L^{p'}(Q)$ are renormalized solutions corresponding to the measure $d\mu = f\,dx\,dt$ and the narrowly convergence implies the $*$-weak convergence, that is, if μ_n converges in the narrow topology of measures to μ then μ_n converges to μ*-weakly in $\mathcal{M}_b(Q)$ and $\mu_n(Q)$ converges to $\mu(Q)$. We now recall the definition of G-convergence related to parabolic operators.

Definition 2.13 Let $(a_n)_{n\in\mathbb{N}}$ and a_0 be Carathéodory functions satisfying $H(c_0, c_1, b_0)$, and let $A_n(u) = -\text{div}(a_n(t, x, \nabla u))$, $n \in \mathbb{N}$, and $A_0(u) = -\text{div}(a_0(t, x, \nabla u))$ be the corresponding operators between the spaces $L^p(0, T; W_0^{1,p}(\Omega))$ and $L^{p'}(0, T; W^{-1,p'}(\Omega))$. We say that A_n G-converges to A_0 if for every $F \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ and for every $z \in L^p(0, T; W^{1,p}(\Omega))$ the (variational) solutions v_n of problems

$$\begin{cases} (v_n)_t - \operatorname{div}(a_n(t,x,\nabla v_n)) = F_n \text{ in } Q = (0,T)\times\Omega, \\ v_n(0,x) = u_0^n \text{ in } \Omega, \quad v_n(t,x) = z \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{2.10}$$

satisfy

$$\begin{cases} v_n \rightharpoonup v \text{ weakly in } L^p(0,T;W^{1,p}(\Omega)); \\ a_n(t,x,\nabla v_n) \rightharpoonup a_0(t,x,\nabla v) \text{ weakly in } (L^{p'}(Q))^N; \end{cases}$$

where v is the (variational) solution of problem

$$\begin{cases} v_t - \operatorname{div}(a_0(t,x,\nabla v)) = F \text{ in } Q = (0,T)\times\Omega, \\ v(0,x) = u_0 \text{ in } \Omega, \quad v(t,x) = z \text{ on } (0,T)\times\partial\Omega. \end{cases} \tag{2.11}$$

2.5 Renormalized solutions

The main idea of renormalized solutions consists on multiplying the pointwise equation by a test function in dependence of u (any smooth function with compact support). Let $W^{2,\infty}(\mathbb{R})$ is the set of all Lipschitz continuous functions $h:\mathbb{R}\to\mathbb{R}$ whose derivative h' has compact support, i.e., every function $h\in W^{2,\infty}(\mathbb{R})$ is constant outside the support of its derivative, so that we can define $h(0)=0$, $u_g = u - g$ where g_t is the time derivative part of μ_0 and $\tilde{\mu}_0 = \mu - g_t - \mu_s = f - \operatorname{div}(G)$.

Definition 2.14 Let a be Carathéodory function satisfying $H(c_0, c_1, b_0)$, and let μ be a measure in $\mathcal{M}_b(Q)$, decomposed as $\mu = \mu_0 + \mu_s$, $\mu_0 \in \mathcal{M}_0(Q)$, $\mu_s \in \mathcal{M}_s(Q)$. A function u is a renormalized solution of problem

$$\begin{cases} u_t - \operatorname{div}(a(t,x,\nabla u)) = \mu \text{ in } Q = (0,T)\times\Omega, \\ u(0,x) = u_0 \text{ in } \Omega, \quad u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{2.12}$$

if the following conditions hold

(a) $u \in \mathcal{T}_0^{1,p}(Q)$;
(b) $|\nabla u|^{p-1}$ belongs to $L^q(Q)$ for every $q < p - \frac{N}{N+1}$;
(c) For every $h \in W^{2,\infty}(\mathbb{R})$ one has

$$\begin{cases} -\displaystyle\int_\Omega h(u_0)\varphi(0)\mathrm{d}x - \int_0^T \langle \varphi_t, h(u_g)\rangle \mathrm{d}t + \int_Q h'(u_g)a(t,x,\nabla u)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t \\ +\displaystyle\int_Q h''(u_g)a(t,x,\nabla u)\cdot\nabla u_g \varphi \mathrm{d}x\mathrm{d}t = \int_Q h'(u_g)\varphi d\tilde{\mu}_0, \end{cases} \tag{2.13}$$

for every $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, with $\varphi(T,x)=0$, such that $h'(u_g)\varphi \in L^p(0,T;W_0^{1,p}(\Omega))$. Moreover, for every $\psi \in C(\overline{Q})$ we have

$$\begin{cases} \displaystyle\lim_{n\to+\infty}\frac{1}{n}\int_{\{n\le v<2n\}} a(t,x,\nabla u)\cdot\nabla u_g \psi \mathrm{d}x\mathrm{d}t = \int_Q \psi \mathrm{d}\mu_s^+, \\ \displaystyle\lim_{n\to+\infty}\frac{1}{n}\int_{\{-2n<v\le n\}} a(t,x,\nabla u)\cdot\nabla u_g \psi \mathrm{d}x\mathrm{d}t = \int_Q \psi \mathrm{d}\mu_s^-, \end{cases}$$

where μ_s^+ and μ_s^- are respectively the positive and the negative parts of the singular part μ_s.

Let us point out that the existence of a renormalized solution of (2.12) is proved in [61] (see also [63] for another proof), the uniqueness of the solution for datum $\mu = \mu_0 \in \mathcal{M}_0(Q)$ is proved in [31, Sect 3] (see also [61,67]), while the uniqueness of the renormalized solution for general $\mu \in \mathcal{M}_b(Q)$ and initial $u_0 \in L^1(\Omega)$ is still open. The following equivalence is proved in [63, Sect 4] (using an analogous definition).

Theorem 2.15 *Let u be a function satisfying $(a)-(b)$ of Definition 2.14. Then u is a renormalized solution of problem (2.12) if and only if for every $k > 0$ there exist a sequence of non-negative measures $(\lambda_k) \in \mathcal{M}_b(Q)$ such that*

(i) $\lambda_k \overrightarrow{k \to \infty} \mu_s$ *in the narrow topology of measures;*
(ii) *the truncations $T_k(u)$ satisfy*

$$-\int_Q T_k(u)v_t \mathrm{d}x\mathrm{d}t+\int_Q a(t,x,\nabla T_k(u))\cdot\nabla v \mathrm{d}x\mathrm{d}t = \int_Q \tilde{v}\mathrm{d}\mu_0+\int_Q \tilde{v}d\lambda_k+\int_\Omega T_k(u_0)v(0)\mathrm{d}x \tag{2.14}$$

for every $v \in W \cap L^\infty(Q)$ such that $v(T) = 0$ (with $\tilde{v}$ being the unique cap$_p$-quasi continuous representative of v).

3 Statements of the main results

Let $(a_n)_{n\in\mathbb{N}}$ and a_0 be Carathéodory functions satisfying $H(c_0, c_1, b_0)$, and let $A_n(u) = -\mathrm{div}\,(a_n(t,x,\nabla u_n))$, $A_0(u) = \mathrm{div}(a_0(t,x,\nabla u))$ be the corresponding operators between $L^p(0,T; W_0^{1,p}(\Omega))$ and $L^{p'}(0,T; W^{-1,p'}(\Omega))$.

Theorem 3.1 *Let $u_0^n \in L^1(\Omega)$ and assume that the sequence of operators $(A_n)_{n\in\mathbb{N}}$ G-converge to A_0 and that the sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ converges to μ in the sense of (2.2). If u_n is a sequence of renormlized solutions of problem*

$$\begin{cases} (u_n)_t - \mathrm{div}(a_n(t,x,\nabla u_n)) = \mu_n \text{ in } Q = (0,T)\times\Omega, \\ u_n(0,x) = u_0^n \text{ in } \Omega, \quad u_n(t,x) = 0 \text{ on } (0,T)\times\partial\Omega. \end{cases} \tag{3.1}$$

Then, up to subsequences, (u_n) converges a.e. in Q to a function $u \in \mathcal{T}_0^{1,p}(Q)$ renormalized solution of the problem

$$\begin{cases} u_t - \mathrm{div}(a_0(t,x,\nabla u)) = \mu \text{ in } Q = (0,T)\times\Omega, \\ u(0,x) = u_0 \text{ in } \Omega, \quad u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega. \end{cases} \tag{3.2}$$

Moreover, we have, for every $k > 0$, the truncation functions $T_k(u_n)$ satisfy

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } L^p(0,T; W_0^{1,p}(\Omega)); \tag{3.3}$$

$$a_n(t,x,\nabla T_k(u_n)) \rightharpoonup a_0(t,x,\nabla T_k(u)) \text{ weakly in } (L^{p'}(Q))^N. \tag{3.4}$$

It suffices to use the definition of renormalized solution of u_n to get

$$-\int_\Omega h(u_0^n)\varphi(0)\mathrm{d}x - \int_0^T \langle \varphi_t, h(u_{g,n})\rangle \mathrm{d}t + \int_Q h(u_{g,n})a(t,x,\nabla u_n)\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t$$
$$+\int_Q h''(u_{g,n})a(t,x,\nabla u_n)\cdot\nabla u_{g,n}\varphi \mathrm{d}x\mathrm{d}t = \int_Q h'(u_{g,n})\varphi d\tilde{\mu}_0^n \tag{3.5}$$

for every $h \in W^{2,\infty}(\mathbb{R})$ and for every $\varphi \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, $\varphi_t \in L^{p'}(0,T; W^{-1,p'}(\Omega))$ with $\varphi(T,x) = 0$ such that $h'(u_{g,n})\varphi \in L^p(0,T; W_0^{1,p}(\Omega))$. In addition, using Theorem 2.15, for every $n \in \mathbb{N}$ and $k > 0$ there exist a sequence of nonnegative measures $(\lambda_{n,k}) \in \mathcal{M}_b(Q)$ satisfying

(i) $\lambda_{n,k} \underset{\substack{n\to+\infty\\ k\to+\infty}}{\longrightarrow} \mu_s$ in the narrow topology of measures;

(ii) the truncations $T_k(u_n)$ satisfy

$$-\int_Q T_k(u_n)v_t \mathrm{d}x\mathrm{d}t + \int_Q a_n(t,x,\nabla T_k(u_n))\cdot\nabla v \mathrm{d}x\mathrm{d}t = \int_Q \tilde{v}\mathrm{d}\mu_0^n + \int_Q \tilde{v}d\lambda_{n,k} + \int_\Omega T_k(u_0^n)v(0)\mathrm{d}x \tag{3.6}$$

for every $v \in W \cap L^\infty(Q)$ such that $v(T) = 0$ (with $\tilde{v}$ being the unique cap_p-quasi continuous representative of v).

Remark 3.2 If we prove that (u_n) converges to u a.e. in Q and that u is a renormalized solution to (3.2). Then using Theorem 2.15 and for every $k > 0$, the truncation functions are variational solutions of problems

$$\begin{cases} -(T_k(u_n)_t - \mathrm{div}(a_n(t,x,\nabla T_k(u_n))) = \mu_{n,k} \text{ in } Q = (0,T)\times\Omega, \\ T_k(u_n)(0,x) = T_k(u_0^n) \text{ in } \Omega, \quad T_k(u_n)(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases}$$

and

$$\begin{cases} -(T_k(u)_t - \mathrm{div}(a_0(t,x,\nabla T_k(u))) = \mu_k \text{ in } Q = (0,T)\times\Omega, \\ T_k(u)(0,x) = T_k(u_0) \text{ in } \Omega, \quad T_k(u)(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases}$$

for suitable measures $\mu_{n,k}, \mu_k \in \mathcal{M}_0(Q)$. Let us finally remark that Eqs. (3.3)–(3.4) are not consequences of the G-convergence of the operators because of the varying right-hand sides $\mu_{n,k}$ converging in the weak topology to μ_k.

4 Some a priori estimates and convergence results

Let us choose $h(u_n) = T_k(u_n)$ with $\varphi \equiv 1$ in (3.5). Then for every $n \in \mathbb{N}$, we have

$$\begin{aligned}\int_\Omega \Theta_k(u_n)(t)\mathrm{d}x + \int_Q a(t,x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\mathrm{d}x\mathrm{d}t &= \int_Q T_k(u_n)\mathrm{d}\mu_0^n + \int_\Omega \Theta_k(u_0^n)\mathrm{d}x \\ &\le k|\mu_n|(Q) + \|u_0^n\|_{L^1(\Omega)},\end{aligned} \tag{4.1}$$

so that using assumption (2.3) and [1, Prop. 5.2], we get

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \le C \text{ and } \int_Q |\nabla T_k(u_n)|^p \mathrm{d}x\mathrm{d}t \le C_0^{-1}|\mu_n|(Q)k + C. \tag{4.2}$$

Now, by using [29, Proposition 3.1] and the estimate (4.2), there exists $C > 0$, independent of k and n, such that

$$\mathcal{L}(\{|u_n| > k\}) \le Ck^{-(p-\frac{N-p}{N})}, \quad \mathcal{L}(\{|\nabla u_n| > k\}) \le Ck^{-(p-\frac{N}{N+1})}, \tag{4.3}$$

where $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure. Thanks to the second inequality of (4.3)

$$\int_Q |\nabla u_n|^{q(p-1)}\mathrm{d}x\mathrm{d}t \le \overline{C}, \quad \forall q < \frac{Np - N + p}{(N+1)(p-1)} \tag{4.4}$$

where $\overline{C} > 0$ depends on q and not on n (see [59, Sect. 2.2]).

As a consequence of the above estimates we obtain the following theorem.

Theorem 4.1 *Let u_n be a sequence of renormalized solutions of problem (3.1). Then there exist a measurable function $u : Q \to \mathbb{R}$ finite a.e. in Q such that (up to subsequences)*

(i) *u_n converges to u a.e. in Q, $u \in \mathcal{T}_0^{1,p}(Q)$ and $|\nabla u|^{p-1} \in L^q(Q)$ for every $1 \leq q < p - \frac{N}{N-1}$;*

(ii) *$T_k(u_n)$ converges to $T_k(u)$ weakly in $L^p(0,T; W_0^{1,p}(\Omega))$ and there exists $C = C(q) > 0$ such that*

$$\int_Q |\nabla u|^{q(p-1)} \mathrm{d}x\mathrm{d}t \leq C, \quad \forall q < \frac{Np - N + p}{(N+1)(p-1)}; \tag{4.5}$$

(iii) *$T_k(u_n)$ converges to $T_k(u)$ strongly in $L^2(Q, \mu_0)$ and μ_0-a.e. in Q;*

(iv) *$\exists \sigma \in (L^q(Q))^N$ for every $q < p - \frac{N}{N+1}$, such that $\sigma_k = \sigma \chi_{\{|u|<k\}} \in (L^{p'}(Q))^N$ for a.e. $k > 0$, and $(a_n(t,x,\nabla u_n))_{n\in\mathbb{N}}$ converges to σ weakly in $(L^q(Q))^N$ for every $q < p - \frac{N}{N+1}$, while $(a_n(t,x,\nabla T_k(u_n)))_{n\in\mathbb{N}}$ converges to σ_n weakly in $(L^{p'}(Q))^N$.*

Proof The convergence results (i)–(ii) are obtained through similar arguments of [1,61] under the same assumptions but for fixed operators. To see that (iii) is true, it is enough to use (ii) and Lemma 2.7. Now, by setting (u_n) and u be such that (i)–(iii) hold, from (2.4) and (4.4) we have $a_n(t,x,\nabla u_n)$ is bounded in $(L^q(Q))^N$ for every $q < p - \frac{N}{N+1}$. Then (up to subsequences) there exist a function $\sigma \in (L^q(Q))^N$ such that $(a_n(t,x,\nabla u_n))$ converges to σ weakly in $(L^q(Q))^N$. Note that (2.4) and (4.2) ensure that there exist a subsequence $(a_n(t,x,\nabla T_k(u_n)))$ (depending on k) and a function σ_k such that $(a_n(t,x,\nabla T_k(u_n))$ converges to σ_k weakly in $(L^{p'}(Q))^N$. Thus, since $a_n(t,x,0) = 0$ for every $n \in \mathbb{N}$, we have $a_n(t,x,\nabla T_k(u_n)) = a_n(t,x,\nabla u_n)\chi_{\{|u_n|<k\}}$, and, by (i), for almost every $k > 0$ the functions $\chi_{\{|u_n|<k\}}$ converges to $\chi_{\{|u|<k\}}$ a.e. in Q. It is easy to see that $\sigma_k = \sigma\chi_{\{|u|<k\}}$ by Remark 2.5. Finally, the sequence of subsequences $(a_n(t,x,\nabla T_k(u_n)))$ converges to σ_k weakly in $(L^{p'}(Q))^N$ for every $k > 0$. □

Remark 4.2 Observe that the function $\frac{T_k(u_n)}{k} \in L^p(0,T; W_0^{1,p}(\Omega))$ and satisfies $\frac{T_k(u_n)}{k} = 1$ a.e. in $\{|u_n| > k\}$, then by using the result of [64, Theorem 1.2] and the estimate (4.2) we get

$$\begin{aligned}
\mathrm{cap}_p(\{|u_n| > k\}, Q) &\leq \|\frac{T_k(u_n)}{k}\|_W \\
&\leq C \max\left\{ \left(k\left(\|\mu\|_{\mathcal{M}(Q)} + \|u_0\|_{L^1(Q)}\right)\right)^{-\frac{1}{p}}, \left(k\left(\|\mu\|_{\mathcal{M}(Q)} + \|u_0\|_{L^1(\Omega)}\right)\right)^{-\frac{1}{p'}} \right\} \\
&\leq C\left(\|\mu\|_{\mathcal{M}(Q)}, \|u_0\|_{L^1(Q)}, p\right) \max\left\{ \frac{1}{k^{\frac{1}{p}}}, \frac{1}{k^{\frac{1}{p'}}} \right\}.
\end{aligned}$$

5 Some a priori estimates for measures

As we have seen, we provide a different, and in some sense more natural approach, to deal with nonlinear parabolic problems with measures using G-convergence theory. Before passing to the proof of our main result, let us state some interesting a priori estimates for the measures $\lambda_{n,k}$ using the convergence results of Theorem 4.1.

Lemma 5.1 *For every $\varphi \in C^1(\overline{Q})$ and for every $n \in \mathbb{N}$, there exists $\omega = \omega(n,k)$ satisfying*

$$\left| \int_Q \varphi d\lambda_{n,k} - \int_Q \varphi \mathrm{d}\mu_s \right| \leq \omega. \tag{5.1}$$

Proof Let $k > \delta > 0$, and let $S_{\delta,k}, h_{\delta,k} : \mathbb{R} \to \mathbb{R}$ be two Lipschitz functions defined by (see Fig. 3)

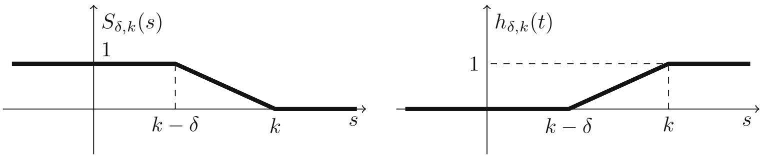


Fig. 3 The functions $S_{\delta,k}(s)$ and $h_{\delta,k}(s)$

$$S_{\delta,k}(s)=\begin{cases}1 & \text{if } s\le k-\delta,\\ \frac{1}{\delta}(k-s) & \text{if } k-\delta<s\le k,\\ 0 & \text{if } s>k,\end{cases}$$

$$h_{\delta,k}(s)=1-S_{\delta,k}(s)=\begin{cases}0 & \text{if } s\le k-\delta,\\ \frac{1}{\delta}(s-k+\delta) & \text{if } k-\delta<s\le k,\\ 1 & \text{if } s>k.\end{cases} \tag{5.2}$$

Since $h_{\delta,k}(u_n)\varphi$ is an admissible test function both in (3.5) and (3.6) for every $\varphi\in C^1(\overline{Q})$ with $\mu_0=f-\operatorname{div}(G)+g_t$, so that using (3.5) we get

$$\begin{aligned}&-\int_\Omega H_{\delta,k}(u_n)\varphi_t\mathrm{d}x\mathrm{d}t+\frac{1}{\delta}\int_{\{k-\delta<u_n<k\}}a(t,x,\nabla u_n)\cdot\nabla u_n\varphi\mathrm{d}x\mathrm{d}t\\ &+\int_Q a_n(t,x,\nabla u_n)\cdot\nabla\varphi h_{\delta,k}(u_n)\mathrm{d}x\mathrm{d}t\\ &=\int_Q h_{\delta,k}(u_n)\varphi\mathrm{d}\mu_0^n+\int_Q\varphi\mathrm{d}\mu_s^n+\int_\Omega H_{\delta,k}(u_0^n)\varphi(0)\mathrm{d}x\end{aligned} \tag{5.3}$$

where $H_{\delta,k}(s)=\int_0^s h_{\delta,k}(r)dr$. On the other hand (3.6) implies

$$\begin{aligned}&-\int_Q H_{\delta,k}(T_k(u_n))\varphi_t\mathrm{d}x\mathrm{d}t+\frac{1}{\delta}\int_{\{k-\delta<u_n<k\}}a(t,x,\nabla T_k(u_n))\cdot\nabla u_n\varphi\mathrm{d}x\mathrm{d}t\\ &+\int_{\{|u_n|<k\}}a_n(t,x,\nabla T_k(u_n))\cdot\nabla\varphi h_{\delta,k}(u_n)\mathrm{d}x\mathrm{d}t=\int_{\{|u_n|<k\}}h_{\delta,k}(u_n)\varphi\mathrm{d}\mu_0^n+\int_Q\varphi d\lambda_{n,k}\\ &+\int_\Omega H_{\delta,k}(T_k(u_0^n))\varphi(0)\mathrm{d}x.\end{aligned} \tag{5.4}$$

It is easy to prove that the first and last terms are in fact equivalent to $\omega(n,k)$ if we use the convergence in $L^1(Q)$ of u_n, $|a_n(t,x,\nabla u_n)|$, $|a_n(t,x,\nabla T_k(u_n))|$ and the properties of φ

$$\int_Q H_{\delta,k}(u_n(t,x))\varphi_t\mathrm{d}x\mathrm{d}t=\omega(n,k),\ \int_Q H_{\delta,k}(T_k(u_n)(t,x))\varphi_t\mathrm{d}x\mathrm{d}t=\omega(n,k).$$

Which yields

$$\int_Q\varphi d\lambda_{n,k}-\int_Q\varphi\mathrm{d}\mu_s=\int_{\{u_n\ge k\}}h_{\delta,k}(u_n)\varphi\mathrm{d}\mu_0^n-\int_{\{u_n\ge k\}}a_n(t,x,\nabla u_n)\cdot\nabla\varphi h_{\delta,k}(u_n)\mathrm{d}x\mathrm{d}t, \tag{5.5}$$

and hence, for $q<p-\frac{N}{N+1}$,

$$\begin{aligned}\left|\int_Q\varphi d\lambda_{n,k}-\int_Q\varphi\mathrm{d}\mu_s\right|&\le\|\varphi\|_{L^\infty(Q)}|\mu_0|(\{u_n\ge k\})\\ &+\|a_n(t,x,\nabla u_n)\|_{(L^q(Q))^N}\|\nabla\varphi\|_{(L^{q'}(\{u_n\ge k\}))^N},\end{aligned} \tag{5.6}$$

for every $\varphi \in C^1(\overline{Q})$. Similarly we get

$$\left|\int_Q \varphi d\lambda_{n,k} - \int_Q \varphi \mathrm{d}\mu_s\right| \leq \|\varphi\|_{L^\infty(Q)} |\mu_0|(\{u_n \leq -k\})$$
$$+\|a_n(t,x,\nabla u_n)\|_{(L^q(Q))^N} \|\nabla\varphi\|_{(L^{q'}(\{u_n \leq -k\}))^N}. \tag{5.7}$$

which is a consequence of (4.3), the absolute continuity of the Lebesgue measure (concerning the term $\|\nabla\varphi\|_{(L^{q'}(\{|u_n|\geq k\}))^N}$) and Remarks 2.2 and 4.2 (for the term $|\mu_0|(\{|u_n| \geq k\})$). Thus if we consider test functions $T_k(u_n)$ in (3.6), we have from (4.1)

$$k\lambda_{n,k}(Q) = \int_\Omega \frac{[T_k(u_n)(T)]^2}{2}\mathrm{d}x - \int_\Omega \frac{[T_k(u_0^n)]^2}{2}\mathrm{d}x + \int_Q a_n(t,x,\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\mathrm{d}x\mathrm{d}t$$
$$- \int_{\{|u_n|<k\}} T_k(u_n)\mathrm{d}\mu_0 \leq k\left[\|\mu\|_{\mathcal{M}_b(Q)} + \|\mu_0\|_{\mathcal{M}_0(Q)} + \|u_0\|_{L^1(\Omega)}\right]. \tag{5.8}$$

Then there exist a sequence of nonnegative measures $\lambda_k \in \mathcal{M}_b(Q)$ such that (up to a subsequence) the sequences $(\lambda_{n,k})$ converges to λ_k in the weak-* topology of $\mathcal{M}_b(Q)$ as n goes to $+\infty$. Moreover, a passage to the limit on n in (5.1) gives

$$\left|\int_Q \varphi d\lambda_k - \int_Q \varphi \mathrm{d}\mu_s\right| \leq \omega(k) \tag{5.9}$$

for every $\varphi \in C_c^\infty(Q)$, that is the sequence (λ_k) converges to μ_s in the weak-* topology of $\mathcal{M}_b(Q)$ as k goes to $+\infty$. □

The reconstruction property of the sequence (λ_k) is essentially played by a technical Lemma.

Lemma 5.2 *Let u and σ be the functions introduced in Theorem 4.1, and let $\lambda_k \in \mathcal{M}_b(Q)$ be the measures introduced above. The λ_k belongs to $\mathcal{M}_0(Q)$, and*

$$-\int_Q T_k(u)v_t\mathrm{d}x\mathrm{d}t + \int_{\{|u|<k\}} \sigma \cdot \nabla v\mathrm{d}x\mathrm{d}t = \int_{\{|u|<k\}} v\mathrm{d}\mu_0 + \int_Q v d\lambda_k + \int_\Omega T_k(u_0)v(0)\mathrm{d}x \tag{5.10}$$

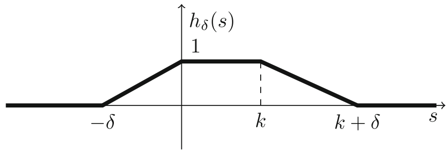
for every $v \in W \cap L^\infty(Q)$ such that $v(T) = 0$ and for a.e. $k > 0$. Moreover there exists a nonnegative measure $\gamma \in \mathcal{M}_b(Q)$ independent of k such that $\lambda_k - \gamma$ belong to $\mathcal{M}_0(Q)$, and

$$-\int_Q T_k(u)v_t\mathrm{d}x\mathrm{d}t + \int_{\{|u|<k\}} \sigma \cdot \nabla v\mathrm{d}x\mathrm{d}t = \int_{\{|u|<k\}} v\mathrm{d}\mu_0 + \int_Q v d(|\lambda_k - \gamma|) + \int_\Omega T_k(u_0)v(0)\mathrm{d}x \tag{5.11}$$

for every $v \in W \cap L^\infty(Q)$ such that $v(T) = 0$ and for a.e. $k > 0$.

Proof For every $k > 0$, by Theorem 4.1, $(a_n(t,x,\nabla T_k(u_n)))$ converges to $\sigma\chi_{\{|u|<k\}}$ weakly in $(L^{p'}(Q))^N$, $(\chi_{\{|u_n|<k\}})$ converges to $\chi_{\{|u|<k\}}$ μ_0-a.e. in Q, and by the fact that λ_k is the weak-* limit of $\lambda_{n,k}$, by passing to the limit in (3.6) as $n \to +\infty$ for every test function $\varphi \in C_c^\infty(Q)$, we get

$$-\int_Q T_k(u)v_t\mathrm{d}x\mathrm{d}t + \int_{\{|u|<k\}} \sigma \cdot \nabla v\mathrm{d}x\mathrm{d}t = \int_{\{|u|<k\}} v\mathrm{d}\mu_0 + \int_Q v d\lambda_k + \int_\Omega T_k(u_0)v(0)\mathrm{d}x. \tag{5.12}$$



$$h_\delta(s) = \begin{cases} 0 & \text{if } s \le -\delta \text{ or } s > k+\delta, \\ \frac{1}{\delta}(s+\delta) & \text{if } -\delta \le s < 0, \\ \frac{1}{\delta}(k+\delta-s) & \text{if } k < s \le k+\delta, \\ 1 & \text{if } 0 \le s \le k. \end{cases} \tag{5.13}$$

Fig. 4 The function $h_\delta(s)$

Using the fact that $\sigma\chi_{\{|u|<k\}}$ belongs to $(L^{p'}(Q))^N$, the measure λ_k belongs to $\mathcal{M}_0(Q)$ and (5.12) can be extended to every test function $\varphi \in W \cap L^\infty(Q)$ such that $\varphi(T) = 0$ by using a standard density argument. Now suppose that the Lebesgue measure of the set $\{u(t,x) = 0\}$ is zero (if it's not, we can replace $u = 0$ with $u = a$ (a is a nonnegative value) where $\mathcal{L}^N(\{u = a\})$), so that for $\delta > 0$ and for the Lipschitz function $h_\delta(s) : \mathbb{R} \to \mathbb{R}$ defined by (see Fig. 4).

If we choose $h_\delta(u_n)\varphi$, with $\varphi \in C_c^\infty(Q)$, as test function in (3.6) for $k > \delta$, we have

$$\begin{aligned} &\int_Q H_\delta(T_k(u_n)(t,x))\varphi_t \mathrm{d}x\mathrm{d}t - \int_\Omega H_\delta(T_k(u_0^n(x))\varphi(0)\mathrm{d}x \\ &\quad + \frac{1}{\delta}\int_{\{-\delta<u_n<0\}} a_n(t,x,\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\varphi \mathrm{d}x\mathrm{d}t \\ &\quad + \int_{\{|u_n|<k\}} a_n(t,x,\nabla T_k(u_n)) \cdot \nabla\varphi h_\delta(u_n)\mathrm{d}x\mathrm{d}t \\ &= \int_{\{|u_n|<k\}} h_\delta(u_n)\varphi \mathrm{d}\mu_0^n + \int_Q \varphi d\lambda_{n,k}. \end{aligned} \tag{5.14}$$

Using also (2.3) and (4.1), for every $\delta > 0$ and for every $n \in \mathbb{N}$

$$\begin{aligned} 0 &\le \frac{1}{\delta}\int_{\{-\delta<u_n<0\}} a_n(t,x,\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\mathrm{d}x\mathrm{d}t \\ &= \frac{1}{\delta}\int_{\{-\delta<u_n<0\}} a_n(t,x,\nabla T_\delta(u_n)) \cdot \nabla T_\delta(u_n)\mathrm{d}x \le \|\mu_n\|_{\mathcal{M}_b(Q)} + \|u_0^n\|_{L^1(\Omega)}, \end{aligned}$$

then there exists a sequence $\delta_h \xrightarrow{h\to\infty} 0$, and a nonnegative measure $\gamma_n \in \mathcal{M}_b(Q)$ such that

$$0 \le \lim_{h\to\infty} \frac{1}{\delta_h}\int_{\{-\delta_h<u_n<0\}} a_n(t,x,\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\varphi \mathrm{d}x\mathrm{d}t = \int_Q \varphi d\gamma_n$$

for every $\varphi \in C_c^\infty(Q)$. Moreover $0 \le \gamma_n(Q) \le |\mu_n|(Q) + |u_0^n|(\Omega)$, so that, up to subsequences, γ_n converges to a nonnegative γ in the weak-* topology of $\mathcal{M}_b(Q)$ allows to pass to the limit in (5.14) as $\delta \to 0$ to obtain

$$\int_{\{0<u_n<k\}} a_n(t,x,\nabla T_k(u_n)) \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t = \int_{\{0<u_n<k\}} \varphi \mathrm{d}\mu_0^n + \int_Q \varphi d(|\lambda_{n,k} - \gamma_n|).$$

Due to the passage to the limit as n tends to $+\infty$, we conclude

$$\int_{\{0<u<k\}} \sigma \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t = \int_{\{0<u<k\}} \varphi \mathrm{d}\mu_0 + \int_Q \varphi d(|\lambda_k - \gamma|) \tag{5.15}$$

for every $\varphi \in C_c^\infty(Q)$. Recall that, since $\sigma\chi_{\{0<u<k\}}$ belongs to $(L^{p'}(Q))^N$, the measure $\lambda_k - \delta$ belongs to $\mathcal{M}_0(Q)$ and (5.15) holds for every test function in $W \cap L^\infty(Q)$. □

6 Proof of Theorem 3.1

Thanks to the above estimates we are able to prove Theorem 3.1. For the sake of simplicity, in what follows, the convergences are all understood to be taken up to a suitable subsequence extraction, even if no explicitly claimed. As usual, u_n and u will be respectively the sequence of renormalized solutions of the associated problems such that all the results in Sects. 4 and 5 hold.

Step .1 The limit equation. Let $w \in W \cap L^\infty(Q)$ be fixed, we take $v = T_k(u_n) - w$ as test function in (3.6) to obtain

$$-\int_0^T \langle T_k(u_n)_t, T_k(u_n) - w\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \mathrm{d}t + \int_Q a_n(t,x,\nabla T_k(u_n)) \cdot \nabla(T_k(u_n) - w)\mathrm{d}x\mathrm{d}t$$
$$= \int_{\{|u_n|<k\}} (T_k(u_n) - w)\mathrm{d}\mu_0^n + \int_Q (k-w) d\lambda_{n,k}. \tag{6.1}$$

Using Lemma 1.4, we can replace $T_k(u_n)_t$ with w_t, we get

$$-\int_0^T \langle w_t, T_k(u_n) - w\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \mathrm{d}t + \int_Q a_n(t,x,\nabla w) \cdot \nabla(T_k(u_n) - w)\mathrm{d}x\mathrm{d}t$$
$$\leq \int_{\{|u_n|<k\}} (T_k(u_n) - w)\mathrm{d}\mu_0^n + \int_Q (k-w) d\lambda_{n,k}. \tag{6.2}$$

Setting $\varphi \in C_c^\infty(Q)$ and using w_n as variational solutions to

$$\begin{cases} (w_n)_t - \mathrm{div}(a_n(t,x,\nabla w_n)) = \varphi_t - \mathrm{div}(a_0(t,x,\nabla\varphi)) \text{ in } Q = (0,T)\times\Omega, \\ w_n(0,x) = u_0^n \text{ in } \Omega,\ \ w(t,x) = 0 \text{ on } (0,T)\times\partial\Omega. \end{cases} \tag{6.3}$$

The hypothesis on G-convergence of the operators implies that w_n converges weakly in $L^p(0,T; W_0^{1,p}(\Omega))$ to the unique solution w_0 of

$$\begin{cases} (w_0)_t - \mathrm{div}(a_0(t,x,\nabla w_0)) = \varphi_t - \mathrm{div}(a_0(t,x,\nabla\varphi)) \text{ in } Q = (0,T)\times\Omega, \\ w_0(0,x) = u_0 \text{ in } \Omega, \quad w_0(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{6.4}$$

that is w_n converges to φ weakly in $L^p(0,T; W_0^{1,p}(\Omega))$. Moreover, since $a_0(t,x,\nabla\varphi)$ belongs to $(L^\infty(Q))^N$, the regularity results of Remark 2.11 imply that (w_n) is equi-Hölder continuous, and hence converges uniformly to φ in Q . Now we choose $w = w_n$ in (6.2) in order to get

$$-\int_0^T \langle (w_n)_t, T_k(u_n) - w_n\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \mathrm{d}t + \int_Q a_n(t,x,\nabla w_n) \cdot \nabla(T_k(u_n) - w_n)\mathrm{d}x\mathrm{d}t$$
$$= \int_{\{|u_n|<k\}} (T_k(u_n) - w_n)\mathrm{d}\mu_0^n + \int_Q (k-w_n) d\lambda_{n,k}. \tag{6.5}$$

Using the equation solved by w_n and (6.5), we have

$$-\int_0^T \langle (w_n)_t, T_k(u_n) - w_n\rangle_{W^{-1,p'}(\Omega), w_0^{1,p}(\Omega)} \mathrm{d}t + \int_Q a_0(t,x,\nabla\varphi) \cdot \nabla(T_k(u_n) - w_n)\mathrm{d}x\mathrm{d}t$$
$$\leq \int_{\{|u_n|<k\}} (T_k(u_n) - w_n)\mathrm{d}\mu_0^n + \int_Q (k-w_n) d\lambda_{n,k}, \tag{6.6}$$

which allows to pass to the limit in (6.6) as n goes to $+\infty$ to obtain

$$-\int_0^T \langle T_k(u)_t, T_k(u)-\varphi\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\mathrm{d}x\mathrm{d}t + \int_Q a_0(t,x,\nabla\varphi)\cdot\nabla(T_k(u)-\varphi)\mathrm{d}x\mathrm{d}t$$
$$\leq \int_{\{|u|<k\}} (T_k(u)-\varphi)\mathrm{d}\mu_0 + \int_Q (k-\varphi)d\lambda_k. \tag{6.7}$$

Recall that in the last two terms we use the lower semi-continuity of the masses of weakly-* converging measures, so that we have from (6.7)

$$-\int_0^T \langle \varphi_t, T_k(u)-\varphi\rangle\mathrm{d}t + \int_Q a_0(t,x,\nabla\varphi)\cdot\nabla(T_k(u)-\varphi)\mathrm{d}x\mathrm{d}t$$
$$\leq \int_{\{|u|<k\}} (T_k(u)-\varphi)\mathrm{d}\mu_0 + \int_Q (T_k(u)-\varphi)d\lambda_k, \tag{6.8}$$

which yields, by Lemma 2.12

$$-\int_0^T \langle T_k(u)_t, T_k(u)-\varphi\rangle\mathrm{d}t + \int_Q a_0(t,x,\nabla T_k(u))\cdot\nabla(T_k(u)-\varphi)\mathrm{d}x\mathrm{d}t$$
$$= \int_{\{|u|<k\}} (T_k(u)-\varphi)\mathrm{d}\mu_0 + \int_Q (T_k(u)-\varphi)d\lambda_k \tag{6.9}$$

for every $\varphi \in C_c^\infty(Q)$. By density arguments and since $\lambda_k \in \mathcal{M}_0(Q)$, (6.9) is valid for test function in $W \cap L^\infty(Q)$. In particular, for $\varphi = T_k(u) - v$, $v \in W \cap L^\infty(Q)$, we obtain

$$-\int_0^T \langle T_k(u)_t, v\rangle\mathrm{d}t + \int_Q a_0(t,x,\nabla T_k(u))\cdot\nabla v\mathrm{d}x\mathrm{d}t = \int_{\{|u|<k\}} \mathrm{d}\mu_0 + \int_Q v d\lambda_k. \tag{6.10}$$

Then by the characterization of Theorem 2.15, u is a renormalized solution of (3.2) where λ_k converge to μ_s in the narrow topology of measures (see Step. 3). So that, by choosing $v = h(u)\varphi$, $u \in W^{1,\infty}(\mathbb{R})$ and $\varphi \in C_c^\infty(Q)$ in (6.10), an easy passage to the limit as $k \to \infty$ leads to

$$\int_Q H(T_k(u))\varphi_t\mathrm{d}x\mathrm{d}t + \int_Q a(t,x,\nabla u)\cdot\nabla(h(u)\varphi)\mathrm{d}x\mathrm{d}t$$
$$= \int_Q h(u)\varphi\mathrm{d}\mu_0 + \int_Q \varphi\mathrm{d}\mu_s + \int_\Omega H(T_k(u_0))\varphi(0)\mathrm{d}x. \tag{6.11}$$

Step 2. Convergence of the momenta. Hereafter, we study the limit of the sequence $a_n(t,x,\nabla v_n)$, this is done by having v_n as variational solution of

$$\begin{cases} (v_n)_t - \operatorname{div}(a_n(t,x,\nabla v_n)) = ((t,x)\cdot\eta)_t - \operatorname{div}(a_0(t,x,\eta)) \text{ in } Q = (0,T)\times\Omega, \\ v_n(0,x) = u_0^n \text{ in } \Omega, \quad v_n(t,x) = (t,x)\cdot\eta \text{ on } (0,T)\times\partial\Omega, \end{cases}$$

where η is a fixed element of $\mathbb{R}^N$, we take advantage of G-convergence properties to get

$$v_n \rightharpoonup (t,x)\cdot\eta \text{ weakly in } L^p(0,T;W^{1,p}),$$
$$a_n(t,x,\nabla v_n) \rightharpoonup a_0(t,x,\eta) \text{ weakly in } (L^{p'}(Q))^N.$$

In addition, by Remark 2.11, (v_n) is equi-Hölder continuous, we can assume that

$$v_n \to (t,x)\cdot\eta \text{ uniformly in } Q. \tag{6.12}$$

The monotonicity assumption (2.5) implies

$$\int_Q (a_n(t,x,\nabla T_k(u_n)) - a_n(t,x,\nabla v_n))\cdot(\nabla T_k(u_n) - \nabla v_n)\varphi \mathrm{d}x\mathrm{d}t \geq 0 \tag{6.13}$$

for every $\varphi \in C_c^\infty(Q)$ with $\varphi \geq 0$. In order to pass to the limit in (6.13), we use the limit integral

$$\begin{aligned}\lim_{n\to+\infty} &\int_Q a_n(t,x,\nabla v_n)\cdot(\nabla T_k(u_n) - \nabla v_n)\varphi \mathrm{d}x\mathrm{d}t \\ &= \int_Q a_0(t,x,\eta)\cdot(\nabla T_k(u) - \eta)\varphi \mathrm{d}x\mathrm{d}t,\end{aligned} \tag{6.14}$$

by compensated compactness (see [51,77]). To complete the passage to the limit in (6.13) we establish the same result for the term

$$\int_Q a(t,x,\nabla T_k(u_n))\cdot(\nabla T_k(u_n) - \nabla v_n)\varphi \mathrm{d}x\mathrm{d}t,$$

where the sequence div $(a_n(t,x,\nabla T_k(u_n)))$ converges in a weak sense, and we get

$$\begin{aligned}&\int_Q a(t,x,\nabla T_k(u_n))\cdot(\nabla T_k(u_n) - \nabla v_n)\varphi \mathrm{d}x\mathrm{d}t \\ &\quad= \langle T_k(u_n)_t - \mathrm{div}(a_n(t,x,\nabla T_k(u_n))), (T_k(u_n) - v_n)\varphi\rangle \\ &\quad\quad - \int_Q (T_k(u_n) - v_n)a_n(t,x,\nabla T_k(u_n))\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t\end{aligned} \tag{6.15}$$

using the formulation (3.6), we obtain

$$\begin{aligned}&\langle T_k(u_n)_t - \mathrm{div}\,(a_n(t,x,\nabla T_k(u_n)))\,, (T_k(u_n) - v_n)\varphi\rangle \\ &\quad= \int_{\{|u_n|<k\}} (T_k(u_n) - v_n)\varphi \mathrm{d}\mu_0^n + \int_Q (k - v_n)\varphi d\lambda_{k,\eta}.\end{aligned} \tag{6.16}$$

Therefore, as $n \to +\infty$ and using the dominated convergence Theorem for the first integral, the fact that v_n converges uniformly in Q and that the measures $\lambda_{k,\eta}$ converge weak-* in $\mathcal{M}_b(Q)$ in other integrals, we obtain

$$\begin{aligned}&\lim_{n\to+\infty} \langle T_k(u_n)_t - \mathrm{div}(a_n(t,x,\nabla T_k(u_n))), (T_k(u_n) - v_n)\varphi\rangle \\ &\quad= \int_{\{|u|<k\}} (T_k(u) - (t,x)\cdot\eta)\varphi \mathrm{d}\mu_0 + \int_Q (k - (t,x)\cdot\eta)\varphi d\lambda_k \\ &\quad= \langle u_t - \mathrm{div}(\sigma_k), (T_k(u) - (t,x)\cdot\eta)\varphi\rangle\end{aligned} \tag{6.17}$$

where σ_k is given in Theorem 4.1 (iv).

Now, since $T_k(u_n)-v_n$ converges strongly to $T_k(u)-(t,x)\cdot\eta$ in $L^p(Q)$, $a_n(t,x,\nabla T_k(u_n))$ converges weakly in $(L^{p'}(Q))^N$ to σ_k and using (6.15) and (6.17), we have

$$\begin{aligned}
&\lim_{n\to+\infty}\int_Q a_n(t,x,\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla v_n)\varphi \mathrm{d}x\mathrm{d}t\\
&\quad=\langle u_t-\mathrm{div}(\sigma),(T_k(u)-(t,x)\cdot\eta)\varphi\rangle-\int_Q (T_k(u)-(t,x)\cdot\eta)\sigma_k\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t \qquad (6.18)\\
&\quad=\int_Q \sigma_k\cdot(\nabla T_k(u)-\eta)\varphi \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

that, together with (6.14) and using also the limit equation of (6.13):

$$\int_Q \varphi(\sigma_k-a_0(t,x,\eta))\cdot(\nabla T_k(u)-\eta)\mathrm{d}x\mathrm{d}t\ge 0$$

for every $\varphi\in C_c^\infty(Q)$ with $\varphi\ge 0$. Hence, for every $\eta\in\mathbb{R}^N$ there exists a set $E(\eta)\subseteq Q$ with Lebesgue measure zero such that

$$(\sigma_k(t,x)-a_0(t,x,\eta))\cdot(\nabla T_k(u)(t,x)-\eta_m)\ge 0,\quad \forall (t,x)\in Q\backslash E(\eta).$$

Now, consider $E=\cup_m E(\eta_m)$ where (η_m) is a countable dense set of $\mathbb{R}^N$, we have

$$(\sigma_k(t,x)-a_0(t,x,\eta_m))\cdot(\nabla T_k(u)(t,x)-\eta_m)\ge 0,\quad \forall m,\ \forall (t,x)\in Q\backslash E$$

in view of the continuity of $a_0(t,x,\cdot)$, we obtain

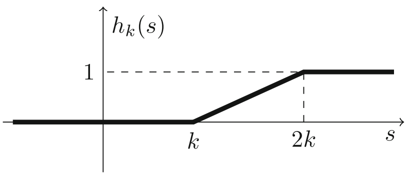
$$(\sigma_k(t,x)-a_0(t,x,\eta))\cdot(\nabla T_k(u)(t,x)-\eta)\ge 0,\quad \forall x\in Q\backslash E. \qquad (6.19)$$

Note that $a_0(t,x,\cdot)$ is continuous and monotone in $\mathbb{R}^N$ with the fact that (6.19), we have $\sigma_k(t,x)=a_0(t,x,\nabla T_k(u))$ a.e. in Q. Using Theorem 4.1 (iv), we deduce that $(a_n(t,x,\nabla T_k(u_n)))$ converges to $a_0(t,x,\nabla T_k(u))$ weakly in $(L^{p'}(Q))^N$ and this concludes the proof of Step 2.

Step 3. End of the proof. Let us prove that λ_k converges to μ_s in the narrow topology of measures. Using estimate (5.9), it is easy to prove that λ_k converge to μ_s in the weak-* topology of measures. As a consequence of Remark 2.9, the narrow convergence follows from the convergence of the measures. Then it's enough to check, since we have $\mu_s(Q)\le \liminf\limits_{k\to+\infty}\lambda_k(Q)$ because of the weak-* convergence,

$$\limsup_{k\to+\infty}\lambda_k(Q)\le\mu_s(Q). \qquad (6.20)$$

Let us define the Lipschitz function $h_k(s):\mathbb{R}\to\mathbb{R}$, $k>0$, by (see Fig. 5).



$$h_k(s)=\begin{cases}0 & \text{if } s\le k,\\ \frac{1}{k}(s-k) & \text{if } k<s\le 2k,\\ 1 & \text{if } s>2k.\end{cases} \qquad (6.21)$$

Fig. 5 The function $h_k(s)$

This step consists in taking $h_k(u)$ as test function in (6.10) corresponding to $2k$

$$-\int_0^T \langle T_k(u)_t, h_k(u)\rangle \mathrm{d}t + \frac{1}{k}\int_{\{k<u<2k\}} a_0(t,x,\nabla u)\cdot\nabla u\,\mathrm{d}x\mathrm{d}t = \int_{\{k<u<2k\}} h_k(u)\mathrm{d}\mu_0 + \lambda_{2k}(Q). \tag{6.22}$$

Since $h_k(u)$ tend to zero μ_0-a.e. in Q and by dominated convergence Theorem, this leads to

$$\limsup_{k\to+\infty} \frac{1}{k}\int_{\{k<u<2k\}} a_0(t,x,\nabla u)\cdot\nabla u\,\mathrm{d}x\mathrm{d}t = \limsup_{k\to+\infty} \lambda_{2k}(Q). \tag{6.23}$$

Due to the definition of h_k (i.e. $h_k(0)=0$), we can take $h_k(u_n)$ as test function in (3.5), we have

$$\frac{1}{k}\int_{\{k<u_n<2k\}} a_n(t,x,\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x\mathrm{d}t = \int_{\{k<u_n<2k\}} h_k(u_n)\mathrm{d}\mu_0^n + \mu_s^n(Q), \tag{6.24}$$

letting $n\to\infty$ then yields

$$\lim_{n\to+\infty}\frac{1}{k}\int_{\{k<u_n<2k\}} a_n(t,x,\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x\mathrm{d}t = \int_{\{k<u<2k\}} h_k(u)\mathrm{d}\mu_0 + \mu_s(Q). \tag{6.25}$$

Now we prove that

$$\lim_{n\to+\infty}\frac{1}{k}\int_{\{k<u_n<2k\}} a_n(t,x,\nabla u_n)\cdot\nabla u_n\varphi\,\mathrm{d}x\mathrm{d}t = \frac{1}{k}\int_{\{k<u<2k\}} a_0(t,x,\nabla u)\cdot\nabla u\varphi\,\mathrm{d}x\mathrm{d}t, \tag{6.26}$$

for every $\varphi\in C_c^\infty(Q)$. It's enough to take $h_k(u_n)\varphi$ as test function in (3.5) and $h_k(u)\varphi$ as test function in (6.11). Subtracting the two equations we obtain

$$\begin{aligned}
&\frac{1}{k}\int_{\{k<u<2k\}} a_n(t,x,\nabla u_n)\cdot\nabla u_n\varphi\mathrm{d}x\mathrm{d}t - \frac{1}{k}\int_{\{k<u<2k\}} a_0(t,x,\nabla u)\cdot\nabla u\varphi\mathrm{d}x\mathrm{d}t \\
&\quad= -\int_Q a_n(t,x,\nabla u_n)\cdot\nabla\varphi h_k(u_n)\mathrm{d}x\mathrm{d}t + \int_Q h_k(u_n)\varphi\mathrm{d}\mu_0^n \\
&\qquad+ \int_Q a_0(t,x,\nabla u)\cdot\nabla\varphi h_k(u)\mathrm{d}x\mathrm{d}t - \int_Q h_k(u)\varphi\mathrm{d}\mu_0.
\end{aligned} \tag{6.27}$$

Comparing (6.10) with (5.10) we deduce that $a_0(t,x,\nabla T_k(u)) = \sigma\chi_{\{|u|<k\}}$ for a.e. $k>0$; then, by Theorem 4.1 (iv), the sequence $(a_n(t,x,\nabla u_n))$ converges to $a_0(t,x,\nabla u)$ weakly in $(L^q(Q))^N$ for $q<p-\frac{N}{N+1}$. By definition of $h_k(s)$, $h_k(u_n)$ converges to $h_k(u)$ both a.e. and μ_0-a.e. in Q, we can pass to the limit as $n\to\infty$ to get (6.26), which implies the weak-* convergence of the sequence of non-negative measures $(\frac{1}{k}a_n(t,x,\nabla u_n))\cdot\nabla u_n\chi_{\{k<u_n<2k\}})$ to the nonnegative measure $\frac{1}{k}a_0(t,x,\nabla u)\cdot\nabla u\chi_{\{k<u<2k\}}$. By means of the semi-continuity of the masses, we get

$$\frac{1}{k}\int_{\{k<u<2k\}} a_0(t,x,\nabla u)\cdot\nabla u\mathrm{d}x\mathrm{d}t \le \liminf_{n\to+\infty} a_n(t,x,\nabla u_n)\cdot\nabla u_n\mathrm{d}x\mathrm{d}t,$$

which yields, from (6.23) and (6.25)

$$\begin{aligned}
\limsup_{k\to+\infty}\lambda_{2k}(Q) &= \limsup_{k\to+\infty}\frac{1}{k}\int_{\{k<u_n<2k\}} a_0(t,x,\nabla u)\cdot\nabla u\mathrm{d}x\mathrm{d}t \\
&\le \limsup_{k\to+\infty}\left(\liminf_{n\to+\infty}\frac{1}{k}\int_{\{k<u_n<2k\}} a_n(t,x,\nabla u_n)\cdot\nabla u_n\mathrm{d}x\mathrm{d}t\right) \\
&= \limsup_{k\to+\infty}\int_{\{k<u<2k\}} h_k(u)\mathrm{d}\mu_0 + \mu_s(Q) = \mu_s(Q),
\end{aligned} \tag{6.28}$$

this proves the assertion (i) of the narrow convergence, and the fact that u is a renormalized solution then follows straightforwardly, so that the proof of Theorem 3.1 is complete.

Compliance with ethical standards

Conflict of interest The authors would like to thank *Pr. Alexander Pankov*, the reviewers for their thoughtful comments towards improving our manuscript. On behalf of all authors, the corresponding author states that there is no conflict of interest.

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